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SCHWARTZ SPACES ASSOCIATED WITH SOME NON-DIFFERENTIAL CONVOLUTION OPERATORS ON HOMOGENEOUS GROUPS

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Introduction. Let $\mathcal{N}$ be a homogeneous group (cf. e.g. [3]) and let P be a homogeneous distribution on $\mathcal{N}$ such that

$$P : C_{\mathrm{c}}^{\infty} \ni f \mapsto f * P \in C^{\infty} \tag{0.1}$$

is the infinitesimal generator of a semigroup of symmetric probability measures μ_t on $\mathcal{N}$ which are absolutely continuous with respect to Haar measure, $d\mu_t(x) = h_t(x)\,dx$. It is well known (cf. e.g. [3]) that if P is supported at the identity, then h_t belongs to the space $\mathcal{S}(\mathcal{N})$ of rapidly decreasing functions. Let

$$Pf = \int\limits_0^{\infty} \lambda\, dE_P(\lambda) f$$

be the spectral resolution of P. In [7] A. Hulanicki has proved that if P is supported at the identity and m is a Schwartz function on $\mathbb{R}^+$, i.e.,

$$\sup_{\lambda} |(1+\lambda)^k m^{(l)}(\lambda)| \le C_{k,l} \quad \text{for all } k, l \in \mathcal{N} \cup \{0\}\,,$$

then

$$\int\limits_0^{\infty} m(\lambda)\, dE_P(\lambda) f = f * \check{m}\,,$$

where $\check{m}$ is in $\mathcal{S}(\mathcal{N})$. This is deduced, by means of a functional calculus, from the fact that for the rapidly decreasing function $m(\lambda) = e^{-\lambda}$ the function $\check{m} = h_1$ is in $\mathcal{S}(\mathcal{N})$.

The aim of this paper is to examine a similar situation where the distribution P is of the form

$$\langle P, f\rangle = \lim_{\varepsilon\to 0} \int\limits_{|x|>\varepsilon} \frac{f(0)-f(x)}{|x|^{Q+1}}\,\Omega(x)\,dx\,, \tag{0.2}$$

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where $\Omega \not\equiv 0$, $\Omega \geq 0$ is a symmetric function smooth on $\mathcal{N} - \{0\}$ and homogeneous of degree 0, $|x|$ is a homogeneous norm on $\mathcal{N}$ smooth away from the origin, and Q is the homogeneous dimension of $\mathcal{N}$. These distributions and the convolution semigroups they generate have been investigated by P. Głowacki in [4] and [5]. The kernels h_t are smooth but their decay at infinity is mild. The basic observation in our present considerations is that if

$$f * q^{(N)} = \int_0^\infty e^{-\lambda^N}\, dE_P(\lambda) f$$

then the decay of $q^{(N)}$ at infinity increases with N (cf. [1]). Thus by working with $e^{-\lambda^N}$ instead of $e^{-\lambda}$ we are able to give a characterization of the functions m such that $\check{m}$ is in $\mathcal{S}(\mathcal{N})$ (cf. Theorem 4.1).

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Preliminaries. A family of *dilations* on a nilpotent Lie algebra $\mathcal{N}$ is a one-parameter group $\{\delta_t\}_{t>0}$ of automorphisms of $\mathcal{N}$ determined by

$$\delta_t e_j = t^{d_j} e_j\,,$$

where $e_1, \ldots, e_n$ is a linear basis for $\mathcal{N}$ and $d_1, \ldots, d_n$ are positive real numbers called the *exponents of homogeneity*. The smallest d_j is assumed to be 1.

If we regard $\mathcal{N}$ as a Lie group with multiplication given by the Campbell–Hausdorff formula, then the dilations δ_t are also automorphisms of the group structure of $\mathcal{N}$ and the nilpotent Lie group $\mathcal{N}$ equipped with these dilations is said to be a *homogeneous group*.

The *homogeneous dimension* of $\mathcal{N}$ is the number Q defined by

$$d(\delta_t x) = t^Q dx\,,$$

where dx is a right-invariant Haar measure on $\mathcal{N}$.

We fix a *homogeneous norm* on $\mathcal{N}$, that is, a continuous positive symmetric function $x \mapsto |x|$ which is, moreover, smooth on $\mathcal{N} - \{0\}$ and satisfies

$$|\delta_t x| = t|x|\,, \qquad |x| = 0 \text{ if and only if } x = 0\,.$$

Let

$$X_j f(x) = \frac{d}{dt}\bigg|_{t=0} f(x \cdot t e_j)$$

be left-invariant basic vector fields. If $I = (i_1, \ldots, i_n)$ is a multi-index ($i_j \in \mathbb{N} \cup \{0\}$), we set

$$X^I f = X_1^{i_1} \ldots X_n^{i_n} f\,, \qquad |I| = i_1 d_1 + \ldots + i_n d_n\,.$$

A distribution R on $\mathcal{N}$ is said to be a *kernel of order* $r \in \mathbb{R}$ if R coincides with a C^∞ function away from the origin, and satisfies

$$\langle R, f \circ \delta_t \rangle = t^r \langle R, f \rangle \quad \text{for } f \in C_c^\infty(\mathcal{N}),\ t > 0 .$$

If R is a kernel of order r then there exists a function Ω_R homogeneous of degree 0 and smooth away from the origin such that

$$\text{(1.1)} \qquad \langle R, f \rangle = -\int_{\mathcal{N}} \frac{\Omega_R(x)}{|x|^{Q+r}} f(x)\, dx \quad \text{for } f \in C_c^\infty(\mathcal{N} - \{0\}) .$$

Note that if R_1 and R_2 are kernels of order $r_1 > 0$, $r_2 > 0$ respectively, then $R_1 * R_2$ is a kernel of order $r_1 + r_2$. Indeed, decompose R_j as $R_j = \psi R_j + (1-\psi)R_j$, $j = 1, 2$, where $\psi \in C_c^\infty(\mathcal{N})$, $\psi \equiv 1$ in a neighbourhood of the origin. Since ψR_j has compact support and $(1-\psi)R_j$ is smooth and belongs (with its all derivatives) to $L^2(\mathcal{N}) \cap L^1(\mathcal{N})$ our statement follows.

We say that a kernel R of order $r > 0$ satisfies the *Rockland condition* if for every non-trivial irreducible unitary representation π of $\mathcal{N}$ the linear operator π_R is injective on the space of C^∞ vectors of π. It is easily seen that if R satisfies the Rockland condition, then $R^N = R * R * \ldots * R$ (N times), has the same property.

If a kernel R of order $r > 0$ has compact support, i.e., $\Omega_R \equiv 0$ (cf. (1.1)), then R is supported at the origin. Hence

$$\text{(1.2)} \qquad R = \sum_{|I|=r} a_I X^I .$$

We call a differential operator R on $\mathcal{N}$ a *Rockland operator* if R is of the form (1.2) and satisfies the Rockland condition.

We say that a function φ on $\mathcal{N}$ belongs to the *Schwartz class* $\mathcal{S}(\mathcal{N})$ if for every $M \ge 0$

$$\text{(1.3)} \qquad \|\varphi\|_{(M)} = \sup_{|I| \le M,\, x \in \mathcal{N}} (1 + |x|)^M |X^I \varphi(x)|$$

is finite.

We denote by $\mathcal{S}(\mathbb{R}^+)$ the space of all functions $m \in C^\infty([0,\infty))$ such that for each $k \ge 0$

$$\sup_{\lambda \in [0,\infty),\, 0 \le l \le k} (1+\lambda)^k |m^{(l)}(\lambda)| < \infty ,$$

where $m^{(l)}(\lambda) = (d^l/d\lambda^l) m(\lambda)$.

Semigroups generated by P^N. Let P be the operator defined by (0.1) and (0.2). Since P is positive and self-adjoint we can investigate, for each natural N, the semigroup $\{T_t^{(N)}\}_{t>0}$ generated by P^N. Obviously

$$T_t^{(N)} f = \int_0^\infty e^{-t\lambda^N}\, dE_P(\lambda) f \,. \tag{2.1}$$

It has been proved by P. Głowacki [5] that the operator P satisfies the following subelliptic estimate:

$$\|X^I f\|_{L^2} \le C_I(\|P^k f\|_{L^2} + \|f\|_{L^2})\,, \tag{2.2}$$

where $|I| \le k$.

Using (2.2) and a standard calculation (cf. [1]) we deduce that there are C^∞ functions $q_t^{(N)}$ on $\mathcal{N}$ such that

$$T_t^{(N)} f = f * q_t^{(N)}\,, \tag{2.3}$$

$$X^I q_t^{(N)} \in L^2 \cap C^\infty(\mathcal{N}) \quad \text{for every multi-index } I\,. \tag{2.4}$$

In virtue of the homogeneity of P, we get

$$q_t^{(N)}(x) = t^{-Q/N} q_1^{(N)}(\delta_{t^{-1/N}} x)\,. \tag{2.5}$$

(2.6) THEOREM. *For every natural $N > 0$ and every multi-index I there is a constant $C_{I,N}$ such that*

$$|X^I q_t^{(N)}(x)| \le C_{I,N}\, t(t^{1/N} + |x|)^{-Q-N-|I|}\,. \tag{2.7}$$

Moreover, if $|\overline{x}| = 1$, then

$$\lim_{t\to\infty} t^{Q+N} q_1^{(N)}(\delta_t \overline{x}) = \Omega_{P^N}(\overline{x})\,. \tag{2.8}$$

Proof. We first assume that $N > Q$. It has been proved in [1] that if I is a multi-index, $k \in \mathbb{N}$ and $\varphi \in C_c^\infty(\mathcal{N} \times \mathbb{R} - \{(0,0)\})$, then

$$\sup_{t>0} \|\varphi X^I P^{Nk} q_t^{(N)}\|_{L^2} < \infty\,. \tag{2.9}$$

Since $P^N q_t^{(N)} = -\partial_t q_t^{(N)}$ the inequality (2.9) implies that

$$|X^I q_t^{(N)}(x)| \le Ct \quad \text{for } 1/2 < |x| < 2\,,\ t \in (0,1)\,. \tag{2.10}$$

Using (2.10) and (2.5), we get

$$|X^I q_1^{(N)}(x)| \le C_{I,N}(1 + |x|)^{-Q-N-|I|} \quad \text{for } N > Q\,, \tag{2.11}$$

which, by (2.5), gives (2.7) for $N > Q$.

In order to show that (2.7) holds for every natural $N > 0$, we use the "principle of subordination". Let l be a natural number such that $2^l N > Q$. Set $M = 2^l N$. Then

$$q_1^{(M/2)}(x) = \int_0^\infty \frac{e^{-s}}{(\pi s)^{1/2}} q_{1/(4s)}^{(M)}(x)ds = \int_0^\infty \frac{e^{-s}}{(\pi s)^{1/2}} (4s)^{Q/M} q_1^{(M)}(\delta_{(4s)^{1/M}} x)ds.$$

Consequently, for every multi-index I

$$X^I q_1^{(M/2)}(x) = \int\limits_0^\infty \frac{e^{-s}}{(\pi s)^{1/2}} (4s)^{(Q+|I|)/M} (X^I q_1^{(M)})(\delta_{(4s)^{1/M}} x)\, ds\,.$$

According to (2.11), we have

$$\begin{aligned}|X^I q_1^{(M/2)}(x)| &\le C \int\limits_0^\varepsilon (4s)^{(Q+|I|)/M} s^{-1/2}\, ds \\ &\quad + C\int\limits_\varepsilon^1 (4s)^{(Q+|I|)/M} s^{-1/2} (4s)^{-(M+Q+|I|)/M} |x|^{-M-Q-|I|}\, ds \\ &\quad + C\int\limits_1^\infty \frac{e^{-s}}{(\pi s)^{1/2}} (4s)^{(Q+|I|)/M} (4s)^{-(M+Q+|I|)/M} |x|^{-M-Q-|I|}\, ds\,.\end{aligned}$$

Setting $\varepsilon = |x|^{-M}$, we obtain

$$|X^I q_1^{(M/2)}(x)| \le C|x|^{-Q-M/2-|I|}\,.$$

By (2.4) and (2.5), we have

$$|X^I q_t^{(M/2)}(x)| \le Ct(t^{2/M} + |x|)^{-Q-M/2-|I|}\,.$$

Iterating the procedure described above, we get (2.7) for every natural $N>0$.

We next show (2.8). From (2.7) and (2.5) it follows that

$$|X^I q_t^{(N)}(x)| \le C_{N,I}\, t \quad \text{for } 1/2 < |x| < 2\,,\ t \in (0,1)\,,\ N > 0\,. \tag{2.13}$$

Note that $(1/t) q_t^{(N)}(x)$ converges weakly to $\Omega_{P^N}(x)/|x|^{Q+N}$ for $1/2 < |x| < 2$ as $t \to 0$. The estimate (2.13) and the Arzelà theorem imply that this convergence is uniform. Applying the Taylor expansion, we get

$$q_t^{(N)}(\overline{x}) = t\Omega_{P^N}(\overline{x}) + o(t)\,. \tag{2.14}$$

From (2.14) and (2.5) (cf. [2]), we obtain (2.8), which completes the proof.

Functional calculus. In this section we introduce some notation and recall some facts we shall need later.

Let $U = \{x \in \mathcal{N} : |x| < 1\}$ and $\tau(x) = \inf\{n \in \mathbb{N} \cup \{0\} : x \in U^n\}$. For every $\alpha \ge 0$ the function $w_\alpha = (1+\tau(x))^\alpha$ is submultiplicative. Moreover, there are constants c, C, a, b such that $a < 1$, $2 < b$ and

$$c\tau(x)^a \le |x| \le C\tau(x)^b \quad \text{for } |x| > 1 \tag{3.1}$$

(cf. [7], Lemma 1.1).

Denote by $\mathcal{M}_\alpha$ the $*$-algebra of Borel measures μ on $\mathcal{N}$ such that $\int_{\mathcal{N}} w_\alpha(x)\, d|\mu|(x) < \infty$.

If A is a self-adjoint operator on $L^2(\mathcal{N})$, E_A is its spectral resolution and m is a bounded function on $\mathbb{R}$, then we denote by $m(A)$ the operator $\int_{\mathbb{R}} m(\lambda)\,dE_A(\lambda)$. If $Af = f * \psi$, then $m(\psi)$ is the abbreviation for $m(A)$.

The following theorem, due to A. Hulanicki (cf. [7]), is the basic tool of the present paper.

(3.2) THEOREM. *Suppose that* $\psi = \psi^* \in \mathcal{M}_\alpha \cap L^2(\mathcal{N})$, $\alpha > \beta + \frac{1}{2}bQ + 2$, $k > 3(\beta + \frac{1}{2}bQ + 3)$, $d > 0$. *Then there is a constant* C *such that for every* $m \in C_c^\infty(-d, d)$ *with* $m(0) = 0$ *there exists a measure* $\nu \in \mathcal{M}_\beta$ *such that* $m(\psi)f = f * \nu$ *and* $\int_{\mathcal{N}} w_\beta(x)\,d|\nu|(x) \le C\|m\|_{C^k}$.

Main result. The main result of this paper is

(4.1) THEOREM. *Let* $m \in \mathcal{S}(\mathbb{R}^+)$. *Then* $m(P)f = f * \check{m}$ *with* $\check{m} \in \mathcal{S}(\mathcal{N})$ *if and only if the function* m *satisfies the following condition*:

$(*)$ *for every natural* $N > 0$ *if* $m^{(N)}(0) \neq 0$ *then* P^N *is a differential operator.*

(4.2) Remark. Note that if $\check{m} \in \mathcal{S}(\mathcal{N})$ for some $m \in L^\infty(\mathbb{R}^+)$, then $m \in C^\infty(0, \infty)$ and $\sup_{\lambda > 1} \lambda^k |m^{(s)}(\lambda)| < \infty$ for every $s, k > 0$. This is a consequence of the following two facts:

(i) $$\int_0^\infty \lambda^N m(\lambda)\,dE_P(\lambda)f = f * (\check{m} * P^N)\,,$$

(ii) $$\frac{d}{dt}\bigg|_{t=1} \int_0^\infty m(t\lambda)\,dE_P(\lambda)f = f * \frac{d}{dt}\bigg|_{t=1} (\check{m}_t)\,,$$

where $\check{m}_t = t^{-Q}\check{m}(\delta_{t^{-1}}x)$.

(4.3) PROPOSITION. *Assume that* $F \in C_c^\infty(\varepsilon, \delta)$, $0 < \varepsilon < \delta < \infty$. *Then there is a unique function* $\check{F} \in \mathcal{S}(\mathcal{N})$ *such that*

$$\int_0^\infty F(\lambda)\,dE_P(\lambda)f = f * \check{F}\,.$$

Moreover, *for each natural* M *there are constants* C, $k = k(M)$ *such that*

$$\|\check{F}\|_{(M)} \le C\|F\|_{C^{k(M)}}\,. \tag{4.4}$$

Proof. By the definition of $\mathcal{S}(\mathcal{N})$ (cf. (1.3)), the proof will be complete if we show (4.4). Let $N > 0$ be a natural number. Then

$$\int_0^\infty F(\lambda)\,dE_P(\lambda)f = \int_0^\infty F_N(\lambda)\,dE_{P^N}(\lambda)f\,, \tag{4.5}$$

where $F_N(\lambda) = F(\lambda^{1/N})$. Put $n(\lambda) = F_N(-\log\lambda)/\lambda$. Clearly, $n \in C_c^\infty(e^{-\delta^N}, e^{-\varepsilon^N})$. Moreover, by (4.5) and (2.3),

$$F(P)f = F_N(P^N)f = T_1^{(N)} n(q_1^{(N)})f = \{n(q_1^{(N)})f\} * q_1^{(N)}\,. \tag{4.6}$$

Applying (4.6), (3.2), (2.7), (3.1) with sufficiently large N, we get (4.4).

P r o o f o f T h e o r e m (4.1). Suppose that $F \in \mathcal{S}(\mathbb{R}^+)$ and $F(\lambda) = 0(\lambda^{l+1})$ as $\lambda \to 0$, for some natural $l > 0$. Let $\zeta(\lambda)$ be a C^∞ function with compact support contained in $(1/2, 2)$ and with

$$\sum_{j=-\infty}^{\infty} \zeta(2^j\lambda) = 1 \quad \text{for } \lambda > 0\,.$$

Let $F_j(\lambda) = \zeta(2^j\lambda)F(\lambda)$, $\widetilde{F}_j(\lambda) = F(2^{-j}\lambda)\zeta(\lambda) = F_j(2^{-j}\lambda)$. Then for each natural $k > 0$ there is a constant C such that

$$\|\widetilde{F}_j\|_{C^k} \le C2^{-(l+1)j} \quad \text{for } j \ge 0\,. \tag{4.7}$$

Since $F \in \mathcal{S}(\mathbb{R}^+)$, we conclude that for every natural k and r there is a constant C such that

$$\|\widetilde{F}_j\|_{C^k} \le C2^{rj} \quad \text{for } j < 0\,. \tag{4.8}$$

Now we turn to proving that for every function m satisfying $(*)$ there exists a function $\check{m}$ in $\mathcal{S}(\mathcal{N})$ such that $m(P)f = f * \check{m}$. It is sufficient to find a function $\check{m}$ on $\mathcal{N}$ such that

(4.9) $m(P)f = f * \check{m}$ and $\|\check{m}\|_{(M)}$ is finite for every $M > 0$.

It has been proved by P. Głowacki [4] that P satisfies the Rockland condition. Hence, if $r \in W_m = \{l \in \mathbb{N} : l > 0, m^{(l)}(0) \neq 0\}$, then by our assumption P^r is a positive Rockland operator. Then a theorem of G. Folland and E. M. Stein (cf. [3]), asserts that $q_t^{(r)}$ belongs to the Schwartz space $\mathcal{S}(\mathcal{N})$. Let N be the smallest element in W_m. Put

$$F(\lambda) = m(\lambda) + \gamma e^{-\lambda^N} + \eta e^{-2\lambda^N}\,, \tag{4.10}$$

where

$$\gamma = \frac{-2m(0)N! - m^{(N)}(0)}{N!}\,, \qquad \eta = \frac{m^{(N)}(0) + m(0)N!}{N!}\,.$$

One can check that

$$F(0) = F'(0) = F^{(2)}(0) = \ldots = F^{(N)}(0) = 0\,, \qquad F \in \mathcal{S}(\mathbb{R}^+)\,.$$

The equality (4.10) and the above-mentioned theorem of Folland and Stein imply that $\check{m} \in \mathcal{S}(\mathcal{N})$ if and only if $\check{F} \in \mathcal{S}(\mathcal{N})$. Note that if $s \in W_F = \{l \in \mathbb{N} : l > 0,\ F^{(l)}(0) \neq 0\}$ then P^s is a Rockland operator. Iterating this procedure, we find that for every $l > 0$ there is a function $F \in \mathcal{S}(\mathbb{R}^+)$ such that

$$F(0) = F'(0) = F^{(2)}(0) = \ldots = F^{(l)}(0) = 0\,, \tag{4.11}$$
$$\|\check{m}\|_{(M)} \text{ is finite if and only if } \|\check{F}\|_{(M)} \text{ is finite}\,.$$

By the homogeneity of P and the definition of F_j, we have

$$\int_0^\infty F(\lambda)\,dE_P(\lambda)f = \sum_{j=-\infty}^{\infty} f * \check{F}_j = \sum_{j=-\infty}^{\infty} f * (\widetilde{F}_j)^\vee_{2^j}\,,$$

where $(\widetilde{F}_j)^\vee_{2^j}(x) = 2^{-jQ}(\widetilde{F}_j)^\vee(\delta_{2^{-j}}x)$. By Proposition (4.3),

$$\sum_{j=-\infty}^{\infty} \|(\widetilde{F}_j)^\vee_{2^j}\|_{(M)} \le \sum_{j\ge 0} \|\widetilde{F}_j\|_{C^{k(M)}}(2^j)^{-Q+M} + \sum_{j<0} \|\widetilde{F}_j\|_{C^{k(M)}}(2^j)^{-Q-M}\,.$$

Using (4.7), (4.8) and (4.11) we get (4.9).

It remains to show that the condition $(*)$ is necessary. Let N be the smallest non-zero natural number such that $m^{(N)}(0) \ne 0$ and P^N is not a differential operator (i.e., the function Ω_{P^N} is non-zero), and let $r = \inf W_m$. We consider two cases: $r = N$ and $r < N$.

For $r = N$ let $F(\lambda)$ be defined by (4.10). We first show that

$$|\check{F}(x)| \le C(1+|x|)^{-Q-N-1/2}\,. \tag{4.12}$$

Indeed,

$$\begin{aligned}
\sup_{x\in\mathcal{N}} |\check{F}(x)|(1+|x|)^{Q+N+1/2} &\le \sup_{x\in\mathcal{N}} \sum_{j=-\infty}^{\infty} |(\widetilde{F}_j)^\vee_{2^j}(x)|(1+|x|)^{Q+N+1/2} \\
&\le C \sup_{x\in\mathcal{N}} \sum_{j=-\infty}^{\infty} |(\widetilde{F}_j)^\vee_{2^j}(x)| + C \sup_{x\in\mathcal{N}} \sum_{j=-\infty}^{\infty} |(\widetilde{F}_j)^\vee_{2^j}(x)|\,|x|^{Q+N+1/2} \\
&\le C \sup_{x\in\mathcal{N}} \sum_{j=-\infty}^{\infty} |(\widetilde{F}_j)^\vee(\delta_{2^{-j}}x)|2^{-jQ} \\
&\quad + C \sup_{x\in\mathcal{N}} \sum_{j=-\infty}^{\infty} |(\widetilde{F}_j)^\vee(\delta_{2^{-j}}x)| \left(\frac{|x|}{2^j}\right)^{Q+N+1/2} 2^{j(Q+N+1/2)}2^{-jQ}\,.
\end{aligned}$$

In virtue of Proposition (4.3) there are constants C and k such that

$$\begin{aligned}
&\sup_{x\in\mathcal{N}} |\check{F}(x)|(1+|x|)^{Q+N+1/2} \\
&\qquad \le C \sum_{j=-\infty}^{\infty} 2^{-jQ}\|\widetilde{F}_j\|_{C^k} + C \sum_{j=-\infty}^{\infty} 2^{j(N+1/2)}\|\widetilde{F}_j\|_{C^k}\,.
\end{aligned}$$

Applying (4.7), (4.8) (with $l = N$), we get (4.12).

On the other hand, there exists $\overline{x}$ such that $|\overline{x}| = 1$ and $\Omega_{P^N}(\overline{x}) \neq 0$. By (4.12) and (4.10), we have

$$\lim_{t\to\infty} t^{Q+N}\check{m}(\delta_t\overline{x}) = -\lim_{t\to\infty} t^{Q+N}[\gamma q_1^{(N)}(\delta_t\overline{x}) + \eta q_2^{(N)}(\delta_t\overline{x})]\,.$$

Using (2.5) and (2.8), we obtain

$$\lim_{t\to\infty} t^{Q+N}\check{m}(\delta_t\overline{x}) = -\gamma\Omega_{P^N}(\overline{x}) - 2\eta\Omega_{P^N}(\overline{x}) = -\Omega_{P^N}(\overline{x})\frac{m^{(N)}(0)}{N!} \neq 0\,.$$

Hence, the function $\check{m}$ does not belong to $\mathcal{S}(\mathcal{N})$.

In the case when $r < N$ set $m_1(\lambda) = m(\lambda) + b_1e^{-\lambda^r} + b_2e^{-2\lambda^r}$, where $b_1 = -2m(0) - m^{(r)}(0)/r!$, $b_2 = m(0) + m^{(r)}(0)/r!$. Then $m_1 \in \mathcal{S}(\mathbb{R}^+)$ and $m_1(0) = m_1'(0) = \ldots = m_1^{(r)}(0) = 0$. Since P^r is a Rockland operator, by the above-mentioned theorem of Folland and Stein (cf. [3, p. 135]), we get that the kernels associated with the multipliers $e^{-\lambda^r}$ and $e^{-2\lambda^r}$ belong to $\mathcal{S}(\mathcal{N})$. Note that $N \in W_{m_1}$, $r \notin W_{m_1}$, and $N = \inf\{l \in W_{m_1} : P^l$ is not a differential operator$\}$. Iterating the above procedure reduces our considerations to the case $r = N$.

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