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Schwarz–Christoffel mappings to unbounded multiply connected polygonal regions

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Abstract

A formula for the generalized Schwarz–Christoffel conformal mapping from a bounded multiply connected circular domain to an unbounded multiply connected polygonal domain is derived. The formula for the derivative of the mapping function is shown to contain a product of powers of Schottky–Klein prime functions associated with the circular preimage domain. Two analytical checks of the new formula are given. First, it is compared with a known formula in the doubly connected case. Second, a new slit mapping formula from a circular domain to the triply connected region exterior to three slits on the real axis is derived using separate arguments. The derivative of this independently-derived slit mapping formula is shown to correspond to a degenerate case of the new Schwarz–Christoffel mapping. The example of the mapping to the triply connected region exterior to three rectangles centred on the real axis is considered in detail.

1. *Introduction*

A Schwarz–Christoffel mapping is a conformal mapping from a simple canonical domain to a polygonal domain having boundaries which are all straight-line segments. Such mappings find wide relevance in applications. The theory and applications of such mappings has been the central topic of a recent monograph by Driscoll and Trefethen [**10**]. There, the role of such formulae in many physical applications, from fluid dynamics and electrostatics to more abstract theoretical applications in approximation theory and the design of digital filters, is expertly surveyed.

The Schwarz–Christoffel mapping (hereafter abbreviated to "S–C mapping") to simply connected polygonal domains dates back to the 1860's [**10**] while a generalized formula for mapping to bounded doubly connected polygonal domains was first presented in the 1920's [**3**] (it appears to have been rediscovered later by Komatu [**16**]). The question of finding more general formulae to polygonal domains of arbitrary finite connectivity is a very natural one, but work on this problem seems to have been largely impeded by implementation problems associated even with the known simply and doubly connected mapping formulae. Before the 1980's, while knowledge of the existence of these formulae was widespread, equally prevalent was the generally held view that application of the formulae in all but the simplest cases was impractical owing to the numerical intractability of solving the so-called "parameter problem" (in combination with various other difficulties such as "crowding" [**10**]). These practical impediments have now been overcome and readily transferable software operating on platforms such as MATLAB are available. Now, S-C mappings of very complicated domains can be constructed at the click of a mouse. Driscoll [**11**] has created a MATLAB package called SC Toolbox based on an earlier Fortran program developed by Trefethen.

Given this history, a long-standing theoretical problem has once again moved to the forefront: that of finding general formulae for S-C mappings to higher connected polygonal domains. Only when they are found can one proceed to transfer and adapt all the mathematical technology devised to negotiate the numerical problems of S-C mapping in the simply and doubly connected cases to the case of arbitrary finite connectivity.

In a recent paper, DeLillo, Elcrat and Pfaltzgraff [**9**] have addressed the multiply connected problem and have derived a formula for a S-C mapping from a finitely-connected unbounded circular preimage region to an unbounded conformally equivalent polygonal region. The derivation relies on an extension of an idea originally presented in [**8**] involving consideration of an infinite sequence of reflections in circles needed to satisfy the relevant argument conditions (on the derivative of the mapping function) on the segments of each preimage circle mapping to the sides of the polygonal region.

This paper complements the work of DeLillo, Elcrat and Pfaltzgraff [**9**] and offers a different perspective. In particular, although it is not mentioned in their paper, the infinite sequence of reflections in circles considered in [**9**] is naturally associated with the theory of classical Schottky groups of Möbius mappings [**5**]. In turn, associated with any such Schottky group is a fundamental function known as the *Schottky–Klein prime function* [**4**]. The key result of this paper is to show that an S-C formula to unbounded polygonal domains (from the canonical class of bounded multiply connected circular preimage regions) can be written, in a natural way, as a product of powers of this prime function. This provides an important link between generalized Schwarz–Christoffel formulae and classical function theory.

The result here is a natural extension, to the case of unbounded domains, of a similar formula for S-C mappings to bounded polygonal domains, recently derived by the present author [**6**]. In Crowdy [**6**], the S-C formula mapping a bounded circular region to a bounded polygonal region is

$$z(\zeta) = A + B\int^{\zeta} S_B(\zeta')\prod_{k=1}^{n_0}\left[\omega\left(\zeta', a_k^{(0)}\right)\right]^{\beta_k^{(0)}}\prod_{j=1}^{M}\prod_{k=1}^{n_j}\left[\omega\left(\zeta', a_k^{(j)}\right)\right]^{\beta_k^{(j)}} d\zeta' \tag{1·1}$$

with

$$S_B(\zeta) \equiv \left(\frac{\omega_\zeta(\zeta,\alpha)\omega(\zeta,\bar{\alpha}^{-1}) - \omega_\zeta(\zeta,\bar{\alpha}^{-1})\omega(\zeta,\alpha)}{\prod_{j=1}^{M}\omega\left(\zeta,\gamma_1^{(j)}\right)\omega\left(\zeta,\gamma_2^{(j)}\right)}\right) \tag{1·2}$$

and where $\omega(\zeta,\gamma)$ is the relevant Schottky–Klein prime function. $M+1$ is the connectivity of the domain. The mapping formula to unbounded domains derived in this paper differs only in that $S_B(\zeta)$ (where the subscript B is chosen to reflect the fact that the image is "bounded") given in (1·2) must be replaced by a function which we call $S_\infty(\zeta)$, the subscript reflecting the fact that the image domain now includes the point at infinity. The final formula derived here is

$$z(\zeta) = A + B\int^{\zeta} S_\infty(\zeta')\prod_{k=1}^{n_0}\left[\omega\left(\zeta', a_k^{(0)}\right)\right]^{\beta_k^{(0)}}\prod_{j=1}^{M}\prod_{k=1}^{n_j}\left[\omega\left(\zeta', a_k^{(j)}\right)\right]^{\beta_k^{(j)}} d\zeta' \tag{1·3}$$

with

$$S_\infty(\zeta) \equiv \frac{S_B(\zeta)}{\omega(\zeta,\zeta_\infty)^2\omega(\zeta,\bar{\zeta}_\infty^{-1})^2} \tag{1·4}$$

and where ζ_∞ is the preimage point in D_ζ mapping to the point at infinity. All other parameters appearing in (1·3) and (1·4) are explained in the main body of the paper.

We are able to provide two analytical checks on this new formula. First, in Section 10 we examine the special case of mappings to unbounded doubly connected polygonal regions and compare with a formula of Akhiezer [**2**] and a related formula resulting from the construction of DeLillo, Elcrat and Pfaltzgraff [**9**]. Second, in Section 11, we explicitly construct an example of an unbounded triply connected domain in order to demonstrate the efficacy of the new formula in practice. The case of the unbounded region exterior to three rectangles centred on the real axis is considered. A special limit of this example consists of three real intervals along the real axis. It turns out that, using independent arguments, an alternative functional form for the mapping from the same circular preimage domain to the unbounded domain exterior to this collection of real intervals can be found. The derivative of this alternative function is then compared to the derivative of (1·3). The two functions are found to be identically equal.

2. *Mathematical formulation*

Let the target region D_z in a complex z-plane be an unbounded $(M+1)$-connected polygonal region. $M = 0$ is the simply connected case. Let D_z be the unbounded region exterior to $M+1$ polygonal regions whose boundaries are denoted $\{P_j \mid j = 0, 1, \ldots, M\}$. Let polygon P_j have n_j edges where $n_j \geqslant 2$ are integers. Let the set of interior angles at each vertex of polygon P_j be

$$\pi\left(\beta_k^{(j)}+1\right), \quad k = 0, \ldots, n_j, \tag{2·1}$$

where the usual S-C conditions [**10**], i.e.,

$$\sum_{k=1}^{n_j} \beta_k^{(j)} = 2, \quad j = 0, 1, \ldots, M \tag{2·2}$$

must hold in order that each boundary is a closed polygon. The parameters $\left\{\beta_k^{(j)} \mid j = 0, 1, \ldots, M\right\}$ are called the *turning angles* [**10**]. Let the straight-line edges of polygon P_j be given by the linear equations

$$\bar{z} = \epsilon_k^{(j)} z + \kappa_k^{(j)} \tag{2·3}$$

where $\epsilon_k^{(j)}$ and $\kappa_k^{(j)}$ are complex constants with $\left|\epsilon_k^{(j)}\right| = 1$. For a given target polygon, $\left\{\epsilon_k^{(j)}, \kappa_k^{(j)}\right\}$ can be determined.

We seek a conformal mapping to D_z from a conformally equivalent bounded multiply connected circular domain D_ζ. (In contrast, DeLillo, Elcrat and Pfaltzgraff [**9**] choose an unbounded circular region as their preimage domain.) Let D_ζ be the unit ζ-circle with M smaller circular discs excised and let the boundaries of these circular discs be $\{C_j \mid j = 1, \ldots, M\}$. The unit circle $|\zeta| = 1$ will be called C_0. The complex numbers $\{\delta_j \mid j = 1, \ldots, M\}$ will denote the centres of the enclosed circular discs, the real numbers $\{q_j \mid j = 1, \ldots, M\}$ will denote their radii. Figure 1 shows a schematic of a typical circular domain. Figure 2 shows a schematic, in the triply connected case, of the mapping from a circular domain to the

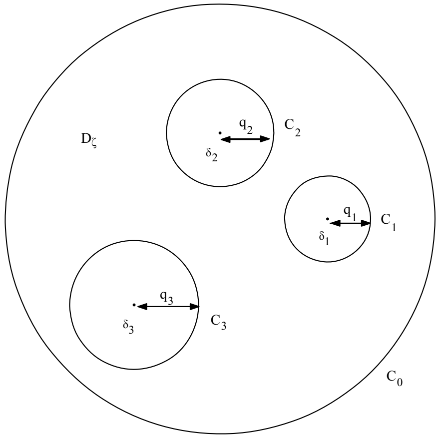

Fig. 1. Schematic of typical multiply connected circular region D_ζ. The case shown, with three enclosed circles, is quadruply connected. C_0 denotes the unit circle. There are M interior circles (the case $M = 3$ is shown here), each labelled $\{C_j \,|\, j = 1, \ldots, M\}$. The centre of circle C_j is δ_j and its radius is q_j.

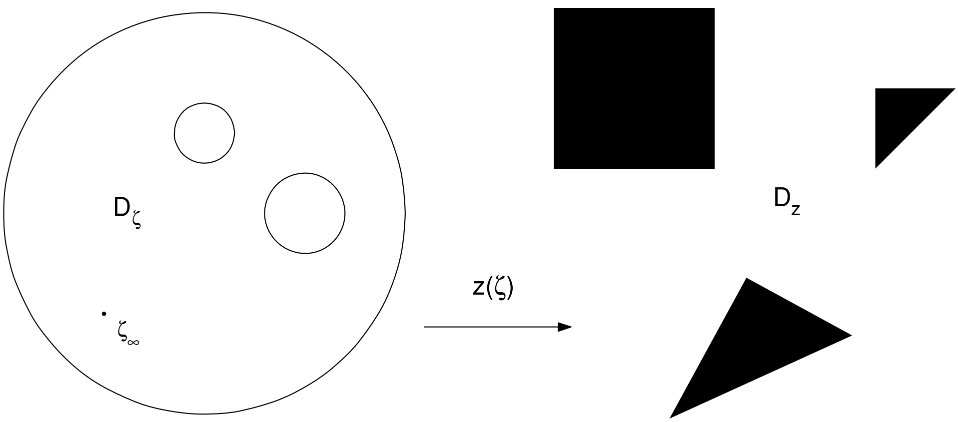

Fig. 2. Schematic of the preimage and target complex planes for the case of a triply connected domain. The bounded circular domain D_ζ in the ζ-plane is to be mapped by the function $z(\zeta)$ to the unbounded domain D_z in the z-plane exterior to the shaded polygonal regions. The point ζ_∞ in D_ζ maps to the point at infinity in the z-plane.

unbounded region exterior to three polygons in a z-plane. It is supposed that the circle C_j maps to P_j.

3. *Schottky groups*

To proceed with the construction, first define M Möbius maps $\{\phi_j \,|\, j = 1, \ldots, M\}$ corresponding, respectively, to the conjugation maps on the circles $\{C_j \,|\, j = 1, \ldots, M\}$. That is, if C_j is defined by the equation

$$|\zeta - \delta_j|^2 = (\zeta - \delta_j)(\bar{\zeta} - \bar{\delta}_j) = q_j^2, \tag{3·1}$$

then

$$\bar{\zeta} = \bar{\delta}_j + \frac{q_j^2}{\zeta - \delta_j}, \tag{3·2}$$

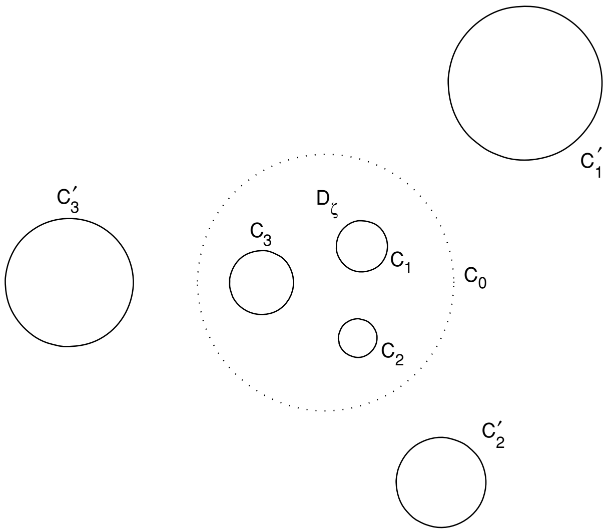

Fig. 3. Schematic of the set of Schottky circles associated with a typical quadruply connected domain D_ζ. The region exterior to all six circles $\{C_j, C'_j | j = 1, 2, 3\}$ is the fundamental region. The part of the fundamental region inside the unit circle C_0 is D_ζ.

and so

$$\phi_j(\zeta) \equiv \bar{\delta}_j + \frac{q_j^2}{\zeta - \delta_j}. \tag{3·3}$$

If ζ is a point on C_j then its complex conjugate is given by

$$\bar{\zeta} = \phi_j(\zeta). \tag{3·4}$$

Next, introduce the Möbius maps

$$\theta_j(\zeta) \equiv \overline{\phi}_j(\zeta^{-1}) = \delta_j + \frac{q_j^2 \zeta}{1 - \bar{\delta}_j \zeta} \tag{3·5}$$

where the conjugate function $\bar{\phi}_j$ is defined by

$$\bar{\phi}_j(\zeta) = \overline{\phi_j(\bar{\zeta})}. \tag{3·6}$$

Let C'_j be the circle obtained by reflection of the circle C_j in the unit circle $|\zeta| = 1$ (i.e. the circle obtained by the transformation $\zeta \mapsto \bar{\zeta}^{-1}$). It is easily verified that the image of the circle C'_j under the transformation θ_j is the circle C_j. Since the M circles $\{C_j | j = 1, \ldots, M\}$ are non-overlapping, so are the M circles $\{C'_j | j = 1, \ldots, M\}$. The (classical) *Schottky group* Θ is defined to be the infinite free group of Möbius mappings generated by compositions of the $2M$ basic Möbius maps $\{\theta_j | j = 1, \ldots, M\}$ and their inverses $\{\theta_j^{-1} | j = 1, \ldots, M\}$. Beardon [**5**] gives a general discussion of such groups. An accessible discussion of Schottky groups and their mathematical properties can also be found in the book by Mumford, Series and Wright [**17**].

Consider the (generally unbounded) region of the plane exterior to the $2M$ circles $\{C_j | j = 1, \ldots, M\}$ and $\{C'_j | j = 1, \ldots, M\}$. A schematic is shown in Figure 3. This region is known as the *fundamental region* associated with the Schottky group generated by the Möbius maps $\{\theta_j | j = 1, \ldots, M\}$ and their inverses. This fundamental region can be understood as having two “halves” – the half that is inside the unit circle but exterior to the circles C_j is the region D_ζ, the region outside the unit circle and exterior to the circles C'_j is the "other" half. The region is called fundamental because the whole complex plane is tesselated by an infinite sequence of "copies" of this region obtainable by conformally mapping points in the fundamental region by elements of the Schottky group. Any point in the plane which can be reached by the action of a finite composition of the basic generating maps on a point in the fundamental region is called an *ordinary point* of the group. Any point not obtainable in this way is a *singular point* of the group. The set of $2M$ circles $\{C_j, C'_j | j = 1, \ldots, M\}$ are known as *Schottky circles*.

There are two important properties of the Möbius maps introduced above. The first is that

$$\theta_j^{-1}(\zeta) = \frac{1}{\phi_j(\zeta)}, \quad \forall \zeta. \tag{3·7}$$

This can be verified using the definitions (3·3) and (3·5) (or, alternatively, by considering the geometrical effect of each map). The second property, which follows from the first, is that

$$\theta_j^{-1}(\zeta^{-1}) = \frac{1}{\phi_j(\zeta^{-1})} = \frac{1}{\overline{\bar{\phi}_j(\bar{\zeta}^{-1})}} = \frac{1}{\overline{\theta_j(\bar{\zeta})}} = \frac{1}{\bar{\theta}_j(\zeta)}, \quad \forall \zeta. \tag{3·8}$$

The full Schottky group will be denoted Θ. The notation Θ'' will be taken to mean all mappings in the group excluding the identity mapping and all inverse mappings meaning that if some mapping θ_k is included in Θ'' then its inverse mapping θ_k^{-1} must be excluded.

4. *The Schottky–Klein prime function*

Following the discussion of Baker [**4**, chapter 12], the *Schottky–Klein prime function* is defined as

$$\omega(\zeta, \gamma) = (\zeta - \gamma)\omega'(\zeta, \gamma) \tag{4·1}$$

where the function $\omega'(\zeta, \gamma)$ is given by

$$\omega'(\zeta, \gamma) = \prod_{\theta_i \in \Theta''} \frac{(\theta_i(\zeta) - \gamma)(\theta_i(\gamma) - \zeta)}{(\theta_i(\zeta) - \zeta)(\theta_i(\gamma) - \gamma)} \tag{4·2}$$

and where the product is over all mappings θ_i in the set Θ''. It is emphasized that the prime notation is not used here to denote differentiation. The function $\omega(\zeta, \gamma)$ is single-valued on the whole ζ-plane, has a zero at γ and all points equivalent to γ under the mappings of the group Θ. Again following Baker [**4**], we proceed under the assumption that the infinite product defining the prime function is convergent. Whether this is true will depend, in general, on the distribution of Schottky circles in the ζ-plane. A basic rule of thumb is that the product is convergent provided the Schottky circles are sufficiently small and well separated in the ζ-plane (similar convergence criteria arise in the construction of [**9**]). Baker [**4**] describes some explicit conditions for convergence. Further comments on the computation of the prime function are given in Section 12.

The Schottky–Klein prime function has some important transformation properties which will be needed in the construction of the S-C mapping. One such property is that it is antisymmetric in its arguments, i.e.,

$$\omega(\zeta, \gamma) = -\omega(\gamma, \zeta). \tag{4·3}$$

This is clear from inspection of (4·1) and (4·2). A second important property is given by

$$\frac{\omega(\theta_j(\zeta), \gamma_1)}{\omega(\theta_j(\zeta), \gamma_2)} = \beta_j(\gamma_1, \gamma_2)\frac{\omega(\zeta, \gamma_1)}{\omega(\zeta, \gamma_2)} \tag{4·4}$$

where θ_j is any one of the basic maps of the Schottky group. A detailed derivation of this result is given in [**4**, chapter 12]. A formula for $\beta_j(\gamma_1, \gamma_2)$ is given explicitly in Crowdy [**6**] but will not be needed here. A third property of $\omega(\zeta, \gamma)$ which will also be useful is

$$\bar{\omega}(\zeta^{-1}, \gamma^{-1}) = -\frac{1}{\zeta\gamma}\omega(\zeta, \gamma) \tag{4·5}$$

where the conjugate function $\overline{\omega}(\zeta, \gamma)$ is defined by

$$\overline{\omega}(\zeta, \gamma) = \overline{\omega(\bar{\zeta}, \bar{\gamma})}. \tag{4·6}$$

The derivation of (4·5) follows directly from the definition of the prime function. Details of the derivation are given in Crowdy [**6**] and will not be repeated here.

5. *Three special functions*

In this section, a series of propositions will outline the properties of three special functions, $\{F_j(\zeta; \zeta_1, \zeta_2) | j = 1, 2, 3\}$, that will be needed in the construction of the S-C mapping formula.

PROPOSITION 1. *If ζ_1 and ζ_2 are any two distinct points on C_0, then the function*

$$F_1(\zeta; \zeta_1, \zeta_2) \equiv \frac{\omega(\zeta, \zeta_1)}{\omega(\zeta, \zeta_2)} \tag{5·1}$$

has constant argument on each of the circles $\{C_j | j = 0, 1, \ldots, M\}$.

PROPOSITION 2. *If ζ_1 and ζ_2 are two distinct points on a particular choice of enclosed circle C_j (for some $j = 1, \ldots, M$) then the function*

$$F_2(\zeta; \zeta_1, \zeta_2) \equiv \frac{\omega(\zeta, \zeta_1)}{\omega(\zeta, \zeta_2)} \tag{5·2}$$

has constant argument on each of the circles $\{C_k | k = 0, 1, \ldots, M\}$.

PROPOSITION 3. *Let ζ_1 and ζ_2 be any two distinct ordinary points of a given Schottky group. Then the function*

$$F_3(\zeta; \zeta_1, \zeta_2) \equiv \frac{\omega(\zeta, \zeta_1)\omega\big(\zeta, \bar{\zeta}_1^{-1}\big)}{\omega(\zeta, \zeta_2)\omega\big(\zeta, \bar{\zeta}_2^{-1}\big)} \tag{5·3}$$

has constant argument on each of the circles $\{C_k | k = 0, 1, \ldots, M\}$.

The proofs of the three propositions are a consequence of relation (3·5) along with repeated application of the two properties (4·4) and (4·5). The reader can find these proofs in Crowdy [**6**]. The functions $\{F_1(\zeta; \zeta_1, \zeta_2) | j = 1, 2, 3\}$ will be used as the basic "building block" functions with which to construct the mapping function in Section 8.

6. *Conformal mapping to circular slit domain*

It is convenient to introduce an intermediate η-plane. Consider a conformal mapping $\eta(\zeta)$ taking the multiply connected circular domain D_ζ to a conformally equivalent circular slit domain D_η consisting of the unit disc in an η-plane with M concentric circular-arc slits inside. Let the image of C_0 under this mapping be the unit circle in the η-plane which will be called L_0. The M circles $\{C_j | j = 1, \ldots, M\}$ will be taken to have concentric circular slit images, centred on $\eta = 0$, and labelled $\{L_j | j = 1, \ldots, M\}$. Let the arc L_j (with $j = 1, \ldots, M$) be specified by the conditions

$$|\eta| = r_j, \quad \arg[\eta] \in \left[\phi_1^{(j)}, \phi_2^{(j)}\right]. \tag{6·1}$$

It is clear that there will be two preimage points on C_j corresponding to the two end-points of the circular slit L_j. These two preimage points, labelled $\gamma_1^{(j)}$ and $\gamma_2^{(j)}$, satisfy the conditions

$$\begin{aligned} \eta\left(\gamma_1^{(j)}\right) &= r_j e^{i\phi_1^{(j)}}, \quad \eta_\zeta\left(\gamma_1^{(j)}\right) = 0, \\ \eta\left(\gamma_2^{(j)}\right) &= r_j e^{i\phi_2^{(j)}}, \quad \eta_\zeta\left(\gamma_2^{(j)}\right) = 0. \end{aligned} \tag{6·2}$$

These $2M$ zeros of η_ζ are all simple since the points $\gamma_1^{(j)}$ and $\gamma_2^{(j)}$ map to the ends of a slit and the arguments of $\eta(\zeta) - \eta\left(\gamma_1^{(j)}\right)$ and $\eta(\zeta) - \eta\left(\gamma_2^{(j)}\right)$ change by 2π as ζ passes through these points. This fact will be important later. The mapping of an M-connected domain to such a circular slit domain was first proposed by Koebe [**15**].

The key idea of the construction of the S-C mapping is to consider conditions on the derivative of the mapping function in the intermediate η-plane. These conditions turn out to be easier to handle than those in the original ζ plane because they take the same functional form on all boundaries (which is not the case in the original ζ-plane). Once these conditions on $z(\eta)$ are satisfied in the η-plane, the functional form of the required mapping function $z(\zeta) = z(\eta(\zeta))$ can be deduced.

The conformal mapping from the circular domain D_ζ in the ζ-plane to the circular slit domain D_η in the η-plane can also be constructed using the Schottky–Klein prime function. Suppose that the point α in the domain D_ζ is to map to $\eta = 0$. α can be chosen arbitrarily. The conformal map $\eta(\zeta)$ taking the circular domain D_ζ to the circular-slit domain D_η is given by

$$\eta(\zeta) = \frac{\omega(\zeta, \alpha)}{|\alpha|\omega(\zeta, \bar{\alpha}^{-1})}. \tag{6·3}$$

That this formula effects the required mapping from D_ζ to D_η can be checked using the properties of the Schottky–Klein prime function given earlier. More details can be found in Crowdy and Marshall [**7**].

7. *Properties of the S-C mapping function*

Let $z(\eta)$ map the circular-slit domain D_η to the bounded polygonal region D_z. In this section, the properties required of this function will be outlined.

First, by definition, $z(\eta)$ must be an analytic function everywhere inside D_η. Furthermore, it must have branch point singularities (i.e. points of non-conformality) on the circular arcs $\{L_j | j = 0, 1, \ldots, M\}$. Define the *prevertices* [**10**] in the ζ-plane to be the points

$$\left\{a_k^{(0)} \middle| k = 1, \ldots, n_0\right\} \tag{7·1}$$

on C_0, and the points

$$\left\{a_k^{(j)} \middle| k = 1, \ldots, n_j\right\} \tag{7·2}$$

on each of the enclosed circles $\{C_j | j = 1, \ldots, M\}$.

Now let the image of the point $a_k^{(j)}$ under the mapping $\eta(\zeta)$ be $\tilde{a}_k^{(j)}$ so that

$$\tilde{a}_k^{(j)} = \eta\big(a_k^{(j)}\big). \tag{7·3}$$

Then, locally, we must have

$$z_\eta\big(\tilde{a}_k^{(j)}\big) = \big(\eta - \tilde{a}_k^{(j)}\big)^{\beta_k^{(j)}} f(\eta) \tag{7·4}$$

where $f(\eta)$ is some function that is analytic at $\eta = \tilde{a}_k^{(j)}$. It follows that the composed function $z(\zeta) = z(\eta(\zeta))$ must have the local behaviour

$$z_\zeta\big(a_k^{(j)}\big) = \big(\zeta - a_k^{(j)}\big)^{\beta_k^{(j)}} g(\zeta) \tag{7·5}$$

where $g(\zeta)$ is some function that is analytic at $\zeta = a_k^{(j)}$. Except for these branch point singularities, the mapping must be analytic at all other points on the circular arcs.

Next, in order that the segments of the circular arcs $\{L_j | j = 0, 1, \ldots, M\}$ between these branch point singularities map to straight-line segments in the z-plane, the mapping function must satisfy the property that the quantity $\eta z_\eta(\eta)$ has piecewise-constant argument on all the circular arcs $\{L_j | j = 0, 1, \ldots, M\}$. To see this, consider the kth line segment of the polygon P_j in the z-plane. On this line, it is known from (2·3) that

$$\bar{z} = \epsilon_k^{(j)} z + \kappa_k^{(j)} \tag{7·6}$$

for some constants $\epsilon_k^{(j)}$ and $\kappa_k^{(j)}$. This means that, on the portion of the circular arc L_j mapping to this line segment, differentiation with respect to z means that we must have

$$\frac{d\bar{z}}{d\eta}\left(\frac{dz}{d\eta}\right)^{-1} = \epsilon_k^{(j)}. \tag{7·7}$$

But, on this portion of L_j, we also have

$$\bar{z} = \overline{z(\eta)} = \bar{z}(\bar{\eta}) = \bar{z}\big(r_j^2 \eta^{-1}\big) \tag{7·8}$$

where $\{r_j | j = 1, \ldots, M\}$ are defined in (6·1) and we stipulate $r_0 = 1$ (since L_0 is the unit circle). Therefore (7·7) becomes

$$-\frac{r_j^2}{\eta^2}\frac{\bar{z}_\eta\big(r_j^2\eta^{-1}\big)}{z_\eta(\eta)} = \epsilon_k^{(j)}. \tag{7·9}$$

Since $\bar{\eta} = r_j^2\eta^{-1}$ on L_j, it can be concluded from (7·9) that, on this portion of L_j, $\eta z_\eta(\eta)$ has constant argument. It follows that the argument of $\eta z_\eta(\eta)$ is piecewise constant on the circular arcs $\{L_j | j = 0, \ldots, M\}$ and hence on the circles $\{C_j | j = 0, \ldots, M\}$.

8. *Construction of the S-C mapping function*

The conformal mapping $z(\zeta)$ from D_ζ to D_z will now be constructed. First, pick an arbitrary point γ_j on each of the circles $\{C_j | j = 0, \ldots, M\}$. It is required to construct a mapping from D_η to D_z satisfying the condition that $\eta z_\eta(\eta)$ has piecewise constant argument on the segments of the circular arcs $\{L_j | j = 0, \ldots, M\}$ between the prevertices $\big\{\tilde{a}_k^{(j)}\big\}$. But this is equivalent to the condition that ηz_η has piecewise constant argument on the segments of the original circles $\{C_j | j = 0, \ldots, M\}$ between the prevertices $\big\{a_k^{(j)}\big\}$. We must also ensure that ηz_η has the requisite branch point singularities on these circles in the ζ-plane.

Consider the function

$$\prod_{k=1}^{n_0}\Big(F_1(\zeta;a_k^{(0)},\gamma_0)\Big)^{\beta_k^{(0)}}\prod_{j=1}^{M}\prod_{k=1}^{n_j}\Big(F_2\big(\zeta;a_k^{(j)},\gamma_j\big)\Big)^{\beta_k^{(j)}}. \tag{8·1}$$

Since this function is a product of various powers of the special functions F_1 and F_2 considered in Propositions 1 and 2 it will have piecewise constant argument on the circles $\{C_j|j=0,\ldots,M\}$. It also has the correct branch point singularities at the points $\{a_k^{(j)}|k=1,\ldots,n_j\}$. However, in addition to the required branch points, on use of the two relations (2·2) this function can be seen to have $M+1$ second order poles at the points $\{\gamma_j|j=0,\ldots,M\}$.

Now multiply (8·1) by the quantity

$$\prod_{j=1}^{M}F_2\big(\zeta;\gamma_j,\gamma_1^{(j)}\big)F_2\big(\zeta;\gamma_j,\gamma_2^{(j)}\big) \tag{8·2}$$

which has a second order zero at the M points $\{\gamma_j|j=1,\ldots,M\}$ and simple poles at the $2M$ points $\{\gamma_1^{(j)},\gamma_2^{(j)}|j=1,\ldots,M\}$. Multiplying (8·1) by this function has the effect of shifting the M second order poles at the arbitrarily chosen points $\{\gamma_j|j=1,\ldots,M\}$ to produce instead $2M$ simple poles of the function at the points $\{\gamma_1^{(j)},\gamma_2^{(j)}|j=1,\ldots,M\}$. Recall from (6·2) that the latter set of points are precisely the positions of the simple zeros of the conformal mapping $\eta_\zeta(\zeta)$ – a fact that will be useful in what follows. It is crucial to note that since (8·2) is a product of F_2-functions, we have effected this shift in the poles of the function without affecting the important property that it has piecewise constant argument on the circles $\{C_j|j=0,\ldots,M\}$.

The new modified function, which can be written

$$\prod_{k=1}^{n_0}\Big(F_1(\zeta;a_k^{(0)},\gamma_0)\Big)^{\beta_k^{(0)}}\prod_{j=1}^{M}F_2(\zeta;\gamma_j,\gamma_1^{(j)})F_2(\zeta;\gamma_j,\gamma_2^{(j)})\prod_{k=1}^{n_j}\Big(F_2\big(\zeta;a_k^{(j)},\gamma_j\big)\Big)^{\beta_k^{(j)}}, \tag{8·3}$$

is now multiplied by a second function given by

$$F_3(\zeta;\gamma_0,\zeta_\infty)F_3(\zeta;\alpha,\zeta_\infty). \tag{8·4}$$

This has the effect of removing the second-order pole at the arbitrarily chosen point γ_0 and replacing it with a second-order pole at the points ζ_∞ and $\bar{\zeta}_\infty^{-1}$. It also adds two simple zeros at the points α and $\bar{\alpha}^{-1}$. Recall that α is the point in the ζ-plane which maps to $\eta=0$ in D_η. Again, because (8·4) is a product of F_3-functions introduced in Proposition 3, this shift in the poles of the function has been effected without sacrificing the property that it has piecewise constant argument on the circles $\{C_j|j=0,\ldots,M\}$. The new function can be rewritten, after cancellations, as

$$S(\zeta)\prod_{k=1}^{n_0}\big[\omega\big(\zeta,a_k^{(0)}\big)\big]^{\beta_k^{(0)}}\prod_{j=1}^{M}\prod_{k=1}^{n_j}\big[\omega\big(\zeta,a_k^{(j)}\big)\big]^{\beta_k^{(j)}}, \tag{8·5}$$

where

$$S(\zeta)\equiv\frac{\omega(\zeta,\alpha)\omega(\zeta,\bar{\alpha}^{-1})}{\omega(\zeta,\zeta_\infty)^2\omega\big(\zeta,\bar{\zeta}_\infty^{-1}\big)^2\prod_{j=1}^{M}\omega\big(\zeta,\gamma_1^{(j)}\big)\omega\big(\zeta,\gamma_2^{(j)}\big)}. \tag{8·6}$$

It will not have escaped the reader's notice that the function in (8·5) has all the properties required of ηz_η. Indeed, it is now easy to prove that ηz_η is indeed some multiple of (8·5). The proof, which relies on an application of Liouville's theorem, is outlined in appendix A. It can be concluded that

$$\eta z_\eta(\eta) = \tilde{B} S(\zeta) \prod_{k=1}^{n_0} \left[\omega\left(\zeta, a_k^{(0)}\right)\right]^{\beta_k^{(0)}} \prod_{j=1}^{M} \prod_{k=1}^{n_j} \left[\omega\left(\zeta, a_k^{(j)}\right)\right]^{\beta_k^{(j)}} \tag{8·7}$$

where $\tilde{B}$ is some complex constant. But, by the chain rule,

$$\eta z_\eta(\eta) = \eta(\zeta) \frac{dz}{d\zeta} \frac{d\zeta}{d\eta} \tag{8·8}$$

which implies the following expression for $dz/d\zeta$:

$$\frac{dz}{d\zeta} = \frac{\tilde{B}}{\eta(\zeta)} \frac{d\eta(\zeta)}{d\zeta} S(\zeta) \prod_{k=1}^{n_0} \left[\omega\left(\zeta, a_k^{(0)}\right)\right]^{\beta_k^{(0)}} \prod_{j=1}^{M} \prod_{k=1}^{n_j} \left[\omega\left(\zeta, a_k^{(j)}\right)\right]^{\beta_k^{(j)}}. \tag{8·9}$$

To check the consistency of the formula, first note that the pole of the right-hand side at $\zeta = \alpha$ (arising because $\eta(\zeta)$ vanishes there) is removable since $S(\zeta)$ vanishes there. Second, the simple zeros of $d\eta/d\zeta$ at $\{\gamma_1^{(j)}, \gamma_2^{(j)} | j = 1, \ldots, M\}$ (cf. Section 2) do not produce unwanted zeros of $dz/d\zeta$ at these points since they are exactly cancelled by the simple poles of $S(\zeta)$. Finally, $dz/d\zeta$ has the required second-order pole at ζ_∞ owing to the presence of a second-order pole of $S(\zeta)$ at this point.

By direct calculation based on the formula (6·3), we obtain

$$\frac{d\eta}{d\zeta} = \frac{1}{|\alpha|} \left(\frac{\omega_\zeta(\zeta, \alpha)\omega(\zeta, \bar{\alpha}^{-1}) - \omega_\zeta(\zeta, \bar{\alpha}^{-1})\omega(\zeta, \alpha)}{\omega(\zeta, \bar{\alpha}^{-1})^2} \right). \tag{8·10}$$

On substitution into (8·9),

$$\frac{dz}{d\zeta} = B S_\infty(\zeta) \prod_{k=1}^{n_0} \left[\omega\left(\zeta, a_k^{(0)}\right)\right]^{\beta_k^{(0)}} \prod_{j=1}^{M} \prod_{k=1}^{n_j} \left[\omega\left(\zeta, a_k^{(j)}\right)\right]^{\beta_k^{(j)}} \tag{8·11}$$

where $S_\infty(\zeta)$ is defined in (1·4) and B is some constant. On integration of (8·11) with respect to ζ, the final formula given in (1·3) is obtained where A is a constant of integration.

While $dz/d\zeta$ must have the general form (8·11) with a second order pole at ζ_∞, its primitive must have a simple pole there. In other words, it is necessary that the residue of $dz/d\zeta$ at ζ_∞ vanishes. This should be considered as an additional condition to be satisfied in the problem of finding the accessory parameters.

9. *The simply connected case*

In the case of a simply connected domain there are no enclosed circles and hence no non-trivial generating Möbius maps. The Schottky group is therefore the trivial group and the associated Schottky–Klein prime function is just

$$\omega(\zeta, \gamma) = (\zeta - \gamma). \tag{9·1}$$

Moreover, if we take $\zeta_\infty = 0$, the function $S_\infty(\zeta)$ reduces to

$$S_\infty(\zeta) = \frac{C}{\zeta^2} \tag{9·2}$$

in this case. Formula (1·3) is then the well-known S-C mapping from the unit disc to an unbounded polygonal region [**10**].

10. *The doubly connected case*

Any doubly connected domain can be obtained by a conformal mapping from some annulus $q < |\zeta| < 1$ in a parametric ζ-plane where the value of the parameter q is determined by the image domain. In this case, $\delta_1 = 0$ and $q_1 = q$, so that the single Möbius map given by (3·5) is

$$\theta_1(\zeta) = q^2\zeta. \tag{10·1}$$

The Schottky group in this case is generated by θ_1 and its inverse. Its elements are all Möbius maps of the form

$$\{\theta_1^j \,|\, j \in \mathbb{Z}\}. \tag{10·2}$$

The associated Schottky–Klein prime function can be shown to be

$$\omega(\zeta, \gamma) = -\frac{\gamma}{D^2} P(\zeta/\gamma, q) \tag{10·3}$$

where

$$P(\zeta, q) \equiv (1-\zeta)\prod_{k=1}^{\infty}(1-q^{2k}\zeta)(1-q^{2k}\zeta^{-1}), \quad D \equiv \prod_{k=1}^{\infty}(1-q^{2k}). \tag{10·4}$$

The transformation properties of $P(\zeta, q)$ corresponding to (4·4) and (4·5) respectively are

$$\begin{gathered}\frac{P\left(q^2\zeta\gamma_1^{-1}, q\right)}{P\left(q^2\zeta\gamma_2^{-1}, q\right)} = \frac{\gamma_1}{\gamma_2}\frac{P\left(\zeta\gamma_1^{-1}, q\right)}{P\left(\zeta\gamma_2^{-1}, q\right)}, \\ P(\zeta^{-1}, q) = -\zeta^{-1}P(\zeta, q).\end{gathered} \tag{10·5}$$

It can also be shown directly from the infinite product definition (10·4) that

$$P(q^2\zeta, q) = -\zeta^{-1}P(\zeta, q). \tag{10·6}$$

By using a rotational degree of freedom in the mapping function we can assume, without loss of generality, that the point α mapping to $\eta = 0$ is real. It follows that $S_\infty(\zeta)$ takes the form

$$S_\infty(\zeta) = \frac{\tilde{S}(\zeta)}{P^2(\zeta\zeta_\infty^{-1}, q)P^2(\zeta\bar{\zeta}_\infty, q)} \tag{10·7}$$

where

$$\tilde{S}(\zeta) = \frac{\alpha^{-1}P_\zeta(\zeta\alpha^{-1}, q)P(\zeta\alpha, q) - \alpha P_\zeta(\zeta\alpha, q)P(\zeta\alpha^{-1}, q)}{\gamma_1\gamma_2 P\left(\zeta\gamma_1^{-1}, q\right)P\left(\zeta\gamma_2^{-1}, q\right)}. \tag{10·8}$$

In fact, as shown by Crowdy [**6**], $\tilde{S}(\zeta)$ can be simplified to

$$\tilde{S}(\zeta) = \frac{C}{\zeta^2} \tag{10·9}$$

where C is some constant. Thus, (8·11) produces the result

$$z_\zeta = \frac{C}{\zeta^2 P^2(\zeta\zeta_\infty^{-1}, q)P^2(\zeta\bar{\zeta}_\infty, q)}\prod_{k=1}^{n_0}\left[P\left(\zeta/a_k^{(0)}, q\right)\right]^{\beta_k^{(0)}}\prod_{k=1}^{n_1}\left[P\left(\zeta/a_k^{(1)}, q\right)\right]^{\beta_k^{(1)}} \tag{10·10}$$

for some constant C. This is the final formula. Note, for use in a moment, that from the transformation property (10·6), as well as (2·2), this can be rewritten as

$$z_\zeta = \frac{B}{P^2(\zeta\zeta_\infty^{-1}, q)P^2(\zeta\bar{\zeta}_\infty, q)} \prod_{k=1}^{n_0} \left[P\left(\zeta/a_k^{(0)}, q\right)\right]^{\beta_k^{(0)}} \prod_{k=1}^{n_1} \left[P\left(q^2\zeta/a_k^{(1)}, q\right)\right]^{\beta_k^{(1)}} \tag{10·11}$$

for some constant B.

Akhiezer [**3**] was apparently the first to derive the mapping of an annulus to an unbounded doubly connected polygonal region. His formula, as reported in [**2**] but transcribed as closely as possible into the notation of the present paper, is

$$c_1 z(\zeta) + c_2 = \int \frac{d\zeta}{\zeta^2} \prod_{k=1}^{N} \left[\Theta_1\left(\frac{\log\zeta - \log a_k}{2\pi i}\right)\right]^{\beta_k} \left[\Theta_1\left(\frac{\log(\zeta/\zeta_\infty)}{2\pi i}\right)\Theta_1\left(\frac{\log(\zeta\zeta_\infty)}{2\pi i}\right)\right]^{-2} \tag{10·12}$$

where Θ_1 is the first Jacobi theta function. One obvious difference between (10·10) and (10·12) is that the latter does not distinguish between branch points occurring on different boundary circles of the annulus, but this is inconsequential. It can be verified that (10·12) is an alternative representation of (10·10) on use of the formula

$$P(\zeta) = -\frac{i e^{\chi/2}}{D\rho^{1/4}}\Theta_1(-i\chi/2) \tag{10·13}$$

where $\chi = \pi^{-1}\log\zeta$. (10·13) provides the connection between $P(\zeta)$ and Θ_1 [**19**].

DeLillo, Elcrat and Pfaltzgraff [**8**] also derive an expression for a mapping to a doubly connected polygonal region from a conformally equivalent unbounded circular region given by the exterior of the unit circle and a circle centred at $c > 1$ and of radius r. Their formula, transcribed in the notation of the present paper, is

$$\begin{aligned} z(\zeta) = A + B\int^{\zeta} \prod_{k=1}^{n_0}\left[\Theta\left(\frac{T(\zeta')}{qT\left(a_k^{(0)}\right)}\right)\right]^{\beta_k^{(0)}} \prod_{k=1}^{n_1}\left[\Theta\left(\frac{qT(\zeta')}{T\left(a_k^{(1)}\right)}\right)\right]^{\beta_k^{(1)}} \\ \times\ (\Theta(-pT(\zeta)/q)\Theta(-T(\zeta)/(qp)))^{-2}\frac{1-p^2}{(1-p\zeta)^2}\,d\zeta' \end{aligned} \tag{10·14}$$

where

$$\Theta(\zeta) \equiv \prod_{k=0}^{\infty}(1 - q^{2k+1}\zeta)(1 - q^{2k+1}\zeta^{-1}), \tag{10·15}$$

A and B are constants, while

$$T(\zeta) = \frac{\zeta - p}{1 - p\zeta}, \quad p = \frac{c}{1 + rq}. \tag{10·16}$$

To see the relationship between these two results, first note that it is easy to check from (10·4) and (10·15) that

$$\Theta(\zeta q^{-1}) = P(\zeta, q). \tag{10·17}$$

The Möbius mapping $T(\zeta)$ is precisely the one taking the unbounded circular region considered in [**9**] to the bounded annulus $q < |\zeta| < 1$. Moreover, we can identify

$$\zeta_\infty = -p^{-1}. \tag{10·18}$$

Combining all this, the integral of (10·11) is identical to (10·14) if the substitution $\zeta'' = T(\zeta')$ is made in the integral of (10·11). This provides an important check on (1·3).

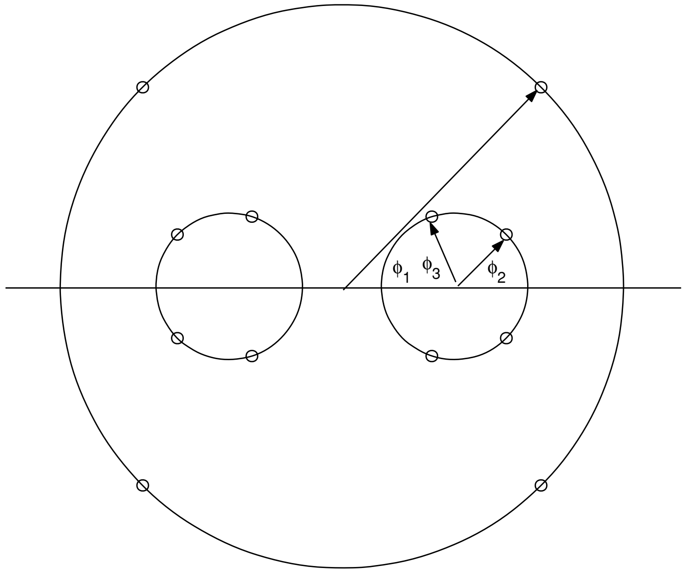

Fig. 4. Definition sketch of the angles ϕ_1, ϕ_2 and ϕ_3 as used in the triply connected example.

11. *The triply connected case*

It is appropriate to check the efficacy of the new formula (8·11) by using it to construct an explicit example. We therefore present a triply connected case. The target domain is chosen to have sufficient geometrical symmetry that the numerical complications of solving the parameter problem are significantly reduced. Consider an unbounded triply connected target region consisting of the complex plane with three rectangles excised. One rectangle is centred at the origin $z = 0$ and is assumed to be reflectionally symmetric about both the real and imaginary axes. Two equal rectangles centred on the positive and negative real axis are positioned to either side of the central rectangle. The domain is taken to be reflectionally symmetric about both the real and imaginary axes in the z-plane. A conformal mapping from a conformally equivalent, triply connected circular domain to this target domain will be constructed based on (1·3).

The parameter A can be chosen so that the origin is correctly placed (i.e., $z(0) = 0$) while B can be thought of as governing the area of the central rectangle. Here we chose B so that the central rectangle extends horizontally between ± 1. By the symmetry of the target domain, we expect

$$q_1 = q_2 = q, \quad \delta_1 = -\delta_2 = \delta, \tag{11·1}$$

where δ is taken to be real so that C_1 and C_2 are centred on the real axis. The two real parameters q and δ will be picked arbitrarily. This can be thought of as specifying the centre and area of the two rectangles at either side of the central rectangle. We also expect, on grounds of symmetry, that we should take $\zeta_\infty = 0$. Further, the prevertices on the unit ζ-circle are also expected to be symmetrically disposed. We therefore take them to be at

$$e^{\pm i\phi_1}, \; e^{\pm i(\pi - \phi_1)} \tag{11·2}$$

where ϕ_1 is an adjustable real parameter. Figure 4 shows a schematic illustrating ϕ_1 as the argument of the branch point on C_0. It can be thought of as governing the aspect ratio (or height) of the central rectangle. Concerning the prevertices on C_1 and C_2, on grounds of

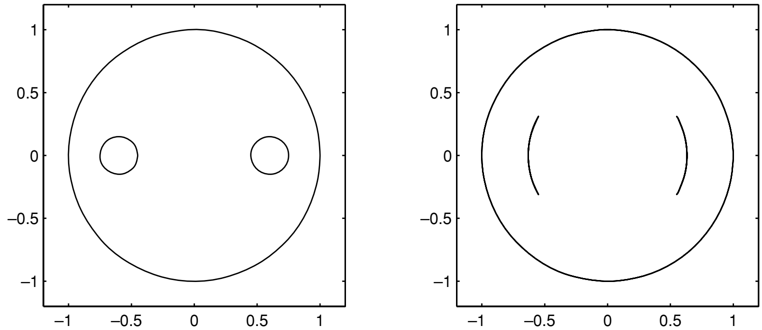

Fig. 5. The circular region D_ζ in the ζ-plane and the circular-slit domain D_η in the η-plane for the triply connected example with $q = 0.15$ and $\delta = 0.6$.

symmetry, we expect it follows from the symmetry that the prevertices on C_1 are at

$$\delta + qe^{\pm i\phi_2}, \quad \delta + qe^{\pm i(\pi-\phi_3)} \tag{11·3}$$

where ϕ_2 and ϕ_3 are real parameters. Only one of these two parameters, ϕ_2 say, should be freely specifiable. The value of ϕ_3 is then determined by the condition that the lengths of the sides of the image polygon should be such that the polygon closes. By the symmetry, we expect the pre-vertices on C_2 to be at

$$-\delta + qe^{\pm i\phi_3}, \quad -\delta + qe^{\pm i(\pi-\phi_2)}. \tag{11·4}$$

See the schematic in Figure 4.

It is easy to deduce from the interior angles of the polygonal region that we must take

$$\begin{aligned} \beta_k^{(0)} &= \frac{1}{2}, \quad k = 1, 2, 3, 4, \\ \beta_k^{(j)} &= \frac{1}{2}, \quad k = 1, 2, 3, 4 \text{ and } j = 1, 2. \end{aligned} \tag{11·5}$$

We also make the choice $\alpha = 0$. In this case, (6·3) is not well-defined and must be replaced by the formula

$$\eta(\zeta) = \frac{\omega(\zeta, 0)}{\omega(\zeta, \infty)} = \frac{\zeta\omega'(\zeta, 0)}{\omega'(\zeta, \infty)} \tag{11·6}$$

which is the appropriate limit of (6·3) as $\alpha \to 0$. Figure 5 shows the effect of this mapping between the ζ and η-planes.

With α, q and δ now specified, the values of $\gamma_1^{(j)}$ and $\gamma_2^{(j)}$ for $j = 1, 2$ can now be determined. This is done using a simple one-dimensional Newton iteration on the argument, $\hat{\phi}$ say, of the point $\gamma_1^{(1)}$ relative to the point δ. The equation to be solved is that η_ζ as given by the derivative of (11·6) vanishes at $\zeta = \delta + qe^{i\hat{\phi}}$. Then it follows from the symmetry that

$$\gamma_1^{(1)} = \delta + qe^{i\hat{\phi}}, \ \gamma_2^{(1)} = \delta + qe^{-i\hat{\phi}}, \ \gamma_1^{(2)} = -\delta + qe^{i(\pi-\hat{\phi})}, \ \gamma_2^{(2)} = -\delta + qe^{-i(\pi-\hat{\phi})}. \tag{11·7}$$

In this way, the symmetries of the target configuration have reduced the parameter problem to that of finding the appropriate value of the single real parameter ϕ_3. Note that the symmetry of the distribution of circles and prevertices in the preimage ζ-plane will automatically ensure that the residue of z_ζ at $\zeta = 0$ vanishes so that there is no need to explicitly impose this condition in this case.

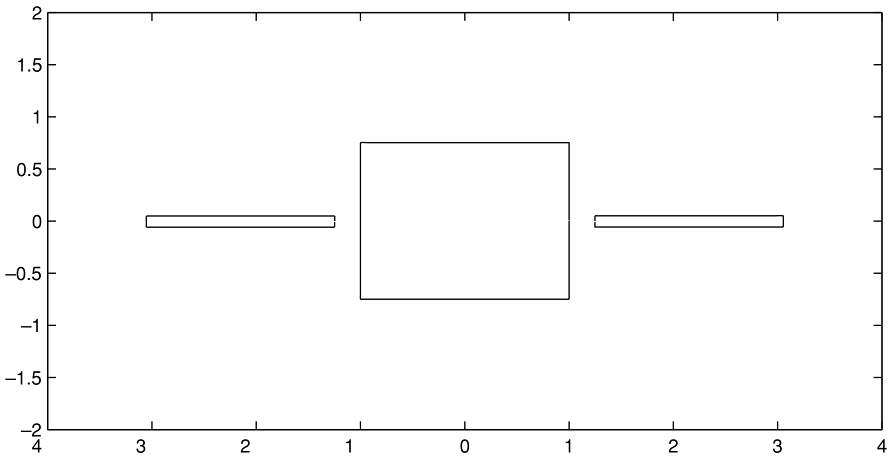

Fig. 6. The unbounded triply connected image domain D_z for the triply connected example. Here $\phi_1 = \pi/4$ and $\phi_2 = 0.4$.

Figure 6 shows the ζ and η-plane for $q = 0.15$ and $\delta = 0.6$ under the mapping (11·6). Figure 6 shows the corresponding triply connected polygonal region for $\phi_2 = 0.4$ with $\phi_1 = \pi/4$.

11·1. *Comparison with a slit mapping formula*

The triply connected example just considered affords us a further non-trivial analytical check on the validity of the derived formula. Consider the degenerate case of the previous mapping in which the target domain consists of three symmetrically-disposed slits on the real axis. This corresponds to

$$\phi_1 = \phi_2 = \phi_3 = 0 \tag{11·8}$$

with z_ζ given by

$$z_\zeta = BS_\infty(\zeta)\omega(\zeta,1)\omega(\zeta,-1)\omega(\zeta,\delta+q)\omega(\zeta,\delta-q)\omega(\zeta,-\delta+q)\omega(\zeta,-\delta-q) \tag{11·9}$$

with

$$S_\infty(\zeta) = \left(\frac{\omega_\zeta(\zeta,\alpha)\omega(\zeta,\bar{\alpha}^{-1}) - \omega_\zeta(\zeta,\bar{\alpha}^{-1})\omega(\zeta,\alpha)}{\omega(\zeta,0)^2\omega(\zeta,\infty)^2\prod_{j=1}^{M}\omega(\zeta,\gamma_1^{(j)})\omega(\zeta,\gamma_2^{(j)})}\right). \tag{11·10}$$

In this degenerate case, it turns out that there is an alternative derivation of the required mapping formula that yields another functional form for $z(\zeta)$ from the same preimage region. It can be shown (see appendix B for details) that an alternative mapping from the preimage domain in Figure 4 to three symmetrically-disposed slits on the real axis is given by

$$z(\zeta) = C\left[\frac{\omega(\zeta,-1)^2 + \omega(\zeta,1)^2}{\omega(\zeta,-1)^2 - \omega(\zeta,1)^2}\right] \tag{11·11}$$

where C is some real constant.See Figure 7. In (11·11), $\zeta = 0$ maps to $z = \infty$ which is also the case for (11·9). Since the pre-image regions D_ζ are also identical it follows that, to within multiplicative constants, the derivative of formula (11·11) must be identically equal

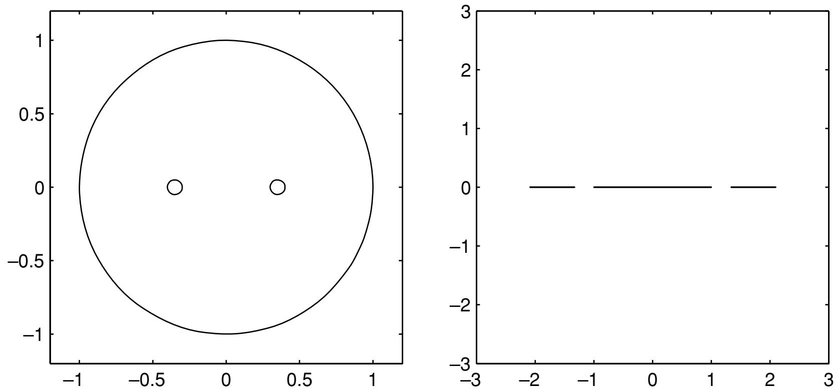

Fig. 7. The degenerate case of mapping to an unbounded triply connected slit domain D_z as given by formula (11·11). On the left, the circular preimage region is shown. The image is the unbounded region exterior to three slits on the real axis as shown on the right. The parameters chosen are $C = 1$, $q_1 = q_2 = 0.05$ and $\delta_1 = -\delta_2 = 0.35$.

to (11·9). That is,

$$\begin{aligned} S_\infty(\zeta)\omega(\zeta,1)\omega(\zeta,-1)\omega(\zeta,\delta+q)\omega(\zeta,\delta-q)\omega(\zeta,-\delta+q)\omega(\zeta,-\delta-q) \\ = D\frac{d}{d\zeta}\left[\frac{\omega(\zeta,-1)^2+\omega(\zeta,1)^2}{\omega(\zeta,-1)^2-\omega(\zeta,1)^2}\right] \end{aligned} \tag{11·12}$$

for some real constant D (determined, for example, by evaluating (11·12) at any arbitrary value of ζ). It can be checked numerically that the identity (11·12) holds for all values of ζ thereby providing a further non-trivial check on the new formula (8·11).

Importantly, the identity (11·12) has practical utility. It can be used to provide an alternative expression for $S_\infty(\zeta)$ which is independent of the parameters α and $\{\gamma_j^{(k)}\}$ – an expression which can be used to replace $S_\infty(\zeta)$ in the general formula (8·11) which also applies, of course, to non-degenerate domains such as those constructed in Figure 6. This is a useful observation since it obviates the need to pick α and then solve for the α-dependent parameters $\{\gamma_j^{(k)}\}$.

The identity (11·12) is also important in providing explicit confirmation of the fact that the function $S_\infty(\zeta)$, while apparently depending on the choice of the arbitrary parameter α and the associated α-dependent parameters $\{\gamma_j^{(k)}\}$, in fact is *independent* of α and depends only on ζ_∞ and the parameters $\{q_j, \delta_j | j = 1, \ldots, M\}$ characterizing the relevant preimage region. We have not yet been able to identify an alternative representation of $S_\infty(\zeta)$ – one which highlights this independence of the function on α – in the general case. While this is desirable, it is no impediment to the direct implementation of the formula (8·11) in practice.

12. *Discussion*

By use of elements of classical function theory, the formula (1·3) for the S-C mapping from a bounded, multiply connected circular domain to an unbounded, multiply connected polygonal domain has been constructed. It reduces to known formulae in the case of simply and doubly connected domains. It also generalizes a new formula derived in [**6**] for the mapping from a bounded, multiply connected circular region to a bounded multiply connected polygonal region. The mapping to an example unbounded triply connected domain has been constructed and checked, in a degenerate case, by comparison with an independent construction of a triply connected slit-mapping formula.

A principal contribution of this paper is the association of the general S-C formula with the Schottky–Klein prime function. We believe this association to be significant, especially when it comes to optimizing the numerical implementation of the S-C mapping formula to multiply connected domains. The Schottky–Klein prime function has intimate connections with the more commonly employed Riemann theta functions which have better convergence properties. Baker [**4**] cites explicit formulae relating Schottky–Klein prime functions to Riemann theta functions. Indeed, as shown in (10·13) the prime function $P(\zeta)$ in the doubly connected case is related to the first Jacobi theta function. Hu [**14**] has found that different representations of the first Jacobi theta function can lead to improved convergence properties when performing a numerical implementation of the doubly connected mapping formula to bounded regions. We expect that similar benefits of convergence may be afforded by rewriting formula (1·3) in terms of Riemann theta functions. This is a subject for future investigation. Indeed, there are many interesting open questions to be answered concerning the numerical issues associated with the construction of multiply connected S-C mappings based on the formula (1·3) or variants thereof.

The formula derived here has been written in terms of the Schottky–Klein prime function which, with a view to finding explicit formulae that can form the basis of a constructive algorithm, has been defined here in terms of a classical infinite-product formula [**4**]. It should be noted, however, that the prime function is a well-defined function on any Riemann surface (see, for example, Hejhal [**12**]) and therefore the final formula derived here (in terms of the prime function) can be shown to be valid even when the infinite product representation of the prime function is not. It is an important matter to be able to find alternative methods of computing the prime function for values of the parameters $\{q_j, \delta_j | j = 1, \ldots, M\}$ for which the infinite product representation (4·2) fails to converge. Work in this direction is currently in progress.

Appendix A: Application of Liouville's theorem

Let

$$U(\eta) \equiv S(\zeta)\prod_{k=1}^{n_0}\left[\omega\left(\zeta, a_k^{(0)}\right)\right]^{\beta_k^{(0)}}\prod_{j=1}^{M}\prod_{k=1}^{n_j}\left[\omega\left(\zeta, a_k^{(j)}\right)\right]^{\beta_k^{(j)}}, \tag{A 1}$$

where

$$S(\zeta) \equiv \frac{\omega(\zeta,\alpha)\omega(\zeta,\bar{\alpha}^{-1})}{\omega(\zeta,\zeta_\infty)^2\omega(\zeta,\bar{\zeta}_\infty^{-1})^2\prod_{j=1}^{M}\omega\left(\zeta,\gamma_1^{(j)}\right)\omega\left(\zeta,\gamma_2^{(j)}\right)}. \tag{A 2}$$

Consider now the function

$$V(\eta) = \frac{U(\eta)}{\eta z_\eta} \tag{A 3}$$

where z_η is the derivative of the S-C mapping we are seeking. First, note that $V(\eta)$ is analytic everywhere inside and on the unit η-circle L_0. This is because: (a) both $U(\eta)$ and ηz_η have the same branch point singularities at the prevertices $\{\tilde{a}_j^{(k)}\}$ on $\{L_j | j = 0, 1, \ldots, M\}$ and so these cancel in the quotient; (b) the zero of the denominator at $\eta = 0$ is removed by the zero of $U(\zeta)$ at $\zeta = \alpha$; (c) since $z(\zeta)$ has a simple pole at ζ_∞ then z_ζ will have a second-order pole there. But $\eta(\zeta)$ is analytic at $\zeta = \zeta_\infty$, so z_η also has a second order pole at ζ_∞ and so the second order pole of $U(\eta)$ at ζ_∞ is removable. Further, by the construction of $U(\eta)$, both $U(\eta)$ and ηz_η have piecewise constant argument on each segment between the branch points on L_0 and have the same changes in argument on passing through the branch points $\{a_j^{(0)} | j = 1, \ldots, n_0\}$. From this we can deduce that, everywhere on L_0, the argument of $V(\eta)$ is a constant. Equivalently,

$$\overline{V(\eta)} = \epsilon V(\eta), \quad \text{on } L_0, \tag{A 4}$$

for some complex constant ϵ. But (A 4) can be written

$$\overline{V}(\eta^{-1}) = \epsilon V(\eta) \tag{A 5}$$

or

$$V(\eta^{-1}) = \overline{\epsilon V(\bar{\eta})} = \bar{\epsilon}\overline{V}(\eta), \tag{A 6}$$

an equation which furnishes the analytic continuation of $V(\eta)$ to the exterior of L_0. In particular, $V(\eta)$ is seen to be analytic everywhere outside the unit η-circle L_0 and is bounded at infinity. $V(\eta)$ is therefore an entire function bounded at infinity and, by Liouville's theorem, is necessarily a constant. It can be concluded that

$$\eta z_\eta(\eta) = \tilde{B} S(\zeta) \prod_{k=1}^{n_0} \left[\omega\left(\zeta, a_k^{(0)}\right)\right]^{\beta_k^{(0)}} \prod_{j=1}^{M} \prod_{k=1}^{n_j} \left[\omega\left(\zeta, a_k^{(j)}\right)\right]^{\beta_k^{(j)}} \tag{A 7}$$

where $\tilde{B}$ is some complex constant.

Appendix B: Alternative slit-mapping formula

To construct the slit-mapping formula (11·11) consider the sequence of conformal mappings given by

$$\begin{aligned} \zeta_1(\zeta) &= -\frac{\omega(\zeta, 1)}{\omega(\zeta, -1)}, \\ \zeta_2(\zeta_1) &= \frac{1 - \zeta_1}{1 + \zeta_1}, \\ z(\zeta_2) &= \frac{C}{2}\left(\zeta_2 + \frac{1}{\zeta_2}\right) \end{aligned} \tag{B 1}$$

where C is any real constant. A schematic illustrating this sequence of mappings is shown in Figure 8. The first mapping takes the circular region D_ζ to the right-half ζ_1-plane with two finite-length slits on the real axis. To see this one makes use of properties (4·4) and (4·5). For example, it is clear that $\zeta = 1$ maps to $\zeta_1 = 0$ while $\zeta = -1$ maps to $\zeta_1 = \infty$. To see that the rest of the unit circle maps to the imaginary ζ_1-axis notice that on C_0 we have

$$\overline{\zeta_1(\zeta)} = -\frac{\overline{\omega}(\zeta^{-1}, 1)}{\overline{\omega}(\zeta^{-1}, -1)} = \frac{\omega(\zeta, 1)}{\omega(\zeta, -1)} = -\zeta_1(\zeta) \tag{B 2}$$

where we have used (4·4). By similar manipulations it is possible to show that the image of C_1 and C_2 under the mapping $\zeta_1(\zeta)$ each has constant argument, equal to zero. Hence each of these two circles maps to a slit on the positive real axis. The second Möbius map takes this slit half-plane in the ζ_1-plane to the unit ζ_2-disc similarly cut along its diameter on the real axis by two finite-length slits. The third Joukowski mapping maps the interior of the unit disc in the ζ_2-plane to the whole of the z-plane exterior to three finite-length slits on the real axis. A composition of the sequence of mappings yields (11·11).

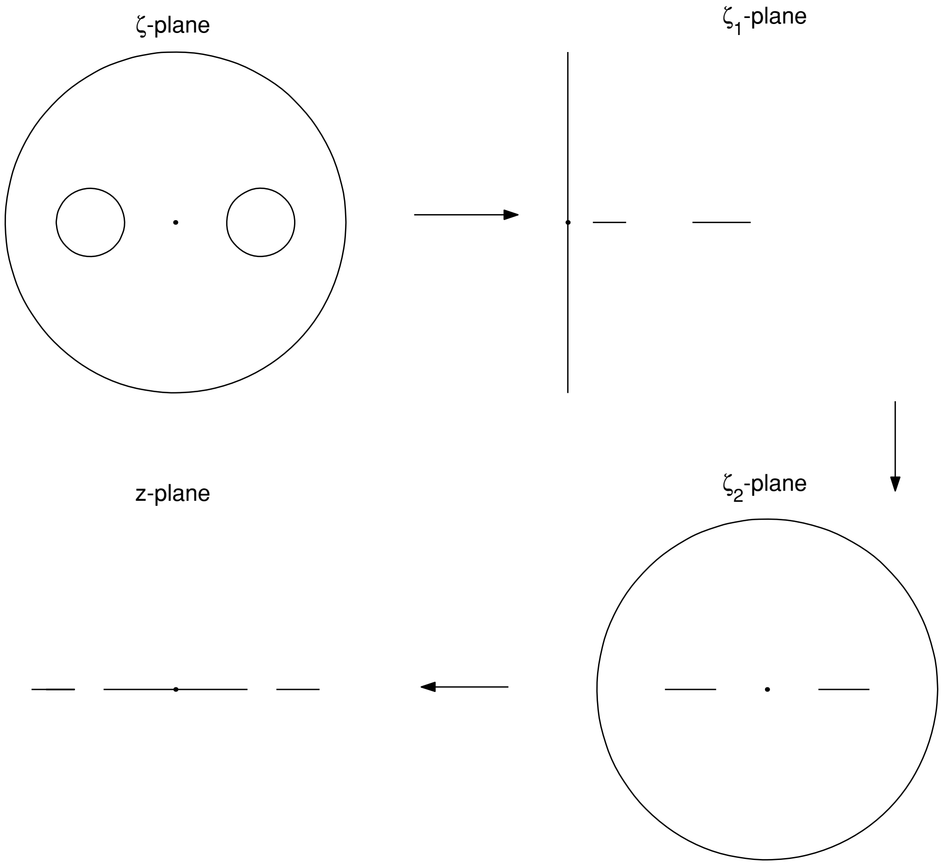

Fig. 8. Schematic illustrating the composition of conformal mappings (B 1) effecting the mapping of a triply connected circular region to the unbounded z-plane with three symmetrically disposed slits on the real axis.

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