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Scoring run-off paradoxes for variable electorates*

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Summary. A no-show paradox occurs each time a single voter or a group of voters can manipulate the outcome by not participating to the election process. Among other voting procedures, the scoring run-off methods, which eliminate progressively the alternatives on the basis of scoring rules, suffer from this flaw. We here estimate how frequent this paradox is for three candidate elections under the classical Impartial Culture and Impartial Anonymous Culture assumptions, for different population sizes. The conditions under which this paradox occurs are also described, as well as the relationships with manipulations for a fixed number of voters.

Keywords and Phrases: Abstention, Paradox, Voting, Manipulation.

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1 Introduction

One of the most intriguing paradox in social choice literature is the *no-show paradox* (a term introduced by Fishburn and Brams, 1983), also called the *abstention paradox*: For some voting rules and some specific voting situations, some voters may get a better result by not participating to the election process. All the Condorcet social choice functions suffer from this flaw (see Moulin, 1988) as well as all the scoring run-off methods (see Smith, 1973; Lepelley, 1989; Saari, 1994; Merlin, 1996). For example, this means that when a society uses two-stage

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plurality to elect the president of the state, a possible manipulation strategy may be the abstention!

Nevertheless, in most of the literature, the existence of a paradox is only proven by providing an example. Few articles go beyond that stage, asking for example whether these paradoxical situations are scarce or generalized, whether it is easy or not to react to an attempt to manipulate, etc. Using recent development in social choice literature, we here intend to analyze precisely whether the no-show paradox is an important flaw for multistage scoring processes. Two approaches will be undertaken. First, using the different statistical techniques that have been developed and improved since the last twenty years, *we will estimate the likelihood of different types of no-show paradoxes for all the scoring run-off methods in three candidate elections, for small electorate as well as large populations*. Secondly, using the tools developed by D. Saari in several articles and in his book, Geometry of Voting (1994), we will propose arguments that tend to prove that, when one wants to manipulate the outcome, the abstention strategy is "dominated" by other strategies, which lead to the same result in a somehow more efficient way. Moreover, this paper can also be viewed as a description of arguments that can be used in order to determine whether a specified paradox for a voting rule is really an important drawback or not, the problem of the abstention paradox for scoring run-off rules being here just an excellent application.

More precisely, when n individuals have to select a candidate among m alternatives, we may first ask each individual to rank all the alternatives without tie according to her preference (i.e. her preference is represented by a linear ordering on $A = \{a_1, \ldots, a_m\}$, the set of alternatives). Next, we shall give w_r^m points to the alternative a_j each time she is ranked in r^{th} position in one individual ordering. We shall assume throughout the paper that $w_r^m \geq w_{r+1}^m$ and $w_1^m > w_m^m$, i.e., the better the position of an alternative is, the higher is the number of points a voter gives to her. The scoring rule g_{w^m}, defined by the scoring vector $w^m = (w_1^m, \ldots, w_r^m, \ldots, w_m^m)$, ranks the alternatives according to the total number of points they receive among the whole population and then selects as a winner the alternative(s) with the highest score.

Scoring methods may also be used in a sequential process. When $m = k_1$ alternatives are to be ranked, we can use a scoring method $g_{w^{k_1}}$ in order to obtain a first ordering. Next, we may only take into consideration the k_2 top ranked alternatives, $k_2 < k_1$, and remove from contention the $k_1 - k_2$ bottom ranked candidates. A new scoring rule $g_{w^{k_2}}$ is then used in order to rank the remaining candidates. The process may continue in this way as long as we want; the final winner is the alternative which is never eliminated and is ranked first with the last scoring rule. This kind of process defines the class of the *scoring run-off* rules, characterized by a sequence of scoring vectors $W = \{w^{k_1}, w^{k_2}, \ldots, w^{k_g}, \ldots, w^{k_h}\}$. k_g is the number of alternatives in contention at the beginning of stage g and h is the number of steps. There are at least 2 stages in the elimination process and $m - 1$ at most.

The scoring methods as well as the scoring run-off rules are used by many committees in order to achieve a social consensus. They are democratic in the sense that they give the same power to any voter and do not favor any candidate[1]. Smith (1973), Young (1975), and Myerson (1995) gave elegant characterizations of the scoring rules, but this issue is still open for scoring run-offs (see Merlin, 1996). Thus, the main argument in favor of the scoring run-off methods is the stability of the social outcome when candidates are added to or dropped from the choice set, as, by definition, the choice on the whole set of candidates depends upon rankings on subsets of candidates.

However, the scoring run-offs also present serious flaws. The key property which characterizes the scoring rules is the *reinforcement* axiom[2]: when two different populations select the same outcome with a common voting rule, the social result should be unchanged when the voting mechanism is directly applied to the whole population. Unfortunately, the sequential use of the scoring vectors does not satisfy this property. More generally Smith (1973) proves that an increasing support in favor of one candidate may disfavor her when scoring run-offs are used. We shall distinguish here the following paradoxes:

- More is Less Paradox (MLP). The winner is ranked higher by some voters (everything else remaining the same) and becomes a loser.
- Less is More Paradox (LMP). A loser is ranked lower by some voters (everything else remaining the same) and becomes a winner.
- Positive Participation Paradox (PPP). Some voters with the winner a_j ranked first are added to the population and a_j becomes a loser.
- Negative Participation Paradox (NPP). Some voters with the loser a_j ranked last are added to the population and a_j becomes a winner.
- Positive Abstention Paradox (PAP). Some voters with a loser a_j ranked first are deleted (or they abstain) and a_j becomes a winner.
- Negative Abstention Paradox (NAP). Some voters with a winner a_j ranked last are deleted (or they abstain) and a_j becomes a loser.

This list of paradoxes calls some comments. MLP and LMP are fixed population paradoxes, while the other ones require the number of voters to vary. It is also easy to check that the Positive Participation Paradox and the Positive Abstention Paradox are equivalent: according to the case, we just add or remove the same voters, who have the same top ranked candidate. This remark also holds for NPP and NAP. These four paradoxes are just the more salient examples of a more general issue, the no show paradox, also called the abstention paradox. A no show paradox situation occurs if a voter or a group of voters obtains a better result by abstaining rather than voting. This strange behavior of some voting mechanisms has been identified first by Fishburn and Brams (1983) for a specific scoring run-off (single transferable vote, also called plurality run-off).

The main objective of this paper is to precise to which extent the no-show paradox is a serious flaw of scoring run-off methods. Lepelley, Chantreuil

[1] However, in a case of tie, we may use tie breaking methods that violate these requirements. For example, the chairman may have the right to select the winner among the tied outcomes, the older candidate or the statu quo may be favored, etc.

[2] This property is called "consistency" by Young (1975) and Saari (1990), and "separability" by Smith (1973). The word "reinforcement" comes from Moulin (1988) and Myerson (1995).

and Berg (1996) have already computed the likelihood of More is Less and Less is More paradoxes of the plurality run-off and antiplurality run-off under a specific assumption for three candidate elections. We here intend to do the same for the four paradoxes implying variable electorate. In Section 2,we first present the conditions under which the different paradoxes occur. These results generalize Ray's characterization (1986) about the voting situations leading to the Negative Participation Paradox. In Section 3, we describe the two main assumptions in Social Choice literature about the occurrence of voting situations, that is the *Impartial Culture* condition (IC) and the *Impartial Anonymous Culture* condition (IAC). Finally, we provide the figures *for the four paradoxes, PPP, NPP, PAP, NAP, and the no-show paradox, for all the scoring run-off methods in three candidate elections, with both assumptions IC and IAC, for small committees as well as for large populations*. The details of the computations, which turn out to be somehow tedious, are presented in Section 5.

The second argument we will present generalizes a previous result about plurality run-off due to D. Lepelley (1995). For a choice among k alternatives, the plurality rule is described by the scoring vector $w^k_{PL} = (1, 0, \ldots, 0) \in \mathbb{R}^k$. The class of the plurality run-off methods is characterized by the fact that, at each stage, a plurality vector is used to select among the candidates. Lepelley showed that, each time a group of voters has an opportunity to manipulate the plurality run-off outcome by abstaining, there exists at least one other strategy which gives the same result without abstaining. In other words, for this particular rule, the set of voting situations which are manipulable through misrepresentation of voters' preferences also contains all the voting situations instable when some voters abstain. In Section 4, using the framework of the geometry of voting (see Saari, 1994), we shall prove that *a similar result holds for all the scoring run-off methods, not only with one but with many strategies*.

2 Characterization of instable voting situations

A way to evaluate the importance of the no-show paradox for scoring run-offs is to estimate the likelihood of participation and abstention paradoxes. More precisely, we will estimate the frequency of the voting situations that may lead to these paradoxes. A real paradox will effectively occur only if, first, the voters who may manipulate the outcome know that they can change it (i.e. have enough information about the individual preferences) and secondly if they are able to coordinate themselves in order to implement this strategy. One should also keep in mind that other voters could react to this abstention strategy, and ruin the chance of a successful manipulation with an appropriate behavior. Thus, even if we speak about the"likelihood of a voting paradox", it should be clear that what we effectively describe are the situations that might give rise to these paradoxes.

2.1 The basic model for three candidates

Finding the exact figures of the likelihood of a voting paradox is a difficult and time consuming task, and, as most of contributions in this field, we shall limit ourselves to the case of three candidate elections. Thus, let $A = \{a_1, a_2, a_3\}$ be the set of candidates. The set of individuals who have to choose collectively among the candidates is $I = \{1, \ldots, i, \ldots, n\}$. We shall assume throughout the paper that all these individuals (or voters) are able to rank all the candidates without tie according to their preferences. In the three candidate case, there exist six possible preference types, labeled as follows:

Type 1	$a_1 \succ a_2 \succ a_3$	*Type* 2	$a_1 \succ a_3 \succ a_2$
Type 3	$a_2 \succ a_1 \succ a_3$	*Type* 4	$a_2 \succ a_3 \succ a_1$
Type 5	$a_3 \succ a_1 \succ a_2$	*Type* 6	$a_3 \succ a_2 \succ a_1$

Let n_i be the number of type i voters and n the size of the population. A voting situation $\tilde{n} = (n_1, n_2, n_3, n_4, n_5, n_6)$ gives the distribution of the voters over the six different possible preference types. Of course, $\sum_{t=1}^{6} n_t = n$.

For three candidate elections, the class of scoring run-off methods is uniquely described by the first vector we use. Without loss of generality, we can use the family of normalized scoring vectors $w_s = (1, s, 0)$, $s \in [0, 1]$. The values $s = 0$, $s = \frac{1}{2}$ and $s = 1$ respectively define the plurality run-off, the Borda run-off (also called Nanson rule or Baldwin rule) and the antiplurality run-off.

2.2 Necessary and sufficient conditions for the occurrence of the paradoxes

Ray (1986) already proposed necessary conditions that may lead to the Negative Participation Paradox[3] for plurality run-off. The next propositions give necessary and sufficient conditions for the occurrence of the four paradoxes and all the scoring run-offs. Without loss of generality, we shall assume in all the propositions that a_3 is removed from contention first, and next a_2. Thus a_1 is the unique winner. The five other cases are symmetric. We also assume that the ties are broken in a way that always leads to the considered paradox, as we don't know a priori which tie-breaking method is used. This may over estimate the likelihood of these paradoxes for small populations, but as the size of population grows, the likelihood of tied outcomes diminishes and this problem vanishes.

Proposition 1 *Let us assume that a_1 wins the second stage against a_2 for the voting situation $\tilde{n} = (n_1, n_2, n_3, n_4, n_5, n_6)$ and the scoring run-off rule g_{w_s}, $s \neq 0$. This situation may lead to a Positive Participation Paradox in favor of a_3 by adding type 2 voters if and only if:*

[3] In Ray's terminology, a "pre-addition situation of a no-show paradox" is a NPP situation and a "post-addition situation of a no-show paradox" is a NAP situation.

$$\begin{cases} n_1 + (1-s)n_2 + sn_3 - sn_4 + (s-1)n_5 + n_6 \geq 0 & (A.1) \\ sn_1 - sn_2 + n_3 + (1-s)n_4 - n_5 + (s-1)n_6 \geq 0 & (A.2) \\ n_1 + n_2 - n_3 - n_4 + n_5 - n_6 \geq 0 & (A.3) \\ -2sn_1 - (1+s)n_3 + (2s-1)n_4 + (1+s)n_5 + n_6 \geq 0 & (A.4) \end{cases}$$

Proof of Proposition 1. Let S_i be the score of alternative a_i with the scoring vector w_s, and M_{ij}, the number of voters who ranks a_i before a_j in their ordering, minus the number of individuals with the opposite preference. It is easy to check that the first three inequalities describe the fact that a_3 is removed from consideration at the first stage ($S_1 > S_3$ $(A.1)$, $S_2 > S_3$ $(A.2)$) and that a_1 wins the pairwise comparison against a_2 ($M_{12} > 0$ $(A.3)$). A PPP occurs if, by adding k type 2 voters, a_3 beats a_2 at the first stage, and then wins the pairwise comparison (notice that adding type 2 voters helps a_1 against a_3 at the first stage, so that a_1 still remains in contention). Thus, a_3 is the new winner if k is such that:

$$\begin{cases} S_2 - sk - S_3 \leq 0 & (A.5) \\ M_{31} - k \geq 0 & (A.6) \end{cases}$$

By multiplying $(A.6)$ by s, and adding it to $(A.5)$, we obtain the constraint $(A.4)$.

Now let us assume that the conditions $(A.1)$ to $(A.4)$ are satisfied for a voting situation $\tilde{n}$: a_1 is elected and a_3 is removed first. First, notice that $(A.4)$ together with $(A.2)$ implies that a_3 beats a_1 in the pairwise comparison for the initial profile. Add k' type 2 voters to the initial situation such that the comparison between a_1 and a_3 ends up into a tie vote:

$$-n_1 - n_2 - k' - n_3 + n_4 + n_5 + n_6 = 0 \; (A.6')$$

By multiplying this condition by s and adding it to $(A.4)$, we obtain:

$$sn_1 - sn_2 - sk' + n_3 + (1-s)n_4 - n_5 + (s-1)n_6 \leq 0 \; (A.2')$$

a_3 now beats a_2 at the first stage, and, provided that the tie is broken in its favor, a_3 becomes the new winner. □

Proposition 2 *Let us assume that a_1 wins the second stage against a_2 for the voting situation $\tilde{n} = (n_1, n_2, n_3, n_4, n_5, n_6)$ and the scoring run-off rule g_{w_s}, $s \neq 1$. This situation may lead to a Negative Abstention Paradox in favor of a_3 by deleting enough type 4 voters if and only if:*

$$\begin{cases} n_1 + (1-s)n_2 + sn_3 - sn_4 + (s-1)n_5 - n_6 \geq 0 & (B.1) \\ sn_1 - sn_2 + n_3 + (1-s)n_4 - n_5 + (s-1)n_6 \geq 0 & (B.2) \\ n_1 + n_2 - n_3 - n_4 + n_5 - n_6 \geq 0 & (B.3) \\ -n_1 + (2s-1)n_2 + (s-2)n_3 + (2-s)n_5 + (2-2s)n_6 \geq 0 & (B.4) \\ sn_1 - sn_2 + n_3 - n_5 + (s-1)n_6 \leq 0 & (B.5) \end{cases}$$

Proof of Proposition 2. The first three inequalities describe the same voting situations as in Proposition 1. A negative abstention paradox in favor of a_3 occurs if and only if, by removing enough type 4 voters, a_3 first beats a_2 and next obtains a majority of votes against a_1. A necessary condition is that, when all the type 4 voters are removed ($n_4 = 0$), the score of a_3 is bigger than a_2's one. This gives condition (B.5). Let $0 \leq k \leq n_4$ be a number of voters such that:

$$\begin{cases} S_2 - (1-s)k - S_3 < 0 & (B.6) \\ M_{31} - k > 0 & (B.7) \end{cases}$$

By multiplying (B.7) by $(1-s)$ and adding it to (B.6) we obtain condition (B.4).

Now let us assume that conditions (B.1) to (B.5) are satisfied for a voting situation $\tilde{n}$. It is easy to check that condition (B.2) and (B.4) imply the fact that a majority of voters prefers a_3 to a_1 ($M_{31} \geq 0$). Let k' be a number of type 4 voters such that $M_{31} - k' = 0$:

$$-n_1 - n_2 - n_3 + n_4 - k' + n_5 + n_6 = 0 \ (B.7')$$

Together with (B.4), this condition implies ($B.6'$). It might be the case that the number k' such as equation ($B.7'$) is satisfied is greater than n_4. However, as (B.5) is satisfied, we can obtain a NAP by removing $k'' < k'$ type 4 voters such as:

$$sn_1 - sn_2 + n_3 + (1-s)n_4 - k''(1-s) - n_5 + (s-1)n_6 \leq 0 \qquad (B.2')$$

Then, for this k'', a_3 reaches the second stage and still beats a_1 as $k'' < k'$. □

The last two propositions concern NPP and PAP.

Proposition 3 *Let us assume that a_1 beats a_2 at the second stage for the voting situation $\tilde{n} = (n_1, n_2, n_3, n_4, n_5, n_6)$ and the scoring run-off rule g_{w_s}, $s \neq 1$. This situation may lead to a Negative Participation Paradox in favor of a_2 by adding type 5 voters if and only if:*

$$\begin{cases} n_1 + (1-s)n_2 + sn_3 - sn_4 + (s-1)n_5 - n_6 \geq 0 & (C.1) \\ (-1+s-s^2)n_1 + (s^2-1)n_2 + (1-2s)n_3 & \\ \quad +(1-s+s^2)n_4 + (2s-s^2)n_6 \geq 0 & (C.2) \\ n_1 + n_2 - n_3 - n_4 + n_5 - n_6 \geq 0 & (C.3) \\ -sn_1 + (2s-2)n_2 + (1-2s)n_3 + n_4 + sn_6 \geq 0 & (C.4) \end{cases}$$

Proposition 4 *Let us assume that a_1 beats a_2 at the second stage for the voting situation $\tilde{n} = (n_1, n_2, n_3, n_4, n_5, n_6)$ and the scoring run-off rule g_{w_s}, $s \neq 0$. This situation may lead to a Positive Abstention Paradox in favor of a_2 by deleting some type 3 voters if and only if:*

$$\begin{cases} n_1 + (1-s)n_2 + sn_3 - sn_4 + (s-1)n_5 - n_6 \geq 0 & (D.1) \\ (s^2-1)n_1 + (-1+s-s^2)n_2 + (2s-s^2)n_4 & \\ \quad +(1-2s)n_5 + (s^2-s+1)n_6 \geq 0 & (D.2) \\ n_1 + n_2 - n_3 - n_4 + n_5 - n_6 \geq 0 & (D.3) \\ (s-1)n_1 - n_2 + 2sn_4 + (1-2s)n_5 + (1-s)n_6 \geq 0 & (D.4) \\ n_1 + (1-s)n_2 - sn_4 + (s-1)n_5 - n_6 \leq 0 & (D.5) \end{cases}$$

As their proofs are similar to the ones of Propositions 1 and 2, we shall just sketch them out.

Proof of Proposition 3. Inequalities (C.1) and (C.3) respectively mean that $S_1 \geq S_3$ and $M_{12} \geq 0$. Inequality (C.2), which is more complex, calls some comments. In the NPP case, the paradox may occur if and only if there exist k voters such as the new score of a_2 is still greater than the score of a_1, while a_3 now beats a_1. This gives the inequalities $S_2 - S_1 - ks \geq 0$ and $S_3 + (1-s)k - S_1 \geq 0$. Combined together, they give the condition $(1-s)(S_2 - S_3) \geq (S_1 - S_3)$, that is ($C$.2). ($C$.4) expresses the following constraints: We must simultaneously add k voters such as the new score of a_1 becomes lower than a_3's one ($S_1 + sk \leq S_3 + k$) and such as a_2 will win the second stage against a_3 ($M_{23} - k \geq 0$). These two conditions, combined together, give $(1-s)M_{23} \geq S_1 - S_3$, that is, ($C$.4). □

Proof of Proposition 4. (D.1) means that, for the initial voting situation, $S_1 \geq S_3$. However, when all the type 3 voters abstain, S_3 becomes greater (D.5). As usual, (D.3) means that $M_{12} \geq 0$. To get a PAP, we must remove k type 3 voters such as:

$$\begin{cases} S_2 - k & \geq & S_3 \\ S_1 - sk & \leq & S_3 \\ M_{23} - k & \geq & 0 \end{cases}$$

Combined together, the first two inequalities give $s(S_2 - S_3) \geq S_1 - S_3$, that is ($D$.2). The last two ones induce $sM_{23} \geq S_1 - S_3$, that is, (D.4). □

2.3 Relationships with the abstention paradox

The negative abstention paradox and the positive abstention paradox are two particular sub cases of the no-show paradox: We just consider either the top ranked alternative or the bottom ranked one. However, for the three candidate case, these two possibilities encompass all the occurences of a no-show paradox.

Proposition 5 *For three candidate elections, suppose that a voting situation may lead to an abstention paradox under a scoring run-off system, that is, a group of voters may successfully manipulate the outcome by abstaining. Furthermore, let us assume that a_3 is eliminated at the first stage, and that a_1 beats a_2 in the pairwise comparison. Then, the two following cases hold:*

- *If $S_2 > S_1 > S_3$ at the first stage, the only type of voter which may manipulate the outcome is type 3, $a_2 \succ a_1 \succ a_3$. A positive abstention paradox will occur in favor of a_2 and against a_1.*
- *If $S_1 \geq S_2 > S_3$ at the first stage, the voter type which has an incentive to abstain is type 4, $a_2 \succ a_3 \succ a_1$. This is a negative abstention paradox situation in favor of a_3 and against a_1.*

The proof is trivial: one just has to check that, in both sub cases, no other type has an incentive to manipulate by abstaining. A consequence of Proposition

5 is that, for $m = 3$ the likelihood of a no-show paradox is the sum of the probabilities of the positive and negative abstention paradoxes. Unfortunately, this result does not generalizes to the case of participation paradoxes: there exist voting situations and scoring run-offs leading to both positive participation and negative participation paradoxes. The profile displayed in Table 1 gives such an example for the Borda run-off. Using the vector $w_B = (2, 1, 0)$, the scores for this initial profile are $S_1 = 49$, $S_2 = 55$ and $S_3 = 46$. Then a_1 beats a_2 with a 10 vote margin. Adding four voters with preference $a_3 \succ a_1 \succ a_2$ changes the scores into $S_1 = 53$, $S_2 = 55$ and $S_3 = 54$. The second stage winner is now a_2, by 35 votes against 19, which leads to a negative participation paradox. Similarly, add 10 voters whose preference type is $a_1 \succ a_3 \succ a_2$ to the initial profile. The new scores are $S_1 = 69$, $S_2 = 55$ and $S_3 = 56$ and a_3 beats a_1 with a two vote margin; This is sufficient to create a positive participation paradox.

Table 1

Preference	Number of voters
$a_1 \succ a_2 \succ a_3$	15
$a_1 \succ a_3 \succ a_2$	1
$a_2 \succ a_1 \succ a_3$	3
$a_2 \succ a_3 \succ a_1$	17
$a_3 \succ a_1 \succ a_2$	14

3 The likelihood of variable electorate paradoxes under IC and IAC assumptions

3.1 Probability models

Propositions 1 to 5 give necessary and sufficient conditions for the existence of abstention and participation paradoxes. However, these results do not provide any information about the likelihood of these situations. In order to obtain such results, we have to set some assumptions about the likelihood of the different voting situations. Moreover, these assumptions should reflect to some extent a concept of "impartiality", and do not favor a priori any alternative.

In Social Choice literature, two assumptions about the occurrence of the different voting situations enable us to formalize more precisely these requirements. The most common one is the Impartial Culture condition, which assumes that each voter selects her preference according to a uniform probability distribution. Thus, each voter has a probability $\frac{1}{6}$ to pick the type t preference and the likelihood of a specified voting situation is described by a multinomial distribution:

$$Prob(\tilde{n} = (n_1, n_2, n_3, n_4, n_5, n_6)) = \frac{n!}{n_1!n_2!n_3!n_4!n_5!n_6!}6^{-n}$$

A drawback of this approach is that it puts more weight on the voting situations close to $\tilde{n} = (n/6, \ldots, n/6)$, and neglects the impact of unanimous or quasi unanimous profiles. On the other hand, the Impartial Anonymous Culture condition considers that each voting situation is equally likely. Thus,

$$Prob(\tilde{n} = (n_1, n_2, n_3, n_4, n_5, n_6)) = \frac{120n!}{(n+5)!} \; \forall \tilde{n}$$

To summarize, the IC assumption could be viewed as a case of extremely splitted opinion, while the IAC takes into account more homogeneous societies. Both are widely used in Social Choice literature, and we shall evaluate the likelihood of the different paradoxes we are concerned with for IC as well as IAC. For more details about these assumptions and their extensive use in the literature, see Berg and Lepelley (1994) and Gehrlein (1997).

3.2 Results for small electorate

Let $P(X, s, n)$ be the likelihood of the voting situations that may lead to the paradox X under the scoring run-off method defined by $w_s = (1, s, 0)$ for the population of size n. X stands for PPP, NPP, PAP, NAP or ABS (ABS denoting the abstention paradox). For small electorate, Propositions 1 to 4 make possible a computer enumeration of the voting situations giving rise to the various paradoxes and -a probability model being given- the computation of $P(X, s, n)$ for every $s \in [0, 1]$ and every $X \in \{PPP, NPP, PAP, NAP\}$; then, from Proposition 5, $P(ABS, s, n)$ is obtained by summing $P(PAP, s, n)$ and $P(NAP, s, n)$. We have done these computations for $s = 1/2$ (Borda run-off), $s = 0$ (plurality run-off) and $s = 1$ (antiplurality run-off). Tables 2 to 4 show the computed values of $P(X, s, n)$ under IAC as well as under IC for $n = 3, 4, 5, ..., 33$. Before providing some comments on these results, two preliminary remarks are in order. First, neither PPP nor PAP can occur under plurality run-off; hence, for every n, $P(PPP, 0, n) = P(PAP, 0, n) = 0$ and $P(ABS, 0, n) = P(NAP, 0, n)$. Similarly, NPP and NAP cannot occur when antiplurality run-off is used: $P(NPP, 1, n) = P(NAP, 1, n) = 0$ and $P(ABS, 1, n) = P(PAP, 1, n)$. Second, it is easily seen from Propositions 1 and 2 that, for $s = \frac{1}{2}$, inequalities $(A.1)$ to $(A.4)$ are equivalent to inequalities $(B.1)$ to $(B.4)$; furthermore, it can be checked that inequalities $(B.1)$ to $(B.4)$ imply $(B.5)$ when $s = \frac{1}{2}$. Consequently, $P(NAP, \frac{1}{2}, n) = P(PPP, \frac{1}{2}, n)$ for every n. For this reason, NAP does not appear in Table 2 although this paradox can occur under Borda run-off.

The main conclusions that emerge from the examination of Tables 2, 3 and 4 are the following:

- First, the values we obtain are more important for the IC case than with the IAC assumption. This is not an unusual feature in Social Choice literature, as the IAC assumption introduces some degree of homogeneity in voters' preferences (the more homogeneous is a society, the less likely is the occurrence of a paradox).
- The likelihood of the various paradoxes can be surprisingly high (more than 50%) for very small values of n, the number of voters. However, when

Table 2. Probabilities of paradoxes, Borda run-off, n voters

	IAC				IC			
n	PPP	NPP	PAP	ABS	PPP	NPP	PAP	ABS
3	0.10714	0.10714	0.10714	0.21428	0.16667	0.16667	0.16667	0.33333
4	0.28571	0.47619	0.33333	0.61905	0.41667	0.66667	0.52778	0.94444
5	0.07143	0.09524	0.07143	0.14286	0.13889	0.16204	0.13889	0.27778
6	0.15584	0.28571	0.19481	0.34990	0.25849	0.45396	0.37423	0.63272
7	0.05303	0.07576	0.05303	0.10606	0.11253	0.14253	0.12153	0.23405
8	0.10256	0.20047	0.13054	0.23310	0.19279	0.35408	0.29982	0.49261
9	0.04795	0.06593	0.04496	0.09291	0.09627	0.12665	0.10978	0.20605
10	0.07393	0.15185	0.09590	0.16983	0.15732	0.29565	0.25630	0.41362
11	0.04258	0.05907	0.03846	0.08104	0.08615	0.11496	0.10143	0.18758
12	0.05915	0.12314	0.07660	0.13575	0.13494	0.10622	0.22732	0.36225
13	0.03782	0.05322	0.03361	0.07143	0.07934	0.10622	0.09530	0.17464
14	0.04954	0.10423	0.06347	0.11300	0.11943	0.22902	0.20631	0.32574
15	0.03483	0.04915	0.03057	0.06540	0.07439	0.09947	0.09060	0.16499
16	0.04275	0.09052	0.05425	0.09701	0.10800	0.20799	0.19022	0.29822
17	0.03235	0.04602	0.02802	0.06038	0.07056	0.09408	0.08683	0.15739
18	0.03834	0.08060	0.04779	0.08612	0.09921	0.19150	0.17741	0.27662
19	0.03021	0.04334	0.02597	0.05618	0.06745	0.08967	0.08371	0.15116
20	0.03501	0.07307	0.04280	0.07781	0.09224	0.17820	0.16691	0.25914
21	0.02864	0.04123	0.02445	0.05309	0.06486	0.08597	0.08105	0.14591
22	0.03233	0.06704	0.03887	0.07120	0.08656	0.16720	0.15809	0.24466
23	0.02729	0.03950	0.02314	0.05043	0.06263	0.08282	0.07874	0.14137
24	0.03031	0.06225	0.03582	0.06614	0.08185	0.15800	0.15057	0.23241
25	0.02610	0.03798	0.02202	0.04812	0.06069	0.08010	0.07670	0.13739
26	0.02867	0.05834	0.03330	0.06197	0.07787	0.15014	0.14404	0.22191
27	0.02518	0.03671	0.02112	0.04630	0.05899	0.07772	0.07488	0.13386
28	0.02728	0.05504	0.03120	0.05847	0.07447	0.14334	0.13832	0.21279
29	0.02437	0.03562	0.02033	0.04470	0.05746	0.07561	0.07323	0.13070
30	0.02615	0.05327	0.02946	0.05561	0.07152	0.13740	0.13325	0.20477
31	0.02363	0.03465	0.01964	0.04327	0.05610	0.07374	0.07173	0.12783
32	0.02519	0.04991	0.02797	0.05316	0.06894	0.13216	0.12871	0.19766
33	0.02302	0.03380	0.01905	0.04208	0.05486	0.07205	0.07036	0.12522

interpreting this conclusion, we have to keep in mind the tie-breaking method we use in our analysis: Ties are always broken in a way that leads to the paradox. This method clearly leads to an over estimation of the probabilities when n is small.

- For each of the paradoxes we have considered, the probabilities of occurrence tend to decrease when the number of voters becomes larger, but this decreasing is not monotonic. Limiting values for these probabilities are provided in the subsequent paragraph.

3.3 Results for large electorate

We here provide results for large populations. By large, we mean that the number of individuals is large enough so that we can approximate the distribution of the voting situations by a multivariate normal law in the IC case, and by a Dirichlet

Table 3. Probabilities of paradoxes, plurality run-off, n voters

	IAC		IC	
n	NPP	NAP(ABS)	NPP	NAP(ABS)
3	0.21429	0.21429	0.33333	0.33333
4	0.57143	0.38095	0.77778	0.55556
5	0.14386	0.07143	0.27778	0.09259
6	0.36364	0.19481	0.68416	0.44753
7	0.12121	0.10606	0.20405	0.18004
8	0.28438	0.13520	0.44410	0.17404
9	0.13786	0.08991	0.27806	0.17004
10	0.27778	0.11988	0.46457	0.27040
11	0.11538	0.07005	0.23252	0.10452
12	0.22204	0.12023	0.45793	0.26090
13	0.10924	0.07633	0.20952	0.14550
14	0.18060	0.08514	0.37171	0.15497
15	0.11378	0.07237	0.24783	0.13734
16	0.17249	0.09317	0.37440	0.19960
17	0.10390	0.06152	0.21815	0.10029
18	0.16298	0.08416	0.38223	0.20074
19	0.10051	0.06592	0.20680	0.12782
20	0.15008	0.07544	0.33181	0.13940
21	0.10289	0.06312	0.23184	0.12207
22	0.14136	0.07462	0.33225	0.16785
23	0.09717	0.05763	0.21059	0.09625
24	0.14015	0.07366	0.43205	0.17036
25	0.09515	0.05962	0.20343	0.11702
26	0.13009	0.06575	0.30709	0.12869
27	0.09648	0.05837	0.22202	0.11283
28	0.12704	0.06780	0.30717	0.14945
29	0.09276	0.05460	0.20557	0.09302
30	0.12468	0.06528	0.31658	0.15188
31	0.09145	0.05610	0.20048	0.10966
32	0.11967	0.06184	0.29019	0.12095
33	0.09235	0.05506	0.21529	0.10650

distribution in the IAC case. For three candidate elections and large populations, the computations can be undertaken and our results provide exact figures (see Section 5 for the details). Table 5 displays the values of $P(X, s, \infty)$ for the plurality run-off (PL), the Borda run-off (BO), and the antiplurality run-off (AP) under the IC and IAC assumptions. In the IC case, we have been able to derive formulas for all the values of $s \in [0, 1]$ and the five paradoxes (these formulas are described in Section 5.1). The corresponding figures are displayed in Table 6.

These figures call some comments. First, the limiting values we obtain for the IC case show a symmetry between *NPP* and *PAP* under Borda run-off. Similarly, we notice that $P(PPP, 1, \infty) = P(NAP, 0, \infty)$ and $P(NPP, 0, \infty) = P(PAP, 1, \infty)$ when IC is assumed. These symmetries are directly related to the fact that the IC assumption describes extremely splitted societies. When we consider all the situations rather than only those close to the situation $\tilde{n} = (\frac{n}{6}, \ldots, \frac{n}{6})$, *i.e.* when we move from IC to IAC, these symmetries vanish.

Table 4. Probabilities of paradoxes, antiplurality run-off, n voters

	IAC		IC	
n	PPP	PAP(ABS)	PPP	PAP(ABS)
3	0.21429	0.21429	0.33333	0.33333
4	0.14286	0.23810	0.11111	0.33333
5	0.16667	0.09524	0.23148	0.13889
6	0.20779	0.25974	0.43724	0.57613
7	0.09091	0.09091	0.11403	0.20405
8	0.14452	0.16317	0.29207	0.39209
9	0.09890	0.09890	0.17229	0.22330
10	0.10390	0.12587	0.15625	0.31671
11	0.09066	0.07418	0.15993	0.17017
12	0.10957	0.13187	0.25463	0.39270
13	0.07073	0.07073	0.10512	0.20001
14	0.09288	0.10578	0.20607	0.33965
15	0.07663	0.07624	0.13802	0.22625
16	0.07814	0.09524	0.14194	0.31025
17	0.07177	0.06448	0.13458	0.18889
18	0.08167	0.09682	0.19545	0.34813
19	0.06183	0.06310	0.09916	0.20529
20	0.07612	0.08549	0.17005	0.31843
21	0.06485	0.06567	0.12212	0.22684
22	0.06748	0.08027	0.13039	0.30362
23	0.06264	0.05977	0.12075	0.20292
24	0.06902	0.08155	0.16602	0.32993
25	0.05751	0.05848	0.09488	0.21046
26	0.06611	0.07483	0.15002	0.30684
27	0.05888	0.06031	0.11260	0.22816
28	0.06118	0.07202	0.12191	0.29816
29	0.05731	0.05643	0.11190	0.20990
30	0.06254	0.07269	0.14822	0.31794
31	0.05399	0.05575	0.09167	0.21530
32	0.06043	0.06855	0.13710	0.29963
33	0.05503	0.05683	0.10611	0.22855

It is well known that the plurality run-offs are the *only* scoring run-off methods which are immune to PPP and PAP, while a symmetric result hold for anti-plurality run-offs with NPP and NAP (see Smith, 1973; Lepelley, 1989; Saari, 1994; Merlin, 1996). But here, we get a stronger result under the IC assumption: The likelihood of PPP and PAP is increasing with s and is maximal for $s = 1$. A similar result holds for the antiplurality run-off with NPP and NAP as s decreases.

The abstention paradoxes can be easily interpreted. The figures give the probability that a group of individuals has an incentive to abstain (whether or not they effectively implement this strategy). Moreover, by summing the NAP and PAP data, we obtain the probability of a no-show paradox. The comparison with the probabilities of More is Less and Less is More paradoxes obtained by Lepelley, Chantreuil and Berg (1996) for the IAC case (see Table 7) indicates similar orders of magnitude. Still with the IAC assumption, the data clearly indicate that Borda run-off does better than plurality or plurality run-off. This is also true with the IC assumption, but the picture is somewhat different: The advantage of Borda

Table 5. Probabilities of paradoxes for plurality run-off, Borda run-off, and antiplurality run-off

	IAC			IC		
Properties	PL	BO	AP	PL	BO	AP
PPP	0	0.014[a]	$\frac{43}{1125}$ (0.0382)	0	0.02198	0.05583
NPP	$\frac{7}{96}$ (0.0729)	0.020[a]	0	0.16230	0.02825	0
PAP	0	0.010[a]	$\frac{49}{1152}$ (0.0425)	0	0.02825	0.16230
NAP	$\frac{47}{1152}$ (0.0408)	0.014[a]	0	0.05583	0.02198	0
ABS	$\frac{35}{648}$ (0.054)	0.0214[a]	$\frac{49}{1152}$ (0.0425)	0.05583	0.05022	0.16230

[a] Estimates obtained by using Monte-Carlo simulations.

run-off over plurality is tiny, and the minimal value for the abstention paradox is obtained with $s \simeq 0.4$.

Giving an interpretation for the participation paradoxes is harder. For example, the figure 16.23% for a NPP with plurality run-off indicates that in these situations, extra voters may hurt themselves by voting. But where do they come from ? Such scenario would imply that some voters have already given their ballots, while others haven't, and that this second group knows the results obtained by the first voters. Although such situations are not completely impossible, it might be difficult to find convincing examples for this scenario.

One last remark is worth noticing: whatever the assumption, the paradox and the rule we consider, our figures are systematically lower than the figures obtained for the likelihood of coalitional manipulations (CM, see Table 7). By coalitional manipulation, we mean that some voters may change the outcome in their favor by misrepresenting their sincere preference, *without abstaining*. However, the data we display are just partial, only concerning the plurality and antiplurality run-offs for three candidate elections. Could it be possible that the situations that lead to a no-show paradox would be also manipulable by coalitions of voters ? The next section will indicate to which extent this comment is funded.

4 Is abstaining the only possible strategy ?

The previous section describes the conditions under which some voters sharing the same preference type may choose to abstain in order to obtain a better result. However, one may wonder whether abstention was the optimal strategy in these

Table 6. The likelihood of paradoxes under IC for large populations as λ varies

λ	$P(PPP, s, \infty)$	$P(NPP, s, \infty)$	$P(PAP, s, \infty)$	$P(NAP, s, \infty)$	$P(ABS, s, \infty)$
1	0.0558269	0	0.1622955	0	0.1622955
0.9	0.0478496	0.0059425	0.1340079	0.0070622	0.1410701
0.8	0.0403399	0.0114469	0.1010195	0.0123198	0.1133393
0.7	0.0335467	0.0163528	0.0684872	0.0160035	0.0844907
0.6	0.0274304	0.0211161	0.0431940	0.0188015	0.0619955
0.5	0.0219771	0.0282491	0.0282491	0.0219771	0.0502262
0.4	0.0188015	0.0431940	0.0211161	0.0274304	0.0485465
0.3	0.0160035	0.0684872	0.0163528	0.0335467	0.0498995
0.2	0.0123198	0.1010195	0.0114469	0.0403399	0.0517868
0.1	0.0070622	0.1340079	0.0059425	0.0478496	0.0537921
0	0	0.1622955	0	0.0558269	0.0558291

Table 7. Other manipulation paradoxes for plurality and antiplurality run-off

Assumption	Paradox	P(X, ∞, 0)	P(X,∞, 1)
	MLP	0.0451[a] $\left(\frac{12}{288}\right)$	0.0556[a] $\left(\frac{7}{18}\right)$
IAC	LMP	0.0197[a] $\left(\frac{17}{864}\right)$	0.0648[a] $\left(\frac{7}{108}\right)$
	CM	0.1111[b] $\left(\frac{1}{9}\right)$	40.43056[b] $\left(\frac{31}{72}\right)$
IC	CM	0.16887[c]	1[d]

[a] Lepelley, Chantreuil and Berg (1996);
[b] Lepelley and Mbih (1994);
[c] Lepelley and Valognes (1999);
[d] Kim and Roush (1997)

situations. Could a "classical manipulation", that is a coordinated change of preferences for a group of voters, give exactly the same result ? Lepelley (1995) showed that the set of voting situations leading to the possibility of a no-show paradox was a proper subset of the set of manipulable situations for plurality run-off. This section will generalize this result to all the scoring run-offs, and a wider class of voting mechanisms. In order to achieve this aim, we shall borrow some tools of the geometric approach, introduced in Social Choice literature by Saari (1994).

4.1 The definition of the domains

Until now, we considered voting procedures as functions from the set of all the possible voting situations into A, the set of possible candidates. For most of the voting rules, it is possible to go a step further and to define them on the simplex $Si(m!)$:

$$Si(m!) = \left\{ p = (p_1, \ldots, p_t, \ldots, p_{m!}) \in \mathbb{R}^{m!} : \sum_{t=1}^{m!} p_t = 1, p_t \geq 0 \right\}$$

where p_t gives the fraction of voters with the type t preference. We do not lose anything by working with profiles p rather than with voting situations $\tilde{n}$ as long as the voting rule is *homogeneous*[4]: when we replicate each individual preference k times, $k \in \mathbb{N}$, to obtain a new population, the social outcome should stay unchanged. For example, the profile on Table 1 can be expressed as $p = (\frac{15}{50}, \frac{1}{50}, \frac{3}{50}, \frac{17}{50}, \frac{14}{50}, 0)$. Similarly, this point in $Si(m!)$ represents $\tilde{n} = (15, 1, 3, 17, 14, 0)$ as well as all the voting situations $k.\tilde{n}$, $k \in \mathbb{N}$.

By defining the set of possible profiles as the set of rational points in the simplex $Si(m!)$, we are now able to use all the tools of geometry and linear algebra for the study of voting procedures. However, there is one limit to this setup. When we state a result for a profile in $Si(m!)$, we know that there exists a voting situation that gives the same result, but we don't know what is the size of this population. It might be the case that the theorem is not true for small numbers of individuals.

In this context, let us consider a voting procedure g as a function from $Si(m!)$ into $2^A \backslash \emptyset$, the set of all the non-empty subsets of A. For g, the *domain of an alternative a* is the set of profiles in $Si(m!)$ which lead to this outcome:

$$D_g(a) = \{p \in Si(m!) : a \in g(p)\}$$

The simplest way to describe a domain is to characterize its boundaries. Fortunately, for most voting processes, the boundaries are just portions of hyperplanes. In particular, this is true for the scoring rules (see Young, 1975; Saari, 1994) as well as for the scoring run-offs. For example, the inequalities $(A.1)$ to $(A.3)$ clearly show that the conditions that lead to a_1's victory are linear (just divide them by n and replace the inequalities by equalities to obtain the exact equations in $Si(m!)$). The linearity of the boundaries is not a characteristic feature of the scoring rules, and this property has been also used for the study of other voting rules like the Copeland method (see Merlin and Saari, 1998) or the Kemeny rule (see Saari and Merlin, 2000).

Table 8. Boundaries of the domain of a_1 for scoring run-off $w_s = (1, s, 0)$

Vectors	Coordinates	Boundary with
M^{12}	$(1, 1, -1, -1, 1, -1)$	$D(a_2)$
M^{13}	$(1, 1, 1, -1, -1, -1)$	$D(a_3)$
S^{12}	$(1-s, 1, s-1, -1, s, -s)$	$D(a_3)$
S^{13}	$(1, 1-s, s, -s, s-1, -1)$	$D(a_2)$
S^{23}	$(s, -s, 1, 1-s, -1, s-1)$	$D(a_3)$
S^{32}	$(-s, s, -1, s-1, 1, 1-s)$	$D(a_2)$

For a specified voting procedure, each linear boundary is characterized by its normal vectors. To precise the shape of a domain, we just have to choose a normal vector pointing toward the interior of the domain for each boundary. As an example, Table 8 displays these vectors for the domain of a_1 and the scoring

[4] This property was first introduced by Young (1975).

run-off $w_s = (1, s, 0)$. Vectors M^{12} and M^{13} describe the hyperplanes where the pairwise outcome is a tie between a_1 and a_2 or a_3. Vectors S^{12} and S^{13} characterize the situations where a_1 shares the last position at the first stage with another alternative. Vectors S^{23} and S^{32} represent the cases where, by changing a_1's second stage opponent, we can end up with a victory for a_1 instead of a defeat.

4.2 Changes of preferences that change the outcome

In $Si(m!)$, any change of preferences by a voter or a group of voters can be represented by a vector $d = p^f - p^i$, where p^i is the initial profile, while p^f is the final profile. By this way, we can describe many senarii involving changes of preferences, like monotonicity, coalitional manipulation, and all the different types of participation and abstention paradoxes.

Let us assume that a_2 is elected and that an abstention paradox occurs in favor of a_1. Moreover, let us consider in a first step that the voters who abstain share the same preference type. Thus, for these voters, $a_1 \succ a_2$. Let E_t be the preference profile where all the coordinates are zero's except the t^{th} one; E_t represents the voting situations for which all the individuals share the same type t preference. Let d be a change of preference which describe the abstention of k type t voters among n individuals. Thus,

$$\tilde{n}^f = \tilde{n}^i - kE_t \quad (1)$$

$$\Leftrightarrow (n-k)p^f = np^i - kE_t \quad (2)$$

$$\Leftrightarrow p^f = \gamma p^i + (1-\gamma)E_t \quad (3)$$

with $\gamma = \frac{n}{n-k}$, $\gamma > 1$. In turns,

$$d = p^f - p^i = (\gamma - 1)(p^i - E_t) \quad (4)$$

As this strategy is successful, the vector d must cross one of the boundaries between $D_g(a_2)$ and $D_g(a_1)$. This is the case if and only if $Nd > 0$ for at least one normal vector N pointing inside the domain of a_1, i.e. if there exists a normal vector N for one of the boundary hyperplanes such that:

$$d.N = (\gamma - 1)Np^i - (\gamma - 1)NE_t > 0 \quad (5)$$

As Np^i is negative by assumption (p^i is in the domain of a_2), a necessary condition for the abstention paradox to occur is that $NE_t < 0$. As an example, one may check in Table 8 that $S^{32}E_1 = -s < 0$, for $s \neq 0$. Thus, if p^i is close enough to the boundary $S^{32}p^i = 0$, it is possible for type 1 voters to manipulate the outcome by abstaining. This leads to a positive abstention paradox in favor of a_1. Similarly, $S^{32}E_4 = s - 1 < 0$, $s \neq 1$ shows that a NPP may also occur in some situations.

Equation (4) describes the abstention of a fraction of $\frac{\gamma-1}{\gamma} = \frac{k}{n}$ voters. It is easy to generalize this equation into $d = (\gamma - 1)(p^i - p^a)$, where p^a describes the preference profiles of the people who abstain, whether or not they share the same preference type. Then, a necessary condition for abstention behavior to be successful for a_1 against a_2 is $Np^a < 0$, with all the individuals in profile p^a preferring a_1 to a_2.

Nevertheless, consider this other scenario: Instead of abstaining, the voters in p^a decide to coordinate themselves in order to report a structure of preference equivalent to the one given by the vector p^i. It is exactly as if the k voters had decided to mimic the average opinion! Thus, the change d is $\frac{(\gamma-1)}{\gamma}(p^i - p^a)$. This "camouflage behavior" is equally successful as the abstention, as we got $N(p^i - p^a) < 0$ in both cases, with just a different scalar. Similarly, instead of mimicking the average opinion or abstaining, these voters could have decided to report unanimously preference t'. Thus the change becomes $d = \frac{(\gamma-1)}{\gamma}(E_{t'} - p^a)$, with $\gamma > 1$.

Proposition 6 *Consider a scoring run-off method characterized by a sequence of scoring vectors $W = \{w^{k_1}, w^{k_2}, \ldots, w^{k_g}, \ldots, w^{k_h}\}$. Let $\tilde{n}$ be a voting situation leading to a no-show paradox. More precisely, assume that k voters who prefer a_1 to a_2 favor a_1 against a_2 by abstaining. Thus, provided that the initial profile is such that only one boundary can be crossed by the coordinated actions of the k voters, they can manipulate the outcome in the same way by pretending of being unanimously a certain type of voter rather than by abstaining. At least half of the possible preference types will lead to this result.*

Proof of Proposition 6. For the profile p^i, the abstention strategy is successful and one hyperplane, corresponding to some equation $S_p = S_q$, is crossed. Thus, $(\gamma - 1)S^{pq}(p^i - p^a) > 0$, with S^{pq} pointing toward the domain of a_1. Assume now that the k voters who abstained decide to report unanimously preference type t instead of abstaining. This shifts the profile from the point p^f to p^m, the new profile, with:

$$np^m = (n-k)p^f + kE_t \tag{6}$$

$$\Leftrightarrow p^m = \frac{1}{\gamma}p^f + \frac{\gamma-1}{\gamma}E_t \tag{7}$$

By definition, $S^{pq}p^f > 0$, as p^f is in $D_g(a_1)$. It is easy to check that $S^{pq}E_t \geq 0$ for at least half of the preference types. This is due to the fact that, for the scoring rule w_g^k, the dot product $S^{pq}E_t$ is equal to the difference of scores between the two alternatives a_p and a_q in the t^{th} preference type reduced to a certain subset of candidates. By interchanging the positions of a_p and a_q in the t^{th} ordering, everything else unchanged, we obtain a new preference type, t'. However, the difference of scores between a_p and a_q has been reversed too. $S^{pq}E_{t'} = -S^{pq}E_t$. Thus at least half of preference type gives $S^{pq}E_t \geq 0$. So, for at least half of the preference types, $S^{pq}p^m > 0$, and provided that only one boundary can be crossed, the new profile stays on the good side of the hyperplane. □

We assume in Proposition 6 that only one boundary can be reached by the k voters when they change their preferences. In fact, if the run is close enough

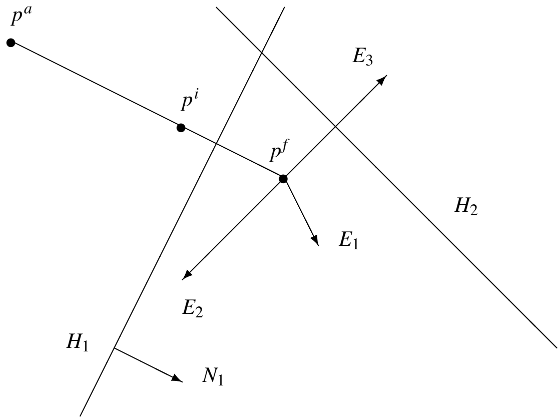

Figure 1

between more than two candidates, it might be the case that a change of preferences crosses several boundaries, leading to an unexpected outcome. Such a situation is described in Figure 1. By abstaining, k individuals change the outcome (hyperplane H_1 is crossed), pushing the outcome from p^i to p^f. If they decide to come back with a new strategy E_t, they will push again the profile in the direction of E_t. Although $N_1 E_3 > 0$, this strategy may not be successful as, by adopting this preference, another boundary, H_2, is also crossed. Similarly, although $N_1 E_2 < 0$, this shift is not sufficient to cross back the hyperplane H_1: this supports the assertion that in some cases, more than half of the possible preferences are possible strategies. When the population is large, it should be possible to apply Proposition 6 almost everywhere, the likelihood of being close to several hyperplanes at the same time being small. Anyway, for small populations, or nearby the equally splitted profile $p = (\frac{1}{m!}, \ldots, \frac{1}{m!})$, situations like the one described on Figure 1 may arise. Thus, can we still find a strategy that is exactly equivalent in its result to an abstention behavior ?

Proposition 7 *Consider a scoring run-off method characterized by a sequence of scoring vectors W. Let $\tilde{n}$ be a voting situation leading to a no-show paradox. More precisely, assume that k voters who prefer a_1 to a_2 favor a_1 against a_2 by abstaining. For any population size, one successful strategy for the k voters is to exhibit as a preference type the reversed order of elimination which leads to the victory of a_1 when they abstain: the alternative removed first is ranked last in their preference, the alternative removed in second is next to the last, etc, and the winner, a_1 is ranked first.*

Proof of Proposition 7. Suppose that the abstention strategy is successful. Then, at each stage the alternative(s) with the lowest score(s) are removed and a_1 always obtains enough points to remain in contention until the end of the elimination process. By adding back the k voters with exactly the reversed order of this elimination process as a preference type, nothing is changed through the elimination process: the first alternative to remove is still eliminated first, etc, and a_1 is still the winner[5]. □

Proposition 7 can be stated without the setup of the geometry of voting. However, Proposition 6 is clearly richer, as it states that in many situations there is not just one, but many different possible ways to manipulate the outcome. In that sense, a slight error in the way the voters coordinate and/or differences in the beliefs about the actual tallies will be benign most of the time. Moreover, the spirit of Proposition 6 can be applied to many other voting rules. As long as the boundaries of domains are hyperplanes H, with the properties that there exists a preference type t' such that $NE_t = -NE_{t'}$ for each $t \in \{1, \ldots, m!\}$ and each N, Proposition 6 applies: The voting situations which may lead to a no-show paradox are also likely to be manipulated by misrepresentation of preferences; Half of the possible preference types may lead to such results. This is especially true for two famous Condorcet voting rules, the Copeland method and the Kemeny rule, which boundaries have been described by Merlin and Saari (1998, 2000). We can even go further for the Kemeny rule, as the argument of Proposition 7 also applies (see Young and Levenglick, 1978).

Table 9

Preference	Number of voters
$a_1 \succ a_2 \succ a_3 \succ a_4$	1
$a_2 \succ a_3 \succ a_1 \succ a_4$	3
$a_2 \succ a_3 \succ a_4 \succ a_1$	1
$a_3 \succ a_4 \succ a_1 \succ a_2$	3
$a_4 \succ a_2 \succ a_1 \succ a_3$	4

This comment clearly raises the question of the existence of voting rules where the abstention is the only possible manipulation behavior for some voting situations. In fact, we can exhibit a voting rule and a voting situation where the only possible strategy is the abstention. The rule is a slight modification of antiplurality run-off we worked with until now; It is the *Balanced Voting*, introduced recently in the literature by Kim and Roush (1997). Instead of eliminating a precise number of alternatives at each stage, they suggest to remove from consideration the alternative(s) the score of which is lower than the average number of points a candidate receives; And all along the elimination process, they use the scoring vector $w = (1, \ldots, 1, 0)$. Consider the example in Table 9, with 12 voters and 4 candidates. First, notice that a_1 loses all the pairwise comparisons: $M_{12} = -4$, $M_{13} = -2$ and $M_{14} = -4$. Nevertheless she gets the highest score with $w = (1, 1, 1, 0)$: $S_1 = 11$, $S_2 = 9$, $S_3 = 8$ and $S_4 = 8$. The only way for a_1 to get elected is to be at the first stage the only candidate who has more points than the average number, $\frac{3\times 12}{4} = 9$, as she will lose all the final confrontations. By changing her preference, the unique voter with the preference $a_1 \succ a_2 \succ a_3 \succ a_4$

[5] A similar argument is used by Merlin (1996) to describe the properties of scoring run-offs.

cannot get this result. On the other hand, if she abstains, the new threshold is 8.25, and the scores of the other candidates are smaller ($S_2 = 8$, $S_3 = 7$ and $S_4 = 8$). There is no second stage, and a_1 is the winner. Of course, such situations may be extremely rare, but Proposition 7 does not apply for the Balanced Voting.

To conclude, notice that the arguments we developed here are also true for the participation paradoxes. Especially, when some voters may hurt themselves by giving their true preference ordering, it is possible for them to adopt the inverse of the initial elimination ordering to keep everything unchanged. And, in most of the cases, half of the possible preference orderings will avoid this paradox. Thus, we strongly believe that the no-show paradox is not an important flaw of the scoring run-off voting systems. As a possible strategic behavior, abstention is clearly dominated by other strategies. It remains possible that in a decision process, some voters realize ex-post that, by non-voting, they could have obtained a better outcome. These situations could be somewhat damaging for the trust the citizen or any member of a collectivity puts in the institutions. However, even in these cases, the figures we obtain suggest that with a limited amount of homogeneity in the distribution of the preferences, such events are unlikely.

5 Remaining Proofs

5.1 Probability computations under IC for large electorate

The technique we use to compute the probability of specific events under IC for large electorate is the one proposed by Saari and Tataru (1999). We just present here the results of the main computations; for more details see also Tataru and Merlin (1997) and Merlin, Tataru and Valognes (2000). The main argument is the following: the conditions which characterize a specified situation, like (A.1) to (A.4), lie in four dimensional subspace of $\mathbb{R}^6$. Their intersection with the unit sphere in $\mathbb{R}^4$ defines a three-dimensional spherical simplex C on its surface. Then, the desired probability is the ratio between the area of the surface of C and the area of the 4-dimensional hyper sphere. This area is evaluated through the Schläfli's formula (1950):

$$dvol_u(C) = \frac{1}{u-1} \sum_{1 \le j < k < l} vol_{u-2}\left(S_j \bigcap S_k\right) d\alpha_{jk};\ vol_0 = 1$$

where u is the dimension of the spherical simplex, l the number of facets, and α_{jk} the dihedral angle formed by the facets S_j and S_k. We provide in the next subsections the results we obtain for the values for the α_{jk}'s and the $S_j \bigcap S_k$'s for the positive participation paradox and the positive abstention paradox. For the IC case, the results for the NAP and NPP are respectively equivalent to the PPP and PAP cases.

5.1.1 Positive participation paradox

The vectors orthogonal to the hyperplanes defining the spherical simplex are:

$$\begin{aligned} N_1 &= (1, 1-s, s, -s, s-1, -1) \\ N_2 &= (s, -s, 1, 1-s, -1, s-1) \\ N_3 &= (1, 1, -1, -1, 1, -1) \\ N_4 &= (-2s, 0, -1-s, 2s-1, 1+s, 1) \end{aligned}$$

Thus, the differential angles between these hyperplanes are:

$$d\alpha_{12} = 0$$

$$d\alpha_{13} = \frac{1-2s}{\sqrt{6s^2-6s+5}\left(2s^2-2s+2\right)}$$

$$d\alpha_{14} = \frac{15s^4-9s^2+s^3+8s-6}{2\sqrt{9s^4+11s^2-12s^3-6s+3}\left(s^2-s+1\right)\left(5s^2+2\right)}$$

$$d\alpha_{23} = \frac{2s-1}{\sqrt{6s^2-6s+5}\left(2s^2-2s+2\right)}$$

$$d\alpha_{24} = \frac{17s^2-18s+10}{2\sqrt{6s^2-6s+5}\left(s^2-s+1\right)\left(5s^2+2\right)}$$

$$d\alpha_{34} = -\frac{2+5s}{\sqrt{14s^2+5+2s}\left(5s^2+2\right)}$$

The volumes of the intersection between the faces are:

$$S_1 \bigcap S_3$$
$$= \arccos\left(\frac{\left(3s^4-5s^3+6s^2-4s+1\right)\sqrt{3}}{\sqrt{\left(3s^2-3s+1\right)\left(s^2-s+1\right)\left(25s^4-28s^3+20s^2-12s+3\right)}}\right)$$

$$S_1 \bigcap S_4$$
$$= \arccos\left(\frac{s^2\sqrt{3}\left(s^2+s-1\right)}{\sqrt{\left(3s^2-3s+1\right)\left(s^2-s+1\right)\left(25s^4-28s^3+20s^2-12s+3\right)}}\right)$$

$$S_2 \bigcap S_3 = \arccos\left(\frac{\sqrt{12s^2-12s+6}}{4\sqrt{\left(s^2-s+1\right)\left(3s^2-3s+1\right)}}\right)$$

$$S_2 \bigcap S_4 = \pi - \arccos\left(\frac{\sqrt{12\,s^2 - 12\,s + 6}}{4\sqrt{\left(s^2 - s + 1\right)\left(3\,s^2 - 3\,s + 1\right)}}\right)$$

$$S_3 \bigcap S_4 = \arccos\left(\frac{(3 - 4\,s)\sqrt{4\,s^2 - 4\,s + 2}}{4\sqrt{25\,s^4 - 28\,s^3 + 20\,s^2 - 12\,s + 3}}\right)$$

Let us define:

$$I_1(s) = \sum_{1 \le i < j \le 4} S_i \bigcap S_j \, d\alpha_{ij}$$

We have to divide this value by 2 (as u = 3 in Schlafli's formula) and by $2\pi^2$, the area of the four dimensional hyper sphere, and then to multiply it by 6. Hence,

$$P(PPP, \lambda, \infty) = \frac{3}{2\pi^2}\int_0^{\lambda} I_1(s)ds$$

5.1.2 Positive abstention paradox

First, notice that the condition (D.5) is trivially satisfied for IC when the population is large. The vectors orthogonal to the hyperplanes defining the spherical simplex C are:

$$\begin{aligned}
N_1 &= (1, 1 - s, s, -s, s - 1, -1)\\
N_2 &= (s^2 - 1, -1 + s - s^2, 0, 2s - s^2, 1 - 2s, s^2 - s + 1)\\
N_3 &= (1, 1, -1, -1, 1, -1)\\
N_4 &= (s - 1, -1, 0, 2s, 1 - 2s, 1 - s)
\end{aligned}$$

Thus, the differential angles between the faces are:

$$d\alpha_{12} = \frac{\sqrt{3}}{2\,s^2 - 2\,s + 2}$$

$$d\alpha_{13} = \frac{1 - 2\,s}{\sqrt{6\,s^2 - 6\,s + 5}\left(2\,s^2 - 2\,s + 2\right)}$$

$$d\alpha_{14} = \frac{(5 - 4\,s)\left(3\,s^2 - 2\,s + 2\right)}{2\sqrt{6\,s^2 - 6\,s + 5}\left(s^2 - s + 1\right)\left(5\,s^2 - 4\,s + 2\right)}$$

$$d\alpha_{23} = \frac{s^2 + 2\,s - 2}{\sqrt{5 - 14\,s + 17\,s^2 - 12\,s^3 + 6\,s^4}\left(s^2 - s + 1\right)}$$

$$d\alpha_{24} = -\frac{18\,s^4 - 32\,s^3 + 25\,s^2 - 10\,s + 2}{\sqrt{\left(3\,s^2 - 4\,s + 2\right)\left(3\,s^2 - 2\,s + 1\right)}\left(s^2 - s + 1\right)\left(5\,s^2 - 4\,s + 2\right)}$$

$$d\alpha_{34} = \frac{7\,s - 4}{\sqrt{14\,s^2 - 14\,s + 5}\left(5\,s^2 - 4\,s + 2\right)}$$

and the volumes:

$$S_1 \bigcap S_2 = \arccos\left(\frac{s\,(s-1)}{3\,s^2 - 3\,s + 1}\right)$$

$$S_1 \bigcap S_3 = \pi - \arccos\left(\frac{\sqrt{12\,s^2 - 12\,s + 6}}{4\,\sqrt{\left(s^2 - s + 1\right)\left(3\,s^2 - 3\,s + 1\right)}}\right)$$

$$S_1 \bigcap S_4 = \arccos\left(\frac{\sqrt{12\,s^2 - 12\,s + 6}}{4\,\sqrt{\left(s^2 - s + 1\right)\left(3\,s^2 - 3\,s + 1\right)}}\right)$$

$$S_2 \bigcap S_3$$
$$= \arccos\left(\frac{s\left(3\,s^2 - 4\,s + 2\right)(s-1)\sqrt{3}}{\sqrt{\left(s^2 - s + 1\right)\left(3\,s^2 - 3\,s + 1\right)\left(25\,s^4 - 56\,s^3 + 52\,s^2 - 24\,s + 5\right)}}\right)$$

$$S_2 \bigcap S_4$$
$$= \arccos\left(\frac{(s-1)\left(s^3 - 4\,s^2 + 3\,s - 1\right)\sqrt{3}}{\sqrt{\left(s^2 - s + 1\right)\left(3\,s^2 - 3\,s + 1\right)\left(25\,s^4 - 56\,s^3 + 52\,s^2 - 24\,s + 5\right)}}\right)$$

$$S_3 \bigcap S_4 = \arccos\left(\frac{(5 - 4\,s)\sqrt{4\,s^2 - 4\,s + 2}}{4\,\sqrt{25\,s^4 - 56\,s^3 + 52\,s^2 - 24\,s + 5}}\right)$$

Let us define:

$$I_2(s) = \sum_{1 \le i < j \le 4} S_i \bigcap S_j \, d\alpha_{ij}$$

Again, we have to divide this value by 2 (as n = 3 in Schlafli's formula) and by $2\pi^2$, the volume of the four dimensional hyper sphere, and then to multiply it by 6. Hence,

$$P(PAP, \lambda, \infty) = \frac{3}{2\pi^2} \int_0^{\lambda} I_2(s) ds$$

5.2 Probability computations under the IAC assumption for large electorate

Although a general representation -similar to the ones we have obtained under IC- cannot be considered under IAC, it is possible to compute under this assumption the limiting probabilities of the paradoxes for specific scoring run-off methods. We give in this paragraph the details of our computations for the plurality run-off and the antiplurality run-off methods. The case of the Borda run-off method turned out to be very intricate; for this reason, we have resorted to simulation techniques in order to derive estimates for the desired probabilities.

5.2.1 Positive participation paradox, antiplurality run-off

Let $p_i = n_i/n$ (this notation is valid for each of the proofs that follow). Assume that a_1 is the antiplurality run-off winner and the electorate is large. From Proposition 1, a PPP in favor of a_3 occurs with s=1 if and only if

$p_1 + p_3 > p_4 + p_6,$
$p_1 + p_3 > p_2 + p_5,$
$p_1 + p_2 + p_5 > 1/2,$
$p_4 + 2p_5 + p_6 > 2p_1 + 2p_3.$

As $\sum p_i = 1$, this set of inequalities is easily seen to be equivalent to:

$$\begin{gathered}\tfrac{1}{3} < p_{13} < \tfrac{1}{2},\\ 1 - 2p_{13} < p_{46} < MIN(p_{13}, 2 - 4p_{13}),\\ \tfrac{2p_{13} - p_{46}}{2} < p_5 < 1 - p_{13} - p_{46},\\ 0 < p_3 < \tfrac{1}{2} - p_{46},\end{gathered}$$

with $p_{ij} = p_i + p_j$. In order to eliminate the MIN argument, we partition the set of situations satisfying the above inequalities in two subsets defined as follows:

$$\begin{gathered}\tfrac{1}{3} < p_{13} < \tfrac{2}{5},\\ 1 - 2p_{13} < p_{46} < p_{13},\\ \tfrac{2p_{13} - p_{46}}{2} < p_5 < 1 - p_{13} - p_{46},\\ 0 < p_3 < \tfrac{1}{2} - p_{46},\end{gathered}$$

and

$$\begin{gathered}\tfrac{2}{5} < p_{13} < \tfrac{1}{2},\\ 1 - 2p_{13} < p_{46} < 2 - 4p_{13},\\ \tfrac{2p_{13} - p_{46}}{2} < p_5 < 1 - p_{13} - p_{46},\\ 0 < p_3 < \tfrac{1}{2} - p_{46}.\end{gathered}$$

The desired probability is then given as:

$$6\left(\int_{p_{13}=1/3}^{2/5}\int_{p_{46}=1-2p_{13}}^{p_{13}}\int_{p_5=(2p_{13}-p_{46})/2}^{1-p_{13}-p_{46}}\int_{p_3=0}^{1/2-p_{46}} 120 p_{46}\,dp_{13}\,dp_{46}\,dp_5\,dp_3\right.$$
$$\left.+\int_{p_{13}=2/5}^{1/2}\int_{p_{46}=1-2p_{13}}^{2-4p_{13}}\int_{p_5=(2p_{13}-p_{46})/2}^{1-p_{13}-p_{46}}\int_{p_3=0}^{1/2-p_{46}} 120 p_{46}\,dp_{13}\,dp_{46}\,dp_5\,dp_3\right) = \frac{43}{1125}.$$

5.2.2 Negative abstention paradox, plurality run-off

Assume that a_1 is the plurality run-off winner. From Proposition 2, a NAP in favor of a_3 occurs with $s = 0$ if and only if:

$p_1 + p_2 > p_5 + p_6,$
$p_3 + p_4 > p_5 + p_6,$
$p_1 + p_2 + p_5 > 1/2,$

$2p_5 + 2p_6 > p_1 + p_2 + 2p_3,$
and
$p_5 + p_6 \geq p_3.$
Clearly, the ante-penultimate inequality implies the last one. Moreover, it can be checked that the four first inequalities are equivalent to:

$$\begin{gathered}\tfrac{1}{6} < p_{56} < \tfrac{1}{3},\\ MAX(p_{56}, 1 - 3p_{56}) < p_{34} < MIN(\tfrac{1}{2}, 1 - 2p_{56}),\\ 0 < p_6 < \tfrac{1}{2} - p_{34},\\ 0 < p_3 < \tfrac{3p_{56}+p_{34}-1}{2},\end{gathered}$$

with $p_{ij} = p_i + p_j$. To eliminate the MAX and MIN arguments, we partition the set of situations satisfying these inequalities in two subsets defined as follows:

$$\begin{gathered}\tfrac{1}{6} < p_{56} < \tfrac{1}{4},\\ 1 - 3p_{56} < p_{34} < \tfrac{1}{2},\\ 0 < p_6 < \tfrac{1}{2} - p_{34},\\ 0 < p_3 < \tfrac{3p_{56}+p_{34}-1}{2},\end{gathered}$$

and

$$\begin{gathered}\tfrac{1}{4} < p_{56} < \tfrac{1}{3},\\ p_{56} < p_{34} < 1 - 2p_{56},\\ 0 < p_6 < \tfrac{1}{2} - p_{34},\\ 0 < p_3 < \tfrac{3p_{56}+p_{34}-1}{2},\end{gathered}$$

Integrating over the domains defined by the above inequalities and multiplying by 6, we obtain $47/1152$.

5.2.3 Negative participation paradox, plurality run-off

Assume that a_1 is the plurality run-off winner. From Proposition 3, a NPP in favor of a_3 occurs with $s = 0$ if and only if
$p_1 + p_2 > p_5 + p_6,$
$p_3 + p_4 > p_1 + p_2,$
$p_1 + p_2 + p_5 > 1/2,$
and
$p_3 + p_4 > 2p_2.$
As $\sum p_i = 1$, this set of inequalities is equivalent to:

$$\begin{gathered}\tfrac{1}{3} < p_{34} < \tfrac{1}{2},\\ \tfrac{1-p_{34}}{2} < p_{12} < p_{34}\\ 0 < p_6 < \tfrac{1}{2} - p_{34},\\ 0 < p_2 < \tfrac{p_{34}}{2}.\end{gathered}$$

Integrating over the domain defined by the above inequalities and multiplying by 6, we obtain $7/96$.

5.2.4 Positive abstention paradox, antiplurality run-off

Assume that a_1 is the plurality runoff winner. From Proposition 4, a PAP in favor of a_3 occurs with s=1 if and only if

$p_1 + p_3 > p_4 + p_6,$
$p_4 + p_6 > p_2 + p_5,$
$p_1 + p_2 + p_5 > 1/2,$
$2p_4 > p_2 + p_5,$

and

$p_4 + p_6 > p_1.$

This set of inequalities is equivalent to:

$$\frac{1}{3} < p_{13} < \frac{1}{2},$$
$$\frac{1-p_{13}}{2} < p_{46} < p_{13}$$
$$p_{13} + p_{46} - \frac{1}{2} < p_1 < p_{46},$$
$$\frac{1-p_{13}-p_{46}}{2} < p_4 < p_{34}.$$

The desired probability immediately follows.

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