

Scrambling Time from Local Perturbations of the Eternal BTZ Black Hole

Andrius Štikonas

University of Edinburgh

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In collaboration with Paweł Caputa, Joan Simón, Tadashi
Takayanagi and Kento Watanabe

Density matrices

Pure state: vector in a Hilbert space

$$|\psi\rangle \in \mathcal{H}$$

Often quantum mechanics

$$|\psi\rangle \in \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

Can we find $|\psi_A\rangle$ corresponding to $|\psi\rangle$?

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Mixed state is probabilistic mixture of pure states:
state $|\Psi_i\rangle$ with probability p_i

Density matrix

$$\rho = \sum_n p_n |\Psi_n\rangle \langle \Psi_n|$$

Density matrices

$$\langle \psi | \mathcal{O}_A \otimes \mathbb{I} | \psi \rangle = \text{Tr}(\mathcal{O}_A \rho_A)$$

Thermal states obey **Boltzmann** distribution

$$p_n = \frac{1}{Z(\beta)} e^{-\beta E_n}$$

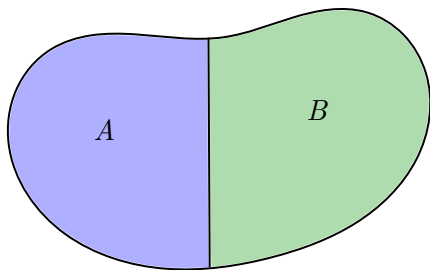
Thermal density matrix

$$\rho = \frac{1}{Z(\beta)} \sum_n e^{-\beta E_n} |\Psi_n\rangle \langle \Psi_n| = \frac{1}{Z(\beta)} e^{-\beta H}$$

Entanglement entropy

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

Trace over one Hilbert space



Reduced density matrix

$$\rho_A = \text{Tr}_B \rho$$

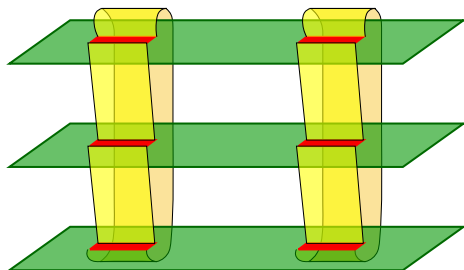
Entanglement entropy is von Neumann entropy of reduced density matrix

Entanglement entropy

$$S_A = -\text{Tr}_A (\rho_A \log \rho_A)$$

Replica trick

- Calculate $S_n(A) = \frac{1}{1-n} \log \text{Tr}_A(\rho_A^n)$ on n -sheeted Riemann surface

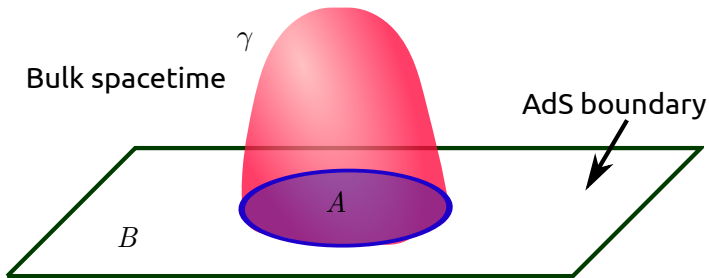


- Continue analytically
- $S_A = \lim_{n \rightarrow 1} S_n(A)$

Holographic entanglement entropy

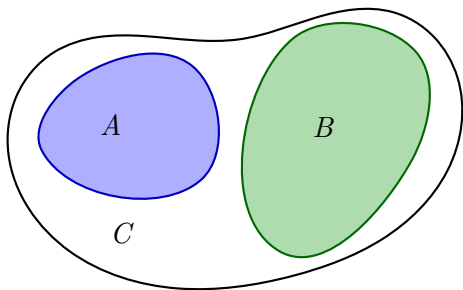
Ryu-Takayanagi formula

$$S_A = \min_{\partial\gamma = \partial A} \frac{\text{Area}(\gamma)}{4G_N}$$



Mutual information

Entanglement between
two systems A and B



Mutual information

$$I_{A:B} = S_A + S_B - S_{A \cup B} \geq \frac{(\langle \mathcal{O}_A \mathcal{O}_B \rangle - \langle \mathcal{O}_A \rangle \langle \mathcal{O}_B \rangle)^2}{2 \|\mathcal{O}_A\|^2 \|\mathcal{O}_B\|^2}$$

Provides an upper bound on correlators between A and B

Scrambling time

A natural question to ask is when correlators and entanglement get destroyed by the perturbation?

Scrambling time is defined by

$$I_{A:B}(t^*) = 0$$

Our goal is to compute the **scrambling time** for thermofield double state

Thermofield double state (TFD)

Two non-interacting 2d CFTs with Hilbert spaces $\mathcal{H}_L \cong \mathcal{H}_R$
Choose a particular **entangled** state

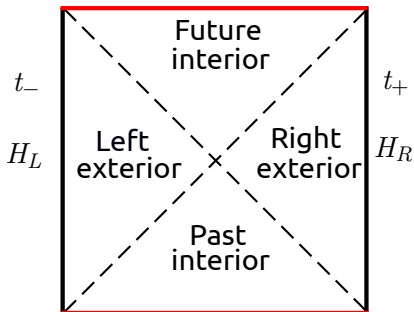
Thermofield double state in $\mathcal{H}_L \otimes \mathcal{H}_R$

$$|\Psi_\beta\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\frac{\beta}{2} E_n} |n\rangle_L |n\rangle_R$$

Tracing over \mathcal{H}_L or \mathcal{H}_R yields *thermal* reduced density matrix

Gravity dual description

Eternal BTZ BH is holographic dual of TFD



Two natural choices for Hamiltonian:

- $H = H_L \otimes \mathbb{I}_R - \mathbb{I}_L \otimes H_R$. $t_- = t_+ = t$
- $H = H_L \otimes \mathbb{I}_L + \mathbb{I}_R \otimes H_R$. $t_- = -t_+ = t$. Approximate description of two AdS BH

Adding a local perturbation

Perturb TFD by a primary operator ψ in the CFT_L at time $-t_\omega$ in the past

$$\rho_L(t) = \mathcal{N} e^{-iH_L t} e^{-\epsilon H_L} \psi(0, -t_\omega) e^{2\epsilon H_L + \beta H_L} \psi^\dagger(0, -t_\omega) e^{-\epsilon H_L} e^{iH_L t}$$

Simpler in Euclidean time $(x + i\tau, x - i\tau)$

$$\mathcal{N} \psi(x_2, \bar{x}_2) e^{\beta H} \psi^\dagger(x_1, \bar{x}_1)$$

where

$$\begin{aligned} x_1 &= t_- + t_\omega + i\epsilon, & x_2 &= t_- + t_\omega - i\epsilon \\ \bar{x}_1 &= -t_- - t_\omega - i\epsilon, & \bar{x}_2 &= -t_- - t_\omega + i\epsilon \end{aligned}$$

Thermal state $\Rightarrow \tau \sim \tau + \beta$

One-sided correlator

$$\begin{aligned} & \langle \Psi_\beta | \mathcal{O}_L(x_1, 0) \mathcal{O}_L^\dagger(x_2, t) | \Psi_\beta \rangle \\ &= \sum_{n,m} e^{-\beta E_n + it(E_n - E_m)} \langle n |_L \mathcal{O}_L(x_1, 0) | m \rangle_L \langle m |_L \mathcal{O}_L^\dagger(x_2, 0) | n \rangle_L \end{aligned}$$

Two-sided correlator

$$\begin{aligned} & \langle \Psi_\beta | \mathcal{O}_L(x_1, 0) \mathcal{O}_R(x_2, t) | \Psi_\beta \rangle \\ &= \sum_{n,m} e^{-\beta E_n + i(t - i\frac{\beta}{2})(E_n - E_m)} \langle n |_L \mathcal{O}_L(x_1, 0) | m \rangle_L \langle m |_R \mathcal{O}_R^\dagger(x_2, 0) | n \rangle_R \end{aligned}$$

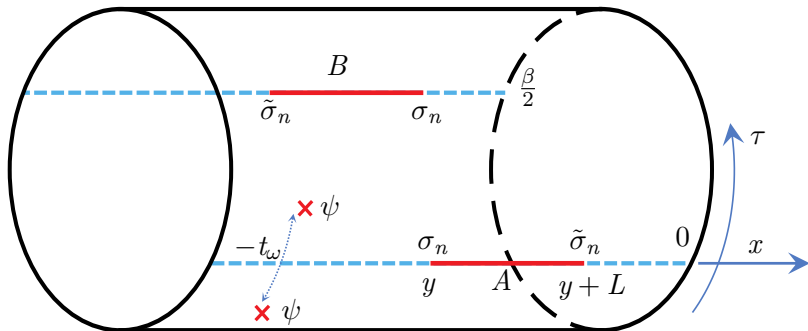
Analytic continuation in TFD

$$\langle \Psi_\beta | \mathcal{O}_L(x_1, 0) \mathcal{O}_L(x_2, t) | \Psi_\beta \rangle = \langle \Psi_\beta | \mathcal{O}_L(x_1, 0) \mathcal{O}_R^\dagger \left(x_2, t - i\frac{\beta}{2} \right) | \Psi_\beta \rangle$$

One-sided and two-sided correlators in TFD are related

$$t \mapsto t + i\frac{\beta}{2}$$

Setup for mutual information



Replica trick for S_A

Replica trick \rightarrow replicate cylinder n times, glue along the cuts
Instead, twist operators can be used

$$\mathrm{Tr} \rho_A^n(t) = \frac{\langle \Psi(x_1, \bar{x}_1) \sigma_n(x_2, \bar{x}_2) \tilde{\sigma}_n(x_3, \bar{x}_3) \Psi^\dagger(x_4, \bar{x}_4) \rangle_{C_n}}{(\langle \psi(x_1, \bar{x}_1) \psi^\dagger(x_4, \bar{x}_4) \rangle_{C_1})^n}$$

$$\begin{aligned} x_1 &= -i\epsilon, & x_2 &= y - t_\omega - t_-, & x_3 &= y + L - t_\omega - t_-, & x_4 &= +i\epsilon \\ \bar{x}_1 &= +i\epsilon, & \bar{x}_2 &= y + t_\omega + t_-, & \bar{x}_3 &= y + L + t_\omega + t_-, & \bar{x}_4 &= -i\epsilon \end{aligned}$$

$$\Psi = \psi_1 \cdot \psi_2 \cdot \dots \cdot \psi_n$$

Ψ has a ψ_i for each copy of the theory

Conformal dimension nh_ψ

Conformal dimension of twist operators is $2H_\sigma$

$$H_\sigma = \frac{c}{24} \left(n - \frac{1}{n} \right)$$

Correlators on the plane

We are in a thermal state on the cylinder

Map to the plane

$$w(x) = e^{\frac{2\pi}{\beta}x}$$

- 2-pt correlators are fixed by conformal symmetry
- 4-pt correlators are expressed via conformal blocks $G(z, \bar{z})$

$$\text{Tr } \rho_A^n(t) = \left| \frac{\beta}{\pi \varepsilon_{UV}} \sinh \left(\frac{\pi(x_2 - x_3)}{\beta} \right) \right|^{-4H_\sigma} |1 - z|^{4H_\sigma} G(z, \bar{z})$$

where

$$z = \frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_3)(w_2 - w_4)}$$

Approximation: $z = 1 + f(y, L, \beta, t)\epsilon + \mathcal{O}(\epsilon^2)$

Evaluating $G(z, \bar{z})$

4-pt function $\langle \Psi(x_1, \bar{x}_1) \sigma_n(x_2, \bar{x}_2) \tilde{\sigma}_n(x_3, \bar{x}_3) \Psi^\dagger(x_4, \bar{x}_4) \rangle_{C_n}$

contains two **light** and two **heavy** operators

Fitzpatrick, Kaplan and Walters considered such setup in

[arXiv:1403.6829](https://arxiv.org/abs/1403.6829)

$$\log G(z, \bar{z}) \simeq -\frac{c(n-1)}{6} \log \frac{z^{\frac{1}{2}(1-\alpha_\psi)} \bar{z}^{\frac{1}{2}(1-\bar{\alpha}_\psi)} (1-z^{\alpha_\psi}) (1-\bar{z}^{\bar{\alpha}_\psi})}{\alpha_\psi \bar{\alpha}_\psi}$$

where

$$\alpha_\psi = \sqrt{1 - \frac{24h_\psi}{c}},$$

Entanglement entropy S_A

$$S_A = S_{\text{thermal}} + \Delta S_A$$

Thermal part of entanglement entropy is

$$S_{\text{thermal}} = \frac{c}{3} \log \left(\frac{\beta}{\pi \epsilon_{UV}} \sinh \frac{\pi L}{\beta} \right)$$

ΔS_A , entanglement entropy due to perturbation

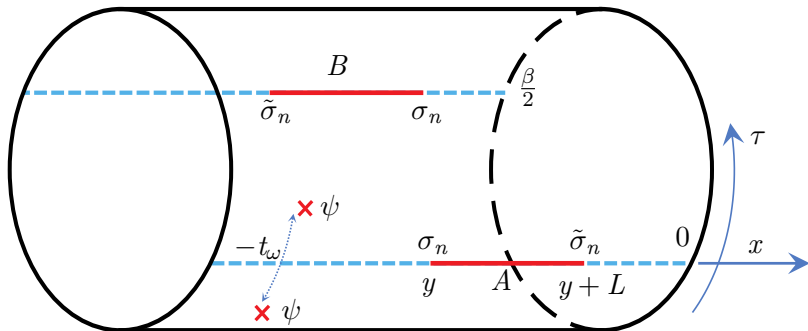
$$0, \quad t_- + t_\omega < y$$

$$\frac{c}{6} \log \left[\frac{\beta \sin \pi \alpha_\psi \sinh \left(\frac{\pi(y+L-t_- - t_\omega)}{\beta} \right) \sinh \left(\frac{\pi(t_- + t_\omega - y)}{\beta} \right)}{\pi \epsilon \alpha_\psi \sinh \left(\frac{\pi L}{\beta} \right)} \right]$$

$$0, \quad t_- + t_\omega > y + L$$

No change before perturbations arrives or after it leaves A

Setup for mutual information



Entanglement entropy S_B

$$\mathrm{Tr} \rho_B^n(t) = \frac{\langle \Psi(x_1, \bar{x}_1) \sigma_n(x_5, \bar{x}_5) \tilde{\sigma}_n(x_6, \bar{x}_6) \Psi^\dagger(x_4, \bar{x}_4) \rangle_{C_n}}{(\langle \psi(x_1, \bar{x}_1) \psi^\dagger(x_4, \bar{x}_4) \rangle_{C_1})^n}$$

$$x_5 = y + L + i\frac{\beta}{2} - t_\omega - t_+, \quad x_6 = y + i\frac{\beta}{2} - t_\omega - t_+$$

$$\bar{x}_5 = y + L - i\frac{\beta}{2} + t_\omega + t_+, \quad \bar{x}_6 = y - i\frac{\beta}{2} + t_\omega + t_+$$

Entanglement entropy remains thermal

$$S_B = \frac{c}{3} \log \left(\frac{\beta}{\pi \epsilon_{UV}} \sinh \frac{\pi L}{\beta} \right)$$

$$\text{Tr } \rho_{A \cup B}^n = \frac{\langle \psi(x_1, \bar{x}_1) \sigma_n(x_2, \bar{x}_2) \tilde{\sigma}_n(x_3, \bar{x}_3) \sigma_n(x_5, \bar{x}_5) \tilde{\sigma}_n(x_6, \bar{x}_6) \psi^\dagger(x_4, \bar{x}_4) \rangle}{(\langle \psi(x_1, \bar{x}_1) \psi^\dagger(x_4, \bar{x}_4) \rangle_{C_1})^n}$$

Map to the plane $w(x) = \exp\left(\frac{2\pi}{\beta} x\right)$ and

$$z(w) = \frac{(w_1 - w)(w_3 - w_4)}{(w_1 - w_3)(w - w_4)}$$

We obtain the following **6-pt** function

$$\langle \psi | \sigma_n(z, \bar{z}) \tilde{\sigma}_n(1, 1) \sigma_n(z_5, \bar{z}_5) \tilde{\sigma}_n(z_6, \bar{z}_6) | \psi \rangle$$

S and T -channels: Resolution of Identity

S -channel

$$\begin{aligned} & \langle \psi | \sigma_n(z, \bar{z}) \tilde{\sigma}_n(1, 1) \sigma_n(z_5, \bar{z}_5) \tilde{\sigma}_n(z_6, \bar{z}_6) | \psi \rangle \\ &= \sum_{\alpha} \langle \psi | \sigma_n(z, \bar{z}) \tilde{\sigma}_n(1, 1) | \alpha \rangle \langle \alpha | \sigma_n(z_5, \bar{z}_5) \tilde{\sigma}_n(z_6, \bar{z}_6) | \psi \rangle \end{aligned}$$

T -channel

$$\begin{aligned} & \langle \psi | \sigma_n(z, \bar{z}) \tilde{\sigma}_n(1, 1) \sigma_n(z_5, \bar{z}_5) \tilde{\sigma}_n(z_6, \bar{z}_6) | \psi \rangle \\ &= \sum_{\alpha} \langle \psi | \sigma_n(z, \bar{z}) \tilde{\sigma}_n(z_6, \bar{z}_6) | \alpha \rangle \langle \alpha | \sigma_n(z_5, \bar{z}_5) \tilde{\sigma}_n(1, 1) | \psi \rangle \end{aligned}$$

S -channel

OPE of twist operators

$$\sigma_n(z, \bar{z}) \tilde{\sigma}_n(1, 1) \sim \mathbb{I} + \mathcal{O}((z-1)^r) \quad r \in \mathbb{N}$$

Orthogonality of 2-pt functions

$$\sum_{\alpha} \langle \psi | \sigma_n(z, \bar{z}) \tilde{\sigma}_n(1, 1) | \alpha \rangle \langle \alpha | \simeq \langle \psi | \sigma_n(z, \bar{z}) \tilde{\sigma}_n(1, 1) | \psi \rangle \langle \psi |$$

So the first 4-pt function is the same as in S_A while the second the same as in S_B

$$S_{A \cup B} = S_A + S_B \quad \text{and}$$

$$I_{A:B} = 0$$

Again, dominant contribution comes from $|\alpha\rangle = |\psi\rangle$.

$$\langle\psi|\sigma_n(z_5, \bar{z}_5)\tilde{\sigma}_n(1, 1)|\psi\rangle = G(z_5, \bar{z}_5)$$

$$\langle\psi|\sigma_n(z, \bar{z})\tilde{\sigma}_n(z_6, \bar{z}_6)|\psi\rangle = |1 - \tilde{z}_2|^{4H_\sigma} |z_2 - z_6|^{-4H_\sigma} G(\tilde{z}_2, \bar{\tilde{z}}_2)$$

Cross-ratios are not the same as in S -channel

Cross-ratios

$$z_5 = 1 - \frac{2\pi i\epsilon}{\beta} \frac{\cosh \frac{\pi(t_- - t_+)}{\beta}}{\sinh \frac{\pi(y+L-t_- - t_\omega)}{\beta} \cosh \frac{\pi(y+L-t_+ - t_\omega)}{\beta}} + \mathcal{O}(\epsilon^2)$$

$$\bar{z}_5 = 1 + \frac{2\pi i\epsilon}{\beta} \frac{\cosh \frac{\pi(t_- - t_+)}{\beta}}{\sinh \frac{\pi(y+L+t_- + t_\omega)}{\beta} \cosh \frac{\pi(y+L+t_+ + t_\omega)}{\beta}} + \mathcal{O}(\epsilon^2)$$

$$\tilde{z}_2 = 1 + \frac{2\pi i\epsilon}{\beta} \frac{\cosh \frac{\pi(t_- - t_+)}{\beta}}{\sinh \frac{\pi(y-t_- - t_\omega)}{\beta} \cosh \frac{\pi(y-t_+ - t_\omega)}{\beta}} + \mathcal{O}(\epsilon^2)$$

$$\tilde{\bar{z}}_2 = 1 - \frac{2\pi i\epsilon}{\beta} \frac{\cosh \frac{\pi(t_- - t_+)}{\beta}}{\sinh \frac{\pi(y+t_- + t_\omega)}{\beta} \cosh \frac{\pi(y+t_+ + t_\omega)}{\beta}} + \mathcal{O}(\epsilon^2)$$

Entanglement entropy S_{AUB}

$$S_{AUB} \simeq \frac{2c}{3} \log \left| \frac{\beta}{\pi \epsilon_{UV}} \cosh \left(\frac{\pi \Delta t}{\beta} \right) \right| \quad t_- + t_w < y$$

$$S_{AUB} \simeq \frac{2c}{3} \log \left| \frac{\beta}{\pi \epsilon_{UV}} \cosh \left(\frac{\pi \Delta t}{\beta} \right) \right| + \frac{c}{6} \log \left(\frac{\beta \sin \pi \alpha_\psi}{\pi \epsilon} \frac{\sinh \frac{\pi(t_- + t_w - y)}{\beta} \cosh \frac{\pi(t_+ + t_w - y)}{\beta}}{\cosh \frac{\pi \Delta t}{\beta}} \right)$$

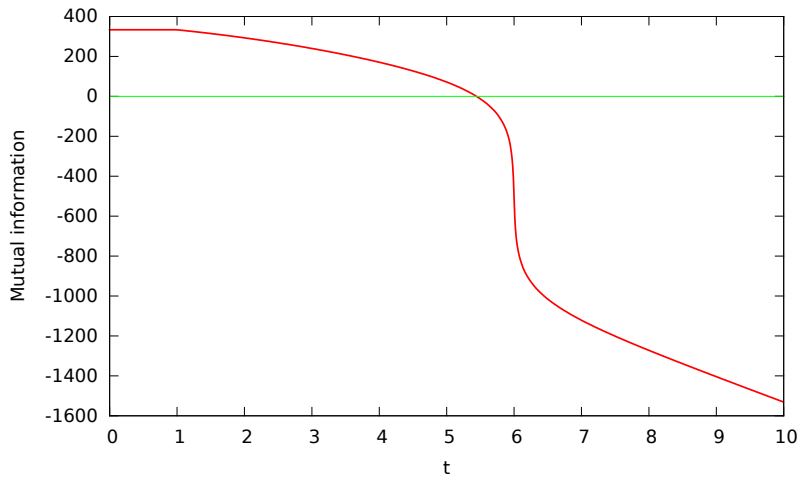
$$y < t_- + t_w < y + L$$

Entanglement entropy S_{AUB}

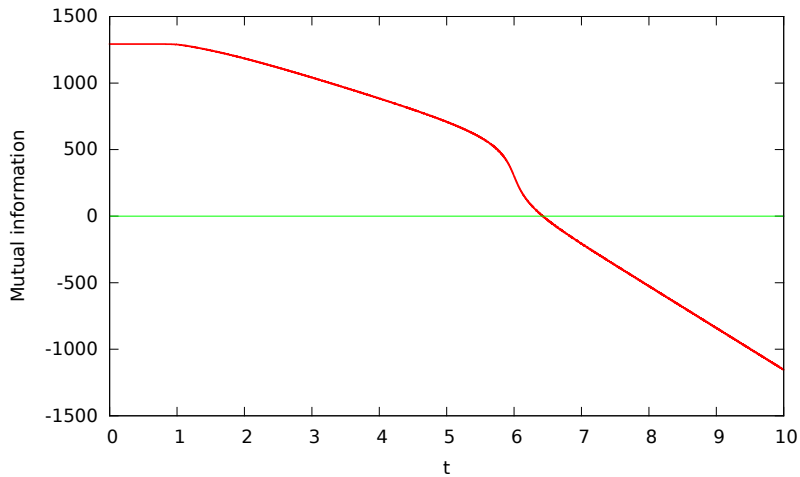
$$\begin{aligned} S_{AUB} \simeq & \frac{2c}{3} \log \left| \frac{\beta}{\pi \epsilon_{UV}} \cosh \left(\frac{\pi \Delta t}{\beta} \right) \right| + \frac{c}{3} \log \left(\frac{\beta \sin \pi \alpha_\psi}{\pi \epsilon \alpha_\psi} \right) \\ & + \frac{c}{6} \log \left(\frac{\sinh \frac{\pi(t_- + t_\omega - y)}{\beta} \cosh \frac{\pi(t_+ + t_\omega - y)}{\beta}}{\cosh \frac{\pi \Delta t}{\beta}} \right) \\ & + \frac{c}{6} \log \left(\frac{\sinh \frac{\pi(t_- + t_\omega - y - L)}{\beta} \cosh \frac{\pi(t_+ + t_\omega - y - L)}{\beta}}{\cosh \frac{\pi \Delta t}{\beta}} \right) \end{aligned}$$

$$t_- + t_\omega > y + L$$

Mutual information



Mutual information



Scrambling time

Scrambling time is defined by

$$I_{A:B}(t_\omega^*) = 0$$

Suppose $t_\omega > y + L$ and $\frac{t_\omega^*}{\beta} \gg 1$

Scrambling time in perturbed TFD

$$t_\omega^* = y + \frac{L}{2} - \frac{\beta}{2\pi} \log \left(\frac{\beta \sin \pi \alpha_\psi}{\pi \epsilon \alpha_\psi} \right) + \frac{\beta}{\pi} \log \left(2 \sinh \frac{\pi L}{\beta} \right)$$

Holographic description

- Gravity dual of TFD is eternal BTZ black hole
- Approximate local perturbation of TFD by a **free falling particle** in the BTZ black hole
- Compute back-reaction of particle on metric of BTZ:
 - Map BTZ Kruskal coordinates to AdS_3
 - Back-reaction in AdS_3 is known: conical singularity or BTZ black hole
 - (Map back-reacted metric back to original coordinates)

Free falling particle

AdS-Schwarzschild patch of the BTZ black hole

$$ds^2 = \frac{R^2}{z^2} \left[- (1 - Mz^2) dt_-^2 + \frac{dz^2}{1 - Mz^2} + d\theta^2 \right], \quad \theta \sim \theta + 2\pi$$

The geodesic of a free falling particle satisfying $z(-t_\omega) = \epsilon$

$$t_- = \tilde{\tau}, \quad \theta = 0, \quad 1 - Mz^2 = (1 - M\epsilon^2) \cosh^{-2} \left(\sqrt{M}(\tilde{\tau} + t_\omega) \right)$$

Energy of the particle

$$E = \frac{m R}{\epsilon} \sqrt{1 - M\epsilon^2}$$

matches energy of CFT perturbation if

$$m = \frac{2h_\psi}{R}$$

Free falling particle in Kruskal coordinates

Metric in Kruskal coordinates

$$ds^2 = R^2 \frac{-4dudv + (-1 + uv)^2 d\phi^2}{(1 + uv)^2}$$

Action of the particle

$$S = -2mR \int \frac{\sqrt{v} du}{1 + uv}$$

$$v(u) = \frac{C_2 + (C_1 + C_2^2)u}{1 + C_2 u}$$

where

$$C_1 = M\epsilon^2 e^{2\sqrt{M}t_w}$$

$$C_2 = -\sqrt{1 - M\epsilon^2} e^{\sqrt{M}t_w}$$

Map to AdS₃

Map to AdS₃ maps dynamic particle to static particle, so we also need to apply boosts

$$r = \frac{R}{\sqrt{M\epsilon}} \left| \frac{1 - uv}{1 + uv} \right| \cdot \sqrt{M\epsilon^2 \sinh^2 \phi + \left(\frac{e^{\sqrt{M}t\omega} u - e^{-\sqrt{M}t\omega} v}{1 - uv} - \sqrt{1 - M\epsilon^2} \cosh \phi \right)^2}$$

$$\tan \tau = \sqrt{M\epsilon} \frac{\frac{e^{\sqrt{M}t\omega} u + e^{-\sqrt{M}t\omega} v}{1 - uv}}{\cosh \phi - \sqrt{1 - M\epsilon^2} \frac{e^{\sqrt{M}t\omega} u - e^{-\sqrt{M}t\omega} v}{1 - uv}}$$

$$\tan \varphi = \sqrt{M\epsilon} \frac{\sinh \phi}{\frac{e^{\sqrt{M}t\omega} u - e^{-\sqrt{M}t\omega} v}{1 - uv} - \sqrt{1 - M\epsilon^2} \cosh \phi}$$

Back-reaction in AdS₃

We mapped point particle to $r = 0, \varphi = 0$ in AdS₃ coordinates

$$ds^2 = - (r^2 + R^2) d\tau^2 + \frac{R^2 dr^2}{r^2 + R^2} + r^2 d\varphi^2$$

Back-reaction in AdS₃ is known:

$$ds^2 = - (r^2 + R^2 - \mu) d\tau^2 + \frac{R^2 dr^2}{r^2 + R^2 - \mu} + r^2 d\varphi^2$$

$\mu = 8G_N R^2 m$ is related to mass of the particle m

Length of geodesics

Length of *spacelike* geodesics in locally AdS_3 space is known

Let X be region with endpoints $(r_\infty^{(1)}, \tau_\infty^{(1)}, \varphi_\infty^{(1)})$ and $(r_\infty^{(2)}, \tau_\infty^{(2)}, \varphi_\infty^{(2)})$

$$S_X = \frac{c}{6} \log \left[\frac{2r_\infty^{(1)} \cdot r_\infty^{(2)} \cos(|\Delta\tau_\infty|a) - \cos(|\Delta\varphi_\infty|a)}{R^2 a^2} \right]$$

where

$$a \equiv \sqrt{1 - \frac{\mu}{R^2}} = \alpha_\psi$$

carries information about perturbation

Holographic entanglement entropy

Pick region A to be an interval with endpoints

$$(t_-, z_\infty, y) \quad \text{and} \quad (t_-, z_\infty, y + L)$$

Region B :

$$(t_+, z_\infty, y) \quad \text{and} \quad (t_+, z_\infty, y + L)$$

S_A and S_B match CFT result

Holographic entanglement entropy

For $A \cup B$ we have two cases: corresponds to S and T -channels

- Endpoints on the same boundary are connected
- Endpoints on different boundaries are connected

