Max-Planck-Institut für Mathematik in den Naturwissenschaften Leipzig

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by

Fortuné Massamba

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SCREEN CONFORMAL LIGHTLIKE GEOMETRY IN INDEFINITE KENMOTSU SPACE FORMS

FORTUNÉ MASSAMBA*

ABSTRACT. In this paper we deal with geometric aspects of lightlike hypersurfaces of indefinite Kenmotsu space forms, tangent to the structure vector field and whose shape operator is conformal to the shape operator of its screen distribution. We show that these hypersufaces are proper totally contact umbilical, semi-parallel and η-Einstein but not Ricci semi-symmetric. They are locally a product of lightlike curves and proper totally umbilical leaves of its screen distributions. Its mean curvature vectors have closed dual differential 1-forms. We also show that there exists an integrable distribution whose leave are space forms, proper totally umbilical, Einstein, locally symmetric and Ricci semi-symmetric. We finally characterize the relative nullity space in a screen conformal lightlike hypersurface of an indefinite Kenmotsu space form.

1. INTRODUCTION

Kenmotsu in [13] studied a class of contact Riemannian manifolds satisfying some special conditions. Such manifolds are called *Kenmotsu manifolds*. Several authors have studied properties of Kenmotsu manifolds since then. In [12], for instance, the authors partially classified Kenmotsu manifolds and considered manifolds admitting a transformation which keeps the Riemannian curvature tensor and Ricci tensor invariant. The contact geometry has significant use in differential equations, phase spaces of dynamical systems (see details in [15] and [24], for instance), but the literature about its lightlike case is very limited. Some specific discussions on this matter can be found in [16], [17], [18], [19], [20], [21], [22], [23] and references therein.

As is well known, the geometry of lightlike submanifolds [3] is different because of the fact that their normal vector bundle intersects with the tangent bundle. Thus, the study becomes more difficult and strikingly different from the study of non-degenerate submanifolds. This means that one cannot use, in the usual way, the classical submanifold theory to define any induced object on a lightlike submanifold. To deal with this anomaly, the lightlike submanifolds were introduced and presented in a book by Duggal and Bejancu [3]. They introduced a non-degenerate screen distribution to construct a nonintersecting lightlike transversal vector bundle of the tangent bundle. Since then, a suitable choice of an integrable screen distribution has produced several new results on lightlike geometry (see, e.g, [9]

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and many more references therein). Also, see [14] for a different approach to deal with lightlike (degenerate) submanifolds. Jin, in a series of papers, studied and characterized the geometry of screen conformal lightlike hypersufaces of semi-Riemannian space forms, for instance Kaehler, Lorentzian space form (see [11] and references therein).

Since the shape operator plays a key role in the geometry of submanifolds [3], the objective of this paper is to study those lightlike submanifolds of codimension 1, of a Kenmotsu space form, tangent to the structure vector field and whose shape operator is conformal to the shape operator of their screen distribution.

The paper is organized as follows. In section 2, we recall some basic definitions for indefinite Kenmotsu manifolds and lightlike hypersurfaces of semi-Riemannian manifolds. In section 3, we consider a screen conformal lightlike hypersurface M of an indefinite Kenmotsu space form $\overline{M}(c)$, tangent to the structure vector field and study the Ricci semi-symmetry and semi-parallelism conditions on this hypersuface. We show that M is proper contact umbilical, η-Einstein, semi-parallel and locally a product manifold $L \times M'$, where L is a lightlike curve and M' is a proper totally umbilical leaf of the screen distribution $S(TM)$. We also show, in the same section, that M cannot be Ricci semi-symmetric and there exists the mean curvature 1-form ϑ on M that is closed. In section 4, we expand the geometry of the leaf M' and study the geometry of other distributions. We show that there is an integrable distribution $\widehat{D}$, subbundle of TM whose leaves are space forms of constant curvature $2\varphi\rho^2$, proper totally umbilical, Einstein, locally symmetric and Ricci semi-symmetric. By Theorem 5.2 in the section 5, we characterize the relative nullity space in a screen conformal lightlike hypersurface of an indefinite Kenmotsu space form.

2. PRELIMINARIES

Let $\overline{M}$ be a $(2n+1)$-dimensional manifold endowed with an almost contact structure $(\overline{\phi}, \xi, \eta)$, i.e. $\overline{\phi}$ is a tensor field of type $(1,1)$, ξ is a vector field, and η is a 1-form satisfying

$$\overline{\phi}^2 = -\mathbb{I} + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \overline{\phi} = 0 \quad \text{and} \quad \overline{\phi}\xi = 0. \tag{2.1}$$

Then $(\overline{\phi}, \xi, \eta, \overline{g})$ is called an almost contact metric structure on $\overline{M}$ if $(\overline{\phi}, \xi, \eta)$ is an almost contact structure on $\overline{M}$ and $\overline{g}$ is a semi-Riemannian metric on $\overline{M}$ such that, for any vector field $\overline{X}, \overline{Y}$ on $\overline{M}$ [2]

$$\eta(\overline{X}) = \overline{g}(\xi, \overline{X}), \quad \overline{g}(\overline{\phi}\,\overline{X}, \overline{\phi}\,\overline{Y}) = \overline{g}(\overline{X}, \overline{Y}) - \eta(\overline{X})\,\eta(\overline{Y}). \tag{2.2}$$

If, moreover, $(\overline{\nabla}_{\overline{X}}\overline{\phi})\overline{Y} = \overline{g}(\overline{\phi}\,\overline{X}, \overline{Y})\xi - \eta(\overline{Y})\overline{\phi}\,\overline{X}$, where $\overline{\nabla}$ is the Levi-Civita connection for the semi-Riemannian metric $\overline{g}$, we call $\overline{M}$ an indefinite Kenmotsu manifold (see [10] for details). Here, without loss of generality, the vector field ξ is assumed to be spacelike, that is, $\overline{g}(\xi, \xi) = 1$.

A plane section σ in $T_p\overline{M}$ is called a $\overline{\phi}$-section if it is spanned by $\overline{X}$ and $\overline{\phi}\,\overline{X}$, where $\overline{X}$ is a unit tangent vector field orthogonal to ξ. Since $\overline{\phi}\,\sigma = \sigma$, the $\overline{\phi}$-section σ is a holomorphic $\overline{\phi}$-section and the sectional curvature of a $\overline{\phi}$-section σ is called a $\overline{\phi}$-holomorphic sectional curvature (see [4], [10] and references therein for more details). If a Kenmotsu manifold $\overline{M}$ has constant $\overline{\phi}$-holomorphic sectional curvature c, then, by virtue of the Proposition 12 in [13], the curvature tensor $\overline{R}$ of $\overline{M}$ is given by

$$\begin{aligned}\overline{R}(\overline{X},\overline{Y})\overline{Z} &= \frac{c-3}{4}\left\{\overline{g}(\overline{Y},\overline{Z})\overline{X} - \overline{g}(\overline{X},\overline{Z})\overline{Y}\right\} + \frac{c+1}{4}\left\{\eta(\overline{X})\eta(\overline{Z})\overline{Y}\right. \\ &-\eta(\overline{Y})\eta(\overline{Z})\overline{X} + \overline{g}(\overline{X},\overline{Z})\eta(\overline{Y})\xi - \overline{g}(\overline{Y},\overline{Z})\eta(\overline{X})\xi + \overline{g}(\overline{\phi}\,\overline{Y},\overline{Z})\overline{\phi}\,\overline{X} \\ &\left. - \overline{g}(\overline{\phi}\,\overline{X},\overline{Z})\overline{\phi}\,\overline{Y} - 2\overline{g}(\overline{\phi}\,\overline{X},\overline{Y})\overline{\phi}\,\overline{Z}\right\}, \quad \overline{X},\overline{Y},\overline{Z} \in \Gamma(T\overline{M}). \end{aligned} \tag{2.3}$$

A Kenmotsu manifold $\overline{M}$ of constant $\overline{\phi}$-sectional curvature c will be called *Kenmotsu space form* and denoted $\overline{M}(c)$.

If a $(2n+1)$-dimensional Kenmotsu manifold $\overline{M}$ has a constant $\overline{\phi}$-sectional curvature c, then the Ricci tensor $\overline{Ric}$ and the scalar curvature $\overline{r}$ are given by [13]

$$\overline{Ric} = \frac{1}{2}\left(n(c-3)+c+1\right)\overline{g} - \frac{1}{2}(n+1)(c+1)\eta\otimes\eta, \tag{2.4}$$

$$\overline{r} = \frac{1}{2}\left(n(2n+1)(c-3) - n(c+1)\right). \tag{2.5}$$

This means that $\overline{M}$ is η-Einstein. Since $\overline{M}$ is Kenmotsu and η-Einstein, by Corollary 9 in [13], $\overline{M}$ is an Einstein one and consequently, $c+1=0$, that is, $c=-1$. So, the Ricci tensor (2.4) becomes $\overline{Ric} = -2n\overline{g}$ and the scalar curvature is given by $\overline{r} = -2n(2n+1)$.

Thus, *if a Kenmotsu manifold $\overline{M}$ is a space form, then it is Einstein and $c=-1$.* This means that, it is a space of constant curvature -1, so, in the Riemannian case $\overline{M}(c=-1)$ is locally isometric to the hyperbolic $\mathbb{H}^{2n+1}(-1)$ and in the proper semi-Riemannian case $\overline{M}(c=-1)$ is locally isometric to the pseudo hyperbolic space $\mathbb{H}^{2n+1}_s(-1)$, s being the index of the metric $\overline{g}$.

Example 2.1. We consider the 7-dimensional manifold

$$\overline{M}^7 = \{(x_1,...,x_7)\in\mathbb{R}^7 : x_7 \neq 0\},$$

where $(x_1,...,x_7)$ are the standard coordinates in $\mathbb{R}^7$. The vectors fields are

$$\begin{aligned} e_1 &= f_1(x_7)\frac{\partial}{\partial x_1} + f_2(x_7)\frac{\partial}{\partial x_2}, \quad e_2 = -f_2(x_7)\frac{\partial}{\partial x_1} + f_1(x_7)\frac{\partial}{\partial x_2}, \\ e_3 &= f_3(x_7)\frac{\partial}{\partial x_3} + f_4(x_7)\frac{\partial}{\partial x_4}, \quad e_4 = -f_4(x_7)\frac{\partial}{\partial x_3} + f_3(x_7)\frac{\partial}{\partial x_4}, \\ e_5 &= f_5(x_7)\frac{\partial}{\partial x_5} + f_6(x_7)\frac{\partial}{\partial x_6}, \quad e_6 = -f_6(x_7)\frac{\partial}{\partial x_5} + f_5(x_7)\frac{\partial}{\partial x_6}, \end{aligned} \tag{2.6}$$

where the functions f_i are given by

$$f_i = k_i e^{-x_7} \quad \text{with} \quad (k_1^2+k_2^2)(k_3^3+k_4^2)(k_5^2+k_6^2) \neq 0, \tag{2.7}$$

for constants c_i. It is obvious that $\{e_1, .., e_7\}$ are linearly independent at each point of $\overline{M}^7$. The vector fields $\frac{\partial}{\partial x_i}$, with $i = 1, 2, ..., 6$, are given, in terms of e_i, by

$$\frac{\partial}{\partial x_i} = \begin{cases} e^{x_7}\kappa_i e_{i+1}, & \text{if} \quad i \quad \text{odd}, \\ e^{x_7}\overline{\kappa}_i e_{i-1}, & \text{if} \quad i \quad \text{even}, \end{cases} \tag{2.8}$$

where

$$\kappa_i = \frac{k_i + k_{i+1}}{k_i^2 + k_{i+1}^2} \quad \text{and} \quad \overline{\kappa}_i = \frac{k_{i-1} - k_i}{k_{i-1}^2 + k_i^2}.$$

Let $\overline{g}$ be the semi-Riemannian metric defined by

$$\overline{g}(e_i, e_j) = 0, \ \forall\, i \neq j, \ i, j = 1, 2, ..., 7$$
$$\overline{g}(e_l, e_l) = 1, \ \forall\, l = 1, 2, 3, 4, 7 \quad \text{and} \quad \overline{g}(e_m, e_m) = -1, \forall\, m = 5, 6.$$

Its tensor product form is given by

$$\overline{g} = \frac{1}{f_1^2 + f_2^2}\{dx_1 \otimes dx_1 + dx_2 \otimes dx_2\} + \frac{1}{f_3^2 + f_4^2}\{dx_3 \otimes dx_3 + dx_4 \otimes dx_4\}$$
$$- \frac{1}{f_5^2 + f_6^2}\{dx_5 \otimes dx_5 + dx_6 \otimes dx_6\} + dx_7 \otimes dx_7. \tag{2.9}$$

Let η be the 1-form defined by $\eta(\overline{X}) = \overline{g}(\overline{X}, e_7)$, for any $\overline{X} \in \Gamma(T\overline{M}^7)$. Let $\overline{\phi}$ be the $(1,\ 1)$ tensor field defined by

$$\overline{\phi}e_1 = -e_2, \ \overline{\phi}e_2 = e_1, \ \overline{\phi}e_3 = -e_4, \ \overline{\phi}e_4 = e_3, \ \overline{\phi}e_5 = -e_6,$$
$$\overline{\phi}e_6 = e_5, \ \overline{\phi}e_7 = 0. \tag{2.10}$$

Then using the linearity of $\overline{\phi}$ and $\overline{g}$, we have $\overline{\phi}^2\overline{X} = -\overline{X} + \eta(\overline{X})e_7$, $\overline{g}(\overline{\phi}\,\overline{X}, \overline{\phi}\,\overline{Y}) = \overline{g}(\overline{X}, \overline{Y}) - \eta(\overline{X})\eta(\overline{Y})$, for any $\overline{X}, \overline{Y} \in \Gamma(T\overline{M}^7)$. Thus, for $e_7 = \xi$, $(\overline{\phi},\ \xi,\ \eta,\ \overline{g})$ defines an almost contact metric structure on $\overline{M}^7$.

Let $\overline{\nabla}$ be the Levi-Civita connection with respect to the metric $\overline{g}$. Since $\frac{\partial f_i}{\partial x_7} = -f_i$, then, we have $[e_i,\ e_7] = e_i$ and $[e_i,\ e_j] = 0$, $\forall\ i \neq j$, $i, j = 1, 2, ..., 6$. The metric connection $\overline{\nabla}$ of the metric $\overline{g}$ is given by

$$2\overline{g}(\overline{\nabla}_{\overline{X}}\overline{Y}, \overline{Z}) = \overline{X}(\overline{g}(\overline{Y}, \overline{Z})) + \overline{Y}(\overline{g}(\overline{Z}, \overline{X})) - \overline{Z}.\overline{g}(\overline{X}, \overline{Y}) - \overline{g}(\overline{X}, [\overline{Y}, \overline{Z}])$$
$$- \overline{g}(\overline{Y}, [\overline{X}, \overline{Z}]) + \overline{g}(\overline{Z}, [\overline{X}, \overline{Y}]),$$

which is known as Koszul's formula. Using this formula, the non-vanishing covariant derivatives are given by

$$\overline{\nabla}_{e_1}e_1 = -e_7, \ \overline{\nabla}_{e_2}e_2 = -e_7, \ \overline{\nabla}_{e_3}e_3 = -e_7, \ \overline{\nabla}_{e_4}e_4 = -e_7, \ \overline{\nabla}_{e_5}e_5 = e_7,$$
$$\overline{\nabla}_{e_6}e_6 = e_7, \ \overline{\nabla}_{e_1}e_7 = e_1, \ \overline{\nabla}_{e_2}e_7 = e_2, \ \overline{\nabla}_{e_3}e_7 = e_3, \ \overline{\nabla}_{e_4}e_7 = e_4,$$
$$\overline{\nabla}_{e_5}e_7 = e_5, \overline{\nabla}_{e_6}e_7 = e_6. \tag{2.11}$$

From these relations, it follows that the manifold $\overline{M}^7$ satisfies $(\overline{\nabla}_{\overline{X}}\overline{\phi})\overline{Y} = \overline{g}(\overline{\phi}\,\overline{X}, \overline{Y}) - \eta(\overline{Y})\overline{\phi}\,\overline{X}$. Hence, $\overline{M}^7$ is indefinite Kenmotsu manifold.

Let $(\overline{M}, \overline{g})$ be a $(2n+1)$-dimensional semi-Riemannian manifold with index s, $0 < s < 2n+1$ and let (M, g) be a hypersurface of $\overline{M}$, with $g = \overline{g}_{|M}$. M is said to be a lightlike hypersurface of $\overline{M}$ if g is of constant rank $2n-1$ and the orthogonal complement $TM^{\perp}$ of tangent space TM, defined as

$$TM^{\perp} = \bigcup_{p\in M} \left\{ Y_p \in T_p\overline{M} : \overline{g}_p(X_p, Y_p) = 0, \; \forall X_p \in T_pM \right\}, \tag{2.12}$$

is a distribution of rank 1 on M [3]: $TM^{\perp} \subset TM$ and then coincides with the radical distribution $\mathrm{Rad}\,TM = TM \cap TM^{\perp}$. A complementary bundle of $TM^{\perp}$ in TM is a rank $2n-1$ non-degenerate distribution over M. It is called a *screen distribution* and is often denoted by $S(TM)$. Existence of $S(TM)$ is secured provided M is paracompact. However, in general, $S(TM)$ is not canonical (thus it is not unique) and the lightlike geometry depends on its choice but it is canonically isomorphic to the vector bundle $TM/\mathrm{Rad}\,TM$ [14].

A lightlike hypersurface endowed with a specific screen distribution is denoted by the triple $(M, g, S(TM))$. As $TM^{\perp}$ lies in the tangent bundle, the following result has an important role in studying the geometry of a lightlike hypersurface.

Theorem 2.2. [3] *Let $(M, g, S(TM))$ be a lightlike hypersurface of $(\overline{M}, \overline{g})$. Then, there exists a unique vector bundle $N(TM)$ of rank 1 over M such that for any non-zero section E of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique section N of $N(TM)$ on $\mathcal{U}$ satisfying*

$$\overline{g}(N, E) = 1 \quad \textit{and} \quad \overline{g}(N, N) = \overline{g}(N, W) = 0, \quad \forall\, W \in \Gamma(S(TM)|_{\mathcal{U}}). \tag{2.13}$$

Throughout the paper, all manifolds are supposed to be paracompact and smooth. We denote by $\perp$ and $\oplus$ the orthogonal and nonorthogonal direct sum of two vector bundles. By Theorem 2.2 we may write down the following decompositions

$$TM = S(TM) \perp TM^{\perp}, \tag{2.14}$$

$$T\overline{M} = TM \oplus N(TM) = S(TM) \perp (TM^{\perp} \oplus N(TM)). \tag{2.15}$$

Let $\overline{\nabla}$ be the Levi-Civita connection on $(\overline{M}, \overline{g})$, then by using decomposition of (2.15) and considering a normalizing pair $\{E, N\}$ as in Theorem 2.2, we have the Gauss and Weingarten formulae in the form,

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.16}$$

$$\text{and} \quad \overline{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{2.17}$$

for any $X, Y \in \Gamma(TM|_{\mathcal{U}})$, $V \in \Gamma(N(TM))$, where $\nabla_X Y$, $A_V X \in \Gamma(TM)$ and $h(X, Y)$, $\nabla^{\perp}_X V \in \Gamma(N(TM))$. ∇ is an induced symmetric linear connection on M, $\nabla^{\perp}$ is a linear connection on the vector bundle $N(TM)$, h is a $\Gamma(N(TM))$-valued symmetric bilinear form and A_V is the shape operator of M concerning V.

Equivalently, consider a normalizing pair $\{E, N\}$ as in Theorem 2.2. Then (2.16) takes the following form,

$$\overline{\nabla}_X Y = \nabla_X Y + B(X,Y)\,N \tag{2.18}$$

$$\text{and}\quad \overline{\nabla}_X N = -A_N X + \tau(X)N, \tag{2.19}$$

where B, A_N, τ and ∇ are called the local second fundamental form, the local shape operator, the transversal differential 1-form and the induced linear torsion-free connection, respectively, on $TM|_{\mathcal{U}}$.

It is important to mention that the second fundamental form B of M is independent of the choice of screen distribution and $B(\cdot, E) = 0$. In fact, from (2.18), we obtain, for any $X, Y \in \Gamma(TM|_{\mathcal{U}})$, $B(X,Y) = \overline{g}(\overline{\nabla}_X Y, E)$ and

$$\tau(X) = \overline{g}(\nabla^{\perp}_X N, E). \tag{2.20}$$

Let P be the projection morphism of TM on $S(TM)$ with respect to the orthogonal decomposition of TM. We have,

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)E, \quad \forall\, X,\ Y \in \Gamma(TM) \tag{2.21}$$

$$\text{and}\quad \nabla_X E = -A^*_E X - \tau(X)E, \quad \forall\, X \in \Gamma(TM),\ E \in \Gamma(TM^{\perp}), \tag{2.22}$$

where $\nabla^*_X PY$ and $A^*_E X$ belong to $\Gamma(S(TM))$. C, A^*_E and ∇^* are called the local second fundamental form, the local shape operator and the induced linear metric connection, respectively, on $S(TM)$. The induced linear connection ∇ is not a metric connection and we have

$$(\nabla_X g)(Y,Z) = B(X,Y)\theta(Z) + B(X,Z)\theta(Y), \tag{2.23}$$

where θ is a differential 1-form locally defined on M by $\theta(\cdot) := \overline{g}(N, \cdot)$. The local second fundamental forms B and C, respectively, of M and on $S(TM)$ are related to their shape operators by

$$g(A^*_E X, PY) = B(X, PY), \quad g(A^*_E X, N) = 0, \tag{2.24}$$

$$g(A_N X, PY) = C(X, PY), \quad g(A_N X, N) = 0. \tag{2.25}$$

We denote by $\overline{R}$, R and R^* the curvature tensors of $\overline{\nabla}$, ∇ and ∇^*, respectively. Using the Gauss-Weingarten equations for M and $S(TM)$, we obtain the Gauss-Codazzi equation for M and $S(TM)$ such that, for any $X, Y, Z, W \in \Gamma(TM)$,

$$\begin{aligned}\overline{g}(\overline{R}(X,Y)Z, PW) &= g(R(X,Y)Z, PW) + B(X,Z)C(Y,PW)\\ &\quad - B(Y,Z)C(X,PW),\end{aligned} \tag{2.26}$$

$$\begin{aligned}\overline{g}(\overline{R}(X,Y)Z, E) &= (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z)\\ &\quad - \tau(Y)B(X,Z),\end{aligned} \tag{2.27}$$

$$\begin{aligned}g(R(X,Y)PZ, PW) &= g(R^*(X,Y)PZ, PW) + C(X,PZ)C(Y,PW)\\ &\quad - C(Y,PZ)C(X,PW),\end{aligned} \tag{2.28}$$

$$\begin{aligned}g(R(X,Y)PZ, N) &= (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) + \tau(Y)C(X,PZ)\\ &\quad - \tau(X)C(Y,PZ).\end{aligned} \tag{2.29}$$

3. SCREEN CONFORMAL LIGHTLIKE HYPERSURFACES OF INDEFINITE KENMOTSU MANIFOLDS

Let $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$ be an indefinite Kenmotsu manifold and (M, g) be a lightlike hypersurface of $(\overline{M}, \overline{g})$, *tangent to the structure vector field* ξ *(*$\xi \in TM$*)*.

If E is a local section of $TM^{\perp}$, it is easy to check that $\overline{\phi}E \neq 0$ and $\overline{g}(\overline{\phi}E, E) = 0$, then $\overline{\phi}E$ is tangent to M. Thus $\overline{\phi}(TM^{\perp})$ is a distribution on M of rank 1 such that $\overline{\phi}(TM^{\perp}) \cap TM^{\perp} = \{0\}$. In fact, if $\overline{\phi}(TM^{\perp}) \cap TM^{\perp} \neq \{0\}$, there exists a non-zero smooth real valued function μ such that $\overline{\phi}E = \mu E$. Applying $\overline{\phi}$ to this and using (2.1), we obtain

$$(\mu^2 + 1)E = 0, \tag{3.1}$$

which implies $\mu^2 + 1 = 0$. It is an impossible case for real submanifold M. Therefore, we have $\overline{\phi}(TM^{\perp}) \cap TM^{\perp} = \{0\}$. This enables us to choose a screen distribution $S(TM)$ such that it contains $\overline{\phi}(TM^{\perp})$ as a vector subbundle. If we consider a local section N of $N(TM)$, we have $\overline{\phi}N \neq 0$. Since $\overline{g}(\overline{\phi}N, E) = -\overline{g}(N, \overline{\phi}E) = 0$, we deduce that $\overline{\phi}E$ belongs to $S(TM)$ and $\overline{\phi}N$ is also tangent to M. At the same time, since $\overline{g}(\overline{\phi}N, N) = 0$, we see that the component of $\overline{\phi}N$, with respect to E, vanishes. Thus $\overline{\phi}N \in \Gamma(S(TM))$, that is, $\overline{\phi}(N(TM))$ is also a vector subbundle of $S(TM)$ of rank 1. We have

Lemma 3.1. *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Kenmotsu manifold $(\overline{M}, \overline{g})$. Then, the distributions $\overline{\phi}(TM^{\perp})$ and $\overline{\phi}(N(TM))$ are vector subbundles of $S(TM)$ of rank* 1.

From (2.1), we have $\overline{g}(\overline{\phi}N, \overline{\phi}E) = 1$. Therefore, $\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))$ is a non-degenerate vector subbundle of $S(TM)$ of rank 2.

If M is tangent to the structure vector field ξ, we may choose $S(TM)$ so that ξ belongs to $S(TM)$. Using this, and since $\overline{g}(\overline{\phi}E, \xi) = \overline{g}(\overline{\phi}N, \xi) = 0$, there exists a non-degenerate distribution D_0 of rank $2n-4$ on M such that

$$S(TM) = \left\{\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))\right\} \perp D_0 \perp < \xi >, \tag{3.2}$$

where $\langle \xi \rangle$ is the distribution spanned by ξ. The distribution D_0 is invariant under $\overline{\phi}$, i.e. $\overline{\phi}(D_0) = D_0$. Moreover, from (2.14) and (3.2) we obtain the decompositions

$$TM = \left\{\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))\right\} \perp D_0 \perp < \xi > \perp TM^{\perp}, \tag{3.3}$$

$$T\overline{M} = \left\{\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))\right\} \perp D_0 \perp < \xi > \perp (TM^{\perp} \oplus N(TM)). \tag{3.4}$$

Example 3.2. Let M be a hypersurface of $(\overline{M}^7, \overline{\phi}, \xi, \eta, \overline{g})$, indefinite Kenmotsu manifold defined in Example 2.1, given by

$$M = \left\{x \in \overline{M}^7 : \quad x_5 = \sqrt{2}(x_2 + x_3), \quad \kappa_2^2 + \kappa_3^2 = 2\kappa_5^2, \ \kappa_5 \neq 0\right\}.$$

Thus, TM is spanned by $\{U_i\}_{1\leq i\leq 6}$, where

$$U_1 = \kappa_3 e_1 - \overline{\kappa}_2 e_4,\ U_2 = e_2,\ U_3 = e_3,\ U_4 = \frac{1}{\sqrt{2}}(\overline{\kappa}_2 e_1 + \kappa_3 e_4) - \kappa_5 e_6,$$
$$U_5 = e_5,\ U_6 = \xi, \tag{3.5}$$

and the 1-dimensional distribution $TM^{\perp}$ of rank 1 is spanned by E, where $E = \frac{1}{\sqrt{2}}(\overline{\kappa}_2 e_1 + \kappa_3 e_4) - \kappa_5 e_6$. It follows that $TM^{\perp} \subset TM$. Then M is a 6-dimensional lightlike hypersurface of $\overline{M}^7$. Also, the transversal bundle $N(TM)$ is spanned by $N = \frac{1}{2\kappa_5^2}\left\{\frac{1}{\sqrt{2}}(\overline{\kappa}_2 e_1 + \kappa_3 e_4) + \kappa_5 e_6\right\}$. Using the almost contact structure of $\overline{M}^7$ and (3.2), the distribution D_0 is spanned by $\{F, \overline{\phi}F\}$, where $F = U_1$, $\overline{\phi}F = -\kappa_3 U_2 - \overline{k}_2 U_3$ and the distributions $\langle\xi\rangle$, $\overline{\phi}(TM^{\perp})$ and $\overline{\phi}(N(TM))$ are spanned, respectively, by ξ, $\overline{\phi}E = \frac{1}{\sqrt{2}}(-\overline{\kappa}_2 U_2 + \kappa_3 U_3) - \kappa_5 U_5$ and $\overline{\phi}N = \frac{1}{2\kappa_5^2}\{\frac{1}{\sqrt{2}}(-\overline{\kappa}_2 U_2 + \kappa_3 U_3) + \kappa_5 U_5\}$. Hence, M is a lightlike hypersurface of $\overline{M}$.

Now, we consider the distributions on M, $D := TM^{\perp} \perp \overline{\phi}(TM^{\perp}) \perp D_0$, $D' := \overline{\phi}(N(TM))$. Then, D is invariant under $\overline{\phi}$ and

$$TM = (D \oplus D') \perp \langle\xi\rangle. \tag{3.6}$$

Let us consider the local lightlike vector fields $U := -\overline{\phi}\,N$, $\quad V := -\overline{\phi}\,E$. Then, from (3.6), any $X \in \Gamma(TM)$ is written as $X = RX + QX + \eta(X)\xi$, $\quad QX = u(X)\,U$, where R and Q are the projection morphisms of TM into D and D', respectively, and u is a differential 1-form locally defined on M by

$$u(X) := g(V, X), \quad \forall\, X \in \Gamma(TM). \tag{3.7}$$

Applying $\overline{\phi}$ to X and (2.1), one obtains

$$\overline{\phi}\,X = \phi\,X + u(X)\,N,$$

where ϕ is a tensor field of type $(1,1)$ defined on M by $\phi\,X := \overline{\phi}\,RX$. In addition, we obtain, $\phi^2\,X = -\,X + \eta(X)\xi + u(X)\,U$ and $\nabla_X\xi = X - \eta(X)\xi$. Using (2.1), we derive

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) - u(Y)v(X) - u(X)v(Y), \tag{3.8}$$

where v is a differential 1-form locally defined on M by $v(\cdot) = g(U, \cdot)$. We have the following identities, for any $X \in \Gamma(TM)$, $\nabla_X\xi = X - \eta(X)\xi$ and

$$B(X, \xi) = 0, \tag{3.9}$$
$$C(X, \xi) = \theta(X), \tag{3.10}$$
$$B(X, U) = C(X, V). \tag{3.11}$$

Although the use of a non-degenerate screen distribution $S(TM)$ has been helpful in defining induced objects on the lightlike spaces, because of the degenerate metric, $S(TM)$ is not unique. Therefore, a lot of induced geometric objects depend on the choice of a screen, which creates a problem. For this reason, it is desirable to look for a unique or canonical screen distribution so that the induced objects on M are well-defined. To clarify this point, we first present a brief review of the dependence on the choice of a screen distribution.

By Theorem 2.2 and relation (2.14), we say that there exists a quasi-orthonormal basis of $\overline{M}$ along M, given by

$$\{E,\, N,\, W_i\}, \quad i \in \{1, ..., 2n-1\}, \tag{3.12}$$

where $\{E\}$, $\{N\}$ and $\{W_i\}$ are the null basis of $TM^{\perp}$, $N(TM)$ and the orthonormal basis of $S(TM)$, respectively. Consider two quasi-orthonormal frames fields $\{E,\, N,\, W_i\}$ and $\{E,\, \widetilde{N},\, \widetilde{W_i}\}$ induced on $\mathcal{U} \subset M$ by $\{S(TM),\ N(TM)\}$ and $\{\widetilde{S(TM)},\ \widetilde{N(TM)}\}$, respectively for the same E. Using (2.13) and (2.15), we obtain

$$\widetilde{W_i} = \sum_{j=1}^{2n-1} W_i^j \left(W_j - \epsilon_j \mathbf{f}_j E\right), \tag{3.13}$$

$$\widetilde{N} = N + \mathbf{f}E + \sum_{i=1}^{2n-1} \mathbf{f}_i W_i, \tag{3.14}$$

where ϵ_i are signature of the orthonormal basis $\{W_i\}$ and W_i^j, c and c_i are smooth functions on $\mathcal{U}$ such that $\{W_i^j\}$ are $(2n-1) \times (2n-1)$ semi-orthogonal matrices. Computing $\overline{g}(N, \widetilde{N}) = 0$ by using (2.13) and $\overline{g}(W_i, W_i) = 1$ we get

$$2\mathbf{f} + \sum_{i=1}^{2n-1} \epsilon_i (\mathbf{f}_i)^2 = 0.$$

Using this in the second relation of the above two equations, we have

$$\widetilde{W_i} = \sum_{j=1}^{2n-1} W_i^j \left(W_j - \epsilon_j \mathbf{f}_j E\right), \tag{3.15}$$

$$\widetilde{N} = N - \frac{1}{2}\{\sum_{i=1}^{2n-1} \epsilon_i (\mathbf{f}_i)^2\}E + \sum_{i=1}^{2n-1} \mathbf{f}_i W_i. \tag{3.16}$$

The above two relations are used to investigate the transformation of the induced objects when one changes the pair $\{S(TM),\ N(TM)\}$ with respect to a change in the basis. Using (2.18) for both screens we have

$$B(X,Y) = \overline{g}(\overline{\nabla}_X Y, E) = \widetilde{B}(X,Y), \quad \forall\, X,\ Y \in \Gamma(TM|_{\mathcal{U}}). \tag{3.17}$$

Thus, $B = \widetilde{B}$ on $\mathcal{U}$. Take $\overline{E} = \alpha E$, for some positive smooth function α on M. Then, it follows that $\overline{N} = (1/\alpha)N$. From (2.18) and (2.19), the associated local fundamental form $\overline{B}$ and 1-form $\overline{\tau}$ are related to B and τ, respectively, by

$$\overline{B} = \alpha B, \tag{3.18}$$

$$\tau(X) = \overline{\tau}(X) + X(\ln \alpha), \tag{3.19}$$

for any $X \in \Gamma(TM|_{\mathcal{U}})$, which proves that B and τ depend on the section E on $\mathcal{U}$. Finally, taking the exterior derivative d on both sides of (3.19) we get $d\tau = d\overline{\tau}$ on $\mathcal{U}$, that is, $d\tau$ is independent of the section E.

Define the Ricci tensor $\overline{Ric}$ of $\overline{M}$ and induced Ricci type tensor $R^{(0,2)}$ of M, respectively, as

$$\overline{Ric}(\overline{X},\overline{Y}) = \mathrm{trace}(\overline{Z} \longrightarrow \overline{R}(\overline{Z},\overline{X})\overline{Y}), \forall\, \overline{X},\, \overline{Y} \in \Gamma(TM), \tag{3.20}$$

$$R^{(0,2)}(X,Y) = \mathrm{trace}(Z \longrightarrow R(Z,X)Y), \forall\, X,\, Y \in \Gamma(TM). \tag{3.21}$$

Since the induced connection ∇ on M is not a Levi-Civita connection, in general, $R^{(0,2)}$ is not symmetric. Therefore, in general, it is just a tensor quantity and has no geometric or physical meaning similar to the symmetric Ricci tensor of $\overline{M}$.

Let consider a local quasi-orthogonal frame field $\{X_0, N, X_i\}_{i=1,\ldots,2n-1}$ on $\overline{M}$ where $\{X_0, X_i\}$ is a local frame field on M with respect to the decomposition (3.4) with N, the unique section of transversal bundle $N(TM)$ satisfying (2.13), and $E = X_0$. It is easy to obtain from (3.21) the following local expression for the Ricci tensor

$$R^{(0,2)}(X,Y) = g^{ij} g(R(X_i,X)Y,X_j) + g(R(X_0,X)Y,N). \tag{3.22}$$

From this we obtain

$$\begin{aligned} R^{(0,2)}(X,Y) - R^{(0,2)}(Y,X) = g^{ij}\{C(Y,X_j)B(X,X_i) - C(X,X_j)B(Y,X_i)\} \\ - \overline{g}(\overline{R}(X,Y)X_0,N). \end{aligned} \tag{3.23}$$

Put $R^{(0,2)}_{ls} := R^{(0,2)}(X_s,X_l)$ and $R^{(0,2)}_{0k} := R^{(0,2)}(X_k,X_0)$. Using the frame field $\{X_0,N,X_i\}$ and replacing X and Y by X_s and X_l respectively, a direct calculation gives locally

$$R^{(0,2)}_{ls} - R^{(0,2)}_{sl} = A^i_s B_{il} - A^i_l B_{is} + \overline{R}^0_{0ls} = 2d\tau(X_l,X_s) \tag{3.24}$$

$$\text{and} \quad R^{(0,2)}_{0k} - R^{(0,2)}_{k0} = -A^i_0 B_{ik} + \overline{R}^0_{00k} = 2d\tau(X_0,X_k), \tag{3.25}$$

where $\overline{R}^0_{ijk} = \overline{g}(\overline{R}(X_k,X_j)X_i,N)$. The Gauss-Codazzi equations are expressed locally by using coefficients of ∇ and local components of h, A_N and τ and they are given by ([3])

$$\overline{R}^0_{0js} = R^0_{0js} = A^i_j B_{is} - A^i_s B_{ij} + 2d\tau(X_j,X_s) \tag{3.26}$$

$$\text{and} \quad \overline{R}^0_{00s} = A^i_0 B_{is} + 2d\tau(X_0,X_s). \tag{3.27}$$

Putting (3.76) and (3.27) into (3.24) and (3.25), respectively, we have

$$R^{(0,2)}_{ls} - R^{(0,2)}_{sl} = 2d\tau(X_l,X_s) \quad \text{and} \quad R^{(0,2)}_{0k} - R^{(0,2)}_{k0} = 2d\tau(X_0,X_k). \tag{3.28}$$

This means that $R^{(0,2)}$ is symmetric on M if and only if $d\tau = 0$ on $\mathcal{U} \subset M$, that is τ is closed. Suppose $R^{(0,2)}$ is a symmetric Ricci tensor Ric. Then, the 1-form τ is closed. Thus there exists a smooth function f on $\mathcal{U}$ such that

$$\tau = df. \tag{3.29}$$

Consequently we get $\tau(X) = X(f)$. This relation, using (3.19), for $\alpha = \exp(f)$, yields

$$\begin{aligned}\tau(X) &= \overline{\tau}(X) + X(\ln \alpha)\\ &= \overline{\tau}(X) + X(f)\\ &= \overline{\tau}(X) + \tau(X),\end{aligned}$$

therefore $\overline{\tau}(X) = 0$, for any $X \in \Gamma(TM|_{\mathcal{U}})$. Then, by taking $\overline{E} = \exp(f)E$, one obtains $\overline{\tau} = 0$ on $\mathcal{U}$. The corresponding $\overline{N}$ is $\overline{N} = (1/\exp(f))N$. We call the pair $\{\xi, N\}$ on $\mathcal{U}$ such that the corresponding 1-form τ vanishes the *canonical null pair of* M.

As it is mentioned above, we observe that the existence of a symmetric Ricci tensor on M is equivalent to $d\tau = 0$, on any $\mathcal{U} \subset M$ and τ need not vanish. Therefore, only vanishing of $d\tau$ is needed to get a symmetric Ricci tensor for M.

If $\overline{M}$ is an indefinite Kenmotsu space form $(\overline{M}(c), \overline{g})$, then, the relation (2.3) becomes, for any X, Y, $Z \in \Gamma(TM)$,

$$\overline{R}(X,Y)Z = g(X,Z)Y - g(Y,Z)X. \tag{3.30}$$

Using (2.26), a direct calculation gives

$$R^{(0,2)}(X,Y) = -(2n-1)g(X,Y) + B(X,Y)trA_N - B(A_N X, Y), \tag{3.31}$$

where trace tr is written with respect to g restricted to $S(TM)$. Note that the Ricci tensor does not depend on the choice of the vector field E of the distribution $TM^{\perp}$. From (3.31), we have

$$R^{(0,2)}(X,Y) - R^{(0,2)}(Y,X) = B(A_N X, Y) - B(A_N Y, X). \tag{3.32}$$

The tensor field $R^{(0,2)}$ of a lightlike hypersurface M of an indefinite Kenmotsu manifold $\overline{M}$ is called induced Ricci tensor [7] if it is symmetric.

For historical reasons, we still call $R^{(0,2)}$ an induced Ricci tensor, but, we denote it by Ric only if it is symmetric. The induced connection ∇ on the lightlike hypersurface M is not metric in general and the Ricci tensor associated is not symmetric, contrary to the case of semi-Riemannian manifolds. Then the relation (3.31) is not symmetric in general. So, only some privileged conditions on the local second fundamental form of M may enable the tensor field $R^{(0,2)}$ to be symmetric. It is easy to check that the tensor field $R^{(0,2)}$ (3.31) of M is symmetric if and only if the shape operator of M is symmetric with respect to the second fundamental form B of M. Also, the tensor field $R^{(0,2)}$ of the induced connection ∇ of any parallel lightlike hypersurface, which becomes totally geodesic and consequently Einstein lightlike hypersurface, is symmetric. Are there any others, with symmetric induced Ricci tensors, but not necessarily totally geodesic or shape operator symmetric with respect to the second fundamental form? Here is one such class. The answer is affirmative for screen conformal lightlike hypersurfaces of a semi-Riemannian manifold of constant curvature (see [9], for details).

A lightlike hypersurface $(M, g, S(TM))$ of a semi-Riemannian manifold is screen locally conformal [9] if the shape operator A_N and A_E^* of M and its screen distribution $S(TM)$, respectively, are related by

$$A_N = \varphi A_E^*, \tag{3.33}$$

or equivalently,

$$C(X, PY) = \varphi B(X, PY), \quad \forall\, X,\, Y \in \Gamma(TM),$$

where φ is a non-vanishing smooth function on $\mathcal{U}$ in M. In case $\mathcal{U} = M$ the screen conformality is said to be global. Such a submanifold has some important and desirable properties, for instance, the integrability of its screen distribution (see [9] for details). We have the following result proved in [9]. *In a locally (or globally) screen conformal lightlike hypersurface M of an indefinite Kenmotsu space form $\overline{M}(c)$ with $\xi \in TM$, admits an induced Ricci tensor.*

Let M be a lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}(c)$ with $\xi \in TM$. Let us consider the pair $\{E, N\}$ on $\mathcal{U} \subset M$. Using (2.3), (2.27) and (2.29), we obtain, for any $X, Y, Z \in \Gamma(TM)$,

$$(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = \tau(Y)B(X, Z) - \tau(X)B(Y, Z), \tag{3.34}$$

and
$$\begin{aligned}(\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) &= g(X, PZ)\theta(Y) - g(Y, PZ)\theta(X)\\ &\quad + \tau(X)C(Y, PZ) - \tau(Y)C(X, PZ).\end{aligned} \tag{3.35}$$

In a screen conformal lightlike hypersurface M of $\overline{M}(c)$ with $\xi \in TM$, the relation between $\overline{R}$ and R is given by

$$\begin{aligned}\overline{R}(X, Y)Z &= R(X, Y)Z + B(X, Z)A_N Y - B(Y, Z)A_N X\\ &\quad + \overline{g}(\overline{R}(X, Y)Z, E)N\\ &= R(X, Y)Z + \varphi B(X, Z)A_E^* Y - \varphi B(Y, Z)A_E^* X.\end{aligned} \tag{3.36}$$

Using (3.30), the curvature tensor R of M is expressed as

$$\begin{aligned}R(X, Y)Z = g(X, Z)Y - g(Y, Z)X + \varphi\{B(Y, Z)A_E^* X\\ - B(X, Z)A_E^* Y\}.\end{aligned} \tag{3.37}$$

Therefore, for any $X, Y, Z, W \in \Gamma(TM)$,

$$\begin{aligned}g(R(X, Y)PZ, PW) &= g(X, PZ)g(Y, PW) - g(Y, PZ)g(X, PW)\\ &\quad + \varphi\{B(Y, Z)B(X, PW) - B(X, Z)B(Y, PW)\}.\end{aligned} \tag{3.38}$$

and

$$\overline{g}(R(X, Y)PZ, N) = g(X, PZ)\theta(Y) - g(Y, PZ)\theta(X). \tag{3.39}$$

Let us consider the following distribution

$$\widehat{D} = \left\{\overline{\phi}(TM^\perp) \oplus \overline{\phi}(N(TM))\right\} \perp D_0 \tag{3.40}$$

so that the tangent space of M is written

$$TM = \widehat{D} \perp \langle\xi\rangle \perp TM^\perp. \tag{3.41}$$

Let $\widehat{P}$ be the morphism of $S(TM)$ on $\widehat{D}$ with respect to the orthogonal decomposition of $S(TM)$ such that

$$\widehat{P}X = PX - \eta(X)\xi, \tag{3.42}$$

for any $X \in \Gamma(TM)$. Using (3.42), one obtains

$$\begin{aligned}\widehat{P}^2X &= \widehat{P}(\widehat{P}X) = \widehat{P}(PX - \eta(X)\xi)\\ &= P(PX - \eta(X)\xi) - \eta(PX - \eta(X)\xi)\xi\\ &= P^2X - \eta(X)\xi = \widehat{P}X.\end{aligned}$$

This means that the morphism $\widehat{P}$ is a projection.

Using the projection morphism $\widehat{P}$, we have the following identities, for any X, Y, $Z \in \Gamma(TM)$,

$$B(X, PY) = B(X, \widehat{P}Y), \tag{3.43}$$

$$\nabla_X PY = \nabla_X \widehat{P}Y + \{X(\eta(Y)) - \eta(X)\eta(Y)\}\xi + \eta(Y)X, \tag{3.44}$$

$$(\nabla_X B)(Y, PZ) = (\nabla_X B)(Y, \widehat{P}Z) - \eta(Z)B(X, Y). \tag{3.45}$$

If M is a screen conformal lightlike hypersurface, then, using (3.45), we have, for any $X, Y, Z \in \Gamma(TM)$,

$$(\nabla_X C)(Y, \widehat{P}Z) = X(\varphi)B(Y, \widehat{P}Z) + \varphi(\nabla_X B)(Y, \widehat{P}Z), \tag{3.46}$$

and using (3.34), the left hand side of (3.35) is given by

$$\begin{aligned}&(\nabla_X C)(Y, \widehat{P}Z) - (\nabla_Y C)(X, \widehat{P}Z) = X(\varphi)B(Y, \widehat{P}Z) - Y(\varphi)B(X, \widehat{P}Z)\\ &+ \varphi\{(\nabla_X B)(Y, \widehat{P}Z) - (\nabla_Y B)(X, \widehat{P}Z)\}\\ &= X(\varphi)B(Y, \widehat{P}Z) - Y(\varphi)B(X, \widehat{P}Z) + \varphi\{\tau(Y)B(X, \widehat{P}Z)\\ &- \tau(X)B(Y, \widehat{P}Z)\}.\end{aligned} \tag{3.47}$$

On the other hand, using (3.42), the relation (3.35) becomes

$$\begin{aligned}&(\nabla_X C)(Y, \widehat{P}Z) - (\nabla_Y C)(X, \widehat{P}Z) = g(X, \widehat{P}Z)\theta(Y) - g(Y, \widehat{P}Z)\theta(X)\\ &+ \varphi\tau(X)B(Y, \widehat{P}Z) - \varphi\tau(Y)B(X, \widehat{P}Z).\end{aligned} \tag{3.48}$$

Puttin the pieces (3.47) and (3.48) together, we have

$$\begin{aligned}&\{X(\varphi) - 2\varphi\tau(X)\}B(Y, \widehat{P}Z) - \{Y(\varphi) - 2\varphi\tau(Y)\}B(X, \widehat{P}Z)\\ &= g(X, \widehat{P}Z)\theta(Y) - g(Y, \widehat{P}Z)\theta(X).\end{aligned} \tag{3.49}$$

Replacing $Y = E$ in (3.49), we get

$$\{E(\varphi) - 2\varphi\tau(E)\}B(X, \widehat{P}Z) = -g(X, \widehat{P}Z). \tag{3.50}$$

Since, for any $Z \in \Gamma(TM)$, $Z = \widehat{P}Z + \eta(Z)\xi + \theta(Z)E$, the relation (3.50) implies

$$\{E(\varphi) - 2\varphi\tau(E)\}B(X, Z) = -\{g(X, Z) - \eta(X)\eta(Z)\}. \tag{3.51}$$

Therefore,

Proposition 3.3. *Let $(M, g, S(TM))$ be a screen conformal lightlike hypersurface of an indefinite Kenmotsu space form $(\overline{M}(c), \overline{g})$ with $\xi \in TM$. Then*

$$\{E(\varphi) - 2\varphi\tau(E)\}B(X, Z) = -\{g(X, Z) - \eta(X)\eta(Z)\}. \tag{3.52}$$

This proposition implies that $E(\varphi) - 2\varphi\tau(E) \neq 0$ and $B \neq 0$. From (3.33) and (3.52), we have, for any $X, Y \in \Gamma(TM)$,

$$B(X, Y) = \rho\,\{g(X, Y) - \eta(X)\eta(Y)\}, \tag{3.53}$$

$$\text{and} \quad C(X, Y) = \varphi\rho\,\{g(X, Y) - \eta(X)\eta(Y)\}, \tag{3.54}$$

where $\rho = -(E(\varphi) - 2\varphi\tau(E))^{-1} \neq 0$.

A lightlike hypersurface (M, g) is said to be totally contact umbilical if its local second fundamental form B satisfies ([17])

$$h(X, Y) = H\,\{g(X, Y) - \eta(X)\eta(Y)\} + \eta(X)h(Y, \xi) + \eta(Y)h(X, \xi), \tag{3.55}$$

for any $X, Y \in \Gamma(TM)$, or equivalently,

$$A^*_E X = \lambda \widehat{P} X, \quad \forall\, X \in \Gamma(TM|_{\mathcal{U}}).$$

where $H = \lambda N$ being the mean curvature vector of M (λ a smooth function on $\mathcal{U} \subset M$). If the function λ is nowhere vanishing on M, then the latter is said to be proper totally contact umbilical. It is easy to check that this an extrinsic notion that is independent on $\mathcal{U}$, of the choice of a screen distribution, E (and hence N as in (2.13)). Using (3.9), it is easy to check that a totally contact umbilical lightlike hypersurface of an indefinite Kenmotsu manifold is *η-totally umbilical*.

Adapting the definition above to screen distribution $S(TM)$ case, the relation (3.54) implies that $S(TM)$ is totally contact umbilical.

Let us assume that the screen distribution $S(TM)$ of M is integrable and let M' be a leaf of $S(TM)$. Then, using (2.18) and (2.21), we obtain

$$\begin{aligned}\overline{\nabla}_X Y &= \nabla^*_X Y + C(X, Y)E + B(X, Y)N \\ &= \nabla'_X Y + h'(X, Y),\end{aligned} \tag{3.56}$$

for any $X, Y \in \Gamma(TM')$, where ∇' and h' are the Levi-Civita connection and the second fundamental form of M' in $\overline{M}$. Thus

$$h'(X, Y) = C(X, Y)E + B(X, Y)N, \quad \forall\, X,\ Y \in \Gamma(TM'). \tag{3.57}$$

Note that, for a screen conformal lightlike hypersurface M, C is symmetric on $S(TM)$. Thus, by Theorem 2.3. in [3, pp. 89], $S(TM)$ is integrable and then M is locally a product manifold $L \times M'$, where L is an open subset of a lightlike geodesic ray in $\overline{M}$ and M' is a leaf of $S(TM)$.

Theorem 3.4. *Let $(M, g, S(TM))$ be a screen conformal lightlike hypersurface of an indefinite Kenmotsu space form $(\overline{M}(c), \overline{g})$ with $\xi \in TM$. Then, M is proper totally contact umbilical and locally a product manifold $L \times M'$, where L is a lightlike curve and M' is a proper totally contact umbilical leaf of $S(TM)$, immersed in $\overline{M}$ as non-degenerate submanifold.*

Proof. Since M is a screen conformal lightlike hypersurface of an indefinite Kenmotsu space form $(\overline{M}(c), \overline{g})$ with $\xi \in TM$, then, as it is mentioned above, M is locally a product manifold $L \times M'$, where L is a lightlike curve and M' is a leaf of $S(TM)$. Since $E(\varphi) - 2\varphi\tau(E) \neq 0$ and $B \neq 0$, that is, M is not totally geodesic. From (3.53), M is totally contact umbilical. The second fundamental form h' (3.57) of M' is given by,

$$h'(X,Y) = (\rho\varphi E + \rho N)\{g(X,Y) - \eta(X)\eta(Y)\}, \tag{3.58}$$

for any $X, Y \in \Gamma(TM')$, which implies that M' is proper totally contact umbilical, since $\rho\varphi \neq 0$ and $h'(X,\xi) = 0$. This completes the proof. □

From (3.53) and (3.54), the shape operators A^*_E and A_N are given by, for any $X \in \Gamma(TM)$,

$$A^*_E X = \rho\widehat{P}X \quad \text{and} \quad A_N X = \rho\varphi\widehat{P}X. \tag{3.59}$$

Let $\{F_i\}_{m=1,\dots,2n-4}$ be the basis of the non-degenerate distribution D_0. Using the relations (3.59), the trace of the shape operator A_N, with respect to g restricted to $S(TM)$, is given by

$$\begin{aligned} trA_N &= \varphi trA^*_E = \varphi g(A^*_E\xi,\xi) + \varphi g(A^*_E V, U) + \varphi g(A^*_E U, V) \\ &\quad + \varphi\sum_{m=1}^{2n-4} \varepsilon_m g(A^*_E F_m, F_m) \\ &= 2(n-1)\varphi\rho. \end{aligned} \tag{3.60}$$

Lemma 3.5. *Let $(M, g, S(TM))$ be a screen conformal lightlike hypersurface of an indefinite Kenmotsu space form $(\overline{M}(c), \overline{g})$ with $\xi \in TM$. Then, the curvature tensor R of M is given by, for any X, Y, $Z \in \Gamma(TM)$,*

$$\begin{aligned} R(X,Y)Z &= g(X,Z)Y - g(Y,Z)X - \varphi\rho^2\{g(X,Z)\widehat{P}Y - g(Y,Z)\widehat{P}X\} \\ &\quad + \varphi\rho^2\{\eta(X)\widehat{P}Y - \eta(Y)\widehat{P}X\}\eta(Z). \end{aligned} \tag{3.61}$$

Proof. Using (3.59), the curvature tensor R of M gives, for any X, Y, $Z \in \Gamma(TM)$,

$$\begin{aligned} R(X,Y)Z &= g(X,Z)Y - g(Y,Z)X + \varphi\{B(Y,Z)A^*_E X \\ &\quad - B(X,Z)A^*_E Y\} \\ &= g(X,Z)Y - g(Y,Z)X - \varphi\rho^2\{g(X,Z)\widehat{P}Y - g(Y,Z)\widehat{P}X\} \\ &\quad + \varphi\rho^2\{\eta(X)\widehat{P}Y - \eta(Y)\widehat{P}X\}\eta(Z), \end{aligned}$$

which completes the proof. □

Using (3.61), then, for any $X, Y \in \Gamma(TM)$,

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X. \tag{3.62}$$

Theorem 3.6. *A screen conformal lightlike hypersurface (M, g) of an indefinite Kenmotsu space form $(\overline{M}(c), \overline{g})$ with $\xi \in TM$, cannot be flat.*

Proof. Let M be a screen conformal lightlike hypersurface of an indefinite Kenmotsu space form $(\overline{M}(c), \overline{g})$ with $\xi \in TM$. If M is flat, then from (3.62), we obtain, for any $Z \in \Gamma(TM)$,

$$\eta(X)g(Y,Z) = \eta(Y)g(X,Z),$$

from which we obtain $\overline{g}(\overline{\phi}Y, \overline{\phi}Z) = 0$, a contradiction. □

The Theorem 3.6 shows that the curvature tensor R and Ricci tensor Ric of a screen conformal lightlike hypersuface M of an indefinite Kenmotsu space form $\overline{M}(c)$ are not vanishing. This allows to avoid some trivial cases in studying, for instance, the symmetry aspects involving curvature and Ricci tensors of such a submanifold.

A lightlike hypersurface M is said to be η-Einstein if its induced Ricci tensor Ric satisfies

$$Ric(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{3.63}$$

where the non-zero functions a and b are not necessarily constant on M.

For η-Einstein lightlike hypersurfaces, due to the symmetry of the induced degenerate metric g, the Ricci tensor is symmetric, and the notion of η-Einstein manifold does not depend on the choice of the screen distribution $S(TM)$.

Theorem 3.7. *Let $(M, g, S(TM))$ be a screen conformal lightlike hypersurface of an indefinite Kenmotsu space form $(\overline{M}(c), \overline{g})$ with $\xi \in TM$. Then, M is η-Einstein.*

Proof. Let $(M, g, S(TM))$ be a screen conformal lightlike hypersurface of an indefinite Kenmotsu space form $(\overline{M}(c), \overline{g})$ with $\xi \in TM$. From (3.59) and using (3.60), the induced Ricci type tensor (3.31) becomes

$$\begin{aligned} Ric(X,Y) &= -(2n-1)g(X,Y) + B(X,Y)trA_N - B(A_N X, Y) \\ &= -(2n-1)g(X,Y) + 2(n-1)\varphi\rho^2\{g(X,Y) - \eta(X)\eta(Y)\} \\ &\quad - \varphi\rho^2\{g(X,Y) - \eta(X)\eta(Y)\} \\ &= ag(X,Y) + b\eta(X)\eta(Y), \end{aligned} \tag{3.64}$$

where $a = -(2n-1) + (2n-3)\varphi\rho^2$ and $b = -(2n-3)\varphi\rho^2$. This induced Ricci type tensor is symmetric and then called an induced Ricci tensor which is satisfied the relation (3.63). Therefore, M is η-Einstein. □

A lightlike submanifold M of a semi-Riemannian manifold $\overline{M}$ is said to be Ricci semi-symmetric if the following condition is satisfied ([6])

$$(R(W_1, W_2) \cdot Ric)(X,Y) = 0, \ \ \forall\, W_1, W_2, X, Y \in \Gamma(TM), \tag{3.65}$$

where R and Ric are induced Riemannian curvature and Ricci tensor on M, respectively. The latter condition is equivalent to

$$-Ric(R(W_1, W_2)X, Y) - Ric(X, R(W_1, W_2)Y) = 0.$$

Let M is a screen conformal lightlike hypersurface of a Kenmotsu space form $\overline{M}(c)$ with $\xi \in TM$. By Theorem 3.7, M is η-Einstein with $a = -(2n-1) +$

$(2n-3)\varphi\rho^2$ and $b = -(2n-3)\varphi\rho^2$. If M is Ricci semi-symmetric, then using (3.61) and (3.64), we have

$$\begin{aligned}
0 &= -Ric(R(W_1, W_2)X, Y) - Ric(X, R(W_1, W_2)Y) \\
&= -ag(W_1, X)g(W_2, Y) - bg(W_1, X)\eta(W_2)\eta(Y) + ag(W_2, X)g(W_1, Y) \\
&+ bg(W_2, X)\eta(W_1)\eta(Y) - a\varphi\rho^2\{g(W_2, X) - \eta(W_2)\eta(X)\}\{g(W_1, Y) \\
&- \eta(W_1)\eta(Y)\} + a\varphi\rho^2\{g(W_1, X) - \eta(W_1)\eta(X)\}\{g(W_2, Y) - \eta(W_2)\eta(Y)\} \\
&- ag(W_1, Y)g(W_2, X) - bg(W_1, Y)\eta(W_2)\eta(X) + ag(W_2, Y)g(W_1, X) \\
&+ bg(W_2, Y)\eta(W_1)\eta(X) - a\varphi\rho^2\{g(W_2, Y) - \eta(W_2)\eta(Y)\}\{g(W_1, X) \\
&- \eta(W_1)\eta(X)\} + a\varphi\rho^2\{g(W_1, Y) - \eta(W_1)\eta(Y)\}\{g(W_2, X) - \eta(W_2)\eta(X)\} \\
&= b\eta(Y)\{g(W_2, X)\eta(W_1) - g(W_1, X)\eta(W_2)\} + b\eta(X)\{g(W_2, Y)\eta(W_1) \\
&- g(W_1, Y)\eta(W_2)\} \qquad (3.66)
\end{aligned}$$

Taking $Y = \xi$ in (3.66), one obtains

$$g(W_2, X)\eta(W_1) = g(W_1, X)\eta(W_2),$$

which implies that $\overline{g}(\overline{\phi}W_1, \overline{\phi}X) = 0$ and this contradicts the fact that $\overline{g}(V, U) = 1$. Therefore,

Theorem 3.8. *There exist no screen conformal lightlike hypersurfaces M of indefinite Kenmotsu space forms $(\overline{M}(c)$ with $\xi \in TM$ and that is Ricci semi-symmetric.*

A submanifold M is said to be semi-parallel if its second fundamental form h satisfies ([20]), for any W_1, W_2, $X, Y \in \Gamma(TM)$,

$$(R(W_1, W_2) \cdot h)(X, Y) = 0, \qquad (3.67)$$

that is , $-h(R(W_1, W_2)X, Y) - h(X, R(W_1, W_2)Y) = 0$.

Theorem 3.9. *Let M be a screen conformal lightlike hypersurface of an indefinite Kenmotsu space form $(\overline{M}(c)$ with $\xi \in TM$. Then M is semi-parallel.*

Proof. Let M is a screen conformal lightlike hypersurface of a Kenmotsu space form $\overline{M}(c)$ with $\xi \in TM$. Then, the second fundamental form B of M satisfies (3.53). Using (3.53) and (3.61), we have, for any W_1, W_2, $X, Y \in \Gamma(TM)$,

$$\begin{aligned}
&(R(W_1, W_2) \cdot h)(X, Y) = -h(R(W_1, W_2)X, Y) - h(X, R(W_1, W_2)Y) \\
&= -g(W_1, X)g(W_2, Y) + g(W_2, X)g(W_1, Y) - \varphi\rho^2\{g(W_2, X) \\
&- \eta(W_2)\eta(X)\}\{g(W_1, Y) - \eta(W_1)\eta(Y)\} + \varphi\rho^2\{g(W_1, X) - \eta(W_1)\eta(X)\} \\
&\times \{g(W_2, Y) - \eta(W_2)\eta(Y)\} - g(W_1, Y)g(W_2, X) + g(W_2, Y)g(W_1, X) \\
&- \varphi\rho^2\{g(W_2, Y) - \eta(W_2)\eta(Y)\}\{g(W_1, X) - \eta(W_1)\eta(X)\} + \varphi\rho^2\{g(W_1, Y) \\
&- \eta(W_1)\eta(Y)\}\{g(W_2, X) - \eta(W_2)\eta(X)\} = 0,
\end{aligned}$$

which implies that M is semi-parallel and this completes the proof. □

In [22], the author showed that if M is totally contact umbilical lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}(c)$ with $\xi \in TM$, that is, the local fundamental form satisfies (3.55), then λ satisfies the partial differential equations

$$E(\lambda) + \lambda\tau(E) - \lambda^2 = 0, \tag{3.68}$$

$$\xi(\lambda) + \lambda(\tau(\xi) + 1) = 0, \tag{3.69}$$

$$\text{and} \quad \widehat{P}X(\lambda) + \lambda\tau(\widehat{P}X) = 0, \quad \forall X \in \Gamma(TM). \tag{3.70}$$

Since M is a screen conformal lightlike hypersurface of a Kenmotsu space form $\overline{M}(c)$ with $\xi \in TM$, then, as it is proven above, second fundamental form B of M satisfies the relation (3.53), that is, M is totally contact umbilical. Therefore, the function ρ satisfies the partial differential equations (3.68), (3.69) and (3.70). Since, any $X \in \Gamma(TM)$ is written as, $X = \widehat{P}X + \eta(X)\xi + \theta(X)E$, using (3.68), (3.69) and (3.70), one obtains

$$X(\rho) + \rho\{\tau(X) + \eta(X)\} = \rho^2\theta(X), \tag{3.71}$$

and the mean curvature vector $H = \rho N$ of M satisfies

$$\nabla^{\perp}_{E}H = \rho^2 N \quad \nabla^{\perp}_{\xi}H = -\rho N$$
$$\text{and} \quad \nabla^{\perp}_{PX}H = 0, \quad PX \neq \xi, \ \forall X \in \Gamma(TM).$$

This means that the mean curvature vector H of M is not parallel, that is, the screen conformal lightlike hypersurface of a Kenmotsu space form, tangent to the structure vector field ξ is not an extrinsic sphere (see [5] and [22] for details).

Let ϑ be the mean curvature 1-form, that is, the dual differential 1-form of the mean curvature vector H of M. Then ϑ is locally defined by

$$\vartheta(X) = \overline{g}(H, X) = \rho\theta(X), \quad \forall\, X \in \Gamma(TM). \tag{3.72}$$

which leads to

$$\tau(X) = \vartheta(X) - X(\ln|\rho|) - \eta(X). \tag{3.73}$$

Lemma 3.10. *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Kenmotsu manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. Then, the differential 1-form η is closed.*

Proof. The covariant derivative of the differential of the 1-form η gives

$$2d\eta(X, Y) = (\nabla_X\eta)Y - (\nabla_Y\eta)X = 0,$$

which completes the proof. □

Note that the closure of the differential 1-form η in this lemma is independent of the definition given by the relation (3.33) and also from Lemma 3.10, then exists a non-zero smooth function δ such that

$$\eta = d\,\delta. \tag{3.74}$$

Now, putting the relation (3.73) together with the relations (3.24) and (3.25), we have

$$\begin{aligned}R^{(0,2)}_{ls} - R^{(0,2)}_{sl} &= 2d\tau(X_l, X_s)\\ &= X_l(\vartheta(X_s)) - X_l(X_s(\ln|\lambda|)) - X_l(\eta(X_s)) - \vartheta(\nabla_{X_l}X_s)\\ &+ \nabla_{X_l}X_s(\ln|\lambda|) + \eta(\nabla_{X_l}X_s) - X_s(\vartheta(X_l)) + X_s(X_l(\ln|\lambda|))\\ &+ X_s(\eta(X_l)) + \vartheta(\nabla_{X_s}X_l) - \nabla_{X_s}X_l(\ln|\lambda|) - \eta(\nabla_{X_s}X_l)\\ &= X_l(\vartheta(X_s)) - X_s(\vartheta(X_l)) - \vartheta([X_l, X_s])\\ &= 2d\vartheta(X_l, X_s), \end{aligned} \tag{3.75}$$

and similarly, we have

$$R^{(0,2)}_{0k} - R^{(0,2)}_{k0} = 2d\vartheta(X_0, X_k), \tag{3.76}$$

Theorem 3.11. *Let $(M, g, S(TM))$ be a screen conformal lightlike hypersurface of an indefinite Kenmotsu space form $(\overline{M}(c), \overline{g})$ with $\xi \in TM$. Then the mean curvature 1-form ϑ is closed.*

This means that the mean curvature 1-form ϑ is exact and from (3.75) we have $\tau - \vartheta$ is closed, that is, there exists a smooth function ε such that $\tau - \vartheta = d\varepsilon$, and using (3.73) and (3.74), the function ε is defined as $\varepsilon = -\ln|\rho| - \delta$. By relation (3.29), the mean curvature 1-form ϑ is now define by $\vartheta = d\beta$, where the some function β is given by $\beta = f + \ln|\rho| + \delta$.

To study the dependence of the induced objects $\{\tau,\, A^*_E,\, \nabla\}$ on the screen distribution $S(TM)$, let $\{\widetilde{\tau},\, \widetilde{A^*_E},\, \widetilde{\nabla}\}$ be another set of induced objects with respect to another screen distribution $\widetilde{S(TM)}$ and its transversal $\widetilde{N(TM)}$. Consider two quasi-orthonormal frames fields $\{E,\, N,\, W_i\}$ and $\{E,\, \widetilde{N},\, \widetilde{W_i}\}$ induced on $\mathcal{U} \subset M$ by $\{S(TM),\, N(TM)\}$ and $\{\widetilde{S(TM)},\, \widetilde{N(TM)}\}$, respectively. Using the transformation equations (3.17) and (3.19), we obtain relationship between the geometrical objects induced by the Gauss-Weingarten equations with respect to $S(TM)$ and $\widetilde{S(TM)}$ as follows:

$$\widetilde{\tau}(X) = \tau(X) + B(X, \widetilde{N} - N), \tag{3.77}$$

$$\widetilde{A^*_E}X = A^*_EX + B(X, N - \widetilde{N})E, \tag{3.78}$$

$$\widetilde{\nabla}_XY = \nabla_XY + B(X, Y)\{\frac{1}{2}(\sum_{i=1}^{2n-1}\epsilon_i(\mathbf{f}_i)^2)E - \sum_{i=1}^{2n-1}\mathbf{f}_iW_i\}, \tag{3.79}$$

for any $X, Y \in \Gamma(TM|_{\mathcal{U}})$. Denote by ω is the dual 1-form of $W = \sum_{i=1}^{2n-1}\mathbf{f}_iW_i$, characteristic vector field of the screen change, with respect to the induced metric g of M, that is

$$\omega(X) = g(X, W), \quad \forall\, X \in \Gamma(TM). \tag{3.80}$$

Let P and $\widetilde{P}$ be projections of TM on $S(TM)$ and $\widetilde{S(TM)}$, respectively with respect to the orthogonal decomposition of TM. Using (3.16), it is easy to check that $\widetilde{P}X = PX - \omega(X)E$ and $\widetilde{C}(X, \widetilde{P}Y) = \widetilde{C}(X, PY)$, for any $X, Y \in \Gamma(TM)$.

The relationship between the second fundamental forms C and $\widetilde{C}$ of the screen distribution $S(TM)$ and $\widetilde{S(TM)}$, respectively, is given by (using (3.16) and (3.79))

$$\widetilde{C}(X,PY) = C(X,PY) - \frac{1}{2}\omega(\nabla_X PY + B(X,Y)W), \qquad (3.81)$$

Let M is a screen conformal lightlike hypersurface of a Kenmotsu space form $\overline{M}(c)$ with $\xi \in TM$. As it is mentioned above, M is proper totally contact umbilical. Therefore, M is not totally geodesic. Since $\widetilde{\theta}(X) = \theta(X) + \omega(X)$, for any $X \in \Gamma(TM)$, the 1-forms $\widetilde{\tau}$ and τ, and the shape operators $\widetilde{A_E^*}$ and A_E^* are related, respectively, as

$$\widetilde{\tau}(X) = \tau(X) + \rho\omega(X) \quad \text{and} \quad \widetilde{A_E^*}X = A_E^*X - \rho\omega(X)E. \qquad (3.82)$$

The dual differential 1-form ϑ of the mean curvature vector H of M depends on the subbundle $N(TM)$ and letting $\widetilde{\vartheta}$ be another induced object with respect to another transversal subbundle $\widetilde{N(TM)}$. Then, the mean curvature 1-forms $\widetilde{\vartheta}$ and ϑ are related as

$$\widetilde{\vartheta}(X) = \vartheta(X) + \rho\omega(X), \quad \forall\, X \in \Gamma(TM). \qquad (3.83)$$

Theorem 3.12. *Let $(M, g, S(TM))$ be a screen conformal lightlike hypersurface of an indefinite Kenmotsu manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. The 1-form τ in (2.20), the mean curvature 1-form ϑ in (3.72) and the shape operator A_E^* in (2.22) all three are independent of $S(TM)$ if and only if the 1-form ω in (3.80) vanishes identically on M.*

4. Geometry of leaves of integrable distributions

Let $(M, g, S(TM))$ be a screen conformal lightlike hypersurface of an indefinite Kenmotsu space form $(\overline{M}(c), \overline{g})$ with $\xi \in TM$. In this section, we investigate the geometry of leaves of some integrable distributions with specific attention to those of screen distribution $S(TM)$, the distributions $\widehat{D}$ and $D \perp \langle\xi\rangle$. It is known that the screen distribution $S(TM)$ of a screen conformal lightlike hypersurface M is integrable [9, pp. 204]. Let M' be a leaf of $S(TM)$. By Theorem 3.4, the leaf M' is totally contact we have, for any $X, Y \in \Gamma(TM)$,

$$\overline{\nabla}_X Y = \nabla'_X Y + H'\{g(X,Y) - \eta(X)\eta(Y)\}, \qquad (4.1)$$

where $H' = \rho\varphi E + \rho N$ is the mean curvature vector of the leaf M'.

Using (3.30) and (3.36), we have

$$\begin{aligned} g(R(X,Y)PZ, PW) &= g(X,PZ)g(Y,PW) - g(Y,PZ)g(X,PW) \\ &+ \varphi\{B(Y,PZ)B(X,PW) - B(X,PZ)B(Y,PW)\} \\ &= g(X,PZ)g(Y,PW) - g(Y,PZ)g(X,PW) \\ &+ \varphi\rho^2\{g(Y,PZ) - \eta(Y)\eta(PZ)\}\{g(X,PW) - \eta(X)\eta(PW)\} \\ &- \varphi\rho^2\{g(X,PZ) - \eta(X)\eta(PZ)\}\{g(Y,PW) - \eta(Y)\eta(PW)\}, \qquad (4.2) \end{aligned}$$

for any X, Y, Z, $W \in \Gamma(TM)$. On the other hand, using the Gauss and Weingarten equations, the curvature tensors R and R^* of ∇ and ∇^*, respectively, are related by

$$\begin{aligned} R(X,Y)PZ &= R^*(X,Y)PZ + C(X,PZ)A^*_E Y - C(Y,PZ)A^*_E X \\ &+ \{(\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) + \tau(Y)C(X,PZ) \\ &- \tau(X)C(Y,PZ)\}E, \end{aligned} \tag{4.3}$$

where $(\nabla_X C)(Y,PZ) = X(C(Y,PZ)) - C(\nabla^*_X Y,PZ) - C(Y,\nabla^*_X PZ)$. Consequently,

$$\begin{aligned} &g(R(X,Y)PZ,PW) = g(R^*(X,Y)PZ,PW) - \varphi\{B(Y,PZ)B(X,PW) \\ &- B(X,PZ)B(Y,PW)\} \\ &= g(R^*(X,Y)PZ,PW) - \varphi\rho^2\{g(Y,PZ) - \eta(Y)\eta(PZ)\}\{g(X,PW)- \\ &\eta(X)\eta(PW)\} + \varphi\rho^2\{g(X,PZ) - \eta(X)\eta(PZ)\}\{g(Y,PW) \\ &- \eta(Y)\eta(PW)\}. \end{aligned} \tag{4.4}$$

From (4.2) and (4.4) the curvature tensor R' of M' is given by

$$\begin{aligned} R'(X,Y)Z &= g(X,Z)Y - g(Y,Z)X + 2\varphi\rho^2\{g(Y,Z) - \eta(Y)\eta(Z)\}\widehat{P}X \\ &- 2\varphi\rho^2\{g(X,Z) - \eta(X)\eta(Z)\}\widehat{P}Y, \end{aligned} \tag{4.5}$$

for any $X, Y \in \Gamma(TM')$, and he non-zero functions ρ and φ satisfy

$$X(\rho) + \rho(\tau(X) + \eta(X)) = 0 \quad \text{and} \quad X(\varphi) - 2\varphi\tau(X) = 0, \tag{4.6}$$

for any $X \in \Gamma(TM')$. Using this, a direct calculation of the Ricci type tensor Ric' of the leaf M' gives

$$\begin{aligned} Ric'(X,Y) &= \{-(2n-1) + 4(n-1)\varphi\rho^2\}g(X,Y) \\ &- 4(n-1)\varphi\rho^2\eta(X)\eta(Y). \end{aligned}$$

Therefore, we have

Theorem 4.1. *Let $(M, g, S(TM))$ be a screen conformal lightlike hypersurface of an indefinite Kenmotsu space form $(\overline{M}(c), \overline{g})$ with $\xi \in TM$. Let M' be a leaf of the distribution $S(TM)$. Then, M' is η-Einstein.*

From the realtions (4.6), one obtains, for any $X \in \Gamma(TM')$,

$$\overline{g}(\nabla'^{\perp}_X H', E) = X(\rho) + \rho\tau(X) = -\rho\eta(X)$$

and

$$\overline{g}(\nabla'^{\perp}_X H', N) = \varphi X(\rho) + \rho\{X(\varphi) - \varphi\tau(X)\} = -\rho\varphi\eta(X),$$

where $\nabla'^{\perp}$ is a linear connection on $N(TM)\oplus TM^{\perp}$ along M' defined by $\nabla'^{\perp}_X E = \nabla^{*\perp}_X E = -\tau(X)E$ and $\nabla'^{\perp}_X N = \nabla^{\perp}_X N = \tau(X)N$. This means that,

$$\overline{g}(\nabla'^{\perp}_X H', E) \neq 0 \quad \text{and} \quad \overline{g}(\nabla'^{\perp}_X H', N) \neq 0,$$

for any $X \in \Gamma(TM)$. That is, the mean curvatuure vector H' of the leaf M' is not parallel. We have,

Theorem 4.2. *Let $(M, g, S(TM))$ be a screen conformal lightlike hypersurface of an indefinite Kenmostu space form $(\overline{M}(c), \overline{g})$ with $\xi \in TM$. Let M' be a leaf of $S(TM)$. Then M' is not an extrinsic sphere.*

The result of this theorem on screen conformal lightlike is similar to the one found in [22, Theorem 5.10].

Now, referring to the decomposition (3.41), for any $X \in \Gamma(TM)$, $Y \in \Gamma(\widehat{D})$, we have

$$\nabla_X Y = \widehat{\nabla}_X Y + \widehat{h}(X, Y), \tag{4.7}$$

where $\widehat{\nabla}$ is a linear connection on the bundle $\widehat{D}$ and

$$\widehat{h} : \Gamma(TM) \times \Gamma(\widehat{D}) \longrightarrow \Gamma(\langle\xi\rangle \perp TM^{\perp})$$

is $\mathcal{F}(M)$-bilinear. Let $\mathcal{U} \subset M$ be a coordinate neighborhood. Then, using (3.41), (4.7) can be rewritten (locally) in the following way:

$$\begin{aligned} \nabla_X Y &= \widehat{\nabla}_X Y + g(\nabla_X Y, \xi)\xi + g(\nabla_X Y, N)E \\ &= \widehat{\nabla}_X Y - g(X, Y)\xi + C(X, Y)E, \end{aligned} \tag{4.8}$$

and the local expression of $\widehat{h}$ is defined as

$$\widehat{h}(X, Y) = -g(X, Y)\xi + C(X, Y)E.$$

The tensor $\widehat{h}$ is not symmetric, in general. Using (4.8), then, the distribution $\widehat{D}$ is integrable if and only if

$$C(X, Y) = C(Y, X), \quad \forall X, Y \in \Gamma(\widehat{D}).$$

This means that $\widehat{h}$ is symmetric if and only if the distribution $\widehat{D}$ is integrable.

By relation (3.50), since $X = \widehat{P}X + \eta(X)\xi + \theta(X)E$ and $B(\cdot, \xi) = 0$, we have

$$\{E(\varphi) - 2\varphi\tau(E)\}B(\widehat{P}X, \widehat{P}Z) = -g(\widehat{P}X, \widehat{P}Z). \tag{4.9}$$

This implies that $B \neq 0$ along $\widehat{M'}$, and the local fundamental forms B and C of M and $S(TM)$, respectively, along $\widehat{M'}$ are given by $B(\widehat{P}X, \widehat{P}Z) = \rho g(\widehat{P}X, \widehat{P}Z)$ and $C(\widehat{P}X, \widehat{P}Z) = \varphi\rho g(\widehat{P}X, \widehat{P}Z)$. The latter means that the distribution $\widehat{D}$ is totally umbilical, then integrable.

Let $\widehat{M'}$ be a leaf of $\widehat{D}$, then, by combining the first equations of (2.18) and (2.21), we obtain

$$\begin{aligned} \overline{\nabla}_X Y &= \widehat{\nabla}_X Y - g(X, Y)\xi + C(X, Y)E + B(X, Y)N \\ &= \widehat{\nabla}'_X Y + \widehat{h}'(X, Y), \end{aligned} \tag{4.10}$$

for any $X, Y \in \Gamma(\widehat{M'})$, where $\widehat{\nabla}'$ and $\widehat{h}'$ are the Levi-Civita connection and second fundamental form of $\widehat{M'}$ in $\overline{M}$, respectively. The second fundamental form $\widehat{h}'$ of $\widehat{M'}$ defined in (4.10) is deduced as

$$\widehat{h}'(X, Y) = \widehat{H}' g(X, Y), \quad \forall\, X,\ Y \in \Gamma(T\widehat{M'}), \tag{4.11}$$

where $\widehat{H}' = -\xi + \rho(\varphi E + N)$ is the mean curvature vector of the leaf $\widehat{M}'$. It is easy to see that $\widehat{H}' \neq 0$, that is, $\widehat{M}'$ is not totally geodesic. Therefore, $\widehat{M}'$ is proper totally umbilical. The relation (4.10) becomes

$$\overline{\nabla}_X Y = \widehat{\nabla}'_X Y + \widehat{H}' g(X, Y), \tag{4.12}$$

which implies

$$\begin{aligned}\overline{\nabla}_X \overline{\nabla}_Y Z &= \widehat{\nabla}'_X \widehat{\nabla}'_Y Z + \widehat{H}' g(X, \widehat{\nabla}'_Y Z) + (\overline{\nabla}_X \widehat{H}') g(Y, Z)\\ &\quad + \widehat{H}' X(g(Y, Z)),\end{aligned} \tag{4.13}$$

$$\text{and} \quad \overline{\nabla}_{[X,Y]} Z = \widehat{\nabla}'_{[X,Y]} Z + \widehat{H}' g([X, Y], Z). \tag{4.14}$$

Form (4.13) and (4.14), we have

$$\overline{R}(X, Y)Z = \widehat{R}'(X, Y)Z + (\overline{\nabla}_X \widehat{H}') g(Y, Z) - (\overline{\nabla}_Y \widehat{H}') g(X, Z). \tag{4.15}$$

Since $\widehat{P}X = X$ and $\eta(X) = 0$, $\forall\, X \in \Gamma(T\widehat{M}')$, the relation (4.15) reduces, for any $X, Y, Z \in \Gamma(T\widehat{M}')$,

$$\overline{R}(X, Y)Z = \widehat{R}'(X, Y)Z - (1 + 2\varphi\rho^2)\{g(Y, Z)X - g(X, Z)Y\}, \tag{4.16}$$

and

$$X(\rho) + \rho\tau(X) = 0 \quad \text{and} \quad X(\varphi) - 2\varphi\tau(X) = 0. \tag{4.17}$$

On the other hand, we have

$$\overline{R}(X, Y)Z = R(X, Y)Z + \varphi\rho^2 \{g(X, Z)Y - g(Y, Z)X\}. \tag{4.18}$$

Putting (4.16) and (4.18) together, we obtain

$$R(X, Y)Z = \widehat{R}'(X, Y)Z - (1 + \varphi\rho^2)\{g(Y, Z)X - g(X, Z)Y\}. \tag{4.19}$$

Also, using (3.61), the curvature R is expressed along the leaf $\widehat{M}'$ as

$$R(X, Y)Z = -(1 - \varphi\rho^2)\{g(Y, Z)X - g(X, Z)Y\}. \tag{4.20}$$

Using this and (4.19), the curvature tensor $\widehat{R}'$ of $\widehat{M}'$ is given by

$$\widehat{R}'(X, Y)Z = 2\varphi\rho^2 \{g(Y, Z)X - g(X, Z)Y\}. \tag{4.21}$$

Therefore, $\widehat{M}'$ is a semi-Riemannian manifold of constant curvature $2\varphi\rho^2$.

Theorem 4.3. *Let $(M, g, S(TM))$ be a screen conformal lightlike hypersurface of an indefinite Kenmotsu space form $(\overline{M}(c), \overline{g})$ with $\xi \in TM$. Let $\widehat{M}'$ be a leaf of the integrable distribution $\widehat{D}$, immersed in $\overline{M}$ as non-degenerate submanifold. Then the following assertions hold*

(i) *$\widehat{M}'$ is a space form of constant curvature $2\varphi\rho^2$,*
(ii) *$\widehat{M}'$ is Einstein,*
(iii) *$\widehat{M}'$ is locally symmetric, and*
(iv) *$\widehat{M}'$ is Ricci semi-symmetric.*

Proof. Using (4.21), the Ricci tensor $\widehat{Ric}'$ of the leaf $\widehat{M}'$ is given by

$$\widehat{Ric}'(X,Y) = \sum_{m=1}^{2n-4} \varepsilon_m g(R'(F_m,X)Y,F_m) + g(R'(\overline{\phi}E,X)Y,\overline{\phi}N) + g(R'(\overline{\phi}N,X)Y,\overline{\phi}E) = 2(2n-3)\varphi\rho^2 g(X,Y), \tag{4.22}$$

for any W, X, Y, $Z \in \Gamma(T\widehat{M}')$. The covariant derivative of $\widehat{R}'$ is

$$\begin{aligned}
(\widehat{\nabla}'_W \widehat{R}')(X,Y)Z &= \widehat{\nabla}'_W \widehat{R}'(X,Y)Z - \widehat{R}'(\widehat{\nabla}'_W X,Y)Z - \widehat{R}'(X,\widehat{\nabla}'_W Y)Z \\
&- \widehat{R}'(X,Y)\widehat{\nabla}'_W Z \\
&= 2W(\varphi\rho^2)\{g(Y,Z)X - g(X,Z)Y\} + 2\varphi\rho^2\{W(g(Y,Z))X + g(Y,Z)\widehat{\nabla}'_W X \\
&- W(g(X,Z))Y - g(X,Z)\widehat{\nabla}'_W Y\} - 2\varphi\rho^2\{g(Y,Z)\widehat{\nabla}'_W X - g(\widehat{\nabla}'_W X,Z)Y\} \\
&- 2\varphi\rho^2\{g(\widehat{\nabla}'_W Y,Z)X - g(X,Z)\widehat{\nabla}'_W Y\} - 2\varphi\rho^2\{g(Y,\widehat{\nabla}'_W Z)X \\
&- g(X,\widehat{\nabla}'_W Z)Y\} = 2W(\varphi\rho^2)\{g(Y,Z)X - g(X,Z)Y\}.
\end{aligned} \tag{4.23}$$

Using (4.17), $W(\varphi\rho^2) = 2\varphi\rho^2\tau(W) - 2\varphi\rho^2\tau(W) = 0$ and, for any W, X, Y, $Z \in \Gamma(T\widehat{M}')$,

$$(\widehat{\nabla}'_W \widehat{R}')(X,Y)Z = 0, \tag{4.24}$$

that is, the leaf $\widehat{M}'$ is locally symmetric. Now we want to show that

$$(\widehat{R}'(W_1,W_2)\cdot \widehat{Ric}')(X,Y) = 0, \quad \forall\, W_1,\ W_2,\ X,\ Y \in \Gamma(TM), \tag{4.25}$$

where

$$(\widehat{R}'(W_1,W_2)\cdot \widehat{Ric}')(X,Y) = -\widehat{Ric}'(\widehat{R}'(W_1,W_2)X,Y) - \widehat{Ric}'(X,\widehat{R}'(W_1,W_2)Y). \tag{4.26}$$

From (4.22), one obtains

$$\begin{aligned}
\widehat{Ric}'(\widehat{R}'(W_1,W_2)X,Y) &= 2\varphi\rho^2\{g(W_2,X)\widehat{Ric}'(W_1,Y) \\
&- g(W_1,X)\widehat{Ric}'(W_2,Y)\} \\
&= 4(2n-3)(\varphi\rho^2)^2\{g(W_1,Y)g(W_2,X) - g(W_2,Y)g(W_1,X)\}
\end{aligned} \tag{4.27}$$

and

$$\begin{aligned}
\widehat{Ric}'(X,\widehat{R}'(W_1,W_2)Y) &= 2\varphi\rho^2\{g(W_2,Y)\widehat{Ric}'(W_1,X) \\
&- g(W_1,Y)\widehat{Ric}'(W_2,X)\} \\
&= 4(2n-3)(\varphi\rho^2)^2\{g(W_1,X)g(W_2,Y) - g(W_2,X)g(W_1,Y)\}
\end{aligned} \tag{4.28}$$

Putting the pieces (4.27) and (4.28) together into (4.26), one obtains that

$$(\widehat{R}'(W_1,W_2)\cdot \widehat{Ric}')(X,Y) = 0, \ \ \forall\, W_1,W_2,X,Y \in \Gamma(T\widehat{M}'), \tag{4.29}$$

that is, the leaf $\widehat{M}'$ is Ricci semi-symmetric. □

Let $\widehat{\nabla}'^{\perp}$ be a linear connection on $N(TM) \oplus TM^{\perp}$ along $\widehat{M}'$ defined by $\widehat{\nabla}'^{\perp}_X E = \nabla^{*\perp}_X E = -\tau(X)E$ and $\widehat{\nabla}'^{\perp}_X N = \nabla^{\perp}_X N = \tau(X)N$, for any $X \in \Gamma(TM')$. Using the relations (4.17), the covariant derivative of the mean curvature vector $\widehat{H}'$ of the leaf $\widehat{M}'$ satisfies

$$\overline{g}(\widehat{\nabla}'^{\perp}_X \widehat{H}', E) = 0 \quad \text{and} \quad \overline{g}(\widehat{\nabla}'^{\perp}_X \widehat{H}', N) = 0. \tag{4.30}$$

This means that the mean curvature vector $\widehat{H}'$ of the leaf $\widehat{M}'$ is parallel and therefore all the integrable manifolds of $\widehat{D}$ are extrinsic spheres.

Next we deal with the geometry of the distribution $D \perp \langle\xi\rangle$ in (3.6). As is known the screen distribution $S(TM)$ of a screen conformal lightlike hypersurface M is integrable. Let Φ be the fundamental 2-form on M, locally defined by

$$\Phi(X, Y) = \overline{g}(X, \overline{\phi}Y).$$

Note that the differential 1-form u in (3.7) is related to the fundamental Φ as

$$u(X) = -\Phi(X, E), \quad \forall\, X \in \Gamma(TM).$$

Suppose that the distribution $D \perp \langle\xi\rangle$ is integrable. Let M^* be a leaf of $D \perp \langle\xi\rangle$. Using the decomposition (3.4) and for any $X \in \Gamma(TM)$, $Y \in \Gamma(D \perp \langle\xi\rangle)$, we have

$$\overline{\nabla}_X Y = \nabla^{D\perp\langle\xi\rangle}_X Y + h^{D\perp\langle\xi\rangle}(X, Y), \tag{4.31}$$

where $\nabla^{D\perp\langle\xi\rangle}$ is a linear connection on $D \perp \langle\xi\rangle$ and $h^{D\perp\langle\xi\rangle} : \Gamma(TM) \times \Gamma(D \perp \langle\xi\rangle) \longrightarrow D' \oplus N(TM)$ is $\mathcal{F}(M)$-bilinear. Let $\mathcal{U} \subset M$ be a coordinate neighborhood as fixed in Theorem 2.2. By (3.4), for any $X \in \Gamma(TM)$, $Y \in \Gamma(D \perp \langle\xi\rangle)$, we have

$$\begin{aligned}
\overline{\nabla}_X Y &= \nabla^{D\perp\langle\xi\rangle}_X Y + \overline{g}(\overline{\nabla}_X Y, E)N + \overline{g}(\overline{\nabla}_X Y, V)U \\
&= \nabla^{D\perp\langle\xi\rangle}_X Y + B(X, Y)N + B(X, \phi Y)U \\
&= \nabla^{D\perp\langle\xi\rangle *}_X Y + h^{D\perp\langle\xi\rangle *}(X, Y),
\end{aligned} \tag{4.32}$$

where $h^{D\perp\langle\xi\rangle *}(X, Y) = B(X, Y)N + B(X, \phi Y)U$ is the second fundamental form of the leaf M^*.

Theorem 4.4. *Let $(M, g, S(TM))$ be a screen conformal lightlike hypersurface of an indefinite Kenmotsu space form $(\overline{M}(c), \overline{g})$ with $\xi \in TM$. If the fundamental 2-form Φ vanishes on $D \perp \langle\xi\rangle$, then the following statements hold:*

(i) *the distribution $D \perp \langle\xi\rangle$ is integrable;*
(ii) *the distribution $D \perp \langle\xi\rangle$ is auto-parallel with respect to the induced connection ∇;*
(iii) *M is locally a product $M^* \times C$, where M^* is a proper totally contact leaf of $D \perp \langle\xi\rangle$ and C is a lightlike curve tangent to the distribution $\overline{\phi}(N(TM))$.*

Proof. Using the second relation in (2.2) and (3.53), we have, for any $X,\ Y \in \Gamma(D \perp \langle\xi\rangle)$,

$$2\Phi(X,Y) = \overline{g}(X,\overline{\phi}Y) - \overline{g}(\overline{\phi}X,Y) = \frac{1}{\rho}\{\overline{g}(h(X,\overline{\phi}Y),E) - \overline{g}(h(\overline{\phi}X,Y),E)\}$$
$$= \frac{1}{\rho}u([X,Y]). \tag{4.33}$$

If Φ vanishes on $D \perp \langle\xi\rangle$, then $u([X,Y]) = 0$, for any $X,\ Y \in \Gamma(D \perp \langle\xi\rangle)$, that is, the distribution $D \perp \langle\xi\rangle$ is integrable (i). To prove (ii), we need to check $g(\nabla_X E, V) = 0$, $g(\nabla_X V, V) = 0$, $g(\nabla_X Y_0, V) = 0$ and $g(\nabla_X \xi, V) = 0$, for any $X \in \Gamma(D \perp \langle\xi\rangle)$ and $Y_0 \in \Gamma(D_0)$. Hence, using (3.53), we obtain

$$g(\nabla_X E, V) = -\overline{g}(\overline{\nabla}_X E, \overline{\phi}E) = \rho\Phi(X,E) = 0,$$
$$g(\nabla_X V, V) = -\overline{g}(\overline{\nabla}_X \overline{\phi}^2 E, E) = g(\nabla_X E, E) = 0,$$
$$g(\nabla_X Y_0, V) = \overline{g}(\overline{\phi}(\overline{\nabla}_X Y_0), E) = \rho\Phi(X, Y_0) = 0,$$
$$g(\nabla_X \xi, V) = \overline{g}(\overline{\nabla}_X \xi, V) = \Phi(X,E) = 0.$$

Finally, from (i) we deduce that $D \perp \langle\xi\rangle$ determines a foliation. $D' := \overline{\phi}(N(TM))$ being a 1-dimensional distribution, it defines a foliation. Let M^* be a leaf of $D \perp \langle\xi\rangle$. Then, by (3.53) and for any $X,\ Y \in \Gamma(TM^*)$, the second fundamental form $h^{D\perp\langle\xi\rangle *}$ in (4.32) of M^* reduces

$$h^{D\perp\langle\xi\rangle *}(X,Y) = B(X,Y)N + B(X,\phi Y)U$$
$$= \rho\{g(X,Y) - \eta(X)\eta(Y)\}N + \rho\overline{g}(X,\overline{\phi}Y)U$$
$$= \rho\{g(X,Y) - \eta(X)\eta(Y)\}N + \rho\Phi(X,Y)U. \tag{4.34}$$

If Φ vanishes on $D \perp \langle\xi\rangle$, $h^{D\perp\langle\xi\rangle *}$ becomes $h^{D\perp\langle\xi\rangle *}(X,Y) = \rho\{g(X,Y) - \eta(X)\eta(Y)\}N$. Since $h^{D\perp\langle\xi\rangle *}(X,\xi) = 0$, the leaf M^* of $D \perp \langle\xi\rangle$ is totally contact umbilical. So being $TM = (D \perp \langle\xi\rangle) \oplus D'$, we obtain (iii). □

5. RELATIVE NULLITY FOLIATIONS OF SCREEN CONFORMAL LIGHTLIKE HYPERSURFACES

Let M be a lightlike hypersurface of indefinite Kenmotsu space form $\overline{M}(c)$ with $\xi \in TM$. The relative nullity space at a point x is defined by

$$T^{*0}(x) = \{X \in T_xM : A^*_E X = 0,\ \ \forall\ E \in T_xM^\perp\}. \tag{5.1}$$

The dimension $\nu(x)$ of $T^{*0}(x)$ is called the index of relative nullity at x. The value $\nu_0 = \min_{x\in M}\nu(x)$ is called the index of minimum relative nullity [5].

Writing A^*_E as, for any $X \in \Gamma(TM)$,

$$A^*_E X = \sum_{i=1}^{2n-4} \frac{B(X,F_i)}{g(F_i,F_i)} F_i + B(X,V)U + B(X,U)V, \tag{5.2}$$

with $g(F_i,F_i) \neq 0$ and using $B(.,\xi) = 0$, it is easy to check that

$$A^*_E \xi = A^*_E E = 0.$$

Therefore, $\dim T^{*0}(x) \geq 2, \ \ \forall\, x \in M$. Moreover

$$T_x M^{\perp} \perp \langle \xi \rangle_x \subset T^{*0}(x). \tag{5.3}$$

Hence, $T^{*0}(x)$ is a degenerate distribution along M and $\nu_0 = 2$.

The orthogonal complement $(T^{*0}(x))^{\perp}$ of $T^{*0}(x)$ in T_xM is denoted by $T^{*1}(x)$.

Proposition 5.1. *Let M be a lightlike hypersurface of indefinite Kenmotsu space form $\overline{M}(c)$ with $\xi \in TM$. The orthogonal complement $T^{*1}(x)$ of $T^{*0}(x)$ in T_xM is given by*

$$T^{*1}(x) = \mathrm{span}\{A^*_E Y, \ \ Y \in T_x M, \ E \in T_x M^{\perp}\} \perp T_x M^{\perp}.$$

Proof. It is obvious to check that $T_xM^{\perp} \subset T^{*1}(x)$. Then, there exists a set $\Delta(x)$ such that

$$T^{*1}(x) = \Delta(x) \perp T_x M^{\perp}.$$

Now we want to show that $\Delta(x) = \mathrm{span}\{A^*_E Y\}$. Given any $E \in T_xM^{\perp}$, $Y \in T_xM$ and $X \in T^{*0}(x)$,

$$g(X, A^*_E Y) = g(A^*_E X, Y) = 0,$$

so, $A^*_E Y \in \Delta(x)$. On the other hand, let $Z \in \mathrm{span}\{A^*_E Y\}^{\perp_S}$ and $Y \in T_xM$, where $\perp_S$ denotes the orthogonality symbol in the screen distribution $S(TM)$. We have

$$0 = g(Z, A^*_E Y) = g(A^*_E Z, Y), \forall\, Y \in T_x M.$$

Then, $A^*_E Z \in S(TM) \cap T_xM^{\perp} = \{0\}$, that is, $A^*_E Z = 0$ and $Z \in T^{*0}(x)$.

Thus $\mathrm{span}\{A^*_E Y\}^{\perp_S} \subset T^{*0}(x)$ and $T^{*1}(x) \subset \mathrm{span}\{A^*_E Y\}$. Since $A^*_E Y \notin T_xM^{\perp}$, then $\Delta(x) \subset \mathrm{span}\{A^*_E Y\}$ which completes the proof. □

Let G be the set of points in M where $\nu(x) = \nu_0$. By Theorem 4.4 in [21], G is an open set in M.

We now show that the relative nullity space $T^{*1}(x)$ is a smooth distribution. Let x_0 be an element of G. From (5.3), we have

$$T^{*0}(x_0) = P(T^{*0}(x_0)) \perp T_{x_0} M^{\perp} \perp \langle \xi \rangle_{x_0}. \tag{5.4}$$

Let $\perp_S$ denotes the orthogonality symbol in the screen distribution $S(TM)$. For $Y \in T_{x_0}M$, $E \in T_{x_0}M^{\perp}$ and $X \in P(T^{*0}(x_0))$, we have

$$g(A^*_E Y, X) = g(Y, A^*_E X) = 0,$$

so we obtain,

$$\mathrm{span}\{A^*_E Y\} \subset P(T^{*0}(x_0))^{\perp_S}.$$

Let $Z \in \mathrm{span}\{A^*_E Y\}^{\perp_S}$ and $Y \in T_{x_0}M$. We have $0 = g(Z, A^*_E Y) = g(A^*_E Z, Y)$, $\forall\ Y \in T_xM$. Then $A^*_E Z \in S(TM) \cap T_{x_0}M^{\perp} = \{0\}$, that is, $A^*_E Z = 0$ and $Z \in P(T^{*0}(x_0))$. Thus

$$\mathrm{span}\{A^*_E Y\}^{\perp_S} \subset P(T^{*0}(x_0)) \ \text{ and } \ P(T^{*0}(x_0))^{\perp_S} \subset \mathrm{span}\{A^*_E Y\}.$$

Consequently $P(T^{*0}(x_0))^{\perp_S} = \mathrm{span}\{A^*_E Y\}$ and $T^{*1}(x_0) = \mathrm{span}\{A^*_E Y\} \perp T_{x_0}M^{\perp}$. There exist vector fields $Y_1, ..., Y_{2n-\nu+1} \in T_{x_0}M$ such that

$$\{E(x_0), A^*_{E(x_0)} Y_1, ..., A^*_{E(x_0)} Y_{2n-\nu+1}\},$$

represent a basis of $T^{*1}(x)$.

Take smooth local extensions of $E(x_0)$ and $Y_1, ..., Y_{2n-\nu+1} \in T_{x_0}M$ in $TM^\perp$ and TM respectively. By continuity, the vector fields $\{E(x_0), Y_1, ..., Y_{2n-\nu+1}\}$ remain linearly independent in a neighborhood $\mathcal{V} \subset G$ of x_0 and then T^{*1} is a smooth distribution. Consequently, T^{*0} is smooth distribution.

Suppose that M is a screen conformal lightlike hypersurface of indefinite Kenmotsu space form $\overline{M}(c)$ with $\xi \in TM$. Let x be an element of G. If $X \in T^{*0}(x)$, then $A^*_E X = 0$. Using the fact that $X = PX + \theta(X)E$ and $A^*_E E = 0$, we get $A^*_E PX = 0$, which implies that

$$B(PX, PY) = 0, \forall\, Y \in T_xM. \tag{5.5}$$

Since $B \neq 0$ on M and ξ is the only vector field in $S(TM)$ such that $B(\xi, \cdot) = 0$, the relation (5.5) implies that PX is proportional to ξ, that is $PX = \eta(X)\xi$. Thus, the vector field is now

$$X = \eta(X)\xi + \theta(X)E. \tag{5.6}$$

That is, $X \in T_xM^\perp \perp \langle\xi\rangle_x$ and $P(T^{*0}(x)) = \{0\}$. Therefore

$$T_xM^\perp \perp \langle\xi\rangle_x \subset T^{*0}(x). \tag{5.7}$$

we have the following result.

Theorem 5.2. *Let $(M, g, S(TM))$ be a screen conformal lightlike hypersurface of an indefinite Kenmotsu space form $(\overline{M}(c), \overline{g})$ with $\xi \in TM$. Then, on G*

$$T^{*0} = TM^\perp \perp \langle\xi\rangle. \tag{5.8}$$

Moreover, the relative nullity distribution T^{*0} is integrable and the leaves are totally geodesic in M and $\overline{M}$.

Proof. From (5.3) and (5.7), we obtain the relation (5.8). From Gauss and Codazzi equations, we have, for any $E \in \Gamma(TM^\perp)$ and $X, Y, Z \in \Gamma(TM)$,

$$\overline{g}(\overline{R}(X, Y)Z, E) = \overline{g}((\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z), E). \tag{5.9}$$

Take $X \in \Gamma(TM)$ and $Y, Z \in T^{*0}(x)$, $x \in G$. Since $(\nabla_X h)(Y, Z) = \nabla^\perp_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$, then

$$\begin{aligned}\overline{g}((\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z), E) &= X.B(Y, Z) - Y.B(X, Z)\\ &- \tau(X)B(Y, Z) + \tau(Y)B(X, Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z)\\ &+ B(\nabla_Y X, Z) + B(X, \nabla_Y Z).\end{aligned} \tag{5.10}$$

Using (3.30) the left hand side of (5.9) vanishes and the relation (5.10) becomes

$$\begin{aligned}0 &= X.B(Y, Z) - Y.B(X, Z) - \tau(X)B(Y, Z) + \tau(Y)B(X, Z)\\ &- B(\nabla_X Y, Z) - B(Y, \nabla_X Z) + B(\nabla_Y X, Z) + B(X, \nabla_Y Z).\end{aligned} \tag{5.11}$$

From (5.6), $Z \in T^{*0}(x)$ implies that Z takes the form $Z = \eta(Z)\xi + \theta(Z)E$ and $B(Y, Z) = \eta(Z)B(\xi, PY) + \theta(Z)B(E, PY) = 0$. Similarly, $B(X, Z) = 0$.

On the other hand, since $B(X, A^*_E Y) = B(A^*_E X, Y)$, we have

$$\begin{aligned} B(\nabla_X Y, Z) &= B(\nabla_X PY, Z) + X.\theta(Y)B(Z, E) + \theta(Y)B(\nabla_X E, Z) \\ &= B(\nabla^*_X PY, Z) + \theta(Y)B(X, A^*_E Z) = 0, \end{aligned} \tag{5.12}$$

for $Z \in T^{*0}(x)$. Also $B(\nabla_Y X, Z) = 0$.

The relation (5.11) becomes $B(X, \nabla_Y Z) - B(Y, \nabla_X Z) = 0$. But

$$\begin{aligned} B(Y, \nabla_X Z) &= B(Y, \nabla_X PZ) + \theta(X)B(Y, \nabla_X E) \\ &= B(Y, \nabla^*_X PZ) - \theta(X)B(A^*_E Y, X) = 0. \end{aligned}$$

Consequently $h(\nabla_Y Z, PX) = 0$, for any $X \in \Gamma(TM)$. Since M is not parallel, we deduce that $\nabla_Y Z \in T_x M^\perp \perp \langle\xi\rangle_x = T^{*0}(x)$, that is, $\nabla_Y X \in T^{*0}(x)$. This implies that $T^{*0}(x)$ is involutive with totally geodesic leaves in both M and $\overline{M}$. □

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* DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BOTSWANA
PRIVATE BAG 0022
GABORONE
BOTSWANA
E-mail address: massfort@yahoo.fr, massambaf@mopipi.ub.bw