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H. Evangelaras
National Technical University of Athens, harris11@central.ntua.gr

Christos Koukouvinos
National Technical University of Athens

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Screening Properties And Design Selection Of Certain Two-Level Designs

H. Evangelaras C. Koukouvinos
Department of Mathematics
National Technical University of Athens

Screening designs are useful for situations where a large number of factors (q) is examined but only few (k) of these are expected to be important. It is of practical interest for a given k to know all the inequivalent projections of the design into the k dimensions. In this paper we give all the inequivalent projections of inequivalent Hadamard matrices of order 28 into k=3 and 4 dimensions and furthermore, we give partial results for k=5. Then, we sort these projections according to their generalized resolution and their generalized aberration.

Key words: Hadamard matrices, inequivalent projections, screening designs, factorial designs, generalized resolution, generalized aberration, generalized wordlength pattern.

Introduction

In the early stages of an experimental situation, a large number of factors is likely to have been identified as possibly having an influence on the response. However, it is believed that only a few of these actually have a substantial effect, a situation known as factor sparsity. The small number of active factors can be identified through a screening experiment. Screening designs are frequently used by experimenters to help understand the impact of a large number of factors in relatively few trials. Traditionally Hadamard matrices have been used for this purpose. A lot of work has been done in this area (see [7, 10, 11, 16]).

A design suitable for screening out the k relevant factors from the total factors is called a screening design, see [2, 7, 11]. An n-dimensional Hadamard matrix is an n by n matrix of 1's and -1's with $H^T H = HH^T = nI_n$.

Correspondence concerning this article should be addressed to H. Evangelaras, Department of Mathematics, National Technical University of Athens, Zografou 15773, Athens, Greece. Email: harris11@central.ntua.gr.

A Hadamard matrix is said to be *normalized* if it has its first row and column all 1's. If not we can normalize the Hadamard matrix by multiplying rows and columns by -1 where is needed. In these matrices, n is necessarily 2 or a multiple of 4. Two Hadamard matrices H_1 and H_2 are called equivalent (or H-equivalent) if one can be obtained from the other by a sequence of row negations, row permutations, column negations and column permutations.

Their usefulness in statistical analysis is as follows. There are two general questions to be answered. (i) If q factors are to be studied, which q columns should be assigned to the q factors? Since any set of q columns are orthogonal, we must compare them in terms of their ability in entertaining m two-factor interactions in addition to the q main effects. (ii) For each assignment, main effect analysis may reveal that only k factors (i.e. k columns), $k \leq q$ are significant.

We can then raise the question (i) for these k factors. Since the projection onto k columns varies with the outcome of the analysis, it will be desirable to study this problem for all (or most) projections. The information obtained will be useful for experimenters in contemplating the choice of designs. The choice of k factors is equivalent to the choice of a $n \times k$ submatrix of a Hadamard matrix of order n. Two such matrices are said to be (combinatorially) equivalent if one can be obtained from the other by permutation of rows, columns and sign changes in columns. In the context of design theory we refer to this equivalence as (combinatorial) equivalence of two factor assignments.

Classification Criteria

Orthogonal factorial designs can be classified into two categories: the regular fractional factorials, that have simple aliasing structure in which any two effects are either orthogonal or fully aliased and the non-regular fractional factorials, that have complex aliasing structure in which effects are neither orthogonal nor fully aliased.

Fractional factorial designs are the most popular experimental designs used in various fields. There are many useful criteria for comparing and ranking fractional factorial designs, such as resolution [2], minimum aberration [6], estimation capacity [3] and uniformity [5]. Among them, the minimum aberration is the most used criterion, but it can be applied only to regular factorials.

It is of practical use to rank and compare non-regular factorial designs in a systematic manner. Deng and Tang [4] proposed *generalized resolution* as a criterion to rank such designs in a similar way as the resolution criterion is used for regular designs. According to this criterion, an orthogonal design is regarded as a set of m columns $D=\{d_1,...,d_m\}$. Then, for $1 \le k \le m$ and any k-subset $s=\{d_{j1},...,d_{jk}\}$ define

$$J_k(s)=|\sum d_{ij1}...d_{ijk}|.$$

If r is the smallest integer such that $max_{|s|=r}J_r(s)>0$ and the maximization is over all the subsets of r distinct columns of D, then the generalized resolution of D is defined to be:

$$R(D)=r+[1-max_{|s|=r}J_r(s)/n].$$

Then, using simple calculations, we are able to calculate the generalized resolution of any fractional factorial design and therefore we can rank and compare any set of inequivalent projections of Hadamard matrices in any order $n=0(mod4)$ and especially when n is not a power of 2. Designs with greater generalized resolution from the others are preferred.

The previously stated criterion of generalized resolution is not strong enough to rank such designs since there are cases where two or more fractional factorial designs have the same generalized resolution (see Table 4, where there are 3 such designs with the same generalized resolution). Ma and Fang [12] proposed a stronger criterion that can be applied to all regular and non-regular factorials. Let D be a fractional factorial design with n runs and s factors, each factor in q levels. The new criterion appends to the design D its *generalized wordlength pattern*, which is defined by: $W^g(D)=\{A_1^g(D),...,A_s^g(D)\}$ where

$$A_i^g(D)=\frac{1}{n(q-1)}\sum_{j=0}^{s}P_i(j;s)E_j(D), \quad \text{i=1,…,s}$$

$P_i(j;s)$ are the Krawtchouk polynomials and $E_j(D)$, $j=0,...,s$ is the distance distribution of D, defined -in a similar way with Hamming distance- as:

$$E_i(D)=\frac{\#\{(c,d)\,|,c,d\in D, d_H(c,d)=i\}}{n}$$

where $d_H(\boldsymbol{c},\boldsymbol{d})$ is the Hamming distance between two runs **c** and **d** of D. For the undefined terms in coding theory, we refer the interested reader to [13] and [15].

Let now D_1 and D_2 be two inequivalent designs. Let t be the smallest integer for which $A_t^g(D_1) \neq A_t^g(D_2)$ in their generalized wordlength patterns. Then, if $A_t^g(D_1) < A_t^g(D_2)$ we say that D_1 has less generalized aberration from D_2 and hence it is preferred. A design D has minimum generalized aberration if no other design has less generalized aberration than it.

By an algorithm which relies on the definition, we have found all the inequivalent projections for n=28, k=3, 4 and 5 as well as their frequencies. Then by simple computations, we sort these projections according to their generalized resolution and aberration in order to present the best classification.

Inequivalent Hadamard Matrices Of Order 28 And Their Projections

We know that by adding a column of 1's to a Plackett and Burman design [14], we obtain a Hadamard matrix H which satisfies $H^T H=nI$. For n=12, H is unique, but for higher n this is not true. Inequivalent Hadamard matrices have different projection properties.

For n=28 there are 487 inequivalent Hadamard matrices [8, 9] but only one of them corresponds to a Plackett and Burman design designated as H28.487 here, that is, only one provides a 28-run design of the type whose projections are widely known and studied [1], [11]. We will now discuss the projection patterns of all the types, which we designate as *H*28.1, *H*28.2 ... *H*28.487 as found in http://www.research.att.com/~njas/hadamard/.

From now on, in this paper we will denote each projection with (k.#) where k are the factors included in the projection and # is the number of the projection. We present each projection as a set of k vectors to save space. In each such vector we have used the letters from A to Z to denote the position of the +1 in each column but since these letters are 26, we need two more characters for the positions 27 and 28. So, we used # for position 27 and * for position 28. For example, the vector ABEGIJLORUVXZ* applies to the ++--+-+-++-+--+--+--++-+-+-+ column.

For k=3 there are three different possible projections listed in Table 1. All of them contain a 2^3 full factorial design.

Table 1: Inequivalent projections of all 28-run inequivalent Hadamard matrices into k=3 dimensions.

No.	Projection
(3.1)	ABEGIJLORUVXZ*, ACDFIKLORTWYZ*, AHIJKLMNOPQRS*
(3.2)	ABEFGKLMOPSTUW, ADEFGHIJNOPWXY, AHIJKLMNOPQRS*
(3.3)	ACEFGKLMNQRVXY, ADEFGHIJNOPWXY, AHIJKLMNOPQRS*

Table 2 shows the generalized resolution and the generalized wordlength pattern of the three inequivalent projections of Hadamard matrices of order 28 in 3 factors. Projection (3.2) has the best properties than the other two and hence it is preferred from the others.

The frequencies of appearance of each projection in every Hadamard matrix are available on request. It is worth mentioning that the Plackett and Burman design does not provide us the projection (3.1).

Table 2: Sorting of the inequivalent projections of Hadamard matrices of order 28 in 3 dimensions according to their generalized resolution and their generalized wordlength pattern.

Projection number	Generalized Resolution	Generalized Wordlength Pattern
(3.2)	3.856	(0, 0, 0.2)
(3.3)	3.571	(0, 0, 0.18)
(3.1)	3.286	(0, 0, 0.51)

For k=4 there are seven different possible projections listed in Table 3. Projection (4.6) contains a full 2^4 factorial design while projections (4.2) and (4.5) contain a half fraction of the full 2^4 factorial design with defining relation $I=ABCD$ contrary to the projections (4.3) and (4.7) that contain a half fraction with defining relation $I=-ABCD$. Finally, projections (4.1) and (4.4) do not have any geometrical property.

The frequencies of appearance of each projection in every Hadamard matrix are available on request. The Plackett and Burman design does not provide us the projections (4.1) and (4.2). It is also worth to mentioning that over the 90% of the projections in each Hadamard matrix contain a half fraction of the full 2^4 factorial design and furthermore, projection (4.6), which is the best under geometric approach as it contains a full 2^4 factorial design, can be recognized in more than 50% out of the whole 17550 possible projections of the 27 columns of each Hadamard matrix of order 28 in 4 factors.

Table 3: Inequivalent projections of all 28-run inequivalent Hadamard matrices into k=4 dimensions.

Number	Projection
(4.1)	ABEGIJLORUVXZ*, ACDFIKLORTWYZ*, ADEFGHIJNOPWXY, AHIJKLMNOPQRS*
(4.2)	ADEFIMNOQTUV#*, ADEGJKPRSTVY#*, ABCDFHJLNPRTUV, ABCDGHIKOQSUVY
(4.3)	ADEGJKPRSTVY#*, ABCDEIJMQRSTWX, ABCDGHIKOQSUVY, ACEFGKLMNQRVXY
(4.4)	ABCDFHJLNPRTUV, ACEFGKLMNQRVXY, ADEFGHIJNOPWXY, AHIJKLMNOPQRS*
(4.5)	ABCDGHIKOQSUVY, ABEFGKLMOPSTUW, ACEFGKLMNQRVXY, ADEFGHIJNOPWXY
(4.6)	ABCDGHIKOQSUVY, ABEFGKLMOPSTUW, ADEFGHIJNOPWXY, AHIJKLMNOPQRS*
(4.7)	ABEFGKLMOPSTUW, ACEFGKLMNQRVXY, ADEFGHIJNOPWXY, AHIJKLMNOPQRS*

Table 4 shows the generalized resolution and the generalized wordlength pattern of the seven inequivalent projections of Hadamard matrices of order 28 in 4 factors. The classification has been made firstly by their generalized resolution and then by their generalized wordlength pattern. So, there are three projections with generalized resolution equal to 3.857 but projection (4.6) is the best since it has better generalized wordlength pattern. On the other hand, projection (4.1) is the worst since it has the least generalized resolution among all.

Table 4: Sorting of the inequivalent projections of Hadamard matrices of order 28 in 4 dimensions according to their generalized resolution and their generalized wordlength pattern.

Projection number	Generalized Resolution	Generalized Wordlength Pattern
(4.6)	3.857	(0, 0, 0.08, 0.02)
(4.5)	3.857	(0, 0, 0.08, 0.18)
(4.2)	3.857	(0, 0, 0.08, 0.51)
(4.7)	3.571	(0, 0, 0.24, 0.02)
(4.3)	3.571	(0, 0, 0.24, 0.18)
(4.4)	3.571	(0, 0, 0.41, 0.02)
(4.1)	3.286	(0, 0, 0.57, 0.02)

For k=5, we give partial results since the combinatorial equivalence algorithm we applied requires vast computational time which increases rapidly as the number of factors enlarges. In particular, we have studied the problem for only the first thirty matrices listed in http://www.research.att.com/~njas/hadamard/. From these Hadamard matrices, 126 inequivalent projections arise and they are listed in Table 5. It is worth mentioning that projections (5.91) and (5.101) contain a 2^{5-1}_V fraction with defining relations I=-$ABCDE$ and I=$ABCDE$ respectively.

The classification of these 126 projections under the generalized resolution and aberration criteria is presented in Table 6. From this table one can notice that projection (5.124) is the best under the classification criteria concerned and on the other hand, projections (5.2) and (5.29) are the worst ones under the same criteria. It is worth mentioning that several inequivalent projections have the same generalized resolution and wordlength pattern.

Table 5: Inequivalent projections of all 28-run inequivalent Hadamard matrices into k=5 dimensions.

Number	Projection
(5.1)	ABEFHIKNQSTWZ#, ABDGILMNSTXYZ*, ABCDGHIKNQRUVY, ADEFGHIJNOPWXY, AHIJKLMNOPQRS*
(5.2)	AHIJKLMTUVWXY#, ABEFGKLMNOPTUV, ACEFGKLMQRSWXY, ADEFGHIJNOPWXY, AHIJKLMNOPQRS*
(5.4)	ABEGHJMOQSUYZ#, ABCDFHJLOQSTVX, ACEFGKLMQRSWXY, ADEFGHIJNOPWXY, AHIJKLMNOPQRS*
(5.5)	ACDEHLMNORVWZ#, ACDGJKMNPQTXZ#, ABDFJKMNSVWYZ*, ABCFIMNOQUWX#*, ABCGHLNPSTWY#*
(5.6)	ACDGJKMNPQTXZ#, ABDEHKLPQUWXZ*, ABCGHLNPSTWY#*, ABCDEIJMPRSTUW, AHIJKLMNOPQRS*
(5.7)	ACDGJKMNPQTXZ#, ABDFJKMNSVWYZ*, ABCGHLNPSTWY#*, ABCDEIJMPRSTUW, ABCDFHJLOQSTVX
(5.8)	ABDEHKLPQUWXZ*, ACEFHIMPQTVYZ*, ABCGHLNPSTWY#*, ADFGHMPRSUVX#*, AHIJKLMNOPQRS*
(5.9)	ABDEHKLPQUWXZ*, ACFGHJKORTUWZ*, ADFGHMPRSUVX#*, ABCDFHJLOQSTVX, ADEFGHIJNOPWXY
(5.10)	ABDEHKLPQUWXZ*, ADFGHMPRSUVX#*, ABCDEIJMPRSTUW, ACEFGKLMQRSWXY, AHIJKLMNOPQRS*

(5.11)	ABDFJKMNSVWYZ*, ACEGIJLNSUVXZ*, ABCGHLNPSTWY#*, ADEGIKOQSTVW#*, ADFGHMPRSUVX#*
(5.12)	ABDFJKMNSVWYZ*, ABCEJKOPRVXY#*, ADEFJLNQRTUY#*, ABCDGHIKNQRUVY, ADEFGHIJNOPWXY
(5.13)	ABDGILMORTXYZ*, ACEFHIMPQTVYZ*, ABCFIMNOQUWX#*, ADEGIKOQSTVW#*, AHIJKLMNOPQRS*
(5.14)	ABDGILMORTXYZ*, ACFGHJKORTUWZ*, ABCEJKOPRVXY#*, ADEGIKOQSTVW#*, AHIJKLMNOPQRS*
(5.15)	ABDGILMORTXYZ*, ABCEJKOPRVXY#*, ABCFIMNOQUWX#*, ADEFJLNQRTUY#*, ABCDFHJLOQSTVX
(5.16)	ABDGILMORTXYZ*, ABCFIMNOQUWX#*, ABCGHLNPSTWY#*, ADEFJLNQRTUY#*, ADFGHMPRSUVX#*
(5.17)	ACEFHIMPQTVYZ*, ACEGIJLNSUVXZ*, ABCEJKOPRVXY#*, ADEGIKOQSTVW#*, ABCDEIJMPRSTUW
(5.18)	ACEFHIMPQTVYZ*, ACEGIJLNSUVXZ*, ADEFJLNQRTUY#*, ADFGHMPRSUVX#*, AHIJKLMNOPQRS*
(5.19)	ACEFHIMPQTVYZ*, ACFGHJKORTUWZ*, ABCEJKOPRVXY#*, ABCFIMNOQUWX#*, ABCGHLNPSTWY#*
(5.20)	ACEFHIMPQTVYZ*, ACFGHJKORTUWZ*, ABCGHLNPSTWY#*, ABCDFHJLOQSTVX, ADEFGHIJNOPWXY

(5.21)	ACEFHIMPQTVYZ*, ADFGHMPRSUVX#*, ABCDEIJMPRSTUW, ABEFGKLMNOPTUV, AHIJKLMNOPQRS*
(5.22)	ACEGIJLNSUVXZ*, ACFGHJKORTUWZ*, ABCEJKOPRVXY#*, ABCGHLNPSTWY#*, AHIJKLMNOPQRS*
(5.23)	ACEGIJLNSUVXZ*, ABCEJKOPRVXY#*, ABCFIMNOQUWX#*, ABCDFHJLOQSTVX, ACEFGKLMQRSWXY
(5.24)	ACEGIJLNSUVXZ*, ABCEJKOPRVXY#*, ADEFJLNQRTUY#*, ABCDFHJLOQSTVX, ABEFGKLMNOPTUV
(5.26)	ACEGIJLNSUVXZ*, ABCGHLNPSTWY#*, ACEFGKLMQRSWXY, ADEFGHIJNOPWXY, AHIJKLMNOPQRS*
(5.27)	ACFGHJKORTUWZ*, ABCEJKOPRVXY#*, ABCFIMNOQUWX#*, ADFGHMPRSUVX#*, AHIJKLMNOPQRS*
(5.28)	ACFGHJKORTUWZ*, ABCEJKOPRVXY#*, ABCGHLNPSTWY#*, ADEGIKOQSTVW#*, AHIJKLMNOPQRS*
(5.29)	ACFGHJKORTUWZ*, ABCEJKOPRVXY#*, ACEFGKLMQRSWXY, ADEFGHIJNOPWXY, AHIJKLMNOPQRS*
(5.30)	ACFGHJKORTUWZ*, ABCFIMNOQUWX#*, ADEFJLNQRTUY#*, ADFGHMPRSUVX#*, ABCDGHIKNQRUVY
(5.31)	ACFGHJKORTUWZ*, ABCGHLNPSTWY#*, ADEGIKOQSTVW#*, ADFGHMPRSUVX#*, ABCDGHIKNQRUVY

(5.32)	ACFGHJKORTUWZ*, ABCGHLNPSTWY#*, ADEGIKOQSTVW#*, ADFGHMPRSUVX#*, ABEFGKLMNOPTUV
(5.33)	ACFGHJKORTUWZ*, ABCGHLNPSTWY#*, ADFGHMPRSUVX#*, ABCDFHJLOQSTVX, ADEFGHIJNOPWXY
(5.34)	ABCEJKOPRVXY#*, ABCFIMNOQUWX#*, ADEFJLNQRTUY#*, ABCDEIJMPRSTUW, ADEFGHIJNOPWXY
(5.35)	ABCEJKOPRVXY#*, ABCFIMNOQUWX#*, ADEGIKOQSTVW#*, ABCDEIJMPRSTUW, ABCDFHJLOQSTVX
(5.37)	ABCEJKOPRVXY#*, ABCFIMNOQUWX#*, ABCDFHJLOQSTVX, ABCDGHIKNQRUVY, ACEFGKLMQRSWXY
(5.38)	ABCEJKOPRVXY#*, ABCGHLNPSTWY#*, ADEGIKOQSTVW#*, ABCDGHIKNQRUVY, ADEFGHIJNOPWXY
(5.39)	ABCEJKOPRVXY#*, ADEFJLNQRTUY#*, ADEGIKOQSTVW#*, ABCDEIJMPRSTUW, ABEFGKLMNOPTUV
(5.40)	ABCEJKOPRVXY#*, ADEFJLNQRTUY#*, ADEGIKOQSTVW#*, ABCDFHJLOQSTVX, ACEFGKLMQRSWXY

(5.41)	ABCEJKOPRVXY#*, ADEFJLNQRTUY#*, ADFGHMPRSUVX#*, ABCDEIJMPRSTUW, AHIJKLMNOPQRS*
(5.42)	ABCEJKOPRVXY#*, ABCDGHIKNQRUVY, ACEFGKLMQRSWXY, ADEFGHIJNOPWXY, AHIJKLMNOPQRS*
(5.43)	ABCFIMNOQUWX#*, ABCGHLNPSTWY#*, ADEFJLNQRTUY#*, ADEGIKOQSTVW#*, ADFGHMPRSUVX#*
(5.44)	ABCFIMNOQUWX#*, ABCGHLNPSTWY#*, ADEFJLNQRTUY#*, ABCDEIJMPRSTUW, AHIJKLMNOPQRS*
(5.45)	ABCFIMNOQUWX#*, ABCGHLNPSTWY#*, ADEGIKOQSTVW#*, ADFGHMPRSUVX#*, ACEFGKLMQRSWXY
(5.47)	ABCFIMNOQUWX#*, ABCGHLNPSTWY#*, ABCDFHJLOQSTVX, ACEFGKLMQRSWXY, ADEFGHIJNOPWXY
(5.48)	ABCFIMNOQUWX#*, ADEFJLNQRTUY#*, ADFGHMPRSUVX#*, ABCDEIJMPRSTUW, ABCDGHIKNQRUVY
(5.49)	ABCFIMNOQUWX#*, ADEGIKOQSTVW#*, ADFGHMPRSUVX#*, ABCDEIJMPRSTUW, AHIJKLMNOPQRS*
(5.50)	ABCFIMNOQUWX#*, ADEGIKOQSTVW#*, ADFGHMPRSUVX#*, ABCDGHIKNQRUVY, ABEFGKLMNOPTUV

(5.51)	ABCFIMNOQUWX#*, ADEGIKOQSTVW#*, ABCDEIJMPRSTUW, ABCDGHIKNQRUVY, ADEFGHIJNOPWXY
(5.52)	ABCFIMNOQUWX#*, ADEGIKOQSTVW#*, ABCDEIJMPRSTUW, ADEFGHIJNOPWXY, AHIJKLMNOPQRS*
(5.53)	ABCFIMNOQUWX#*, ADEGIKOQSTVW#*, ABCDFHJLOQSTVX, ABEFGKLMNOPTUV, ACEFGKLMQRSWXY
(5.54)	ABCFIMNOQUWX#*, ADEGIKOQSTVW#*, ABCDFHJLOQSTVX, ABEFGKLMNOPTUV, AHIJKLMNOPQRS*
(5.55)	ABCFIMNOQUWX#*, ADFGHMPRSUVX#*, ABCDFHJLOQSTVX, ABCDGHIKNQRUVY, ACEFGKLMQRSWXY
(5.56)	ABCFIMNOQUWX#*, ADFGHMPRSUVX#*, ABCDFHJLOQSTVX, ACEFGKLMQRSWXY, ADEFGHIJNOPWXY
(5.58)	ABCGHLNPSTWY#*, ADEFJLNQRTUY#*, ADFGHMPRSUVX#*, ABCDEIJMPRSTUW, ADEFGHIJNOPWXY
(5.59)	ABCGHLNPSTWY#*, ADEFJLNQRTUY#*, ADFGHMPRSUVX#*, ABCDEIJMPRSTUW, AHIJKLMNOPQRS*
(5.60)	ABCGHLNPSTWY#*, ADEFJLNQRTUY#*, ABCDEIJMPRSTUW, ABCDGHIKNQRUVY, ADEFGHIJNOPWXY
(5.61)	ABCGHLNPSTWY#*, ADEFJLNQRTUY#*, ABCDFHJLOQSTVX, ACEFGKLMQRSWXY, ADEFGHIJNOPWXY

(5.62)	ABCGHLNPSTWY#*, ADEGIKOQSTVW#*, ADFGHMPRSUVX#*, ABCDEIJMPRSTUW, ABCDFHJLOQSTVX
(5.63)	ABCGHLNPSTWY#*, ADEGIKOQSTVW#*, ADFGHMPRSUVX#*, ABCDEIJMPRSTUW, ADEFGHIJNOPWXY
(5.64)	ABCGHLNPSTWY#*, ADEGIKOQSTVW#*, ADFGHMPRSUVX#*, ABCDGHIKNQRUVY, ACEFGKLMQRSWXY
(5.65)	ABCGHLNPSTWY#*, ADEGIKOQSTVW#*, ADFGHMPRSUVX#*, ABCDGHIKNQRUVY, ADEFGHIJNOPWXY
(5.66)	ABCGHLNPSTWY#*, ADEGIKOQSTVW#*, ADFGHMPRSUVX#*, ABCDGHIKNQRUVY, AHIJKLMNOPQRS*
(5.67)	ABCGHLNPSTWY#*, ADEGIKOQSTVW#*, ABCDEIJMPRSTUW, ABCDFHJLOQSTVX, ADEFGHIJNOPWXY
(5.69)	ABCGHLNPSTWY#*, ADEGIKOQSTVW#*, ABCDFHJLOQSTVX, ABCDGHIKNQRUVY, ADEFGHIJNOPWXY
(5.70)	ABCGHLNPSTWY#*, ADEGIKOQSTVW#*, ABCDGHIKNQRUVY, ABEFGKLMNOPTUV, ADEFGHIJNOPWXY
(5.71)	ABCGHLNPSTWY#*, ADFGHMPRSUVX#*, ABCDEIJMPRSTUW, ABCDFHJLOQSTVX, ADEFGHIJNOPWXY

(5.72)	ABCGHLNPSTWY#*, ADFGHMPRSUVX#*, ABCDEIJMPRSTUW, ABCDFHJLOQSTVX, AHIJKLMNOPQRS*
(5.73)	ABCGHLNPSTWY#*, ADFGHMPRSUVX#*, ABCDEIJMPRSTUW, ABCDGHIKNQRUVY, AHIJKLMNOPQRS*
(5.74)	ABCGHLNPSTWY#*, ADFGHMPRSUVX#*, ABCDEIJMPRSTUW, ABEFGKLMNOPTUV, AHIJKLMNOPQRS*
(5.75)	ABCGHLNPSTWY#*, ADFGHMPRSUVX#*, ABCDEIJMPRSTUW, ACEFGKLMQRSWXY, AHIJKLMNOPQRS*
(5.76)	ABCGHLNPSTWY#*, ADFGHMPRSUVX#*, ABCDFHJLOQSTVX, ABCDGHIKNQRUVY, ADEFGHIJNOPWXY
(5.77)	ABCGHLNPSTWY#*, ADFGHMPRSUVX#*, ABCDFHJLOQSTVX, ABEFGKLMNOPTUV, ADEFGHIJNOPWXY
(5.78)	ABCGHLNPSTWY#*, ADFGHMPRSUVX#*, ABCDFHJLOQSTVX, ACEFGKLMQRSWXY, AHIJKLMNOPQRS*
(5.79)	ABCGHLNPSTWY#*, ADFGHMPRSUVX#*, ABCDFHJLOQSTVX, ADEFGHIJNOPWXY, AHIJKLMNOPQRS*
(5.80)	ABCGHLNPSTWY#*, ADFGHMPRSUVX#*, ABCDGHIKNQRUVY, ADEFGHIJNOPWXY, AHIJKLMNOPQRS*

(5.81)	ABCGHLNPSTWY#*, ABCDEIJMPRSTUW, ABCDFHJLOQSTVX, ABCDGHIKNQRUVY, ADEFGHIJNOPWXY
(5.82)	ABCGHLNPSTWY#*, ABCDEIJMPRSTUW, ABCDFHJLOQSTVX, ACEFGKLMQRSWXY, ADEFGHIJNOPWXY
(5.83)	ABCGHLNPSTWY#*, ABCDEIJMPRSTUW, ABCDFHJLOQSTVX, ADEFGHIJNOPWXY, AHIJKLMNOPQRS*
(5.84)	ABCGHLNPSTWY#*, ABCDFHJLOQSTVX, ABCDGHIKNQRUVY, ACEFGKLMQRSWXY, ADEFGHIJNOPWXY
(5.85)	ABCGHLNPSTWY#*, ABCDFHJLOQSTVX, ABCDGHIKNQRUVY, ADEFGHIJNOPWXY, AHIJKLMNOPQRS*
(5.86)	ABCGHLNPSTWY#*, ABCDGHIKNQRUVY, ABEFGKLMNOPTUV, ACEFGKLMQRSWXY, ADEFGHIJNOPWXY
(5.87)	ABCGHLNPSTWY#*, ABCDGHIKNQRUVY, ACEFGKLMQRSWXY, ADEFGHIJNOPWXY, AHIJKLMNOPQRS*
(5.88)	ADEFJLNQRTUY#*, ADFGHMPRSUVX#*, ABCDEIJMPRSTUW, ABCDGHIKNQRUVY, ACEFGKLMQRSWXY
(5.89)	ADEFJLNQRTUY#*, ADFGHMPRSUVX#*, ABCDEIJMPRSTUW, ABCDGHIKNQRUVY, AHIJKLMNOPQRS*
(5.90)	ADEFJLNQRTUY#*, ADFGHMPRSUVX#*, ABCDFHJLOQSTVX, ACEFGKLMQRSWXY, ADEFGHIJNOPWXY

(5.91)	ADEFJLNQRTUY#*, ABCDFHJLOQSTVX, ABCDGHIKNQRUVY, ADEFGHIJNOPWXY, AHIJKLMNOPQRS*
(5.92)	ADEGIKOQSTVW#*, ADFGHMPRSUVX#*, ABCDEIJMPRSTUW, ABCDFHJLOQSTVX, ABEFGKLMNOPTUV
(5.93)	ADEGIKOQSTVW#*, ADFGHMPRSUVX#*, ABCDEIJMPRSTUW, ABCDFHJLOQSTVX, ACEFGKLMQRSWXY
(5.94)	ADEGIKOQSTVW#*, ADFGHMPRSUVX#*, ABCDEIJMPRSTUW, ABCDFHJLOQSTVX, ADEFGHIJNOPWXY
(5.95)	ADEGIKOQSTVW#*, ADFGHMPRSUVX#*, ABCDEIJMPRSTUW, ABCDFHJLOQSTVX, AHIJKLMNOPQRS*
(5.96)	ADEGIKOQSTVW#*, ADFGHMPRSUVX#*, ABCDEIJMPRSTUW, ABCDGHIKNQRUVY, AHIJKLMNOPQRS*
(5.97)	ADEGIKOQSTVW#*, ADFGHMPRSUVX#*, ABCDEIJMPRSTUW, ACEFGKLMQRSWXY, ADEFGHIJNOPWXY
(5.98)	ADEGIKOQSTVW#*, ADFGHMPRSUVX#*, ABCDEIJMPRSTUW, ADEFGHIJNOPWXY, AHIJKLMNOPQRS*
(5.99)	ADEGIKOQSTVW#*, ABCDEIJMPRSTUW, ABCDGHIKNQRUVY, ACEFGKLMQRSWXY, ADEFGHIJNOPWXY

(5.100)	ADEGIKOQSTVW#*, ABCDEIJMPRSTUW, ABCDGHIKNQRUVY, ADEFGHIJNOPWXY, AHIJKLMNOPQRS*
(5.101)	ADEGIKOQSTVW#*, ABCDFHJLOQSTVX, ABEFGKLMNOPTUV, ADEFGHIJNOPWXY, AHIJKLMNOPQRS*
(5.102)	ADFGHMPRSUVX#*, ABCDEIJMPRSTUW, ABCDFHJLOQSTVX, ABCDGHIKNQRUVY, ABEFGKLMNOPTUV
(5.103)	ADFGHMPRSUVX#*, ABCDEIJMPRSTUW, ABCDFHJLOQSTVX, ABCDGHIKNQRUVY, ADEFGHIJNOPWXY
(5.104)	ADFGHMPRSUVX#*, ABCDEIJMPRSTUW, ABCDFHJLOQSTVX, ABEFGKLMNOPTUV, ACEFGKLMQRSWXY
(5.105)	ADFGHMPRSUVX#*, ABCDEIJMPRSTUW, ABCDFHJLOQSTVX, ADEFGHIJNOPWXY, AHIJKLMNOPQRS*
(5.106)	ADFGHMPRSUVX#*, ABCDEIJMPRSTUW, ABCDGHIKNQRUVY, ABEFGKLMNOPTUV, AHIJKLMNOPQRS*
(5.107)	ADFGHMPRSUVX#*, ABCDEIJMPRSTUW, ABCDGHIKNQRUVY, ACEFGKLMQRSWXY, AHIJKLMNOPQRS*
(5.108)	ADFGHMPRSUVX#*, ABCDEIJMPRSTUW, ABCDGHIKNQRUVY, ADEFGHIJNOPWXY, AHIJKLMNOPQRS*
(5.109)	ADFGHMPRSUVX#*, ABCDEIJMPRSTUW, ABEFGKLMNOPTUV, ACEFGKLMQRSWXY, ADEFGHIJNOPWXY

(5.110)	ADFGHMPRSUVX#*, ABCDEIJMPRSTUW, ABEFGKLMNOPTUV, ACEFGKLMQRSWXY, AHIJKLMNOPQRS*
(5.111)	ADFGHMPRSUVX#*, ABCDEIJMPRSTUW, ABEFGKLMNOPTUV, ADEFGHIJNOPWXY, AHIJKLMNOPQRS*
(5.112)	ADFGHMPRSUVX#*, ABCDEIJMPRSTUW, ACEFGKLMQRSWXY, ADEFGHIJNOPWXY, AHIJKLMNOPQRS*
(5.113)	ADFGHMPRSUVX#*, ABCDFHJLOQSTVX, ABCDGHIKNQRUVY, ABEFGKLMNOPTUV, ADEFGHIJNOPWXY
(5.114)	ADFGHMPRSUVX#*, ABCDFHJLOQSTVX, ABCDGHIKNQRUVY, ACEFGKLMQRSWXY, AHIJKLMNOPQRS*
(5.115)	ADFGHMPRSUVX#*, ABCDFHJLOQSTVX, ABCDGHIKNQRUVY, ADEFGHIJNOPWXY, AHIJKLMNOPQRS*
(5.116)	ADFGHMPRSUVX#*, ABCDFHJLOQSTVX, ABEFGKLMNOPTUV, ACEFGKLMQRSWXY, ADEFGHIJNOPWXY
(5.117)	ADFGHMPRSUVX#*, ABCDFHJLOQSTVX, ABEFGKLMNOPTUV, ADEFGHIJNOPWXY, AHIJKLMNOPQRS*
(5.118)	ADFGHMPRSUVX#*, ABCDFHJLOQSTVX, ACEFGKLMQRSWXY, ADEFGHIJNOPWXY, AHIJKLMNOPQRS*
(5.119)	ADFGHMPRSUVX#*, ABCDGHIKNQRUVY, ABEFGKLMNOPTUV, ACEFGKLMQRSWXY, ADEFGHIJNOPWXY

(5.120)	ADFGHMPRSUVX#*, ABEFGKLMNOPTUV, ACEFGKLMQRSWXY, ADEFGHIJNOPWXY, AHIJKLMNOPQRS*
(5.121)	ABCDEIJMPRSTUW, ABCDFHJLOQSTVX, ABCDGHIKNQRUVY, ABEFGKLMNOPTUV, AHIJKLMNOPQRS*
(5.122)	ABCDEIJMPRSTUW, ABCDFHJLOQSTVX, ABCDGHIKNQRUVY, ADEFGHIJNOPWXY, AHIJKLMNOPQRS*
(5.123)	ABCDFHJLOQSTVX, ABCDGHIKNQRUVY, ABEFGKLMNOPTUV, ACEFGKLMQRSWXY, ADEFGHIJNOPWXY
(5.124)	ABCDFHJLOQSTVX, ABCDGHIKNQRUVY, ABEFGKLMNOPTUV, ADEFGHIJNOPWXY, AHIJKLMNOPQRS*
(5.125)	ABCDFHJLOQSTVX ABCDGHIKNQRUVY ACEFGKLMQRSWXY ADEFGHIJNOPWXY AHIJKLMNOPQRS*
(5.126)	ABCDGHIKNQRUVY ABEFGKLMNOPTUV ACEFGKLMQRSWXY ADEFGHIJNOPWXY AHIJKLMNOPQRS*

Table 6: Sorting of the inequivalent projections of Hadamard matrices of order 28 in 5 dimensions according to their generalized resolution and their generalized wordlength pattern.

Projection number	Generalized Resolution	Generalized Wordlength Pattern
(5.124)	3.857	(0, 0, 0.204, 0.102, 0)
(5.91)	3.857	(0, 0, 0.204, 0.102, 0.082)
(5.104)	3.857	(0, 0, 0.204, 0.102, 0.082)
(5.101)	3.857	(0, 0, 0.204, 0.102, 0.327)
(5.102)	3.857	(0, 0, 0.204, 0.265, 0)
(5.119)	3.857	(0, 0, 0.204, 0.265, 0.082)
(5.122)	3.857	(0, 0, 0.204, 0.265, 0.082)
(5.121)	3.857	(0, 0, 0.204, 0.265, 0.327)
(5.114)	3.857	(0, 0, 0.204, 0.429, 0)
(5.123)	3.857	(0, 0, 0.204, 0.429, 0)
(5.109)	3.857	(0, 0, 0.204, 0.429, 0.082)
(5.93)	3.857	(0, 0, 0.204, 0.592, 0)
(5.88)	3.857	(0, 0, 0.204, 0.592, 0.082)
(5.92)	3.857	(0, 0, 0.204, 0.592, 0.082)
(5.117)	3.571	(0, 0, 0.367, 0.102, 0)
(5.61)	3.571	(0, 0, 0.367, 0.102, 0.082)
(5.108)	3.571	(0, 0, 0.367, 0.102, 0.082)
(5.113)	3.571	(0, 0, 0.367, 0.102, 0.082)
(5.34)	3.571	(0, 0, 0.367, 0.102, 0.327)
(5.40)	3.571	(0, 0, 0.367, 0.102, 0.327)
(5.86)	3.571	(0, 0, 0.367, 0.265, 0)
(5.90)	3.571	(0, 0, 0.367, 0.265, 0)
(5.106)	3.571	(0, 0, 0.367, 0.265, 0)
(5.107)	3.571	(0, 0, 0.367, 0.265, 0)
(5.116)	3.571	(0, 0, 0.367, 0.265, 0)
(5.58)	3.571	(0, 0, 0.367, 0.265, 0.082)
(5.97)	3.571	(0, 0, 0.367, 0.265, 0.082)
(5.103)	3.571	(0, 0, 0.367, 0.265, 0.082)
(5.111)	3.571	(0, 0, 0.367, 0.265, 0.082)
(5.115)	3.571	(0, 0, 0.367, 0.265, 0.082)
(5.78)	3.571	(0, 0, 0.367, 0.429, 0)
(5.96)	3.571	(0, 0, 0.367, 0.429, 0)
(5.41)	3.571	(0, 0, 0.367, 0.429, 0.082)
(5.99)	3.571	(0, 0, 0.367, 0.429, 0.082)
(5.100)	3.571	(0, 0, 0.367, 0.429, 0.082)
(5.110)	3.571	(0, 0, 0.367, 0.429, 0.082)
(5.45)	3.571	(0, 0, 0.367, 0.429, 0.327)
(5.89)	3.571	(0, 0, 0.367, 0.592, 0)
(5.95)	3.571	(0, 0, 0.367, 0.592, 0)
(5.47)	3.571	(0, 0, 0.531, 0.102, 0)
(5.73)	3.571	(0, 0, 0.531, 0.102, 0)
(5.83)	3.571	(0, 0, 0.531, 0.102, 0)
(5.84)	3.571	(0, 0, 0.531, 0.102, 0)
(5.98)	3.571	(0, 0, 0.531, 0.102, 0)

(5.49)	3.571	(0, 0, 0.531, 0.102, 0.082)
(5.80)	3.571	(0, 0, 0.531, 0.102, 0.082)
(5.82)	3.571	(0, 0, 0.531, 0.102, 0.082)
(5.85)	3.571	(0, 0, 0.531, 0.102, 0.082)
(5.105)	3.571	(0, 0, 0.531, 0.102, 0.082)
(5.50)	3.571	(0, 0, 0.531, 0.265, 0)
(5.52)	3.571	(0, 0, 0.531, 0.265, 0)
(5.74)	3.571	(0, 0, 0.531, 0.265, 0)
(5.77)	3.571	(0, 0, 0.531, 0.265, 0)
(5.35)	3.571	(0, 0, 0.531, 0.265, 0.082)
(5.44)	3.571	(0, 0, 0.531, 0.265, 0.082)
(5.63)	3.571	(0, 0, 0.531, 0.265, 0.082)
(5.70)	3.571	(0, 0, 0.531, 0.265, 0.082)
(5.75)	3.571	(0, 0, 0.531, 0.265, 0.082)
(5.43)	3.571	(0, 0, 0.531, 0.429, 0)
(5.46)	3.571	(0, 0, 0.531, 0.429, 0)
(5.54)	3.571	(0, 0, 0.531, 0.429, 0)
(5.55)	3.571	(0, 0, 0.531, 0.429, 0)
(5.56)	3.571	(0, 0, 0.531, 0.429, 0)
(5.57)	3.571	(0, 0, 0.531, 0.429, 0)
(5.64)	3.571	(0, 0, 0.531, 0.429, 0)
(5.51)	3.571	(0, 0, 0.531, 0.429, 0.082)
(5.53)	3.571	(0, 0, 0.531, 0.429, 0.082)
(5.60)	3.571	(0, 0, 0.531, 0.429, 0.082)
(5.62)	3.571	(0, 0, 0.531, 0.592, 0)
(5.18)	3.571	(0, 0, 0.531, 0.592, 0.082)
(5.94)	3.571	(0, 0, 0.531, 0.592, 0.082)
(5.59)	3.571	(0, 0, 0.694, 0.102, 0)
(5.76)	3.571	(0, 0, 0.694, 0.102, 0)
(5.17)	3.571	(0, 0, 0.694, 0.102, 0.082)
(5.32)	3.571	(0, 0, 0.694, 0.102, 0.082)
(5.33)	3.571	(0, 0, 0.694, 0.102, 0.082)
(5.38)	3.571	(0, 0, 0.694, 0.102, 0.082)
(5.13)	3.571	(0, 0, 0.694, 0.102, 0.327)
(5.36)	3.571	(0, 0, 0.694, 0.265, 0)
(5.65)	3.571	(0, 0, 0.694, 0.265, 0)
(5.66)	3.571	(0, 0, 0.694, 0.265, 0)
(5.67)	3.571	(0, 0, 0.694, 0.265, 0)
(5.69)	3.571	(0, 0, 0.694, 0.265, 0)
(5.72)	3.571	(0, 0, 0.694, 0.265, 0)
(5.79)	3.571	(0, 0, 0.694, 0.265, 0)
(5.11)	3.571	(0, 0, 0.694, 0.265, 0.082)
(5.37)	3.571	(0, 0, 0.694, 0.265, 0.082)
(5.71)	3.571	(0, 0, 0.694, 0.265, 0.082)
(5.68)	3.571	(0, 0, 0.694, 0.429, 0)
(5.81)	3.571	(0, 0, 0.694, 0.429, 0)
(5.6)	3.571	(0, 0, 0.694, 0.429, 0.082)
(5.10)	3.571	(0, 0, 0.694, 0.429, 0.082)
(5.16)	3.571	(0, 0, 0.694, 0.429, 0.082)

(5.19)	3.571	(0, 0, 0.694, 0.592, 0)
(5.39)	3.571	(0, 0, 0.694, 0.592, 0)
(5.24)	3.571	(0, 0, 0.857, 0.102, 0)
(5.30)	3.571	(0, 0, 0.857, 0.102, 0)
(5.31)	3.571	(0, 0, 0.857, 0.102, 0.082)
(5.25)	3.571	(0, 0, 0.857, 0.265, 0)
(5.15)	3.571	(0, 0, 0.857, 0.265, 0.082)
(5.9)	3.571	(0, 0, 0.857, 0.429, 0)
(5.20)	3.571	(0, 0, 0.857, 0.429, 0)
(5.7)	3.571	(0, 0, 0.857, 0.429, 0.082)
(5.21)	3.571	(0, 0, 0.857, 0.429, 0.082)
(5.48)	3.571	(0, 0, 0.857, 0.592, 0)
(5.8)	3.571	(0, 0, 0.857, 0.592, 0.082)
(5.23)	3.571	(0, 0, 0.857, 0.592, 0.082)
(5.5)	3.571	(0, 0, 1.02, 0.102, 0.082)
(5.120)	3.286	(0, 0, 0.694, 0.102, 0)
(5.27)	3.286	(0, 0, 0.694, 0.102, 0.082)
(5.125)	3.286	(0, 0, 0.694, 0.102, 0.082)
(5.126)	3.286	(0, 0, 0.694, 0.265, 0)
(5.118)	3.286	(0, 0, 0.857, 0.102, 0)
(5.87)	3.286	(0, 0, 0.857, 0.265, 0)
(5.14)	3.286	(0, 0, 0.857, 0.265, 0.082)
(5.28)	3.286	(0, 0, 0.857, 0.265, 0.082)
(5.112)	3.286	(0, 0, 0.857, 0.429, 0)
(5.22)	3.286	(0, 0, 0.857, 0.592, 0)
(5.42)	3.286	(0, 0, 1.02, 0.265, 0)
(5.12)	3.286	(0, 0, 1.02, 0.429, 0)
(5.3)	3.286	(0, 0, 1.02, 0.429, 0.082)
(5.26)	3.286	(0, 0, 1.02, 0.429, 0.082)
(5.4)	3.286	(0, 0, 1.184, 0.265, 0)
(5.1)	3.286	(0, 0, 1.184, 0.592, 0)
(5.2)	3.286	(0, 0, 1.184, 0.592, 0.082)
(5.29)	3.286	(0, 0, 1.184, 0.592, 0.082)

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