Search with Adverse Selection[*]

Stephan Lauermann[†] and Asher Wolinsky[‡]

This Version: April 2011

Abstract

This paper analyzes a search model with asymmetric information of the common values variety. The basic features of this environment resemble those of a common values (procurement) auction, except that the searcher in our model, who is the counterpart of the auctioneer in the auction model, encounters trading partners through costly sequential search. The main objective is to understand how the combination of search activity and information asymmetry affects prices and welfare. We specifically inquire about the extent of information aggregation by the price —how close the equilibrium prices are to the full information prices—when the search frictions are small. Roughly speaking, we conclude that information is aggregated less well in the search environment than it is in the corresponding auction environment. We trace this to a stronger form of winner's curse that is present in the search scenario. This understanding is a central qualitative insight of this paper, which is likely to have implications beyond the narrow confines of our model. We also look at the efficiency perspective and examine the relations between total surplus and the informativenss of the signal technology available to the uninformed. We conclude that total surplus is not monotone in the quality of the signals.

JEL Classifications: D44, D82, D83

Keywords: Adverse Selection, Winner's Curse, Search Theory, Auctions, Information Aggregation.

[*]We are grateful to Lones Smith and Gábor Virág for their comments and to Guannan Luo for research assistance. Wolinsky gratefully acknowledges support from the NSF. The authors also thank the Hausdorff Research Institute (HIM) in Bonn for hospitality, where parts of this paper have been written.

[†]University of Michigan, Department of Economics, slauerma@umich.edu.

[‡]Northwestern University, Department of Economics, a-wolinsky@northwestern.edu.

1 Introduction

This paper analyzes a search model with asymmetric information of the common value variety. The basic features of this environment resemble those of a common value (procurement) auction, except that the searcher in our model, who is the counterpart of the auctioneer in the auction model, encounters trading partners through costly sequential search.

The main objective of this paper is to understand how the combination of search activity and information asymmetry affects prices and welfare. We specifically inquire about the extent of information aggregation by the price —how close the equilibrium prices are to the full information prices—when the search frictions are small. Roughly speaking, we conclude that information is aggregated less well in the search environment than it is in the corresponding auction environment. We trace this to a stronger form of the winner's curse that is present in the search scenario. This understanding is a central qualitative insight of this paper, which is likely to have implications beyond the narrow confines of our model. We also look at the efficiency perspective and examine the relations between total surplus and the informativeness of the signal technology available to the uninformed. We conclude that total surplus is not monotone in the quality of the signals.

The searching agent—the buyer—samples sequentially alternative trading partners—sellers—for a potential transaction that involves information asymmetry. To have a concrete story in mind, one can think of a simple procurement scenario in which a homeowner (the buyer) needs a repair service and searches among potential providers (the sellers)[1]. The buyer has private information that determines the cost to sellers. We assume that the buyer can be one of two types, L and H, which stand for low and high, respectively. A seller's cost of providing the service, c_w, is the same for all sellers and it depends on the buyer's type $w \in \{L, H\}$ with $c_H > c_L$. Upon being sampled, a seller observes a noisy signal of the cost. Then the buyer and that seller bargain over the price. In the main model the bargaining takes the form of a take-it-or-leave-it price offer by the buyer, but later in the paper we consider other forms as well. If sellers had complete information, search would end immediately with a transaction at a price equal to the true cost. Under incomplete information, however, the prices that a seller accepts depend on his belief about the type of the buyer conditional on being sampled and conditional on the observed signal. Since the signal is noisy, different sellers observe different realizations of the signal and, therefore, have different beliefs. The buyer has an incentive to search for sellers who receive a favorable signal in order to trade at a lower price, but this incentive and hence the search intensity might vary across the different types of the buyer. Sellers cannot observe the buyer's search history, but they understand the buyer's search behavior and take it into account when interpreting their signals. This effect of the anticipated search behavior on sellers' beliefs and hence on the price distinguishes the information aggregation

[1] Alternatively, one may reverse the roles of what we call buyer and sellers to obtain an alternative story of sale of an object of uncertain quality by an informed seller.

in the search environment from that in the corresponding auction environment.

The equilibrium concept is Perfect Bayesian supplemented by a refinement that restricts the beliefs following moves off the path. Equilibrium prices aggregate the information perfectly if they are equal to the true cost. We show that when the sampling cost is small, the extent of information aggregation by equilibrium prices depends on the informativeness of the signal technology in the following way.

Let F_w denote the distribution of the signal x on a common support $[\underline{x}, \overline{x}]$ when the buyer is of type w. Low realizations of the signal indicate a higher likelihood of the low cost type, that is, the likelihood ratio f_L/f_H is decreasing. Subject to a regularity condition, we show that the equilibrium outcome is determined by the tail properties of the distribution of the likelihood ratios. The equilibrium is revealing if and only if the following two conditions hold. First, there are values in the support of the signal distributions that are arbitrarily informative about L, i.e., $f_L(x)/f_H(x) \longrightarrow_{x\to\underline{x}} \infty$. Second, conditional on the buyer being of the low cost type, the probability of "exceedingly informative" signals is sufficiently high. Specifically, the distribution of the likelihood ratios must have a "thick tail" in a sense that will be made precise later in the paper. If these conditions fail, the limit equilibrium prices do not aggregate the information perfectly. It might be completely pooling, in the sense that both of the buyer types end up trading at the same price, or semi-pooling in the sense that the different types end up trading at different expected prices which do not coincide with the expected costs. This is caused by excessive search of the bad type of the buyer, which diminishes the informative value of signals.

In our base model, welfare coincides with (the negative of) the accumulated search costs (the buyer always buys while the price is just a transfer). We evaluate welfare in the limit as the sampling cost goes to zero. This limit is not monotone in the informativeness of the signal technology. The limit outcome is nearly efficient when the signal technology is very informative or very uninformative and it is inefficient for intermediate levels of informativeness. The efficiency losses are more significant when the signal technology is of intermediate quality, since this is the case in which the high cost type invests the most effort in trying to mimic the low cost type. In contrast, when the signals are very uninformative, both types do not search much, and when they are very informative, only the low cost type who is the more efficient searcher undertakes significant search.

We compare our results to their counterparts in a setting in which a buyer can commit to a procurement auction (with the same signal structure) and we look at the case in which the number of bidders is large. As shown by Milgrom (1979) and Wilson (1977), the equilibrium winning bid in the limit auction (as the number of bidders increases indefinitely) aggregates the information perfectly if and only if there exist arbitrarily informative signals, i.e., $f_L(x)/f_H(x) \longrightarrow_{x\to\underline{x}} \infty$. In contrast, as mentioned above, nearly perfect information aggregation in the search model requires an additional condition guaranteeing that exceedingly informative signals are sufficiently likely. Indeed, in our search model, the outcome can involve complete pooling even when there are arbitrarily informative signals (i.e., the condition $f_L(x)/f_H(x) \longrightarrow_{x \to \underline{x}} \infty$ holds). Furthermore, the limit auction equilibrium outcome never involves complete pooling, even when the most informative signals are of bounded strength (i.e., $\lim_{x \to \underline{x}} f_L(x)/f_H(x) < \infty$), whereas in our search model these situations will necessarily involve complete pooling. These are the senses in which the search model aggregates information more poorly than the corresponding auction model. The reason for this difference is that, in the auction model, the buyer commits to consider only the bids of a given number of sellers, even when their signals are not favorable. In the search model, the buyer may continue to sample sellers until encountering one with sufficiently favorable signal. This exacerbates the winner's curse in the search model relative to its counterpart in the corresponding auction model and impedes the aggregation of information by prices.

This paper is related to two bodies of work. One deals with the question of information aggregation in the interaction of a large group of players. We have already mentioned Milgrom (1979) and Wilson (1977) who addressed this question in the context of a single unit auction. Feddersen and Pesendorfer (1997) and Duggan and Martinelli (2001) consider information aggregation in the context of voting model, Smith and Sorenson (2000) consider it in the context of social learning, Pesendorfer and Swinkels (1997) consider it in the context of a multi-unit auction.

Another related body of work deals with search with adverse selection, e.g., Inderst (2005), Moreno and Wooders (2010), Guerrieri, Shimer and Wright (2010), and Hörner and Vieille (2009). While these papers are not directly related to ours in terms of the model and questions, they do have in common with our paper the idea that, in a search model, the distribution of types is determined endogenously. For example, in Inderst (2005) the distribution of types adjusts to sustain the Rothschild-Stiglitz best separating outcome as an equilibrium (which would not necessarily be the case for an arbitrary exogenous distribution of types).

There are a few papers in the intersection of these literatures. Wolinsky (1990) and Blouin and Serrano (2001) show that, in a two sided search model with binary signals and actions, information is not aggregated perfectly even as the frictions are made negligible. Duffie and Manso (2007) and Duffie, Malamud and Manso (2009) characterize information percolation in markets where agents truthfully exchange their information with each other whenever they are matched.

2 The Model

We present a lean version of the model without apologies and relegate discussion of possible variations and extensions to Section 8 below.

A buyer samples sequentially from a continuum of identical sellers in search for a single transaction, incurring a cost $s > 0$ ("search cost") for each seller sampled. The set of sellers is an interval and the buyer's draws from this set are independent and uniformly distributed.

To have a concrete story in mind, one may think of a procurement scenario in which the buyer needs a repair service and samples providers sequentially to select one to perform it.[2]

A seller's cost of providing the service, c_w, is the same for all sellers and depends on the buyer's type $w \in \{L, H\}$ with $c_H > c_L$. The prior probabilities of L and H are g_L and g_H respectively ($g_L + g_H = 1$). The type w is known to the buyer but not to the sellers.

Upon meeting the buyer, the seller obtains a signal $x \in X = [\underline{x}, \overline{x}]$ from a distribution F_w, $w = L, H$, with continuously differentiable density f_w strictly positive on $(\underline{x}, \overline{x})$. A lower value of the signal indicates a higher likelihood of L and F_w has the monotone likelihood ratio property, i.e., $f_L(x)/f_H(x)$ is strictly decreasing in x on $(\underline{x}, \overline{x})$. The signal is observed by the buyer as well. Conditional on the state, the signals are independent across sellers.

After a seller was sampled and the signal wass observed, the buyer and this seller have an opportunity to agree whether to transact and at what price. A transaction means that the seller provides the buyer with a service of known value u independent of the buyer's type. In our base model, we assume that the buyer makes a take-it-or-leave-it price offer to the seller who then either accepts or rejects.

The buyer's payoff from transacting at a price p after sampling n sellers is

$$u - p - ns.$$

The seller's payoff is

$$\begin{cases} p - c_w & if \quad \text{transacts,} \\ 0 & \quad\quad\;\; \text{otherwise.} \end{cases}$$

We assume that u is sufficiently larger than $c_H + s$, so that the buyer of either type would like to participate.

The buyer observes the entire search history. But sellers observe only what occurs in their own encounters with the buyer: being sampled, the signal and the price offer. In particular, a seller does not know how many other sellers were sampled by the buyer prior to being contacted.

[2] Alternatively, one may reverse the roles of what we call buyer and sellers to obtain an even more standard story of sale of an object of uncertain quality w.

The buyer's behavior is described by a (Markovian pure[3]) strategy $P = (P_L, P_H)$, where $P_w(x) \in [0, u]$ is the price[4] offered by w after signal x.

The seller's strategy prescribes an acceptance probability $A(p, x) \in [0, 1]$ of price p after signal x.

Consider a symmetric situation in which the buyer employs the Markovian offer strategy P and the sellers employ the acceptance strategy A. Let $V_w(P, A)$ denote the expected payoffs of buyer $w = L, H$, given P and A. It is defined recursively by

$$V_w = \int_{\underline{x}}^{\bar{x}} \left[A\left(P_w\left(x\right), x\right)\left(u - P_w\left(x\right)\right) + \left(1 - A\left(P_w\left(x\right), x\right)\right) V_w\right] dF_w\left(x\right) - s.$$

Let $\widetilde{n}(P, A)$ denote the random number of sellers sampled by the buyer until a trade takes place, and let $n_w(P, A) = E(\widetilde{n}(P, A) \mid w)$ be the expected number of sellers sampled by buyer type w until a trade takes place.

Let $\beta_I(x, P, A)$ denote the interim belief that $w = H$ held by a seller with signal x who is sampled by the buyer (but has not yet received a price offer), given that the buyer employs strategy P and all sellers employ strategy A. Here and later on the subscript "I" indicates the interim stage (after the signal x was realized but before the price was offered). In the appendix we derive the following expression for $x \in (\underline{x}, \overline{x})$

$$\beta_I\left(x, P, A\right) = \frac{g_H f_H\left(x\right) n_H(P, A)}{g_H f_H\left(x\right) n_H(P, A) + g_L f_L\left(x\right) n_L(P, A)} = \frac{1}{1 + \frac{g_L}{g_H} \frac{f_L(x)}{f_H(x)} \frac{n_L(P, A)}{n_H(P, A)}}. \quad (1)$$

and we set $\beta_I(\underline{x}, P, A) = \lim_{x \to \underline{x}^+} \beta_I(x, P, A)$ and $\beta_I(\bar{x}, P, A) = \lim_{x \to \bar{x}^-} \beta_I(x, P, A)$.

To understand this expression intuitively, suppose momentarily that there is a finite number N of sellers and that the strategies and uniform sampling are as above. Let ρ_w be the probability that an encounter between buyer type w and a seller ends with disagreement, i.e., $\rho_w = \Pr\{x : A(P_w(x), x) = 0\}$. Observe that, when sampling without replacement, $\Pr\{\text{seller } i \text{ is sampled} \mid w\}$ is equal to

[3] Since the buyer does not learn anything from the history and sellers do not observe it, the restriction to Markovian does not seem to exclude anything of interest. The restriction to pure strategies does not matter for the qualitative insights of the paper and it simplifies the refinement arguments we present later. We will return to discuss it in more detail in Section 7 below.

[4] Note that the strategy does not include a "quit" option. Nothing would change if we include such an option and associate the payoff $(-ns)$ with quitting after sampling n sellers. Since $u > c_H + s$ quitting would never occur in equilibrium anyway.

$$\frac{1}{N}+(1-\frac{1}{N})\rho_w\frac{1}{N-1}+(1-\frac{1}{N})(1-\frac{1}{N-1})\rho_w^2\frac{1}{N-2}+...+(1-\frac{1}{N})(1-\frac{1}{N-1})...(1-\frac{1}{2})\rho_w^{N-1}$$
$$= \frac{1}{N}(1+\rho_w+\rho_w^2+...+\rho_w^{N-1})=\frac{1}{N}\frac{1-\rho_w^N}{1-\rho_w}.$$

Therefore, $\Pr\{H \mid \text{seller } i \text{ is sampled}, x\}$ is equal to

$$\frac{g_H f_H(x)\frac{1-\rho_H^N}{1-\rho_H}}{g_H f_H(x)\frac{1-\rho_H^N}{1-\rho_H}+g_L f_L(x)\frac{1-\rho_L^N}{1-\rho_L}} \underset{N\to\infty}{\longrightarrow} \frac{g_H f_H(x)\, n_H(P,A)}{g_H f_H(x)\, n_H(P,A)+g_L f_L(x)\, n_L(P,A)},$$

which coincides with RHS(1). In our model the set of sellers is not finite, so this calculation does not apply directly. The appendix presents the appropriate derivation for the infinite case. It involves a small subtlety, since owing to the continuum of sellers, it relies on probabilities that are conditional on the zero probability event of a particular seller being sampled.

Notice that β_I depends on (P,A) only through the ratio $\frac{n_L(P,A)}{n_H(P,A)}$. This ratio will play a central role in the analysis and the accompanying intuition. It captures the effect of the differential sampling behavior of the two buyer's types on the sellers' beliefs.

An **equilibrium** is a Markovian offer strategy $P=(P_L,P_H)$, an acceptance strategy $A(p,x)$ and beliefs $\beta(p,x)$ s.t.

(i) $P_w(x)\in Arg\max_{p'}\{A(p',x)(u-p')+(1-A(p',x))V_w(P,A)\}$.

(ii) $p \gtrless \beta(p,x)\, c_H+(1-\beta(p,x))\, c_L$ implies $A(p,x)=1$ and 0 respectively.

(iii) $\beta(P_w(x),x)$ is Bayesian updating of $\beta_I(x,P,A)$.

Obviously, the equilibrium is not unique: the freedom in choosing beliefs off the path allows to sustain a large set of equilibria. We will focus on a specific type of equilibrium which we call "undefeated" equilibrium.

2.1 "Undefeated" Equilibria

Given the values $V_w(P,A)$, the interaction between the buyer and any seller after signal x is a simple take-it-or-leave-it bargaining game with asymmetric information. An overall equilibrium induces a sequential equilibrium in the individual bargaining component after any x. There are two kinds of pure sequential equilibrium outcomes: pooling and separating. In a pooling equilibrium, both types trade at a common price. In the separating equilibrium, L offers c_L and H offers c_H, and acceptance may be probabilistic.[5]

[5] The fact that the equilibrium separating prices coincide with c_L and c_H will be proven below.

We would like to confine attention to equilibria of this bargaining component that do not rely on "non-credible" beliefs. We do so by confining attention to a set of equilibria in which, after any x, type L's payoff is at least as high as the payoff in the pooling outcome. The intuitive idea is that it is L who is generally interested in revealing itself, while it is H who wishes to masquerade as L. So, when L prefers to pool, separation could be forced only by an "unnatural" belief that we would like to avoid. In Section 7.1 we present two alternative formal arguments that support this selection: (i) it coincides with the known Sequential Equilibrium refinement of Undefeated Equilibrium (Mailath, Okuno, Postlewaite (1993)); (ii) it contains the set of equilibria that will arise if we let the uniformed sellers make the offers (which will completely bypass the belief formation issue but will require that two or more sellers offer simultaneously to avoid the "Diamond Paradox"). We also discuss in that section other known equilibrium refinements.

We name the selected equilibria "undefeated" where the quotation marks are a reminder that our equilibrium concept is defined for the whole process rather than just for the bargaining component for which the original undefeated refinement is defined.

To avoid a detour, the "undefeated"equilibrium is defined here directly in terms of a simple condition stated below and, as promised, we will return later to present the arguments that justify this selection. Formally, let $E_I[c|x, P, A]$ denote the (interim) expected cost of a seller

$$E_I[c|x, P, A] = \beta_I(x, P, A)\, c_H + (1 - \beta_I(x, P, A))\, c_L\,. \tag{2}$$

Since β_I depends on (P, A) only through the ratio $n_L(P, A)/n_H(P, A)$, so does $E_I[c|x, P, A]$. The buyer's pooling payoff is $u - E_I[c|x, P, A]$.

Definition: An "undefeated" equilibrium (P, A) is such that, for any x,

$$A(P_L(x), x)(u - P_L(x)) + [1 - A(P_L(x), x)]V_L(P, A) \geq u - E_I[c|x, P, A]\,. \tag{3}$$

3 "Undefeated" Equilibria: Characterization and Existence

To streamline the notation we will write n_w, V_w, $E_I[c|x]$ and $\beta_I(x)$, omitting the arguments (P, A) though these magnitudes depend of course on (P, A). But our arguments never go across different equilibria so this simplification should create no confusion.

Let $\bar{a} = \bar{a}(P, A)$ be the maximal probability $a \in [0, 1]$ satisfying

$$a(u - c_L) + (1 - a)V_H \leq \max\{u - c_H, V_H\}\,. \tag{4}$$

This is the maximal probability with which the price offer c_L can be accepted in any separating equilibrium of the bargaining component, since any higher probability would induce H to also offer c_L.

Define the cutoff point $x^* = x^*(P, A)$ as the solution to the equation

$$\bar{a}(u - c_L) + (1 - \bar{a}) V_L = u - E_I[c|x^*], \tag{5}$$

if it exists. Otherwise, let $x^* = \underline{x}$ if the LHS is always larger and let $x^* = \overline{x}$ if the LHS is always smaller. That is, at x^* type L is indifferent between trading at the expected interim cost, $E_I[c|x^*]$, and taking the lottery that with probability $\bar{a}$ results in trade at price c_L and with $(1 - \bar{a})$ results in continued search.

Define the cutoff $x^{**} = x^{**}(P, A)$ by the solution to

$$V_L = u - E_I[c|x^{**}], \tag{6}$$

if such cutoff exists. Otherwise, let $x^{**} = \underline{x}$ if the LHS is always larger and let $x^{**} = \overline{x}$ if the LHS is always smaller. That is, at x^{**} type L is indifferent between trading at the expected interim cost, $E_I[c|x^{**}]$, and just continuing the search.

Since obviously $V_L < u - c_L$ in equilibrium and $E_I[c|x]$ is monotonically increasing in x, it follows that $x^* \leq x^{**}$. Also, if $\bar{a} = 0$, then $x^* = x^{**}$.

Again, to keep the notation simple, we follow the convention we have adopted for other equilibrium magnitudes and write $\bar{a}$, x^* and x^{**} omitting the arguments (P, A) though of course these magnitudes are dependent on the strategies being played.

Claim 1 *: If (P, A) is an "undefeated" equilibrium, then $V_L > V_H$ and (up to irrelevant differences*[6]*) its outcome satisfies the following.*

- *After $x \leq x^*$, both types of the buyer pool on price $= E_I[c|x, P, A]$ and the price is accepted,*

$$P_L(x) = P_H(x) = E_I[c|x] \text{ and } A(E_I[c|x]) = 1.$$

- *After $x > x^{**}$, the buyer types separate*

$$P_L(x) = c_L, \quad P_H(x) = c_H,$$

[6] These "irrelevant differences" concern zero probability events and the description of situations in which both buyer types disagree with the seller. We describe those situations as separating with zero acceptance probabilities, but they can also be described in equivalent ways, e.g., as pooling on a price below cost.

and the acceptance probabilities satisfy

$$\frac{(u - E_I[c|x]) - V_L}{(u - c_L) - V_L} \leq A(c_L, x) \leq \max\{0, \frac{(u - c_H) - V_H}{(u - c_L) - V_H}\} \tag{7}$$

$$A(c_H, x) = \begin{cases} 0 & \text{if} \quad V_H > u - c_H, \\ 1 & \text{if} \quad V_H < u - c_H. \end{cases} \tag{8}$$

- *After $x \in (x^*, x^{**})$, the two types either pool or separate as above.*

Conversely, if an equilibrium (P, A) satisfies the above for every x, it is "undefeated".

The proof is relegated to the appendix. But let us comment briefly on some of the main steps. The observations on $A(c_w, x)$ summarized by (7) and (8) are derived from

$$\begin{aligned} A(c_L, x)(u - c_L) + (1 - A(c_L, x)) V_L &\geq u - E_I[c|x] \quad \text{(Undomination by pooling)} \\ A(c_L, x)(u - c_L) + (1 - A(c_L, x)) V_H &\leq \max\{u - c_H, V_H\} \quad (H\text{'s IC}) \\ A(c_H, x)(u - c_H) + (1 - A(c_H, x)) V_H &\geq \max\{u - c_H, V_H\} \quad (H\text{'s IR}). \end{aligned}$$

Notice also that (7) and (8) imply $A(c_L, x) \leq A(c_H, x)$.

After $x < x^*$, type L prefers pooling to separating (by the definition of x^*), hence the equilibrium must involve pooling. After $x > x^*$, type L prefers separating to pooling (as implied by the LHS of (7), but nevertheless after $x \in (x^*, x^{**})$ the equilibrium may be pooling. Only after $x > x^{**}$, the equilibrium is necessarily separating (since by the definition of x^{**} type L prefers even continued search to pooling).

Claim 1 provides the complete characterization of the "undefeated" equilibria. For example, if there is an "undefeated" equilibrium with $V_H \geq u - c_H$ (which will turn out to always be the case when s is small enough), it follows from the claim and from (5) that L will search till it generates a signal below the threshold $x^* > \underline{x}$ and that H will either do the same (if $V_H > u - c_H$) or will just be indifferent (if $V_H = u - c_H$) between doing the same or settling immediately for the price c_H. Therefore, the equilibrium payoffs are

$$\begin{aligned} V_L &= u - \int_{\underline{x}}^{x^*} E_I[c|x] \frac{dF_L(x)}{F_L(x^*)} - \frac{s}{F_L(x^*)}, \quad &(9) \\ V_H &= u - \int_{\underline{x}}^{x^*} E_I[c|x] \frac{dF_H(x)}{F_H(x^*)} - \frac{s}{F_H(x^*)}. \quad &(10) \end{aligned}$$

The expected search durations are $n_L = \frac{1}{F_L(x^*)}$ and $n_H \in [1, \frac{1}{F_H(x^*)}]$, where $n_H = \frac{1}{F_H(x^*)}$ if $V_H > u - c_H$, while n_H may assume any value in the interval if $V_H = u - c_H$. Thus,

$$\frac{F_H(x^*)}{F_L(x^*)} \leq \frac{n_L}{n_H} \leq \frac{1}{F_L(x^*)}; \tag{11}$$

These outcomes can be sustained by beliefs off the equilibrium path that place all the weight on type H, i.e., $\beta(p, x) = 1$ for $p \neq P_w(x)$, so only $p \geq c_H$ are accepted off path.

Finally, for completeness observe that the "undefeated" equilibrium exists.

Claim 2 *: An "undefeated" equilibrium exists.*

The (routine) proof is outlined in the appendix. Since from now on attention will be confined almost exclusively to "undefeated" equilibria (except for parts of the discussion section that explicitly discuss other equilibria), we will omit most of the time the word "undefeated" and refer to it simply as an equilibrium.

4 Information Aggregation

The question is to what extent information is aggregated into the equilibrium prices when s is small. Aggregation is maximal if the prices that types L and H pay are close to c_L and c_H, respectively. Aggregation is minimal when the two buyer types pay the same price(s).

Recall that the literature on auctions considered a related question. It inquired to what extent the equilibrium price in a common value auction reflects the correct information when the number of bidders is made arbitrarily large (Wilson (1977) and Milgrom (1979)). Milgrom's result translated to an auction version of our model is that the price approaches the true value iff $\lim_{x \to \underline{x}} \frac{f_L(x)}{f_H(x)} = \infty$. That is, when there are signals that are exceedingly more likely when the true state is L than when it is H. In our model the number of "bidders" is endogenously determined through the sampling, and the counterpart of increasing the number of bidders is reduction of the sampling cost s.

Formally, consider a sequence $s_k \to 0$ and a sequence of equilibria (P_k, A_k) associated with it. Let x_k^*, x_k^{**}, $\bar{a}_k$, V_{wk}, etc. denote magnitudes associated with the equilibrium (P_k, A_k). Let p_{wk} denote the expected price paid by type w in equilibrium (P_k, A_k) and let S_{wk} denote the expected search costs incurred by type w in equilibrium (P_k, A_k). For example, if in the equilibrium $x_k^{**} = x_k^*$ and $\bar{a}_k = 0$, then $p_{wk} = E_I[c|x \leq x_k^*, w]$ and $S_{wk} = s_k/F_w(x_k^*)$. Let

$$\overline{p}_w = \lim_{k \to \infty} p_{wk} \text{ and } \overline{S}_w = \lim_{k \to \infty} S_{wk},$$

if these limits exist. The following analysis will investigate $\overline{p}_w$ and $\overline{S}_w$. In particular, it will inquire about the extent to which $\overline{p}_w$ aggregates the information and total welfare.

4.1 Preliminaries and the Main Result

The following claim collects some observations about limits that will be used repeatedly in the subsequent analysis.

Claim 3 *: (i) $x_k^* \to \underline{x}$; (ii) $\lim \bar{a}_k = 0$ and $\lim x_k^{**} = \underline{x}$; (iii) $x_k^* > \underline{x}$ for all k; (iv) $\lim s_k / F_L(x_k^*) = 0$ and $\lim \bar{a}_k / F_L(x_k^*) = 0$; (v) $\overline{S}_L = 0$.*

Besides its usefulness for the coming analysis, this claim also exposes some central features of the "undefeated" equilibria. The range of pooling signals shrinks to the bottom of the support. Type L bears negligible search costs while essentially searching till it gets a signal below x_k^*.

Consider the expression

$$\int_{\underline{x}}^{x^*} \left(\frac{f_L(x)}{f_H(x)} - \frac{f_L(x^*)}{f_H(x^*)} \right) \frac{f_L(x)}{F_L(x^*)} dx, \tag{12}$$

which measures by how much signals $x < x^*$ are more informative (on average) than x^*. In a sense, this is the rate at which the informativeness of signals improves as they decrease. Obviously, it is related to $\frac{d}{dx}\left(\frac{f_L(x)}{f_H(x)}\right)$ – other things equal, the steeper is $\left(\frac{f_L(x)}{f_H(x)}\right)$ over $(x^*, \underline{x})$, the larger (12) will be. Assume that $\lim_{x^* \to \underline{x}}(12)$ exists and let

$$\lambda \triangleq \lim_{x^* \to \underline{x}} \int_{\underline{x}}^{x^*} \left(\frac{f_L(x)}{f_H(x)} - \frac{f_L(x^*)}{f_H(x^*)} \right) \frac{f_L(x)}{F_L(x^*)} dx \tag{13}$$

which can take the value ∞ as well (the case in which this limit does not exist is taken up later). A large λ means that the informativeness of lower signals increases sharply as x approaches $\underline{x}$. When the informativeness of the signals is bounded, i.e., $\lim_{x \to \underline{x}} \frac{f_L(x)}{f_H(x)} < \infty$, then $\lambda = 0$. The parameter λ will have a prominent role in the characterization results that follow.

Proposition 1 *: Suppose that the limit λ exists and consider a sequence $s_k \to 0$ and a sequence (P_k, A_k) of corresponding equilibria. Then the limit prices exist and are*

$$\bar{p}_L = \begin{cases} (1 - \frac{1}{\lambda})c_L + \frac{1}{\lambda}c_H & \text{if} \quad \lambda \in [\frac{1}{g_H}, \infty], \\ g_L c_L + g_H c_H & \text{if} \quad \lambda \leq \frac{1}{g_H}, \end{cases}$$

$$\bar{p}_H = \begin{cases} \frac{1}{\lambda}\frac{g_L}{g_H}c_L + (1 - \frac{1}{\lambda}\frac{g_L}{g_H})c_H & \text{if} \quad \lambda \in [\frac{1}{g_H}, \infty], \\ g_L c_L + g_H c_H & \text{if} \quad \lambda \leq \frac{1}{g_H}. \end{cases}$$

The proof is relegated to the appendix, but a shorter outline of it is provided in subsection 4.3 below. If λ is small enough, there is complete pooling in the unique limit outcome: both types end up paying the same expected price $g_L c_L + g_H c_H$ which is the ex-ante expected cost. This includes of course the case in which the informativeness of the signals is bounded, i.e., $\lim_{x \to \underline{x}} \frac{f_L(x)}{f_H(x)} < \infty$, since then $\lambda = 0$. For larger values of λ the pooling is partial:

$$\bar{p}_L = (1 - \frac{1}{\lambda})c_L + \frac{1}{\lambda}c_H < g_L c_L + g_H c_H < \frac{1}{\lambda}\frac{g_L}{g_H}c_L + (1 - \frac{1}{\lambda}\frac{g_L}{g_H})c_H = \bar{p}_H.$$

The two types end up paying different expected prices but these prices are away from the respective true costs. In the extreme, when λ is very large, these prices are close to the respective costs and hence nearly aggregate all the information.

It is useful to understand the form of the equilibria that are associated with different values of λ. If $\lambda < \frac{1}{g_H}$, then far enough in the sequence the equilibria are such that both types search till they generate a signal below x_k^* and trade at the corresponding interim expected cost. To make it worthwhile for type H this requires that the cutoff x_k^* does not converge to $\underline{x}$ too quickly. What keeps x_k^* from converging too quickly to $\underline{x}$ is the credibility requirement of the "undefeated" equilibrium which prevents sellers from rejecting pooling offers that are profitable for both L and H and the relatively poor informativeness of the signals (captured by a small λ). Since both L and H end up transacting, the overall expected cost must be the ex-ante expected cost $g_H c_H + g_L c_L$. Therefore, since in the limit both types end up transacting at the same price, and since the "undefeated" equilibrium leaves the sellers with zero profit, the common expected price must be equal to that expected cost.

When $\lambda \in (1/g_H, \infty)$, type L is still "essentially" searching for a signal below x_k^* as above to trade at the interim expected cost, which in this case is not converging to $g_L c_L + g_H c_H$. Type H is just indifferent between mimicking L's behavior or settling immediately at c_H. Type H "mixes": with probability $\frac{1}{(\lambda-1)}\frac{g_L}{g_H}$ it ends up mimicking L and paying $\bar{p}_L$; with the complementary probability it ends up settling on c_H. This "mixing" exactly brings $\bar{p}_L$ to the level that maintains H's indifference. We say "mixing" since H is using a pure strategy, so the "mixing" is purified by having a set of signal realizations after which H offers c_H, which is accepted, and another set after which H offers a price just below (or equal to) c_H which is then rejected. In the extreme, when λ is close to ∞, the probability with which H mimics L is near 0 and the equilibrium outcome is nearly separating.

4.2 More about λ

The central role of λ calls for better understanding of its meaning. Recall that λ measures the rate at which informativeness of the signals improves as x approaches $\underline{x}$. In fact, this statement can be formalized as follows.

Claim 4 *: If*

$$\lim_{x \to \underline{x}^+} \frac{-\frac{d}{dx}\left(\frac{f_L(x)}{f_H(x)}\right)}{\frac{f_L(x)}{F_L(x)}} \tag{14}$$

exists, then it is equal to λ as defined by (13).

The proof of this claim is relegated to the appendix. Thus, λ is the rate at which $\frac{f_L(x)}{f_H(x)}$ increases as x approaches $\underline{x}$, corrected by $\frac{f_L(x)}{F_L(x)}$. This correction captures the fact that the smaller is $\frac{f_L(x)}{F_L(x)}$ the larger is the relative weight on smaller values of x and hence the more significant is the effect of the increase in $\frac{f_L(x)}{f_H(x)}$. This perhaps further clarifies the sense in which a larger λ corresponds to a more informative signal structure. Viewing λ as a measure of informativeness, the proposition then establishes that the extent of information revealed by the equilibrium prices increases in the informativeness of the signal in this sense. Observe that, if the signals are boundedly informative, i.e.,

$$\lim_{x \to \underline{x}} \frac{f_L(x)}{f_H(x)} < \infty,$$

then $\lambda = 0$ and the unique limit outcome is complete pooling. If $\lambda = \infty$, the limit prices aggregate the information perfectly. This requires $\lim_{x \to \underline{x}} \frac{f_L(x)}{f_H(x)} = \infty$ but the unbounded likelihood ratio is not sufficient for full revelation. All values of λ in $(0, \infty)$ are associated with $\lim_{x \to \underline{x}} \frac{f_L(x)}{f_H(x)} = \infty$, though they give rise to full pooling or partial revelation depending on the magnitude of λ.

In subsection 10.3 of the appendix we will present yet another way to think about λ by relating it to the properties of the distribution of the likelihood ratios.

Finally, let us turn to the case in which the limit denoted by λ does not exist. To extend the results of Proposition 1 to this case, let

$$\overline{\lambda} = \lim_{x^* \to \underline{x}} \sup \int_{\underline{x}}^{x^*} \left(\frac{f_L(x)}{f_H(x)} - \frac{f_L(x^*)}{f_H(x^*)}\right) \frac{f_L(x)}{F_L(x^*)} dx,$$

and let $\underline{\lambda}$ denote the $\liminf$.

Proposition 2 *:(i) Consider a sequence $s_k \to 0$ and a sequence (P_k, A_k) of corresponding equilibria such that $\bar{p}_L = \lim p_{kL}$ exists. Then $\bar{p}_H = \lim p_{kH}$ exists and there exists a $\lambda \in [\underline{\lambda}, \overline{\lambda}]$ such that*

$$\bar{p}_L = \begin{cases} (1-\frac{1}{\lambda})c_L + \frac{1}{\lambda}c_H & \text{if } \lambda \in (\frac{1}{g_H}, \infty], \\ g_Lc_L + g_Hc_H & \text{if } \lambda \leq \frac{1}{g_H}, \end{cases}$$
$$\bar{p}_H = \begin{cases} \frac{1}{\lambda}\frac{g_L}{g_H}c_L + (1-\frac{1}{\lambda}\frac{g_L}{g_H})c_H & \text{if } \lambda \in (\frac{1}{g_H}, \infty], \\ g_Lc_L + g_Hc_H & \text{if } \lambda \leq \frac{1}{g_H}. \end{cases}$$

(ii) For any $\lambda \in [\underline{\lambda}, \overline{\lambda}]$, there exists a sequence $s_k \to 0$ and a sequence (P_k, A_k) of corresponding equilibria such that $\bar{p}_L = \lim p_{kL}$ and $\bar{p}_H = \lim p_{kH}$ exist and are of the above form.

Thus, if $\underline{\lambda} \neq \overline{\lambda}$, there are multiple equilibrium limit points. This does NOT imply that there are multiple equilibria for the same s_k, though it might be the case. The proof is in the appendix.

We provide two parametrized examples to further illustrate λ in Section 4.4. As explained before, every signal distribution for which the likelihood ratios have bounded support has $\lambda = 0$. The examples consider families of distributions for which the likelihood ratios are unbounded.

4.3 Outline of the Proof of Proposition 1

The main steps of the proof of Proposition 1 for the case of $\lambda < \infty$ are as follows.

First, when s_k is sufficiently small, L transacts at equilibrium only after generating signal $x \leq x_k^*$. So the equilibrium is described by (9)-(11). Second, the "undefeatedness" refinement prevents rejection of pooling offers profitable for both types. Therefore, x_k^* is determined by L's indifference between trading at x_k^* and continuing to search

$$\frac{s_k}{F_L(x_k^*)} = (E_I[c|x_k^*] - E_I[c|x \leq x_k^*, L]).$$

H's cost of mimicking L's behavior, $\frac{s_k}{F_H(x_k^*)}$, is obtained by multiplying both sides by $\frac{F_L(x_k^*)}{F_H(x_k^*)}$

$$\frac{s_k}{F_H(x_k^*)} = \frac{F_L(x_k^*)}{F_H(x_k^*)}(E_I[c|x_k^*] - E_I[c|x \leq x_k^*, L]). \tag{15}$$

Let

$$\eta_k(x) \triangleq \frac{g_L}{g_H}\frac{f_L(x)}{f_H(x)}\frac{n_{Lk}}{n_{Hk}}$$

–the **effective likelihood ratio**[7] for a seller who observes signal x, in the equilibrium $(P_k,\ A_k)$–and

[7] It augments the likelihood ratio based on the signal distributions alone, $\frac{f_L(x)}{f_H(x)}$, with the likelihood of being sampled by the different types.

use it together with (2) and (1) to write the interim expected cost $E_I\left[c|x\right]$

$$E_I\left[c|x\right] = \frac{c_H + \frac{g_L}{g_H}\frac{f_L(x)}{f_H(x)}\frac{n_{Lk}}{n_{Hk}}c_L}{1 + \frac{g_L}{g_H}\frac{f_L(x)}{f_H(x)}\frac{n_{Lk}}{n_{Hk}}} \equiv \frac{c_H + \eta_k(x)c_L}{1+\eta_k(x)}. \tag{16}$$

(15) can be written as

$$\frac{s_k}{F_H\left(x_k^*\right)} = (c_H - c_L)\frac{f_H\left(x_k^*\right)}{f_L\left(x_k^*\right)}\frac{F_L\left(x_k^*\right)}{F_H\left(x_k^*\right)}\eta_k(x_k^*)\int_{\underline{x}}^{x_k^*}\frac{\frac{f_L(x)}{f_H(x)} - \frac{f_L\left(x_k^*\right)}{f_H\left(x_k^*\right)}}{\left(1+\eta_k(x_k^*)\right)\left(1+\eta_k(x)\right)}\frac{f_L\left(x\right)}{F_L\left(x_k^*\right)}dx.$$

To obtain the limit form of this equation it is shown in the proof that $\lim\frac{f_H\left(x_k^*\right)}{f_L\left(x_k^*\right)}\frac{F_L\left(x_k^*\right)}{F_H\left(x_k^*\right)} = 1$ which together with the definition of λ yields

$$\lim\frac{s_k}{F_H\left(x_k^*\right)} = (c_H - c_L)\frac{\lim\eta(x_k^*)}{[1+\lim\eta(x_k^*)]^2}\lambda. \tag{17}$$

Ignoring costs, H's benefit from searching to $x \leq x_k^*$ (rather than settle on c_H) is

$$c_H - E_I\left[c|x \leq x_k^*, H\right].$$

Hence, H decides whether to search or settle immediately according to whether

$$\frac{s_k}{F_H\left(x_k^*\right)} \lesseqgtr c_H - E_I\left[c|x \leq x_k^*, H\right]. \tag{18}$$

With $\lambda < \infty$ it may not be that at (or near) the limit H prefers to settle immediately. If H were to settle immediately, then $n_H = 1$. This would imply that $\lim\eta(x_k^*) = \infty$, since $\eta(x_k^*) \equiv \frac{g_L}{g_H}\frac{f_L\left(x_k^*\right)}{f_H\left(x_k^*\right)}\frac{n_{Lk}}{n_{Hk}}$, $\frac{f_L\left(x_k^*\right)}{f_H\left(x_k^*\right)} \to \infty$, and $n_{Lk} \to \infty$. This would further imply via (17) that $\lim\frac{s_k}{F_H\left(x_k^*\right)} = 0$, and via (18) that H would actually prefer searching rather than settling, contradicting the hypothesis that H prefers to settle. Therefore, at (or near) the limit, equation (18) holds with "$\leq$".

Now, if λ is sufficiently small so that

$$\lim\frac{s_k}{F_H\left(x_k^*\right)} < c_H - (g_H c_H + g_L c_L),$$

then at (or near) the limit H mimics L, transactions take place only after $x \leq x_k^*$ at a price near $g_H c_H + g_L c_L$ which as evident from the above inequality compensates H for the search cost.

If λ is larger so that

$$\lim\frac{s_k}{F_H\left(x_k^*\right)} > c_H - (g_H c_H + g_L c_L),$$

then near the limit we may not have complete pooling since then H would prefer to settle immediately for c_H rather than search for the complete pooling price $g_H c_H + g_L c_L$. In this case the equilibrium involves partial pooling. H settles after some signal realizations above x_k^* and continues searching after other. $E_I\left[c|x \leq x_k^*, H\right]$ adjusts to achieve equality in (18) which keeps H indifferent between these two options. Since $E_I\left[c|x \leq x_k^*, H\right]$ is increasing in the probability with which H searches in equilibrium and since $\lim \frac{s_k}{F_H(x_k^*)}$ is increasing in λ, it follows that smaller values of λ in the partial pooling range are associated with more search by H.

4.4 Examples of Families of Signal Distributions

This subsection presents two specific families of signal distribution that for different parameter values exhibit different cases with respect to information aggregation. The purpose is just to illustrate how the extent of information aggregation is related to the parameters of concrete distributions. The reader may skip this part as it is unimportant for the subsequent analysis and discussion.

Example I. *The parameters of the following family of signal distributions are $\alpha > 0$ and $\mu > 0$,* such that $(\alpha, \mu) \in [1, \infty) \times (0, \infty) \cup (0, 1) \times [1, \infty)$. *Given α and μ, let $r \geq 0$ be the solution to $\mu \int_{-\infty}^{-r} (-t)^{-\alpha} e^{t+r} dt = 1$ and define*

$$
\begin{aligned}
F_L(x) &= \begin{cases} e^{x+r} & \text{if} \quad x \leq -r, \\ 1 & \text{if} \quad x > -r, \end{cases} \\
F_H(x) &= \begin{cases} \mu \int_{-\infty}^{-x} (-t)^{-\alpha} e^{t+r} dt & \text{if} \quad x \leq -r, \\ 1 & \text{if} \quad x > -r. \end{cases}
\end{aligned}
$$

The restriction on μ and the definition of r ensure that F_L and F_H are proper distribution functions[8].

The likelihood ratio is given by $\frac{f_L(x)}{f_H(x)} = \frac{1}{\mu}(-x)^{\alpha}$. Using Claim (4) the parameter λ defined by (13) is

$$
\lambda = \begin{cases} \infty & \text{if} \quad \alpha > 1, \\ \frac{1}{\mu} & if \quad \alpha = 1, \\ 0 & \text{if} \quad \alpha < 1. \end{cases}
$$

Thus, Proposition 1 implies that prices perfectly aggregate the information if $\alpha > 1$, there will be perfect pooling if $\alpha < 1$ and there will be partial aggregation if $\alpha = 1$. In contrast, since $\lim_{x \to \underline{x}} \frac{f_L(x)}{f_H(x)} = \infty$, *for all $\alpha > 0$, the limit outcome of the corresponding auction setting always exhibits complete information aggregation.*

[8] *An alternative to the restriction on μ when $\alpha < 1$ would be to allow for an atom of F_H at $x = -r$.*

Example II: *Consider the family of distributions*

$$F_L(x) = \begin{cases} 1 & \text{for} \quad x \geq r(\lambda), \\ e^{-\frac{1}{\lambda}\frac{g_H}{g_L}\left(\frac{1}{x}-\frac{1}{r}\right)} & \text{for} \quad 0 \leq x \leq r(\lambda), \\ 0 & \text{for} \quad x = 0, \end{cases} \tag{19}$$

and

$$F_H(x) = \begin{cases} 1 & \text{for} \quad x \geq r(\lambda), \\ \frac{g_L}{g_H}\int_0^x \frac{\tau}{1-\tau} f_L(\tau)d\tau & \text{for} \quad 0 \leq x \leq r(\lambda), \\ 0 & \text{for} \quad x = 0, \end{cases}$$

which is parameterized by $\lambda \in (0, \infty)$. The number $r(\lambda) \in (g_H, 1)$ is the r solution to $\frac{g_L}{g_H}\int_0^r \frac{x}{1-x} f_L(x)dx = 1$. The choice of $r(\lambda)$ guarantees that F_H is a proper distribution function on the same support. Notice that this example is such that the value of the signal coincides with the posterior probability conditional on that value,

$$x = \frac{f_H(x)\, g_H}{f_H(x)\, g_H + f_L(x)\, g_L}. \tag{20}$$

It is fairly straightforward to verify that the λ parameter of these distributions is the λ defined by (13). So depending on the choice of the parameter λ, this family exhibits limit outcomes that range from complete pooling to arbitrarily close to perfect aggregation. Since for this family $\frac{f_L(x)}{f_H(x)} = \frac{g_H}{g_L}\frac{1-x}{x}$ (see (20) above) and $\underline{x} = 0$, $\lim_{x\to\underline{x}} \frac{f_L(x)}{f_H(x)} = \infty$ and hence the limit outcome of the corresponding auction setting always exhibits complete information aggregation.

5 Welfare

Since trade is always beneficial and always takes place in this model, the expected surplus is fully determined by the expected search cost incurred by the buyer. The following proposition characterizes the expected search costs $\overline{S}_w$, $w = L, H$, arising in the equilibria in the limit as s becomes negligible.

Proposition 3 *: (i) $\overline{S}_L = 0$ in all cases; (ii) If $\lim_{x\to\underline{x}} \frac{f_L(x)}{f_H(x)} < \infty$, then $\overline{S}_H = 0$; (iii) If $\lim_{x\to\underline{x}} \frac{f_L(x)}{f_H(x)} = \infty$, then*

$$\overline{S}_H = \begin{cases} (c_H - c_L)\frac{1}{\lambda}\frac{g_L}{g_H} & \textit{if} \quad \lambda \in [\frac{1}{g_H}, \infty], \\ (c_H - c_L) g_H g_L \lambda & \textit{if} \quad \lambda < \frac{1}{g_H}. \end{cases} \tag{21}$$

Proof: (i) Proved in Claim 3-(V). (ii)+(iii) By definition, $V_{Hk} = u - p_{Hk} - S_{Hk}$ and hence $\overline{S}_H = u - \bar{p}_H - \lim V_{Hk}$. From the proof of Proposition 1,

$$\lim V_{Hk} = \begin{cases} u - \bar{p}_H - \lim \frac{s_k}{F_H(x_k^*)} & \text{iff} \quad \lambda < \frac{1}{g_H}, \\ u - c_H & \text{iff} \quad \lambda \in [\frac{1}{g_H} \infty]. \end{cases} \tag{22}$$

Hence, using the characterization of $\bar{p}_H$ and (35),

$$\overline{S}_H = u - \bar{p}_H - \lim V_{Hk} = \begin{cases} (c_H - c_L)\, g_L g_H \lambda & \text{if} \quad \lambda < \frac{1}{g_H}, \\ \frac{1}{\lambda}\frac{g_L}{g_H}(c_H - c_L) & \text{if} \quad \lambda \geq \frac{1}{g_H}. \end{cases}$$

■

The limit of a sequence (P_k, A_k) of equilibria (corresponding to a sequence $s_k \to 0$) is **efficient** if $\overline{S}_w = 0$, for $w = L, H$. The limit is **ε-efficient** if $\overline{S}_w < \varepsilon$.

Thus, in the limit as $s_k \longrightarrow 0$, the equilibrium is efficient when $\lambda = 0$, e.g., when $\lim_{x \to \underline{x}} \frac{f_L(x)}{f_H(x)} < \infty$. When $\lim_{x \to \underline{x}} \frac{f_L(x)}{f_H(x)} = \infty$ it is nearly efficient if λ is sufficiently small or sufficiently large (i.e., for any $\varepsilon > 0$, the limit is ε-efficient for sufficiently small and sufficiently large λ's).

As we have already pointed out, the parameter λ is a measure of the informativeness of the signal technology, with lower values corresponding to lower informativeness. Proposition 3 shows that welfare in not monotone in the informativeness of the signal technology and welfare. If, for example $\lambda \leq 1/g_H$, a more informative signal technology decreases welfare.

Proposition 3 can be used to construct examples of signal distributions for which a lower search cost results in lower welfare.

Example: *Let* $[\underline{x}, \overline{x}] = [-\infty, -\hat{r}]$ *where* $\hat{r} \geq 0$ *satisfies* $\int_{-\infty}^{-\hat{r}} (-x)\, e^{x+\hat{r}} dx < 1$ *and consider the distribution functions*

$$\begin{aligned} F_L(x) &= e^{x+\hat{r}}, \\ F_H(x) &= \begin{cases} \int_{-\infty}^{-x} \frac{1}{-t} e^{t+\hat{r}} dt & \text{if} \quad x < \hat{r}, \\ 1 & \text{if} \quad x = \hat{r}. \end{cases} \end{aligned}$$

Thus, F_H *has an atom at* $-\hat{r}$ *(*$dF_H(-\widehat{r})$*), while* F_L *is atomless. For a given* $s > 0$*, consider the following behavior: after signal* $x < -\widehat{r}$*, both* H *and* L *offer the pooling price* $E_I[c|x] = \frac{c_H + x\frac{g_L}{g_H} c_L}{1 + x\frac{g_L}{g_H}}$*, which sellers accept; after signal* $-\widehat{r}$*, type* H *offers* c_H*, which is accepted, while type* L *offers* c_L*, which is rejected. Thus, both* H *and* L *transact with the first seller they sample (*L *continues the search after* $x = -\hat{r}$ *which occurs with* 0*-probability with* F_L*).*

For this behavior to constitute the path of an equilibrium it has to be that: (i) After $x < -\widehat{r}$, type L prefers transacting at $E_I[c|x]$ to continued search, a sufficient condition for which is

$$\frac{c_H + \widehat{r}\frac{g_L}{g_H}c_L}{1+\widehat{r}\frac{g_L}{g_H}} \leq c_L + s,$$

and (ii) after $x = -\widehat{r}$, H prefers settling for c_H to continued search, a sufficient condition for which is

$$c_H \leq c_L + \frac{s}{1 - dF_H(-\widehat{r})}.$$

Observe that, for any s, there is a sufficiently large $\hat{r}$ such that both conditions hold. Therefore, for such $\hat{r}$ (and the associated F_w's), there is an equilibrium with this path. In this equilibrium the ex ante expected search cost is s. In contrast, since in this case $\lambda(F_L, F_H) = 1$ (see Section 4.4 above), when we look at a sequence $\{s_k\}_{k=1}^{\infty}$ that converges to 0 and any sequence of corresponding equilibria $\{P_k, A_k\}$, the ex-ante expected search cost in the limit is $g_L\overline{S}_L + g_H\overline{S}_H = (c_H - c_L)g_H^2 g_L$ (from (21) and $\overline{S}_L = 0$).Thus, if s is chosen so that $s < (c_H - c_L)g_H^2 g_L = g_L\overline{S}_L + g_H\overline{S}_H$, then welfare is higher with s than with negligible search costs.■

When s is not too low, the signal is sufficiently informative to generate the equilibrium partial pooling outcome that is better for both types than continued search. therefore, no search takes place and welfare is at the efficient level, $u - g_Lc_L - g_Hc_H - s$. But at the same time the signal is not highly informative at very low realizations (as capture by $\lambda = 1$). Therefore, when s is small, the equilibrium involves complete pooling which means wasteful search by H. Owing to the externalities present in a search model, it is not entirely surprising that the extent of wasteful search might increase when the search cost becomes smaller. Still this example is interesting since it is rarely the case in the search literature that the welfare decreases when the absolute level of search costs vanishes altogether (see, for example, Gale (1987) or Lauermann (2011)).

As noted above, the basic assumptions guarantee that trade is always beneficial and always takes place in equilibrium, so welfare is fully determined by the expected search cost. However, it is fairly straightforward to modify the basic model in a way that will introduce efficiency considerations regarding the volume of trade as well. Suppose that the model is as above except that $u \in (g_Lc_L + g_Hc_H, c_H)$ and the buyer has the option to quit searching. Quitting after n samples yields the payoff $-ns$.

The analysis will remain essentially the same: those cases in which in type H settles for c_H with positive probability will be translated in the present version to type H quitting the search with the same probability. Thus, for sufficiently large u, the limiting equilibrium will be completely pooling (with type H searching with certainty) and hence the volume of trade will be inefficient. The nearly separating equilibrium will be nearly efficient since type H would essentially not trade.

Thus, in this version, the near efficiency associated with small λ in our basic model will be replaced with (possibly substantial) inefficiency. The expected search costs incurred by H would still be low, but excessive trading will take place.

6 Intuition: Sampling Curse vs Winner's Curse

This section brings a more substantive discussion of the results. Some modeling issues will be discussed in the following section.

6.1 Intuitive Explanation

The basic consideration behind the aggregation of information is how costly it is for H to mimic L's search for a favorable signal. The case of $\lim_{x\to\underline{x}} \frac{f_L(x)}{f_H(x)} < \infty$ is simple. If the limit were (nearly) separating, type L would trade (nearly) at c_L while type H would trade (nearly) at c_H. Since $\overline{S}_L = 0$ (Claim 3), L would find prices close to c_L at almost no cost. The separation implies that the search cost that H would incur by mimicking L must be large enough to prevent H from strictly preferring this option to trading at c_H. But since L is just at most $\lim_{x\to\underline{x}} \frac{f_L(x)}{f_H(x)} < \infty$ times more likely than H to generate a signal near $\underline{x}$ that would get a price near c_L, the expected length of search by H to secure such price is at most that many times longer than the expected length of the search by L. This translates to a negligible cost for H to mimic L. So any (even partial) separation cannot survive with $\lim_{x\to\underline{x}} \frac{f_L(x)}{f_H(x)} < \infty$.

In the case of $\lim_{x\to\underline{x}} \frac{f_L(x)}{f_H(x)} = \infty$ there is no bound on the informativeness of signals, so the above argument does not apply. Here the nature of information aggregation depends on the speed with which $\frac{f_L(x)}{f_H(x)}$ is increasing (when x approaches $\underline{x}$) which is captured by λ. A relatively smaller λ means that L's incentive to search is relatively weaker. This is because the incentive to search after a given signal realization x is higher the more significant is the decrease in the expected interim costs (and hence prices) associated with still lower signals. The latter depends on the rate at which $\frac{f_L(x)}{f_H(x)}$ is increasing (when x is decreasing), which is captured by λ. Now, a relatively weaker incentive to search translates to a relatively higher cutoff x^* (i.e., x_k^* goes to 0 more slowly when s_k goes to 0) which makes it easier for H to pool. In the extreme, when λ is very small, the convergence of x_k^* to 0 is so slow that the cost for H of pooling is nearly 0. Notice that, since in this case $\frac{f_L(x_k^*)}{f_H(x_k^*)} \to \infty$, type H's expected search duration grows without a bound relative to type L's expected search duration when H tries to mimic L. Yet it still grows slowly relative to the rate at which s_k decreases. Therefore, the total expected search cost that H incurs in mimicking L goes to 0 and precludes separation.

Conversely, a relatively large λ means that L has a stronger incentive to search which translates to a lower x_k^* (or rather faster convergence of x_k^* to 0). This makes it too costly for H to pool when the reward to search is the pooling price. Therefore, in the equilibrium H searches only with a certain probability. As a result the expected price obtained by a searcher is lower than the complete pooling price and is just sufficiently low to make H indifferent between searching and not. In the extreme when λ is very large, type H is almost not searching so the expected price paid by a searcher is near c_L. In this case the expected search cost borne by type H in the event it searches is relatively high (close to $c_H - c_L$) but H's search takes place only with small probability and hence the overall expected search cost is low.

6.2 Information Aggregation: Search vs. Auction—Enhanced Winner's Curse

We have already mentioned that the literature on auctions addressed a closely related question concerning the extent to which the equilibrium price in a common value auction reflects the correct information when the number of bidders is made arbitrarily large (Wilson (1977) and Milgrom (1979)). In the auction version of our model, instead of searching, the buyer assembles n sellers for an (procurement) auction. Each of the sellers gets a signal from the same distribution as above. They submit bids simultaneously and the lowest bidder is selected for the transaction. Milgrom's result translated to an auction version of our model is that the equilibrium price of the auction approaches the true cost when $n \to \infty$, iff $\lim_{x \to \underline{x}} \frac{f_L(x)}{f_H(x)} = \infty$. That is, when there are signals that are exceedingly more likely when the true state is L than when it is H.

In our model the number of "bidders" is endogenously determined through the sampling. If we think of reduction of the sampling cost s in our model as the counterpart of increasing the number of bidders n in the auction, then our results concerning information aggregation via search differ from those in the auction literature. When $\lim_{x \to \underline{x}} \frac{f_L(x)}{f_H(x)} = \infty$ the search equilibrium price is not always near the true cost when $s \to 0$, but only when a further condition on the informativeness of the signals–that the parameter λ is large–is met.

Thus, information aggregation is more difficult in the search environment than in the auction environment. One way to explain this difference is from the perspective of the winner's curse. In the auction a bidder who gets a good signal x sheds its bid appropriately to defend against the winner's curse. Let $\Pr(H \mid \text{winning}, x)$ be the probability that such bidder assigns to type H, conditional on signal x and being the winner in a monotone equilibrium

$$\Pr(H \mid \text{winning}, x) = \frac{g_H f_H(x)\left[1 - F_H(x)\right]^{n-1}}{g_H f_H(x)\left[1 - F_H(x)\right]^{n-1} + g_L f_L(x)\left[1 - F_L(x)\right]^{n-1}} = \frac{g_H}{g_H + g_L \frac{f_L(x)}{f_H(x)} \frac{[1-F_L(x)]^{n-1}}{[1-F_H(x)]^{n-1}}}.$$

Observe that $\Pr(H \mid \text{winning}, x)$ is jointly determined by the "signal effect", $\frac{f_L(x)}{f_H(x)}$, and the "winner's curse effect", $\frac{[1-F_L(x)]^{n-1}}{[1-F_H(x)]^{n-1}}$. The signal effect goes to ∞ as $x \to \underline{x}$; the magnitude of the winner's curse effect depends on the rate at which $x \to \underline{x}$ as $n \to \infty$. If we focus on a sequence of signals

x_n at which the probability of winning in a monotone equilibrium, $[1 - F_L(x_n)]^{n-1}$, exceeds some $\varepsilon > 0$, then the winner's curse effect is bounded away from 0 and the signal effect overwhelms it, $\lim_{n\to\infty} \frac{f_L(x_n)}{f_H(x_n)} \frac{[1-F_L(x_n)]^{n-1}}{[1-F_H(x_n)]^{n-1}} = \infty$. It follows that, for large n, $\Pr(H \mid \text{winning auction}, x_n) \approx 0$. That is, after allowing for the winner's curse, the winner is almost certain that the buyer is of type L.

In the search environment the mere fact of being sampled already implies a form of a winner's curse. The probability of the buyer being type H held by a seller just conditional on being sampled, and before the signal x is observed, is $\frac{g_H n_H}{g_H n_H + g_L n_L}$. In an equilibrium in which H mimics L with high probability, H's search duration n_H could be significantly larger than L's search duration n_L. Therefore, this probability might be substantially higher than the prior. After signal x is observed, the probability of type H becomes

$$\beta(x; P, A) = \frac{g_H f_H(x) n_H}{g_H f_H(x) n_H + g_L f_L(x) n_L} = \frac{g_H}{g_H + g_L \frac{f_L(x)}{f_H(x)} \frac{n_L}{n_H}}.$$

That is, it is determined jointly by a "signal effect" $\frac{f_L(x)}{f_H(x)}$ and a "winner's curse effect" $\frac{n_L}{n_H}$. In contrast to the auction scenario in which the signal effect prevails, here the winner's curse effect $\frac{n_L}{n_H}$ might offset the signal effect $\frac{f_L(x)}{f_H(x)}$ even when $\lim_{x\to 0} \frac{f_L(x)}{f_H(x)} = \infty$. For example, if both H and L search till they generate a signal below x^*, then $\frac{n_L}{n_H} = \frac{F_H(x^*)}{F_L(x^*)}$ so that $\lim_{x^*\to \underline{x}} \frac{f_L(x^*)}{f_H(x^*)} \frac{n_L}{n_H} = \lim_{x^*\to \underline{x}} \frac{f_L(x^*)}{f_H(x^*)} \frac{F_H(x^*)}{F_L(x^*)} = 1$ and $\beta(x; P, A) \to g_H$. Thus, the probability of H remains significant and hence the expected cost in the eyes of a seller remains well above c_L even when x is very informative.

In both the search and auction models, revelation of the information requires signals that make L exceedingly more likely to counteract the winner's curse. But the more substantial winner's curse in the search model requires that there are signals that separate L from H even in a more pronounced way than in the large auction model.

Observe that the extent of information aggregation differs across these two scenarios also in the case of boundedly informative signals, $\lim_{x\to \underline{x}} \frac{f_L(x)}{f_H(x)} < \infty$. The equilibrium of search model exhibits complete pooling, whereas in the corresponding auction model the equilibrium outcome exhibits only partial pooling.

Notice that, while in the search process the number of "bidders" depends on the buyer's type, the auction to which it is compared above has n bidders independently of the type. One may think of an alternative auction environment in which the buyer decides on the number of bidders (bearing a cost per bidder) which the bidders do not observe. Such an auction model will be closer to the search model in the sense that a bidder learns some information from the mere fact of being selected (in fact, it can be thought of as a simultaneous search model). In a companion paper, Lauermann and Wolinsky (2010), we consider this scenario. Preliminary results suggest that partial separation is possible in equilibrium even when signals are boundedly informative. Thus, in terms of information aggregation, this model with a small search cost behaves somewhat similar to the auction with a large commonly known n considered in the discussion above rather than like the search model with negligible costs.

7 Discussion—Equilibrium Selection

One difficulty that many search models have to deal with is avoidance of the familiar Diamond Paradox (Diamond (1971)). Roughly speaking, if the searcher incurs the search cost before observing a price offer, then the monopoly price emerges in equilibrium regardless of how small the search cost is. While it is useful to understand this effect, it is also useful to note that in richer environments it is mitigated by a number of factors. So a frequent challenge in search models is how to keep the model simple enough without getting bogged down with Diamond's paradox. In our model, the Diamond effect is avoided by letting the buyer make the offers. But since the buyer has private information, this gives rise to multiplicity through the freedom of selecting off path beliefs in perfect Bayesian equilibria. Thus, to avoid one modeling problem we have to deal with another.

Below, we first explain the multiplicity and then discuss our refinement. We also show that a similar selection is implied by a modification of the bargaining component.

7.1 Equilibrium Multiplicity

We have already noted that this model has many equilibria. This subsection explains this statement in more detail. First, there is always a trivial pooling equilibrium in which both of the buyer's types offer c_H, which sellers accept, and lower prices trigger a belief that the offerer is type H resulting in a rejection. Second, when the sampling cost is sufficiently small, there are always equilibria that exhibit nearly perfect separation regardless of the magnitude of λ. To construct a sequence of equilibria of the latter type, consider the strategies defined by a threshold level $\widehat{x}_k$ as follows. Type L searches till it generates signal $x \leq \widehat{x}_k$, while type H settles in the first visit be it at a price $E_I[c|x]$ if $x \leq \widehat{x}_k$ or at the price c_H if $x > \widehat{x}_k$. With these strategies $n_{Lk} = 1/F_L(\widehat{x}_k)$, $n_{Hk} = 1$. Formally

$$\begin{aligned} \text{for } x &\leq \widehat{x}_k, \quad P_L(x) = P_H(x) = E_I[c|x] \text{ and } A(E_I[c|x]) = 1; \\ \text{for } x &> \widehat{x}_k, \quad P_L(x) = 0;\ P_H(x) = c_H \text{ and } A(p,x) = 1 \text{ iff } p \geq c_H, \end{aligned}$$

where

$$E_I[c|x] = \frac{c_H + \frac{g_L}{g_H}\frac{f_L(x)}{f_H(x)}\frac{1}{F_L(\widehat{x}_k)}c_L}{1 + \frac{g_L}{g_H}\frac{f_L(x)}{f_H(x)}\frac{1}{F_L(\widehat{x}_k)}}.$$

These strategies are just like the "undefeated" strategies in the case $\bar{a}_k = 0$ with $\widehat{x}_k$ in the role of x^*_k. Let $\widehat{x}_k$ be chosen to satisfy $s_k/F_H(\widehat{x}_k) = c_H - E_I[c|x \leq \widehat{x}_k, H]$, which obviously exists. Then the above strategies constitute an equilibrium (when off path beliefs are chosen appropriately).

When $s_k \to 0$, the cutoff $\widehat{x}_k \to \underline{x}$ and this sequence of equilibria approaches full separation. That is, the expected price paid by Type L, $E_I[c|x \leq \widehat{x}_k, L] \to c_L$, the expected price paid by Type H approaches c_H. If $\frac{f_L(\underline{x})}{f_H(\underline{x})}$ is sufficiently large, the total search costs of both types are close to zero as well (Type L's expected search cost approaches $\lim s_k/F_L(\widehat{x}_k) = \lim(F_H(\widehat{x}_k)/F_L(\widehat{x}_k))(s_k/F_H(\widehat{x}_k)) = (c_H - c_L)\lim_{\widehat{x}_k \to \underline{x}} \frac{f_H(\widehat{x}_k)}{f_L(\widehat{x}_k)}$. If $\lim_{\widehat{x}_k \to \underline{x}} \frac{f_H(x)}{f_L(\underline{x})} = 0$, the limit is equal to zero; if $\frac{f_H(\underline{x})}{f_L(\underline{x})}$ is bounded but sufficiently small, the limit is positive, but close to zero, too).

This construction and hence the full separation are independent of the magnitude of λ. But these equilibria rely on "unattractive" beliefs off the path. At signals x just above $\widehat{x}_k$, the seller is supposed to reject any price below c_H, though there might be a price that would be beneficial for both types of the buyer and for the seller if its beliefs remain at $\beta_I(x)$.

In the light of this multiplicity and the fact that there are always equilibria that aggregate the information well, one may wonder why we focus on the "undefeated" equilibria and their performance with regard to information aggregation. The answer is that we are interested in information aggregation under "natural" trading conditions. For example, in the related environment of an auction with common values, it is not difficult to construct mechanisms that will aggregate the information very well. Yet we are interested in the extent of information aggregation under natural trading/bidding conditions. What we mean here by the term "natural" is an equilibrium that does not use off-path beliefs to introduce an artificial wedge between prices and expected costs. In other words, we do not want the seller to reject a price offer that is beneficial for both buyer types (relative to the putative equilibrium) and profitable for the seller (at the interim expected cost) because the very act of offering it triggers an unfavorable belief.

Following are two more formal arguments that single out an "undefeated" equilibrium.

7.2 A Modified Bargaining Component

Consider a variation of the model that differs in its bargaining component. Now, in each period the buyer samples $K \geq 2$ sellers who observe the same signal and simultaneously offer prices. The buyer then either trades with the seller who offered the lower price (or chooses at random in case of a tie) or continues to search. All other details of the model remain the same, so the only change is in the bargaining component.

The main implication of this change is that, since the uninformed sellers are the ones who make the offers, out of equilibrium beliefs play no role and do not generate the multiplicity noted above. Since the two sellers observe the same signal, they play a one-shot Bertrand game, which drives the prices to the expected costs conditional on the buyer's acceptance decision. It is easy to see that the unique equilibrium outcome in this version is an "undefeated" outcome with pooling for all $x \leq x^{**}$ and no trade with L at higher x's. That is, for $x \leq x^{**}$ the price will be the interim expected cost conditional on x, $E_I[c|x]$, and, for $x > x^{**}$, the price will be c_H, which type L would reject and continue searching.

In a sense this version of the model gives the results in a cleaner way. Its drawback is the somewhat artificial assumption of $K \geq 2$ sellers who observe the same signal and bid simultaneously.

7.3 The Equilibrium Refinement

Consider an equilibrium (P, A). Given the equilibrium values $V_w(P, A)$ and beliefs $\beta_I(x, P, A)$, the bargaining game between the buyer and a seller after each realization x can be viewed in isolation, and the overall equilibrium induces a sequential equilibrium in each of them. Notice that the isolated bargaining game in our model is a special sort of signaling game in which the messages—the price offers—do not entail up-front costs (like the cost of education in Spence's model) and the types differ only in their payoffs following a rejection (in case $V_L(P, A) \neq V_H(P, A)$). This is not a cheap talk game either, since in the event of acceptance the buyer's payoff depends on the price offer, but in this event the payoffs of both types are identical.

This implies immediately that the trivial pooling equilibrium in which both types offer c_H after any x cannot be eliminated by any of the forward induction refinements (e.g., Cho and Kreps (1987)'s Intuitive-Criterion and D1, and Banks and Sobel (1987)'s Divinity) since in it $V_L(P, A) = V_H(P, A)$. Furthermore, even if we are willing to restrict attention to equilibria in which $V_L(P, A) > V_H(P, A)$, the Intuitive-Criterion and Divinity would not have a bite since any offer that for some best responses benefits type L also benefits type H for a non-empty subset of those responses. The D1 refinement will not help either since it is inconsistent with $V_L(P, A) > V_H(P, A)$. This is because, if $V_L(P, A) > V_H(P, A)$, then D1 would select the same separating equilibrium in every bargaining game, independently of x. But the payoffs' independence of the distribution of x is inconsistent with $V_L(P, A) > V_H(P, A)$[9].

Thus, we need a refinement that respects the forward induction logic but also prevents a seller from adopting pessimistic beliefs in the face of an offer that is preferred by both of the buyer's types. This combination is featured by the undefeated equilibrium refinement (Mailath, Okuno-Fujiwara and Postlewaite (1993)).

A sequential equilibrium in the bargaining game is defeated if: (i) there is an out-of-equilibrium offer p that is used in an alternative sequential equilibrium by a subset K of buyer's types; (ii) the payoffs of all types in K after offering p in the alternative equilibrium are larger than their payoff at the considered equilibrium with a strict inequality for at least one type; (iii) the beliefs after p

[9] Thus, the only equilibria consistent with D1 are such that $V_L(P, A) \leq V_H(P, A)$, like the trivial pooling on c_H equilibrium.

in the alternative equilibrium differ from the beliefs after p in the considered equilibrium (in the alternative equilibrium they coincide with the sellers' interim belief conditional on the buyer's type being in K).

A sequential equilibrium of the bargaining game is undefeated if it is not defeated.

It turns out that the selection of "undefeated" equilibria in our model is equivalent to the selection achieved by imposing the undefeated equilibrium refinement (Mailath, Okuno-Fujiwara and Postlewaite (1993)) on the sequential equilibria of the bargaining games. The following claim (proved in the appendix) establishes this equivalence.

Claim 5 *: (i) If an equilibrium (P, A) induces an undefeated sequential equilibrium in the bargaining games after each x, then (P, A) is "undefeated". (ii) Conversely, if (P, A) is an "undefeated" equilibrium, then in each of the bargaining games there is an undefeated sequential equilibrium that yields the same outcome.*

8 Discussion—Variations on the Model

This section collects brief remarks about possible variations and extensions. Some of these remarks are backed by actual analysis and some are more conjectural. We try to make this distinction clearly.

BUYER OFFERS MECHANISM

The model remains as above except for the bargaining component. Now, instead of a price the buyer offers a direct mechanism $m = [p_L, q_L; p_H, q_H]$, which the seller either accepts or rejects. If m is accepted, the buyer reports his type $r \in \{L, H\}$ and trade takes place at price p_r with probability q_r. If either m is rejected or it prescribes no trade, the search continues just like following a rejection in the original model.

The main reason for considering this variation is to assure the reader that the simple take-or-leave-it price in the role of the bargaining component does not eliminate equilibria that do not support our results. In principle, of course, the more general setting of a mechanism offer might be expected to give rise to additional equilibrium outcomes (satisfying the same refinement) which might not support our results concerning information aggregation.

Without loss of generality (Myerson (1983)), attention may be restricted to equilibria that involve *truthful* reporting, are *inscrutable* (i.e., the same mechanism m is offered by both types), and sure *acceptance* of equilibrium offers by sellers. An *"undefeated" equilibrium* for this environment is such that, for any signal x, the offered mechanism yields higher payoffs for L than any other admissible mechanism given the seller's belief.[10]

It turns out that the set of "undefeated" equilibrium outcomes in this version is contained in the set of undefeated outcomes arising in the take-it-or-leave-it price offer game. More formally, the counterpart of Claim 5 above is

Claim[11]**.** In an "undefeated" equilibrium, $V_L > V_H$ and the offered mechanism m satisfies: (i) After $x < x^*$, $m = [E_I[c|x], 1, E_I[c|x], 1]$; (ii) After $x > x^*$, $m = [c_L, a_L, c_H, a_H]$, where $a_L = \max\{0, \frac{(u-c_H)-V_H}{(u-c_L)-V_H}\}$ and $V_H \gtrless u - c_H$ implies $a_H = 0$ and 1, respectively.

Conversely, if an equilibrium satisfies (i) and (ii) for every x, it is "undefeated".

The important conclusion is that our decision to focus on a take-or-leave-it bargaining game does not eliminate equilibria that do not support our results concerning information aggregation.

MORE THAN TWO COST TYPES

While we have analyzed only the case of two buyer types, it seems that the generalization to any finite number of types would follow along similar lines. Suppose that the buyer has three types $\{L, M, H\}$, where $c_L < c_M < c_H$. The sellers' interim belief in type w would be $\beta_I^w(x) = \frac{g_w f_w(x) n_w(P,A)}{g_H f_H(x) n_H(P,A) + g_M f_M(x) n_M(P,A) + g_L f_L(x) n_L(P,A)}$. The natural conjecture is that an equilibrium with (partial) separation would be characterized by two cutoff signals, x_L^* and x_M^*. After $x \leq x_L^*$ all three types would trade at the interim expected cost, after $x \in (x_L^*, x_M^*]$ only types M and H would trade at the interim expected cost conditional on M and H, and after $x > x_M^*$ only type H might trade. Information aggregation would require conditions on the relative informativeness of the signals for any two adjacent types. A natural conjecture is that defining $\lambda(F_L, F_M)$ and $\lambda(F_M, F_H)$ in the same way λ was defined by (13) for F_L vs F_H would give rise to analogous conditions concerning the separation of adjacent types in the limit of outcome of the search equilibrium. However, without actually performing this analysis, it is difficult to say how demanding it would be. But conceptually the case of a larger than two but finite number of types does not seem different than the case of two.

BUYER DOES NOT OBSERVE SIGNAL

We have been assuming that the buyer observes the realization of the signal in each encounter. In an alternative specification, the buyer does not observe the signal. We have not analyzed this alternative in detail, but we conjecture that the results concerning information aggregation would be qualitatively similar. Since the buyer does not observe the signal, price offers would be a function of the type alone. At equilibrium, there will be at most two offers P_L and $P_H = c_H$. Type L would offer P_L; Type H would mix between the two prices. Sellers would accept P_L provided the signal is below a threshold x^* (the counterpart of that threshold in the above analysis). In a

[10] We forgo here the full formal presentation of strategies and equilibrium as we are just outlining this variation.

[11] We do not provide the proof in this paper since it is fairly long and requires further development of the notation.

pooling equilibrium both types offer P_L. In a partially separating equilibrium, type H is indifferent between offering P_L and settling for c_H. The probability with which type H offers P_L is such that the expected interim cost after x^* is exactly equal to P_L. Again, we have not performed this analysis, but from the outset this variation does not seem to lead to qualitatively different insights.

BUYER DOES NOT KNOW OWN TYPE

In another possible variation the buyer does not observe its own type but does observe the realization of the signals (as the model currently assumes). We have not analyzed this variation, but we conjecture that the results concerning information aggregation would be qualitatively similar. When the search costs are small, the buyer could learn fairly quickly its type with high probability and the subsequent behavior should resemble what it is in the present version. Of course, we cannot claim this as a result in the absence of the analysis, but it is hard to imagine that this variation would modify the basic insights. Generally, if the buyer did not know its own type, the sampling behavior of the two types would become more similar to each other. This suggests that separation is even more difficult in this case. The assumption that the buyer is better informed than the sellers about the cost does not seem crucial for our results. The critical assumptions are that the costs are correlated across sellers and that sellers observe different signals.

DISCOUNTING.

In the present model there is no discounting. The only time cost is the sampling cost s. We conjecture that discounting would not change our results. Intuitively, with discounting, the type with the lower expected continuation payoff has a stronger incentive to search. Since it will be the H type who has lower continuation payoffs, one might expect the H type would search even more with discounting than without, thus strengthening the insight concerning the search induced winner's curse.

OBSERVABLE HISTORY

The sellers' inability to observe the buyer's history is a key feature of the model and the environments it represents. It is crucial for generating the search induced winner's curse and the main qualitative insights of this paper. Obviously there are interesting environments in which the searcher's history (or a significant feature of it) is observable. Such might be the case in labor markets where the duration of unemployment is often observable. The present model does not cover these situations.

MIXED STRATEGIES

The model restricted attention to pure buyer strategies. This simplifies the refinement arguments and characterization of equilibrium. With mixed strategies, the analysis is more complicated. There are mixed equilibria with full support (in some or even all subgames), which are naturally immune against any refinement since no out-of-equilibrium prices are left to deviate to.

One way to deal with these difficulties is to introduce a small cost for making offers (which eliminates fully mixed equilibria in certain subgames in which the seller rejects all the offers) and to restrict attention to equilibria in which $V_L > V_H$ (which eliminates fully mixed equilibria in all subgames). Subject to these modifications, we verified that our results continue to hold for the case $\underline{x} > 0$. We believe that the results also continue to hold for the case $\underline{x} = 0$, but we have not performed the analysis for this case. The bottom line is that we believe that our results would survive the admission of mixed strategies, but this would be at the cost of a messier analysis as outlined above.

HETEROGENEOUS VALUATIONS

We have been assuming that the buyer's valuation u is known and independent of the cost type. This is done primarily for convenience and we conjecture that allowing for a two dimensional buyer's type that captures heterogeneity with respect to the valuation as well would not alter our insights in a significant way, though it might add some new ones. As we explain in the end of Section 3, efficiency considerations with respect to the volume of trade can already be generated in the present model by considering the case of $u < c_H$. The introduction of heterogeneity of valuations would facilitate further insights of this type. We have conducted formal analysis of certain subcases of a version with two valuation types. This might be a worthwhile extension which we decided not to pursue in this paper.

9 Conclusion

In a search model with adverse selection, information is not always aggregated perfectly, in the sense that the prices do not always coincide with the full information prices. This is the case even when the search cost is negligible and even when there are exceedingly informative signals. The prices aggregate the information nearly perfectly if and only if the signals of the uninformed can be arbitrarily informative and, in addition, the rate at which the informativeness of these signals increases (as they become more informative) is large.

In the corresponding auction environment, the availability of exceedingly informative signals would translate to nearly perfect aggregation when the number of bidders is large, without the further condition mentioned above. The source of the difference is a stronger winner's curse in the search model that owes to the longer search duration of the bad types. When the search history is unobservable, being sampled is bad news, potentially overwhelming the informative content of signals.

Imperfect information aggregation has welfare consequences through wasteful search activities, which may be significant even when the one time sampling cost is negligible. And, in a variation on the basic model, imperfect information aggregation can generate inefficiently low volume of trade.

10 Appendix

10.1 Derivation of Beliefs

In what follows we derive the formula for the interim beliefs

$$\beta_I(x,P,A) = \frac{g_H f_H(x)\, n_H(P,A)}{g_H f_H(x)\, n_H(P,A) + g_L f_L(x)\, n_L(P,A)} = \frac{1}{1 + \frac{g_L}{g_H}\frac{f_L(x)}{f_H(x)}\frac{n_L(P,A)}{n_H(P,A)}}.$$

In the main body we have suppressed some formalities of the extensive form game. To derive beliefs, we introduce additional notation: The set of histories is denoted by $\mathcal{H}$ and consists of sequences of the form $(w, z_1, ..., z_t)$, $(w, z_1, ..., z_t, i_{t+1})$, $(w, z_1, ..., z_t, i_{t+1}, x_{t+1})$ or $(w, z_1, ..., z_t, i_{t+1}, x_{t+1}, p_{t+1})$, where $z_t = (i_t, x_t, p_t, a_t)$ describes the outcome of an encounter between the buyer and seller i: $x_t \in [\underline{x}, \overline{x}]$ is the realization of the signal, p_t is a price offer $a_t =$"acceptance" or "rejection" of the price p_t by seller i_t. $\mathcal{H}$ also includes the infinite sequences $(w, \{z_t\}_{t=1}^{\infty})$. Notice that we consider here sequences whose elements are generated by this process even after a price was accepted. Any strategy profile (P, A) induces a probability distribution $\Psi(P, A)$ on $\mathcal{H}$: First, $w \in \{L, H\}$ is determined by a chance move, where $w = L$ with probability g_L and $w = H$ with probability g_H. Then, for each t, i_t is determined by a chance move that draws some seller $i \in [0, 1]$ from a uniform distribution and x_t is determined by a chance move that draws signal $x_t \in [\underline{x}, \overline{x}]$ from F_w. The price p is chosen according to P_w, and a_t is chosen according to $A(x_t, p_t)$.

Let $\widetilde{n}(z)$ denote the stopping time associated with the history $z \in \mathcal{H}$, i.e., the $\widetilde{n}(z)$ is the first t such that $a_t =$ "acceptance" (where $\tilde{n}(z) = \infty$ if $a_t =$"rejection" for all t). Let $\widetilde{n}_{-i}(z)$ be the stopping time when z_t's containing i are omitted. Let $(Z(P, A))_i$ or Z_i for short be the subset of *infinite* histories such that i is sampled before the stopping time, i.e., $Z_i = \{(w, z_1, z_2, ..) \mid i_t = i$ for some $t \leq \widetilde{n}(z)\}$. Let $\langle Z_i, x\rangle$ be the set of finite histories that are the first segments of histories from Z_i where i is sample for the first time and observes signal realization x; A typical element of $\langle Z_i, x\rangle$ is $(w, z_1, z_2, ., z_{t-1}, i_t, x_t)$ with $i_t = i$ and $x_t = x$. The set $\langle Z_i, x\rangle$ is the collection of *information sets* of seller i who is sampled for the first time and who has observed signal x, where the collection is taken over all price offers. The probability of $w = H$ conditional on that set corresponds to the interim belief of a seller, before the seller observes the price offer:

$$\beta_I(x, (P, A)) = \Pr(w \mid \langle Z_i, x\rangle).$$

Since $\langle Z_i, x\rangle$ is a zero probability event, what we mean by $\Pr(w \mid Z_i, x)$ is $\lim_{\varepsilon\to 0} \Pr(w \mid \langle Z_j, x\rangle : j \in [i, i+\varepsilon])$ (if $i = 1$, let $j \in [1-\varepsilon, 1]$).[12]

[12] Note that our approach is analogous to the conventional approach to conditioning on a particular realization of a signal from the real line. Indeed, implicitly we consider $\lim_{\varepsilon\to 0} \Pr(w \mid \langle Z_j, x'\rangle : j \in [i, i+\varepsilon], x' \in [x, x+\varepsilon])$. However, the conditioning on a signal from the real line is standard and unlikely to cause confusion; we therefore suppress it here.

Conditioning on $\langle Z_i, x\rangle$ is equivalent to conditioning on a realization from the set of infinite histories Z_i and requiring that the first occurrence of i is accompanied with the signal realization x. Thus,

$$\Pr(w \mid \langle Z_j, x\rangle) = \lim_{\varepsilon\to 0} \frac{\Psi(\{Z_j : j \in [i, i+\varepsilon]\} \mid w) f_w(x) g_w}{\Psi(\{Z_j : j \in [i, i+\varepsilon]\} \mid H) f_H(x) g_H + \Psi(\{Z_j : j \in [i, i+\varepsilon]\} \mid L) f_L(x) g_L}.$$

Notice that

$$\Psi(\{Z_j \ : \ j \in [i, i+\varepsilon]\} \mid w) = \sum_{n\geq 1} \Psi(\{Z_j : j \in [i, i+\varepsilon]\} \mid \widetilde{n}(z) = n, w) \Pr(\widetilde{n}(z) = n \mid w) =$$

$$\Psi(\{Z_j \ : \ j \in [i, i+\varepsilon]\} \mid \widetilde{n}(z) = 1, w) \sum_{n\geq 1} \frac{\Psi(\{Z_j : j \in [i, i+\varepsilon], w\} \mid \widetilde{n}(z) = n, w)}{\Psi(\{Z_j : j \in [i, i+\varepsilon], w\} \mid \widetilde{n}(z) = 1, w)} \Pr(\widetilde{n}(z) = n \mid w).$$

Now,

$$\lim_{\varepsilon\to 0} \frac{\Psi(\{Z_j : j \in [i, i+\varepsilon]\} \mid \widetilde{n}(z) = n, w)}{\Psi(\{Z_j : j \in [i, i+\varepsilon]\} \mid \widetilde{n}(z) = 1, w)} = \lim_{\varepsilon\to 0} \frac{1-(1-\varepsilon)^n}{\varepsilon} = n.$$

Therefore,

$$\begin{aligned}
\Pr(w \mid \langle Z_j, x\rangle) = & \quad \frac{\sum_{n\geq 1} n \Pr(\widetilde{n}_{-i}(z) = n \mid w) f_w(x) g_w}{\sum_{n\geq 1} n \Pr(\widetilde{n}_{-i}(z) = n \mid H) f_H(x) g_H + \sum_{n\geq 1} n \Pr(\widetilde{n}_{-i}(z) = n \mid L) f_L(x) g_L} \\
= & \quad \frac{n_w(P, A) f_w(x) g_w}{n_H(P, A) f_H(x) g_H + n_L(P, A) f_L(x) g_L}.
\end{aligned}$$

This completes the derivation.

10.2 Proofs of Results

Claim 1: If (P, A) is an "undefeated" equilibrium, then $V_L > V_H$ and (up to irrelevant differences[13]) its outcome satisfies the following.

- After $x \leq x^*$, both types of the buyer pool on price $p = E_I[c|x, P, A]$ and the price is accepted,

$$P_L(x) = P_H(x) = E_I[c|x] \text{ and } A(E_I[c|x]) = 1.$$

- After $x > x^{**}$, the buyer types separate

$$P_L(x) = c_L, \ \ P_H(x) = c_H,$$

[13] These "irrelevant differences" concern zero probability events and the description of situations in which both buyer types disagree with the seller. We describe those situations as separating with zero acceptance probabilities, but they can also be described in equivalent ways, e.g., as pooling on a price below cost.

and the acceptance probabilities satisfy

$$\frac{(u - E_I [c|x]) - V_L}{(u - c_L) - V_L} \leq A (c_L, x) \leq \max\{0, \frac{(u - c_H) - V_H}{(u - c_L) - V_H}\},$$

$$A (c_H, x) = \begin{cases} 0 & \text{if} \quad V_H > u - c_H, \\ 1 & \text{if} \quad V_H < u - c_H. \end{cases}$$

- After $x \in (x^*, x^{**})$, the two types either pool or separate as above.

- Conversely, if an equilibrium (P, A) satisfies the above for every x, it is "undefeated".

Proof of 1 If (P, A) is an "undefeated" equilibrium, it determines the values V_H, V_L and induces a sequential equilibrium in the bargaining game following any signal x. A pure sequential equilibrium outcome can be either pooling or separating, and it can either involve trade with positive or zero probability. Without any loss, we will assume that if (P, A) induces a no-trade equilibrium after x, then it is separating with $P_w(x) = c_w$ and $A(c_w, x) = 0$. But of course many other choices of $P_w(x)$ could yield the same result.

Step 1: If the bargaining equilibrium induced by (P, A) after x is pooling (with trade), then $x \leq x^{**}$, $P_H(x) = P_L(x) = E_I [c|x, P, A]$ and $A(E_I [c|x, P, A], x) = 1$ for any $x < x^{**}$.

Proof of Step 1: Clearly, a pooling equilibrium (with trade) exists only if $u - E_I [c|x, P, A] \geq V_L$, i.e., only if $x \leq x^{**}$. It may not be of course that $P_H(x) = P_L(x) < E_I [c|x, P, A]$ for this would mean losses for the seller and the definition of "undefeated" equilibrium rules out the reverse inequality or rejection. Therefore, $P_H(x) = P_L(x) = E_I [c|x, P, A]$ and also $A(E_I [c|x, P, A], x) = 1$ for any $x < x^{**}$.

Step 2: $V_L > V_H$ in every undefeated equilibrium.

Proof of Step 2:

Let $\widetilde{x} = \sup\{x : A (P_H (x)) > 0\}$. The following observations are immediate:

(i) By definition $V_w = \int_{\underline{x}}^{\bar{x}} [A (P_w (x)) (u - P_w (x)) + (1 - A (P_w (x))) V_w] \, dF_w (x) - s$.

(ii) $A (P_w (x)) > 0$ implies $u - P_w (x) \geq V_w$

(iii) $A (P_H (x)) > 0$ implies $P_H (x) \geq E_I [c|x, P, A]$ since this is so regardless of whether the equilibrium after x is pooling or separating.

(iv) $u - E_I [c|x, P, A] > V_H$ for all $x < \widetilde{x}$, since by the definition of $\widetilde{x}$, $\exists x' \in (x, \widetilde{x}]$ such that $A (P_H (x')) > 0$ and hence observation (ii) and (iii) above and the strict monotonicity of $E_I [c|x, P, A]$ in x imply $u - E_I [c|x, P, A] > u - E_I [c|x', P, A] \geq u - P_H (x') \geq V_H$.

From observations (i) and (ii) above and the characterization of "undefeated" equilibrium (3)

$$
\begin{aligned}
V_L &= \int_{\underline{x}}^{\bar{x}} A\left(P_L\left(x\right)\right)\left(u-P_L\left(x\right)\right)+\left(1-A\left(P_L\left(x\right)\right)\right)V_L dF_L\left(x\right)-s \\
&\geq \int_{\underline{x}}^{\widetilde{x}} [u-E_I\left[c|x,P,A\right]]dF_L\left(x\right)+[1-F_L\left(\widetilde{x}\right)]V_L-s.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
V_L &\geq u-\int_{\underline{x}}^{\widetilde{x}} E_I\left[c|x,P,A\right]\frac{dF_L\left(x\right)}{F_L\left(\widetilde{x}\right)}-\frac{s}{F_L\left(\widetilde{x}\right)} \\
&> u-\int_{\underline{x}}^{\widetilde{x}} E_I\left[c|x,P,A\right]\frac{dF_H\left(x\right)}{F_H\left(\widetilde{x}\right)}-\frac{s}{F_H\left(\widetilde{x}\right)} \\
&= \frac{1}{F_H\left(\widetilde{x}\right)}\left(\int_{\underline{x}}^{\widetilde{x}}\left(u-E_I\left[c|x,P,A\right]\right)dF_H\left(x\right)-s\right) \\
&\geq \frac{1}{F_H\left(\widetilde{x}\right)}\left(\int_{\underline{x}}^{\widetilde{x}}\left(A\left(P_H\left(x\right)\right)\left(u-P_H\left(x\right)\right)+\left(1-A\left(P_H\left(x\right)\right)\right)V_H\right)dF_H\left(x\right)-s\right) \\
&= \frac{1}{F_H\left(\widetilde{x}\right)}\left(\int_{\underline{x}}^{\bar{x}}\left(A\left(P_H\left(x\right)\right)\left(u-P_H\left(x\right)\right)+\left(1-A\left(P_H\left(x\right)\right)\right)V_H\right)dF_H\left(x\right)-[1-F_H\left(\widetilde{x}\right)]V_H-s\right) \\
&= \frac{1}{F_H\left(\widetilde{x}\right)}\left(V_H-[1-F_H\left(\widetilde{x}\right)]V_H\right)=V_H,
\end{aligned}
$$

where the one before last equality follows from observation (i); the prior equality from the definition of $\widetilde{x}$; the prior inequality from observations (iii) and (iv); the preceding strict inequality from the monotone likelihood property and from $F_H\left(\widetilde{x}\right)<F_L\left(\widetilde{x}\right)$.

Step 3: If the bargaining equilibrium induced by (P,A) after x is separating (with trade), then $x\geq x^*$, $P_L(x)=c_L$, $P_H(x)=c_H$, $\frac{(u-E_I[c|x,P,A])-V_L}{(u-c_L)-V_L}\leq A\left(c_L,x\right)\leq\max\left\{\frac{u-c_H-V_H}{u-c_L-V_H},0\right\}$ and $A(c_H,x)=0$ if $u-c_H<V_H$ and $A(c_H,x)=1$ if $u-c_H>V_H$ (Hence, $A(c_L,x)\leq A(c_H,x)$, $A(c_L,x)\leq\bar{a}$)

Proof of Step 3: If $P_L(x)>c_L$ in a separating equilibrium, then $A(P_L(x),x)=1$ and $u-P_L(x)\geq\max\{V_L,A(P_H(x),x)(u-P_H(x))+(1-A(P_H(x),x))\,V_L\}$. But this implies that

$$
u-P_L(x)>\max\{V_H,u-P_H(x)\},
$$

since $V_L>V_H$ and $u-P_L(x)\geq u-P_H(x)$ while $P_L(x)\neq P_H(x)$. Therefore, H prefers offering $P_L(x)$ to $P_H(x)$, in contradiction to the separating equilibrium. Hence, $P_L(x)=c_L$.

In a separating equilibrium with trade, it may not be of course that $P_H(x)<c_H$. Since $A(p,x)=1$ for any $p>c_H$, it must be (in a separating equilibrium with trade) that $P_H(x)=c_H$. And if

$u - c_H > V_H$ it must be that $A(c_H, x) = 1$, for otherwise the buyer would benefit from deviating to a slightly higher price. If $u - c_H < V_H$, type H does not trade in a separating equilibrium, so the outcome is equivalent to $P_H(x) = c_H$ and $A(c_H, x) = 0$. The left inequality of $\frac{(u - E_I[c|x,P,A]) - V_L}{(u - c_L) - V_L} \leq A(c_L, x) \leq \max\left\{\frac{u - c_H - V_H}{u - c_L - V_H}, 0\right\}$ follows from the equilibrium being "undefeated"; the right inequality follows from the IC constraint for type H, which must hold in a separating equilibrium. It follows that $A(c_L, x) \leq A(c_H, x)$, $A(c_L, x) \leq \bar{a}$.

By the definition of x^*, for $x < x^*$, $(u - c_L)\bar{a} + V_L(1 - \bar{a}) = u - E_I[c|x^*, P, A] < u - E_I[c|x, P, A]$. Therefore, an "undefeated" equilibrium may not be separating after $x < x^*$.

Step 4: If an equilibrium (P, A) satisfies the conditions stated in the claim for every x, it is "undefeated".

Proof of Step 4: Suppose that (P, A) satisfies the conditions in the claim. It is clear that (P, A) is a (Markovian) equilibrium. To verify that it is "undefeated", we have to establish that, for all x, $A(P_L(x), x)(u - P_L(x)) + [1 - A(P_L(x), x)]V_L \geq u - E_I[c|x, P, A]$. This is clearly true for $x \leq x^*$, $x \geq x^{**}$ and those $x \in (x^*, x^{**})$ after which the equilibrium is pooling. For $x \in (x^*, x^{**})$ after which the equilibrium is separating $\frac{(u - E_I[c|x,P,A]) - V_L}{(u - c_L) - V_L} \leq A(c_L, x)$ which means that L's payoff $A(c_L, x)(u - c_L) + (1 - A(c_L, x))V_L \geq u - E_I[c|x, P, A]$. Therefore, after any value of x, L's payoff exceeds the payoff from the pooling outcome implying that the equilibrium is "undefeated".

This completes the proof of the claim. ■

Claim 2: An "undefeated" equilibrium exists.

Proof of 2: The proof is standard fixed point argument and is brought just for completeness.

If $s \geq c_H - c_L$, then there exists a simple undefeated equilibrium in which both types trade immediately at interim expected costs. So, suppose $s < c_H - c_L$. Let $\varepsilon > 0$ be uniquely defined as the solution to $\frac{s}{F_L(\underline{x} + \varepsilon)} = c_H - c_L$. Define a correspondence from $[\underline{x} + \varepsilon, \overline{x}] \times [0, 1]$ to itself as follows. For each pair (y_1, y_2) in this set, let the strategies P and A satisfy

$$\begin{array}{llll} \text{for} & x \leq y_1 & P_L(x) = P_H(x) = E_I[c|x, P, A] & A(E_I[c|x, P, A], x) = 1; \\ \text{for} & x > y_1 & P_L(x) = c_L; \; P_H(x) = c_H & A(c_L, x) = 0; \; A(c_H, x) = y_2. \end{array}$$

Compute the continuation payoffs $V_w = V_w(P, A)$ that correspond to these strategies. Let

$$y_1' = \begin{cases} \overline{x} & \text{if} \quad V_L < u - E_I[c|x, P, A] \text{ for all } x \in [\underline{x} + \varepsilon, \overline{x}], \\ \underline{x} + \varepsilon & \text{if} \quad V_L > u - E_I[c|x, P, A] \text{ for all } x \in [\underline{x} + \varepsilon, \overline{x}], \\ \text{unique } x & \text{s.t.} \quad V_L = u - E_I[c|x, P, A]. \end{cases}$$

Let

$$y_2' = \begin{cases} 0 & \text{if} \quad V_H - (u - c_H) > 0, \\ [0,1] & \text{if} \quad V_H - (u - c_H) = 0, \\ 1 & \text{if} \quad V_H - (u - c_H) < 0. \end{cases}$$

The correspondence that maps (y_1, y_2) to (y_1', y_2') is upper hemi-continuous and has a fixed point (x^*, a) by Kakutani's theorem. The fixed point cannot be at $x^* = \underline{x} + \varepsilon$: If $y_1 = \underline{x} + \varepsilon$, then the induced payoff $V_L(P, A) \leq u - c_L - \frac{s}{F_L(\underline{x}+\varepsilon)} \leq u - c_H$, which implies $y_1' = \overline{x}$. Thus, any fixed point of the correspondence above describes the path of an "undefeated" equilibrium, which can be completed by specifying pessimistic beliefs and rejection after all $p < c_H$ off the path.■

Claim 3. (i) $x_k^* \to \underline{x}$; (ii) $\lim \bar{a}_k = 0$ and $\lim x_k^{**} = \underline{x}$; (iii) $x_k^* > \underline{x}$ for all k; (iv) $\lim s_k / F_L(x_k^*) = 0$ and $\lim \bar{a}_k / F_L(x_k^*) = 0$; (v) $\overline{S}_L = 0$.

Proof of Claim 3: As a preliminary step observe that the equilibrium conditions imply the following key conditions

$$\begin{aligned} F_L(x_k^{**})(E_I[c|x_k^{**}] - E_I[c|x \leq x_k^{**}, L]) &\leq s_k \\ &\leq F_L(x_k^*)[E_I[c|x_k^*] - E_I[c|x \leq x_k^*, L]] + \frac{\bar{a}_k}{1 - \bar{a}_k}[E_I[c|x_k^*] - c_L], \end{aligned} \tag{23}$$

$$\frac{F_H(x_k^*)}{F_L(x_k^{**}) + (1 - F_L(x_k^{**}))\bar{a}} \leq \frac{n_{Lk}}{n_{Hk}} \leq \frac{1}{F_L(x_k^*)}, \tag{24}$$

$$\bar{a}_k = \max\{\frac{u - c_H - V_{Hk}}{u - c_L - V_{Hk}}, 0\}. \tag{25}$$

To see this, recall that the equilibrium payoff of type L is at most the payoff of pooling for $x \leq x_k^*$ and separating with $A(c_L, x) = \bar{a}_k$ (i.e., the maximum acceptance probability) for $x > x_k^*$. This follows from the definition of x_k^* and the fact that the payoff from pooling is strictly decreasing in x while the payoff from separating is constant in x. Thus,

$$V_{Lk} \leq u - \frac{F_L(x_k^*) E_I[c|x \leq x_k^*, L] + (1 - F_L(x_k^*))\bar{a}_k c_L}{F_L(x_k^*) + (1 - F_L(x_k^*))\bar{a}_k} - \frac{s_k}{F_L(x_k^*) + (1 - F_L(x_k^*))\bar{a}_k}. \tag{26}$$

On the other hand, L's equilibrium payoff is at least the payoff of pooling after $x \leq x_k^{**}$ and continuing the search after $x > x_k^{**}$

$$V_{Lk} \geq u - E_I[c|x \leq x_k^{**}, L] - \frac{s_k}{F_L(x_k^{**})}. \tag{27}$$

Using (26) to substitute out V_{Lk} in (5) and rearranging the resulting inequality yields the RHS of (23). Similarly, using (27) and (6) we get the LHS of (23).

The ex-ante probability that type L will end up transacting with a sampled seller is at least $F_L(x_k^*)$ and at most $F_L(x_k^*)+(1-F_L(x_k^*))\,\bar{a}_k$. Similarly, type H's ex-ante probability of transacting is at least $F_H(x_k^*)$ and at most 1. It follows that $1/\left[F_L(x_k^*)+(1-F_L(x_k^*))\,\bar{a}_k\right] \leq n_L \leq 1/F_L(x_k^*)$ and $1 \leq n_H \leq 1/F_H(x_k^*)$ implying (24).

Finally (25) follows from the definition of $\bar{a}$ in (4).■

(i) Suppose to the contrary that x_k^* is bounded away from $\underline{x}$. Then (24) implies that $\frac{n_{Lk}}{n_{Hk}}$ is bounded away from 0 and ∞. By definition, $x_k^{**} \geq x_k^*$. Hence, using (2) and (1) to express $E_i[c \mid x]$, we have for all $x \in (\underline{x}, x_k^{**})$,

$$E_I[c|x] \equiv \frac{c_H + \frac{g_L}{g_H}\frac{f_L(x)}{f_H(x)}\frac{n_{Lk}}{n_{Hk}}c_L}{1+\frac{g_L}{g_H}\frac{f_L(x)}{f_H(x)}\frac{n_{Lk}}{n_{Hk}}} < \frac{c_H + \frac{g_L}{g_H}\frac{f_L\left(x_k^{**}\right)}{f_H\left(x_k^{**}\right)}\frac{n_{Lk}}{n_{Hk}}c_L}{1+\frac{g_L}{g_H}\frac{f_L\left(x_k^{**}\right)}{f_H\left(x_k^{**}\right)}\frac{n_{Lk}}{n_{Hk}}} \equiv E_I\left[c|x_k^{**}\right].$$

So, if x_k^* is bounded away from $\underline{x}$, $(E_I[c|x_k^{**}] - E_I[c|x \leq x_k^{**}, L])$ and, hence, the LHS of (23) is bounded away from zero in contradiction to $s_k \to 0$.

(ii) If $\bar{a}_k > 0$, then, by definition, $\bar{a}_k(u-c_L)+(1-\bar{a}_k)V_{Hk} = u - c_H$, which together with $V_{Hk} \geq u - c_H - s_k$ implies $\frac{s_k}{c_H-c_L} \geq \frac{\bar{a}_k}{1-\bar{a}_k}$ and hence $\lim \bar{a}_k = 0$. Given this, $\lim x_k^{**} = \lim x_k^* = \underline{x}$ follows immediately from (5) and (6) and (i).

(iii) Suppose to the contrary that $x_k^* = \underline{x}$. Hence, $\bar{a}_k > 0$ and $V_{Lk} \leq u - c_L - \frac{s_k}{\bar{a}_k}$. Since, $\frac{s_k}{c_H-c_L} \geq \frac{\bar{a}_k}{1-\bar{a}_k}$ (see (ii) above) this implies $V_{Lk} \leq u - c_L - \frac{c_H-c_L}{1-\bar{a}_k}$. From definition of $\bar{a}_k$ and (25), $(1-\bar{a}_k) = \frac{c_H-c_L}{u-c_L-V_{Hk}}$, and so $u - c_L - \frac{c_H-c_L}{1-\bar{a}_k} = V_{Hk}$. Thus, $x_k^* = \underline{x}$ implies $V_{Lk} \leq V_{Hk}$ in contradiction to the characterization of payoffs in Claim 1.

(iv) Divide (23) through by $F_L(x_k^*)$ and take limits of both sides of the RHS inequality of (23)

$$\lim \frac{s_k}{F_L\left(x_k^*\right)} \leq \lim\left[E_I[c|x_k^*] - E_I[c|x \leq x_k^*, L]\right] + \lim \frac{\bar{a}_k}{(1-\bar{a}_k)F_L\left(x_k^*\right)}\left[E_I[c|x^*] - c_L\right]. \tag{28}$$

We will use $\eta_k(x)$ as a shorthand for the effective likelihood ratio for a seller who observes signal x, in the equilibrium $(P_k,\, A_k)$

$$\eta_k(x) \triangleq \frac{g_L}{g_H}\frac{f_L(x)}{f_H(x)}\frac{n_{Lk}}{n_{Hk}}$$

and use it together with (2) and (1) to express the interim expected cost

$$E_I[c|x] = \frac{c_H + \frac{g_L}{g_H}\frac{f_L(x)}{f_H(x)}\frac{n_{Lk}}{n_{Hk}}c_L}{1+\frac{g_L}{g_H}\frac{f_L(x)}{f_H(x)}\frac{n_{Lk}}{n_{Hk}}} \equiv \frac{c_H+\eta_k(x)c_L}{1+\eta_k(x)}$$

Consider now the first term on the RHS of (28). If $\frac{f_L(\underline{x})}{f_H(\underline{x})} < \infty$ or if $\lim_{x_k^* \to \underline{x}} \eta_k(x_k^*) = \infty$, then

$$\begin{aligned} \lim_{x_k^* \to \underline{x}} E_I\left[c|x_k^*\right] &= \lim_{x_k^* \to \underline{x}} \frac{c_H + \eta_k(x_k^*) c_L}{1 + \eta_k(x_k^*) \frac{n_{Lk}}{n_{Hk}}} = \lim_{x_k^* \to \underline{x}} \int_{\underline{x}}^{x_k^*} \frac{c_H + \eta_k(x) c_L}{1 + \eta_k(x)} \frac{f_L(x)}{F_L(x_k^*)} dx \qquad (29) \\ &= \lim_{x_k^* \to \underline{x}} E_I\left[c|x \le x_k^*, L\right], \end{aligned}$$

since, when $\frac{f_L(\underline{x})}{f_H(\underline{x})} < \infty$, $\lim_{x_k^* \to \underline{x}} \eta_k(x_k^*) = \frac{g_L}{g_H} \frac{f_L(\underline{x})}{f_H(\underline{x})} \lim \frac{n_{Lk}}{n_{Hk}}$ hence $\lim_{x_k^* \to \underline{x}} E_I\left[c|x_k^*\right] = \lim_{x_k^* \to \underline{x}} E_I\left[c|x \le x_k^*, L\right] = \frac{c_H + \frac{g_L}{g_H} \frac{f_L(\underline{x})}{f_H(\underline{x})} \lim \frac{n_{Lk}}{n_{Hk}} c_L}{1 + \frac{g_L}{g_H} \frac{f_L(\underline{x})}{f_H(\underline{x})} \lim \frac{n_{Lk}}{n_{Hk}}}$ and, when $\lim_{x_k^* \to \underline{x}} \eta_k(x_k^*) = \infty$ then $\lim_{x_k^* \to \underline{x}} E_I\left[c|x_k^*\right] = \lim_{x_k^* \to \underline{x}} E_I\left[c|x \le x_k^*, L\right] = c_L$.

Now, if (far enough) $\bar{a}_k = 0$, for all k, then obviously $\lim \bar{a}_k / F_L(x_k^*) = 0$. In this case, if $\frac{f_L(\underline{x})}{f_H(\underline{x})} < \infty$ or $\lim_{x_k^* \to \underline{x}} \eta_k(x_k^*) = \infty$, then from (29) and (28) $\lim \frac{s_k}{F_L(x_k^*)} = 0$. If instead $\lim_{x \to \underline{x}} \frac{f_L(x)}{f_H(x)} = \infty$ and $\lim_{x_k^* \to \underline{x}} \eta_k(x_k^*) < \infty$, then $\lim \frac{n_{Lk}}{n_{Hk}} = 0$. And since $\overline{S}_H = \lim n_{Hk} s_k = \lim(\frac{n_{Hk}}{n_{Lk}} n_{Lk} s_k) = \lim(\frac{n_{Hk}}{n_{Lk}} \frac{s_k}{F_L(x_k^*)})$, it follows that $\lim \frac{s_k}{F_L(x_k^*)} > 0$ would imply $\overline{S}_H = \infty$ hence $\lim V_{kH} = -\infty$ contradicting $V_{Hk} \ge u - c_H - s_k$. Therefore, $\lim \frac{s_k}{F_L(x_k^*)} = 0$.

Suppose next that there is a subsequence over which $\bar{a}_k > 0$. It then follows from (25) that $V_{Hk} < u - c_H$ hence $n_{Hk} = 1$ and $\frac{n_{Lk}}{n_{Hk}} = \frac{1}{F_L(x_k^{**}) + (1 - F_L(x_k^{**}))\bar{a}}$ which together with (ii) implies $\lim \frac{n_{Lk}}{n_{Hk}} = \infty$. This in turn implies $\lim \eta_k(x_k^*) \equiv \lim_{x_k^* \to \underline{x}} \left(\frac{g_L}{g_H} \frac{f_L(x_k^*)}{f_H(x_k^*)} \frac{n_{Lk}}{n_{Hk}} \right) = \infty$, which as shown above implies $\lim\left[E_I\left[c|x^*\right] = \lim E_I\left[c|x \le x^*, L\right]\right] = c_L$. Now, $\frac{s_k}{c_H - c_L} \ge \frac{\bar{a}_k}{1 - \bar{a}_k}$ (see (ii) above) implies $\frac{\bar{a}_k}{(1 - \bar{a}_k) F_L(x_k^*)} \left[E_I\left[c|x^*\right] - c_L\right] \le \frac{s_k}{F_L(x_k^*)} \frac{E_I[c|x^*] - c_L}{c_H - c_L}$. Substituting this and $\lim\left[E_I\left[c|x^*\right] = \lim E_I\left[c|x \le x^*, L\right]\right] = c_L$ into (28) we get upon rearranging $\lim \frac{s_k}{F_L(x_k^*)} \le 0$. Since $\frac{s_k}{c_H - c_L} \ge \bar{a}_k$ it follows that $\lim \frac{\bar{a}_k}{F_L(x_k^*)} = 0$ as well.

(v) Clearly $S_{Lk} \le \frac{s_k}{F_L(x_k^*)}$. Therefore, $\overline{S}_L \equiv \lim S_{Lk} = 0$ follows from (iv)■

Proposition 1 : Suppose that the limit λ exists and consider a sequence $s_k \to 0$ and a sequence (P_k, A_k) of corresponding equilibria. Then the limit prices exist and are

$$\bar{p}_L = \begin{cases} (1 - \frac{1}{\lambda}) c_L + \frac{1}{\lambda} c_H & \text{if} \quad \lambda \in [\frac{1}{g_H}, \infty], \\ g_L c_L + g_H c_H & \text{if} \quad \lambda \le \frac{1}{g_H}, \end{cases}$$

$$\bar{p}_H = \begin{cases} \frac{1}{\lambda} \frac{g_L}{g_H} c_L + (1 - \frac{1}{\lambda} \frac{g_L}{g_H}) c_H & \text{if} \quad \lambda \in [\frac{1}{g_H}, \infty], \\ g_L c_L + g_H c_H & \text{if} \quad \lambda \le \frac{1}{g_H}. \end{cases}$$

Proof of Proposition 1: (The main arguments of the proof are sketched in Section 4.3 and here is the complete version). Consider a subsequence of $k \to \infty$ such that the limit $\bar{p}_L = \lim_{k \to \infty} p_{Lk}$ exists. Throughout the proof all limits are taken with respect to $k \to \infty$ over this subsequence and we will therefore suppress the $k \to \infty$ and will not mention this repeatedly.

Consider first the case of $\lambda < \infty$ (the case of $\lambda = \infty$ will be taken up only towards the end of the proof). The following two claims are used throughout the proof.

Claim 6 *: If $\lambda < \infty$, then* $\lim_{x_k^* \to \underline{x}} \frac{f_L(x_k^*)}{f_H(x_k^*)} \frac{F_H(x_k^*)}{F_L(x_k^*)} = 1$.

Claim 7 *: If $\lambda < \infty$, then for large enough k*

$$V_{kH} \geq u - c_H.$$

The proofs of these claims follow this proof. We recommend to the reader to go first over the body of the proof below and only then turn to the proofs of these claims. The benefit of establishing $V_{kH} \geq u - c_H$ is that we can use (9) and (11) to express the equilibrium conditions in a relatively simple form, which make it easier to follow the proof. Indeed, from $V_{kH} \geq u - c_H$ and (9), $V_L = u - E_I[c|x \leq x_k^*, L] - \frac{s_k}{F_L(x_k^*)}$, which together with (5) yields

$$\frac{s_k}{F_L(x_k^*)} = \left(E_I[c|x_k^*] - E_I[c|x \leq x_k^*, L]\right). \tag{30}$$

i.e., L must be indifferent between trading with a seller with signal x_k^* and continuing search.

Step 1: Since by Claim **3-(iv)** $\lim \frac{s_k}{F_L(x_k^*)} = 0$ it follows that

$$\lim E_I[c|x \leq x_k^*, L] = \lim E_I[c|x_k^*] \qquad \blacksquare. \tag{31}$$

Let $\eta_k(x)$ denote the effective likelihood ratio for a seller who observes signal x, in the equilibrium $(P_k,\ A_k)$

$$\eta_k(x) \triangleq \frac{g_L}{g_H} \frac{f_L(x)}{f_H(x)} \frac{n_{Lk}}{n_{Hk}} \text{ and } \bar{\eta} \triangleq \lim \eta_k(x_k^*)$$

and use it together with (2) and (1) to write

$$E_I[c|x] \equiv \frac{c_H + \eta_k(x) c_L}{1 + \eta_k(x)} \quad \text{and} \quad \lim E_I[c|x_k^*] = \frac{c_H + \bar{\eta} c_L}{1 + \bar{\eta}} \tag{32}$$

Step 2:

$$\lim \frac{s_k}{F_H(x_k^*)} = (c_H - c_L) \frac{\bar{\eta}}{(1 + \bar{\eta})^2} \lambda. \tag{33}$$

To see this multiply both sides of (30) by $\frac{F_L\left(x_k^*\right)}{F_H\left(x_k^*\right)}$ to get

$$\frac{s_k}{F_H\left(x_k^*\right)}=\frac{F_L\left(x_k^*\right)}{F_H\left(x_k^*\right)}\left(E_I\left[c|x_k^*\right]-E_I\left[c|x\leq x_k^*,L\right]\right).$$

Substituting

$$E_I\left[c|x_k^*\right]-E_I\left[c|x\leq x_k^*,L\right]=\left(c_H-c_L\right)\frac{g_L}{g_H}\frac{n_{Lk}}{n_{Hk}}\int_{\underline{x}}^{x_k^*}\frac{\frac{f_L(x)}{f_H(x)}-\frac{f_L\left(x_k^*\right)}{f_H\left(x_k^*\right)}}{\left(1+\eta(x_k^*)\right)\left(1+\eta_k(x)\right)}\frac{f_L\left(x\right)}{F_L\left(x_k^*\right)}dx, \tag{34}$$

into the previous equation we get

$$\frac{s_k}{F_H\left(x_k^*\right)}=\left(c_H-c_L\right)\frac{F_L\left(x_k^*\right)}{F_H\left(x_k^*\right)}\frac{g_L}{g_H}\frac{n_{Lk}}{n_{Hk}}\int_{x}^{x_k^*}\frac{\left(\frac{f_L(x)}{f_H(x)}-\frac{f_L\left(x_k^*\right)}{f_H\left(x_k^*\right)}\right)}{\left(1+\eta_k(x_k^*)\right)\left(1+\eta_k(x)\right)}\frac{f_L\left(x\right)}{F_L\left(x_k^*\right)}dx. \tag{35}$$

It follows from Step 1 that $\lim E_I\left[c|x_k^*\right]=\lim E_I\left[c|\alpha x_k^*\right]$ for any $\alpha\in(0,1)$ and then from (32) and (16) that

$$\frac{c_H+\bar{\eta}c_L}{1+\bar{\eta}}=\lim E_I\left[c|x_k^*\right]=\lim E_I\left[c|\alpha x_k^*\right]=\frac{c_H+\lim\eta_k(\alpha x_k^*)c_L}{1+\lim\eta_k(\alpha x_k^*)}$$

Therefore, $\lim\eta_k(\alpha x_k^*)=\bar{\eta}$, for any $\alpha\in(0,1)$, and since the integral on the RHS of (35) is bounded (owing to $\lambda<\infty$), it follows that the integral converges to $\lambda/\left(1+\bar{\eta}\right)^2$. Using Lemma 6, $\lim\frac{F_L\left(x_k^*\right)}{F_H\left(x_k^*\right)}\frac{g_L}{g_H}\frac{n_{Lk}}{n_{Hk}}=\lim\frac{f_L\left(x_k^*\right)}{f_H\left(x_k^*\right)}\frac{g_L}{g_H}\frac{n_{Lk}}{n_{Hk}}=\bar{\eta}$. Using these observations to take the limits of (35) yields (33).■

Step 3: $\lim V_{Hk}$ exists and

$$\lim V_{Hk}=u-\frac{c_H+\bar{\eta}c_L}{1+\bar{\eta}}-\left(c_H-c_L\right)\frac{\bar{\eta}}{(1+\bar{\eta})^2}\lambda \tag{36}$$

This is established by recalling from (10) that $V_{Hk}=u-E_I\left[c|x\leq x_k^*,H\right]-\frac{s_k}{F_H\left(x_k^*\right)}$. Since $E_I\left[c|x_k^*\right]\geq E_I\left[c|x\leq x_k^*,H\right]\geq E_I\left[c|x\leq x_k^*,L\right]$, it follows from Step 2 and (32) that

$$\lim E_I\left[c|x\leq x_k^*,H\right]=\frac{c_L+\bar{\eta}c_H}{1+\bar{\eta}}$$

which together with (33) yields (36).■

Claim 7 implies $\lim V_{Hk}\geq u-c_H$. Using (36) this can be rewritten as

$$\frac{\bar{\eta}}{1+\bar{\eta}}\left(c_H-c_L\right)\left(1-\frac{1}{1+\bar{\eta}}\lambda\right)\geq 0. \tag{37}$$

which is equivalent to

$$(1+\bar{\eta}) \geq \lambda$$

This inequality can be used to express $\bar{\eta}$ in terms of the parameters.

Step 4: If $\lambda \leq \frac{1}{g_H}$, then $\bar{\eta} = \frac{g_L}{g_H}$.

This is so since Lemma 6 and (11) together imply that $\bar{\eta} \geq \frac{g_L}{g_H}$. It may not be that $\bar{\eta} > \frac{g_L}{g_H}$, since then $(1+\bar{\eta}) > \lambda$ and hence $V_{Hk} > u - c_H$ sufficiently far in the sequence. But this means that both L and H search until they find a seller with $x \leq x_k^*$, implying

$$\bar{\eta} = \lim \eta(x_k^*) \equiv \lim \frac{g_L}{g_H} \frac{f_L\left(x_k^*\right)}{f_H\left(x_k^*\right)} \frac{n_{kL}}{n_{kH}} = \lim \frac{g_L}{g_H} \frac{f_L\left(x_k^*\right)}{f_H\left(x_k^*\right)} \frac{F_H\left(x_k^*\right)}{F_L\left(x_k^*\right)} = \frac{g_L}{g_H}.$$

where the last equality follows from Lemma 6. Thus, assuming $\bar{\eta} > \frac{g_L}{g_H}$ implies $\bar{\eta} = \frac{g_L}{g_H}$ and contradiction. So, $\bar{\eta} = \frac{g_L}{g_H}$.■

Step 5: If $\frac{1}{g_H} < \lambda < \infty$, then $\bar{\eta} = \lambda - 1$.

This is established by eliminating the possibility of $(1+\bar{\eta}) > \lambda$, since the argument in the proof of Step 4 would imply $\bar{\eta} = \frac{g_L}{g_H}$ and hence $\frac{1+\bar{\eta}}{\lambda} < g_H + g_H \frac{g_L}{g_H} < 1$. Therefore, $(1+\bar{\eta}) = \lambda$ which yields the result.■

Step 6: For any $\lambda \leq \infty$, $\bar{p}_L$ is described in the statement of the proposition.

For $\lambda < \infty$, this follows from substituting $\bar{\eta}$ from Steps 4 and 5 into $\bar{p}_L = \frac{c_H + \bar{\eta} c_L}{1+\bar{\eta}}$.

Consider $\lambda = \infty$ (so far we focused on $\lambda < \infty$ and now we include $\lambda = \infty$). If $\bar{\eta} = \infty$, then $\bar{p}_L = \lim E_I\left[c | x_k^*\right] = \lim \frac{c_H + \eta_k(x_k^*) c_L}{1+\eta_k(x_k^*)} = c_L$ as required. Observe that it may not be that $\bar{\eta} < \infty$. If $\lim V_{kH} < u - c_H$ then far enough in the sequence $n_{kH} = 1$ implying $\bar{\eta} = \infty$. If $\lim V_{kH} \geq u - c_H$, then (11) implies $\frac{F_L\left(x_k^*\right)}{F_H\left(x_k^*\right)} \frac{n_{Lk}}{n_{Hk}} \geq 1$, for all k. Hence, it follows from (35) that $\lim \frac{s_k}{F_H\left(x_k^*\right)} = \infty$, which together with (10) contradicts $\lim V_{kH} \geq u - c_H$. ■

Step 7: For any $\lambda \leq \infty$, $\bar{p}_H$ is described in the statement of the proposition.

To see this let $\Pr(H|\bar{p}_L) \equiv \Pr(\text{type } H | \text{ trade at } \bar{p}_L)$, $\Pr(\bar{p}_L|H) \equiv \Pr(\text{trade at } \bar{p}_L | \text{ type } H)$. Observe that

$$\bar{p}_L = E(c|\text{trade at } \bar{p}_L) = \Pr(H|\bar{p}_L) c_H + \Pr(L|\bar{p}_L) c_L$$

Therefore,

$$\Pr(H|\bar{p}_L) = \begin{cases} \frac{1}{\lambda} & \text{if} \quad \lambda \in [\frac{1}{g_H}, \infty] \\ g_H & \text{if} \quad \lambda < \frac{1}{g_H} \end{cases}$$

Plugging this into Bayes formula

$$\Pr(H|\bar{p}_L) = \frac{\Pr(\bar{p}_L|H) g_H}{g_L[1 - \Pr(\bar{p}_L|H)] + \Pr(\bar{p}_L|H) g_H}$$

and solving yields

$$\Pr(\bar{p}_L|H) = \begin{cases} \frac{g_L}{g_H}\frac{1}{\lambda-1} & \text{if} \quad \lambda \in [\frac{1}{g_H}, \infty] \\ 1 & \text{if} \quad \lambda < \frac{1}{g_H} \end{cases}$$

The result now follows from substituting from above into

$$\bar{p}_H = \Pr(\bar{p}_L|H)\bar{p}_L + (1 - \Pr(\bar{p}_L|H))\, c_H$$

■

This completes the proof of the proposition. QED

Claim 6 For $\lambda < \infty$, $\lim_{x_k^* \to \underline{x}} \frac{f_L(x_k^*)}{f_H(x_k^*)} \frac{F_H(x_k^*)}{F_L(x_k^*)} = 1$.

Proof of Claim 6

The claim is immediate if $\lim \frac{f_L(x_k^*)}{f_H(x_k^*)}$ is bounded. So, suppose $\frac{f_L(x_k^*)}{f_H(x_k^*)} \to \infty$. Note that

$$\frac{f_L(x_k^*)}{f_H(x_k^*)} \frac{F_H(x_k^*)}{F_L(x_k^*)} = \frac{f_L(x_k^*)}{f_H(x_k^*)} \int_{\underline{x}}^{x_k^*} \frac{f_H(x)}{f_L(x)} \frac{f_L(x)}{F_L(x_k^*)} dx \le \frac{f_L(x_k^*)}{f_H(x_k^*)} \int_{\underline{x}}^{x_k^*} \frac{f_H(x_k^*)}{f_L(x_k^*)} \frac{f_L(x)}{F_L(x_k^*)} dx = 1.$$

where the inequality owes to $\frac{f_H(x)}{f_L(x)} < \frac{f_H(x_k^*)}{f_L(x_k^*)}$ by MLRP.

By $\lambda < \infty$, for every $\varepsilon > 0$, there exists $T(\varepsilon) > 0$ such that for all k large enough,

$$\frac{f_L(x)}{f_H(x)} - \frac{f_L(x_k^*)}{f_H(x_k^*)} \ge T(\varepsilon) \Rightarrow \frac{F_L(x)}{F_L(x_k^*)} \le \varepsilon. \tag{38}$$

Otherwise, for some $\varepsilon > 0$, there is a sequence of k's and $T_k \to \infty$ such that

$$\int_{\underline{x}}^{x_k^*} \left(\frac{f_L(x)}{f_H(x)} - \frac{f_L(x_k^*)}{f_H(x_k^*)} \right) \frac{f_L(x)}{F_L(x_k^*)} dx \ge T_k \varepsilon$$

contradicting $\lambda < \infty$. Hence, is equal to

$$\begin{aligned} \frac{f_L(x_k^*)}{f_H(x_k^*)} \frac{F_H(x_k^*)}{F_L(x_k^*)} &= \frac{f_L(x_k^*)}{f_H(x_k^*)} \int_{\underline{x}}^{x_k^*} \frac{f_H(x)}{f_L(x)} \frac{f_L(x)}{F_L(x_k^*)} dx \ge \frac{f_L(x_k^*)}{f_H(x_k^*)} \int_{\left\{x | \frac{f_L(x)}{f_H(x)} - \frac{f_L(x_k^*)}{f_H(x_k^*)} \le T(\varepsilon)\right\}} \frac{f_H(x)}{f_L(x)} \frac{f_L(x)}{F_L(x_k^*)} dx \\ &\ge \frac{f_L(x_k^*)}{f_H(x_k^*)} \frac{1}{\frac{f_L(x_k^*)}{f_H(x_k^*)} + T(\varepsilon)} (1 - \varepsilon) \end{aligned} \tag{39}$$

where the last inequality owes to $\frac{f_H(x)}{f_L(x)} \geq 1/\left(\frac{f_L(x_k^*)}{f_H(x_k^*)} + T(\varepsilon)\right)$ over the range $\left\{x|\frac{f_L(x)}{f_H(x)} - \frac{f_L(x_k^*)}{f_H(x_k^*)} \leq T(\varepsilon)\right\}$, and the previous inequality owes to $\Pr\left(\left\{x|\frac{f_L(x)}{f_H(x)} - \frac{f_L(x_k^*)}{f_H(x_k^*)} \leq T(\varepsilon)\right\}|L\right) \geq 1-\varepsilon$, which follows from (38). Now, for any $\varepsilon > 0$, (39) approaches $1-\varepsilon$ as $\frac{f_L(x_k^*)}{f_H(x_k^*)} \to \infty$. Therefore, $\lim_{x_k^* \to \underline{x}} \frac{f_L(x_k^*)}{f_H(x_k^*)} \frac{F_H(x_k^*)}{F_L(x_k^*)} \geq 1$. ■

Claim 7. Suppose $\lambda < \infty$. Then $V_{kH} \geq u - c_H$ for sufficiently large k.

Proof of Claim 7: Suppose to the contrary that there is a subsequence over which $V_{kH} < u - c_H$ and confine attention to this subsequence. Let us use the shorthand $\eta_k(x)$ for the effective likelihood ratio for a seller who observes signal x, in the equilibrium $(P_k,\, A_k)$

$$\eta_k(x) \triangleq \frac{g_L}{g_H} \frac{f_L(x)}{f_H(x)} \frac{n_{Lk}}{n_{Hk}}$$

and use it together with (2) and (1) to write

$$E_I[c|x] \equiv \frac{c_H + \eta_k(x) c_L}{1 + \eta_k(x)}$$

Observe that $V_{Hk} < u - c_H$ implies $n_{Hk} = 1$. Hence, $\frac{n_{Lk}}{n_{Hk}} = \frac{1}{F_L(x_k^{**}) + (1 - F_L(x_k^{**}))\bar{a}}$ which together with (3)-(ii) implies $\lim \frac{n_{Lk}}{n_{Hk}} = \infty$. This in turn implies $\lim \eta_k(x_k^*) \equiv \lim \left(\frac{g_L}{g_H} \frac{f_L(x_k^*)}{f_H(x_k^*)} \frac{n_{Lk}}{n_{Hk}}\right) = \infty$. Hence, for any $x \leq x_k^*$, $\lim E_I[c|x] \equiv \frac{c_H + \eta_k(x) c_L}{1 + \eta_k(x)} = c_L$ and hence $\lim E_I[c|x_k^*] = \lim E_I[c|x \leq x_k^*, L] = c_L$

We now use these observations to show that $\lim \frac{s}{F_H(x_k^*)} = 0$.

Dividing the RHS of (23) by $F_H(x^*)$,

$$\frac{s}{F_H(x^*)} \leq \frac{F_L(x^*)}{F_H(x^*)} [E_I[c|x^*] - E_I[c|x \leq x^*, L]] + \frac{F_L(x^*)}{F_H(x^*)} \frac{\bar{a}_k}{1 - \bar{a}_k} \frac{1}{F_L(x^*)} [E_I[c|x^*] - c_L]. \quad (40)$$

To evaluate the limit of the RHS, let us start with the second term. Substituting $E_I[c|x_k^*] = \frac{c_H + \eta_k(x_k^*) c_L}{1 + \eta_k(x_k^*)}$ into the 2nd term, it can be rewritten as

$$\text{2nd term on RHS} = \lim \frac{F_L(x^*)}{F_H(x^*)} \frac{\bar{a}_k}{1 - \bar{a}_k} \frac{1}{F_L(x^*)} \frac{c_H - c_L}{1 + \eta_k(x_k^*)}$$

Now,

$$\lim \frac{F_L(x^*)}{F_H(x^*)} \frac{1}{1+\eta_k(x_k^*)} \leq \lim \frac{1}{\frac{F_H(x^*)}{F_L(x^*)} \frac{g_L}{g_H} \frac{f_L\left(x_k^*\right)}{f_H\left(x_k^*\right)} \frac{n_{Lk}}{n_{Hk}}} = \frac{1}{\frac{g_L}{g_H} \lim \left(\frac{F_H(x^*)}{F_L(x^*)} \frac{f_L\left(x_k^*\right)}{f_H\left(x_k^*\right)} \right) \lim \frac{n_{Lk}}{n_{Hk}}} = 0$$

since $\lim \frac{n_{Lk}}{n_{Hk}} = \infty$ (argued above) and $\lim \frac{F_H(x^*)}{F_L(x^*)} \frac{f_L\left(x_k^*\right)}{f_H\left(x_k^*\right)} = 1$ (by Claim 6). Since $\bar{a}_k / F_L(x_k^*) \to 0$ as well, the 2nd term converges to zero.

For the limit of the first term on RHS(40), rewrite

$$\begin{aligned}
& \frac{F_L\left(x_k^*\right)}{F_H\left(x_k^*\right)} \left(E_I\left[c|x_k^*\right] - E_I\left[c|x \leq x_k^*, L\right]\right) \\
= \quad & (c_H - c_L) \frac{F_L\left(x_k^*\right)}{F_H\left(x_k^*\right)} \frac{g_L}{g_H} \frac{n_{Lk}}{n_{Hk}} \int_{\underline{x}}^{x_k^*} \frac{\left(\frac{f_L(x)}{f_H(x)} - \frac{f_L\left(x_k^*\right)}{f_H\left(x_k^*\right)}\right)}{\left(1 + \frac{g_L}{g_H}\frac{f_L\left(x_k^*\right)}{f_H\left(x_k^*\right)}\frac{n_{Lk}}{n_{Hk}}\right)\left(1 + \frac{g_L}{g_H}\frac{f_L(x)}{f_H(x)}\frac{n_{Lk}}{n_{Hk}}\right)} \frac{f_L(x)}{F_L\left(x_k^*\right)} dx \\
\leq \quad & \frac{F_L\left(x_k^*\right)}{F_H\left(x_k^*\right)} \frac{g_L}{g_H} \frac{n_{Lk}}{n_{Hk}} \frac{(c_H - c_L)}{\left(1+\eta_k(x_k^*)\right)^2} \int_{\underline{x}}^{x_k^*} \left(\frac{f_L(x)}{f_H(x)} - \frac{f_L\left(x_k^*\right)}{f_H\left(x_k^*\right)}\right) \frac{f_L(x)}{F_L\left(x_k^*\right)} dx
\end{aligned}$$

But the last expression converges to zero, since (i) $\eta_k(x_k^*) \to \infty$ implies that the terms preceding the integral vanish to zero and (ii) the integral is bounded because $\lambda < \infty$.

Since H can search for $x \leq x_k^*$, $V_{Hk} \geq u - E_I\left[c|x \leq x_k^*, H\right] - s_k/F_H\left(x_k^*\right)$. It was shown above that $\lim E_I\left[c|x \leq x_k^*, H\right] = c_L$ and by (3)-(iv) $\lim\left(s_k/F_H\left(x_k^*\right)\right) = 0$. Therefore, $\lim V_{Hk} \geq u - c_L$. This contradicts the hypothesis $V_{kH} < u - c_H$. Hence, for sufficiently large k, $V_{kH} \geq u - c_H$. ■.

Claim 4: If

$$\lim_{x \to \underline{x}} \frac{-\frac{d}{dx}\left(\frac{f_L(x)}{f_H(x)}\right)}{\frac{f_L(x)}{F_L(x)}}$$

exists, then it is equal to λ as defined by (13).

Proof of Claim 4: Suppose that the limit in the claim exists and is equal to γ. Define

$$t(x, x^*) = \frac{f_L(x)}{f_H(x)} - \frac{f_L(x^*)}{f_H(x^*)}$$

and let $x(t, x^*)$ be its inverse. Of course, $x(0, x^*) = x^*$. We have to show that

$$\lim_{x \to \underline{x}} \int_{\underline{x}}^{x^*} \left(\frac{f_L(x)}{f_H(x)} - \frac{f_L(x^*)}{f_H(x^*)}\right) \frac{f_L(x)}{F_L(x^*)} dx \equiv \lim_{x \to \underline{x}} \int_{\underline{x}}^{x^*} t(x, x^*) \frac{f_L(x)}{F_L(x^*)} dx = \gamma$$

Define the distribution function F_{x^*} with support on $[0,\infty]$ by

$$F_{x^*}(t) \equiv 1 - \frac{F_L(x(t,x^*))}{F_L(x^*)}.$$

Observe that

$$\int_{\underline{x}}^{x^*} t(\xi,x^*) \frac{f_L(\xi)}{F_L(x^*)} d\xi = \int_{t(\underline{x},x^*)}^{t(x^*,x^*)} t(x(\tau,x^*),x^*)(-1)\frac{dF_{x^*}(\tau)}{d\tau} d\tau = \int_0^\infty \tau dF_{x^*}(\tau) \tag{41}$$

The first equality owes to a standard change of integration variable $\xi = x(\tau,x^*)$, since $\frac{dF_{x^*}(\tau)}{d\tau} = -\frac{f_L(x(\tau,x^*))}{F_L(x^*)}\frac{dx(\tau,x^*)}{d\tau}$ and $x(t(x,x^*)) = x$. The second equality follows from $t(x(\tau,x^*),x^*) = \tau$, $t(x^*,x^*) = 0$ and $t(\underline{x},x^*) = \infty$.

Thus, to establish the claim we have to show that

$$\lim_{x^*\to\underline{x}} \int_0^\infty \tau dF_{x^*}(\tau) = \gamma, \tag{42}$$

which will be done by the following two steps.

Step 1. If $\gamma = 0$, then $\lim_{x^*\to\underline{x}} \frac{F_L(x(t,x^*))}{F_L(x(0,x^*))} = 0$ for all t. If $\gamma > 0$, then

$$\lim_{x^*\to\underline{x}} \frac{F_L(x(t,x^*))}{F_L(x(0,x^*))} = e^{-\frac{t}{\gamma}}. \tag{43}$$

To see this, observe that by the mean value theorem

$$\begin{aligned} \ln\frac{F_L(x(t,x^*))}{F_L(x(0,x^*))} &= \ln F_L(x(t,x^*)) - \ln F_L(x(0,x^*)) \\ &= t\frac{d}{d\tau}\ln F_L(x(\tau,x^*))\big|_{\tau=\widehat{\tau}\in(0,t)} = t\frac{\frac{f_L(x(\widehat{\tau},x^*))}{F_L(x(\widehat{\tau},x^*))}}{\frac{d}{dx}\left(\frac{f_L(x(\widehat{\tau},x^*))}{f_H(x(\widehat{\tau},x^*))}\right)} \end{aligned} \tag{44}$$

This implies immediately that, for $\gamma = 0$, $\lim_{x^*\to\underline{x}} \frac{F_L(x(t,x^*))}{F_L(x(0,x^*))} = 0$. Next observe that it follows from the statement of the claim that, for any $\varepsilon > 0$, there is some $X(\varepsilon) > 0$ such that for all $x^* \leq X(\varepsilon)$ and all z,

$$\frac{1}{\gamma} - \varepsilon \leq -\frac{\frac{f_L(x(z,x^*))}{F_L(x(z,x^*))}}{\frac{d}{dx}\left(\frac{f_L(x(z,x^*))}{f_H(x(z,x^*))}\right)} \leq \frac{1}{\gamma} + \varepsilon. \tag{45}$$

This is true because $x(z,x^*) \leq x(0,x^*) \equiv x^*$.

Observe that from (45) and (44) we have that for $x^* < X(\varepsilon)$

$$e^{-t\left(\frac{1}{\gamma}+\varepsilon\right)} \leq \frac{F_L(x(t,x^*))}{F_L(x(0,x^*))} \leq e^{-t\left(\frac{1}{\gamma}-\varepsilon\right)} \tag{46}$$

Since ε can be arbitrarily small, (43) follows.■

Step 2:

$$\lim_{x^* \to \underline{x}} \int_0^\infty t dF_{x^*}(t) = \gamma$$

where γ may be ∞ as well.

To see this observe that, for $\gamma \in (0, \infty)$, by the definition of F_{x^*}, (43), the Dominated Convergence Theorem and Helley's convergence theorem

$$\lim_{x^* \to \underline{x}} \int_0^T t dF_{x^*}(t) = \int_0^T \frac{t}{\gamma} e^{-\frac{t}{\gamma}} dt,$$

for any $T < \infty$. Moreover, since by (46) F_{x^*} is stochastically dominated by $1 - e^{-(\frac{1}{\gamma} - \varepsilon)t}$ for $x^* < X(\varepsilon)$

$$\int_T^\infty t dF_{x^*}(t) \leq \int_T^\infty t(\frac{1}{\gamma} - \varepsilon) e^{-(\frac{1}{\gamma} - \varepsilon)t} = (T + \frac{1}{\frac{1}{\gamma} - \varepsilon}) e^{-(\frac{1}{\gamma} - \varepsilon)T}.$$

Thus, the remainder of the integral on $[T, \infty]$ is uniformly bounded and, for $T \to \infty$, the bound vanishes to zero. Therefore, for $\gamma \in (0, \infty)$,

$$\lim_{x^* \to \underline{x}} \int_0^\infty t dF_{x^*}(t) = \int_0^\infty \frac{t}{\gamma} e^{-\frac{t}{\gamma}} dt = \gamma$$

For $\gamma = 0$, it follows from Step 1 that $F_{x^*}(t)$ converges to mass 1 at 0, which implies the result. For $\gamma = \infty$, it follows from Step 1 that $F_{x^*}(t) \to 0$ for all t, which means that $\lim_{x^* \to \underline{x}} \int_0^\infty t dF_{x^*}(t) = \infty$ as required.■

Step 2 completes the proof. QED

Proposition 2: (i) Consider a sequence $s_k \to 0$ and a sequence (P_k, A_k) of corresponding equilibria such that $\bar{p}_L = \lim p_{kL}$ exists. Then $\bar{p}_H = \lim p_{kH}$ exists and there exists a $\lambda \in [\underline{\lambda}, \overline{\lambda}]$ such that

$$\begin{aligned} \bar{p}_L &= \begin{cases} (1 - \frac{1}{\lambda}) c_L + \frac{1}{\lambda} c_H & \text{if} \quad \lambda \in (\frac{1}{g_H}, \infty], \\ g_L c_L + g_H c_H & \text{if} \quad \lambda \leq \frac{1}{g_H}, \end{cases} \\ \bar{p}_H &= \begin{cases} \frac{1}{\lambda} \frac{g_L}{g_H} c_L + (1 - \frac{1}{\lambda} \frac{g_L}{g_H}) c_H & \text{if} \quad \lambda \in (\frac{1}{g_H}, \infty], \\ g_L c_L + g_H c_H & \text{if} \quad \lambda \leq \frac{1}{g_H}. \end{cases} \end{aligned}$$

(ii) For any $\lambda \in [\underline{\lambda}, \overline{\lambda}]$, there exists a sequence $s_k \to 0$ and a sequence (P_k, A_k) of corresponding equilibria such that $\bar{p}_L = \lim p_{kL}$ and $\bar{p}_H = \lim p_{kH}$ exist and are of the above form.

Proof of Proposition 2:

(i) Consider a sequence $s_k \to 0$ and a sequence (P_k, A_k) of corresponding equilibria such that $\bar{p}_L = \lim p_{kL}$ exists. Consider a subsequence of these equilibria and corresponding cutoffs x_k^* such that there is some $\lambda \in [0, \infty]$ for which

$$\lambda = \lim \int_{\underline{x}}^{x_k^*} \left(\frac{f_L(x)}{f_H(x)} - \frac{f_L(x_k^*)}{f_H(x_k^*)} \right) \frac{f_L(x)}{F_L(x_k^*)} dx.$$

From the proof of Proposition 1, $x_k^* \to \underline{x}$. Hence, $\lambda \in \left[\underline{\lambda}, \overline{\lambda}\right]$, as desired. The claim now follows from repeating the steps in the proof of Proposition 1.

(ii). For any $\lambda \in \left[\underline{\lambda}, \overline{\lambda}\right]$, there exists some sequence $x_k^* \to \underline{x}$ such that

$$\lambda = \lim \int_{\underline{x}}^{x_k^*} \left(\frac{f_L(x)}{f_H(x)} - \frac{f_L(x_k^*)}{f_H(x_k^*)} \right) \frac{f_L(x)}{F_L(x_k^*)} dx.$$

This follows from the continuity of the above integral in x_k^* and the intermediate value theorem.

For each x_k^* and $\beta \in [0,1]$, define $s_k(\beta)$ as follows. Let $n_{Lk} \equiv \frac{1}{F_L(x_k^*)}$ and let

$$n_{Hk}(\beta) \equiv \frac{1}{F_H(x_k^*) + \beta\left(1 - F_H(x_k^*)\right)}.$$

Now, let $E_I[c|x, n_{Hk}(\beta), n_{Lk}]$ denote the expected interim cost for a seller conditional on being sample and observing signal x when the expected search durations of the two buyer types are $n_{Hk}(\beta)$ and n_{Lk} respectively.

$$E_I[c|x, n_{Hk}(\beta), n_{Lk}] = \frac{c_H + \frac{g_L}{g_H}\frac{f_L(x)}{f_H(x)}\frac{n_{Lk}}{n_{Hk}(\beta)} c_L}{1 + \frac{g_L}{g_H}\frac{f_L(x)}{f_H(x)}\frac{n_{Lk}}{n_{Hk}(\beta)}}$$

and let

$$E_I[c|x \le x_k^*, n_{Hk}(\beta), n_{Lk}, w] = \int_{\underline{x}}^{x_k^*} E_I[c|x, n_{Hk}(\beta), n_{Lk}] \frac{f_w(x)}{F_w(x_k^*)} dx$$

Let

$$s_k(\beta) \equiv F_L(x_k^*)\left[E_I[c|x \le x_k^*, n_{Hk}(\beta), n_{Lk}, L] - E_I[c|x_k^*, n_{Hk}(\beta), n_{Lk}]\right],$$

and

$$\Delta_k(\beta) \equiv (u - c_H) - \left[E_I[c|x \le x_k^*, n_{Hk}(\beta), n_{Lk}, H] - E_I[c|x_k^*, n_{Hk}(\beta), n_{Lk}]\right] - \frac{s_k(\beta)}{F_H(x_k^*)}.$$

The function $\Delta_k(\beta)$ is continuous.

Define β_k^* as a solution for $\Delta(\beta_k^*) = 0$ if such exists. Otherwise, if $\Delta_k(\beta) > 0$ for all $\beta \in [0,1]$, let $\beta_k^* = 1$; if $\Delta_k(\beta) < 0$ for all $\beta \in [0,1]$, let $\beta_k^* = 1$. Set $s_k = s_k(\beta_k^*)$. Then the following strategies constitute an equilibrium. Type L offers $E_I[c|x, n_{Hk}(\beta), n_{Lk}]$ after $x \le x_k^*$ and c_L after $x > x_k^*$. trades at $E_I[c|x, n_{Hk}(\beta), n_{Lk}]$. Type H trades after $x \le x_k^*$ at $E_I[c|x, n_{Hk}(\beta), n_{Lk}]$ and offers c_H to sellers with $x > x_k^*$, which is accepted with probability $A_k(c_H, x) = \beta^*$. By construction of s_k, H is indifferent between searching and trading at c_H; L is exactly indifferent between trading at x_k^* and continued search, as required. Therefore, these strategies constitute an equilibrium.

Thus, there exists a sequence of search costs $\{s_k\}$ such that x_k^* is the sequence of equilibrium cutoffs. But now Part I implies that the limit outcome must have the desired form. **QED.**

Claim 5: (i) If an equilibrium (P, A) induces an undefeated sequential equilibrium in the bargaining games after each x, then (P, A) is "undefeated". (ii) Conversely, if (P, A) is an "undefeated" equilibrium, then in each of the bargaining games there is an undefeated sequential equilibrium that yields the same outcome.

Proof of Claim 5: (i) Consider an equilibrium (P, A). Given the associated values $V_w(P, A)$ and beliefs $\beta_I(x, P, A)$, the bargaining game after any signal realization x is well defined. Let $\sigma_x = \sigma_x(P, A)$ be the sequential equilibrium induced by (P, A) in the bargaining game after x, and suppose that $\sigma_x(P, A)$ is undefeated for all x.

Step 1. If $\sigma_x(P, A)$ is pooling with trade, i.e., $P_L(x) = P_H(x) = p$ and $A(p, x) > 0$, then $p = E_I[c|x, P, A]$ and $A(p, x) = 1$.

Proof of Step 1: It may not be that $p < E_I[c|x, P, A]$ and $A(p, x) > 0$, or else the seller will incur a loss. If $p > E_I[c|x, P, A]$, choose $p' \in (E_I[c|x, P, A], p)$ and construct a sequential equilibrium σ' in the bargaining game following x in which both buyers pool at p' and the seller accepts (it can be supported by beliefs that assign probability 1 to H after all $p \neq p'$). Clearly, both L and H prefer σ' to $\sigma_x(P, A)$, and it is easy to check that σ' defeats $\sigma_x(P, A)$.

Step 2. $V_L(P, A) \ge V_H(P, A)$.

Proof of Step 2: Suppose to the contrary that $V_H(P, A) > V_L(P, A)$. It follows that $V_H(P, A) > u - c_H - s$. Therefore there is no separating equilibrium with trade after any x since in such a case the price H pays would be c_H. Thus, the only trade takes place at pooling equilibria with trade after some x's. From Step 1 such trade takes place at $E_I[c|x, P, A]$ with probability 1. The set of those x's is an interval that contains the bottom of the support $\underline{x}$, since otherwise there would be x_1 and x_2 such that $x_1 < x_2$, after x_2 trade takes place at $E_I[c|x_2, P, A]$ while after x_1 no trade takes place. This implies that $V_w(P, A) \le u - E_I[c|x_2, P, A] < u - E_I[c|x_1, P, A]$. But then the no trade equilibrium after x_1 is defeated by a pooling equilibrium at which trade takes place at price $p \in (E_I[c|x_1, P, A], E_I[c|x_2, P, A])$. But since the distribution of x conditional on H stochastically dominates that of x conditional on L, it follows immediately that $V_H(P, A) < V_L(P, A)$ —

contradiction.

Step 3. (P, A) is "undefeated", i.e., $A(P_L(x), x)(u - P_L(x)) + [1 - A(P_L(x), x)]V_L \geq u - E_I[c|x, P, A]$.

Proof of Step 3: Suppose to the contrary that there is an x such that $A(P_L(x), x)(u - P_L(x)) + [1 - A(P_L(x), x)]V_L < u - E_I[c|x, P, A]$. Take a $p' \notin \{P_L(x), P_H(x)\}$ such that $p' > E_I[c|x, P, A]$ and $A(P_L(x), x)(u - P_L(x)) + [1 - A(P_L(x), x)]V_L < u - p'$ (which obviously exists.) Construct a sequential equilibrium σ' in the bargaining game following x in which both buyers pool at p' and the seller accepts (it can be supported by beliefs that assign probability 1 to H after all $p \neq p'$). By construction L prefers σ' to the $\sigma_x(P, A)$. Step 1 implies that $\sigma_x(P, A)$ is not pooling with trade. If $\sigma_x(P, A)$ is with no trade, then H prefers σ' as well since $u - p' > V_L(P, A) \geq V_H(P, A)$, where the last inequality is by Step 2. If $\sigma_x(P, A)$ is separating with trade, then $P_H(x) = c_H$ and

$$
\begin{aligned}
A(c_H, x)(u - c_H) + [1 - A(c_H, x)]V_H &\leq A(c_H, x)(u - c_H) + [1 - A(c_H, x)]V_L \\
&\leq A(P_L(x), x)(u - P_L(x)) + [1 - A(P_L(x), x)]V_L < u - p',
\end{aligned}
$$

where the first inequality follows from $V_L(P, A) \geq V_H(P, A)$ and the second from L's IC condition. The conclusion is that in this case too H prefers σ'. Given that both types prefer σ', it is easy to check that σ' defeats $\sigma_x(P, A)$.

(ii) Let (P, A) be an "undefeated" equilibrium and $\sigma_x(P, A)$ be the sequential equilibrium induced by (P, A) in the bargaining game after x.

If $\sigma_x(P, A)$ is pooling, then $x \leq x^{**}$. Consider a sequential equilibrium $\widetilde{\sigma}$ in the bargaining game after x that coincides with $\sigma_x(P, A)$ everywhere except perhaps that the belief after a price offer c_L is that the seller is type L. It is to verify that $\widetilde{\sigma}$ is indeed a sequential equilibrium with the same outcome as $\sigma_x(P, A)$. Observe that $\widetilde{\sigma}$ cannot be defeated by any other pooling equilibrium since the price is already $E_I[c|x, P, A]$. It also may not be defeated by a separating equilibrium with trade, since in such an equilibrium L offers c_L after which the seller believes in type L with probability 1 which coincides with the belief in $\widetilde{\sigma}$. Therefore, by definition, the separating equilibrium does not defeat $\widetilde{\sigma}$.

If $\sigma_x(P, A)$ is separating then $x \geq x^*$. We may assume that $P_w(x) = c_w$ (since otherwise there is a sequential equilibrium with the same outcome in which type w offers c_w). Consider a sequential equilibrium $\widetilde{\sigma}$ in the bargaining game after x that coincides with $\sigma_x(P, A)$ everywhere except perhaps that the belief after the off equilibrium price $E_I[c|x, P, A]$ is the interim belief $\beta_I(x, P, A)$. Obviously, $\widetilde{\sigma}$ has the same outcome as $\sigma_x(P, A)$. Observe that $\widetilde{\sigma}$ cannot be defeated by a pooling equilibrium with pooling price $p > E_I[c|x, P, A]$, since owing to $\widetilde{\sigma}$'s "undefeated" status L's payoff is at least as high as $u - E_I[c|x, P, A] > u - p$. It cannot be defeated by a pooling equilibrium with pooling price $p = E_I[c|x, P, A]$, since by construction the belief after $E_I[c|x, P, A]$

coincides with the interim belief and hence condition (iii) in the definition of a defeating equilibrium is not satisfied. Observe that $\widetilde{\sigma}$ cannot be defeated by a separating equilibrium with trade, since in any separating equilibrium with trade the prices are c_w, which are already used in $\widetilde{\sigma}$. It cannot be defeated by an equilibrium with no trade, since such an equilibrium cannot strictly increase the payoff of either of the types. ■

10.3 λ and the distribution of the likelihood ratio

Further insights into the meaning of λ are obtained by relating it to the properties of the distribution of the likelihood ratios. Specifically, the inverse of the expression (14) is equal to the hazard rate of the distribution of the likelihood ratios in the low state. To see this, let $\tilde{F}_L(l)$ be the cumulative distribution of the likelihood ratios conditional on $w = L$, that is, $\tilde{F}_L(l) = Prob[\frac{f_L(x)}{f_H(x)} \leq l | w = L]$. Then

$$\frac{\frac{d}{dl}\tilde{F}_L(l)}{1 - \tilde{F}(l)} = -\frac{f_L(x)}{F_L(x)} \Big/ \frac{d}{dx}\left(\frac{f_L(x)}{f_H(x)}\right),$$

at $l = \frac{f_L(x)}{F_L(x)}$.

The distribution of the likelihood ratios captures all the properties of the signal that are relevant for the model. Any two signal distributions that induce the same distribution of likelihood ratios will have the same set of equilibrium outcomes. Claim 4 and Proposition 1 together imply that the limit outcome can be uniquely characterized by the limiting hazard rate of the distribution of likelihood ratios, if the limit exists. The smaller the hazard rate of the distribution, the more weight is put on large realizations of the likelihood ratios. In particular, if the limiting hazard rate exists, information is aggregated if and only if the hazard rate vanishes to zero, that is, if and only if the distribution of the likelihood ratio is heavy-tailed (the right tail of the distribution is heavier than those of any exponential distribution).

We illustrate this further using two parametric examples in Section 4.4. In example 4.4

$$F_L(x) = \begin{cases} e^{x+r} & \text{if} \quad x \leq -r \\ 1 & \text{if} \quad x > -r \end{cases} \qquad F_H(x) = \begin{cases} \mu \int_{-\infty}^{-x} (-t)^{-\alpha} e^{t+r} dt & \text{if} \quad x \leq -r \\ 1 & \text{if} \quad x > -r \end{cases}$$

and the likelihood ratio is given by $\frac{f_L(x)}{f_H(x)} = \frac{1}{\mu}(-x)^{\alpha}$. The distribution of the likelihood ratio in the low state is

$$Prob\left[\frac{f_L(x)}{f_H(x)} \leq l | w = L\right] = 1 - e^{-(\mu l)^{\frac{1}{\alpha}} + r},$$

that is, the distribution of the likelihood ratio follows a Weibull distribution with scale parameter $\frac{1}{\mu}$ and (inverse) shape parameter α. The parameter α determines the weight of the tail: If α is larger than one, the distribution is heavy-tailed and the hazard rate vanishes to zero. If α is equal to one, the likelihood ratio is exponentially distributed with a constant hazard rate μ; if α is less than one, the distribution is light-tailed and the hazard rate diverges to infinity.

In example 4.4 where

$$F_L(x) = \begin{cases} 1 & \text{for} \quad x \geq r(\lambda), \\ e^{-\frac{1}{\lambda}\frac{g_H}{g_L}\left(\frac{1}{x}-\frac{1}{r}\right)} & \text{for} \quad 0 \leq x \leq r(\lambda) \\ 0 & \text{for} \quad x = 0, \end{cases}$$

the λ parameter is exactly equal to the expression in *(14)* and hence is the λ defined by (13). Conditional on $w = L$, the likelihood ratios are exponentially distributed with hazard rate λ^{-1}. The hazard rate determines the limit outcome uniquely.

11 References

Banks, Jeffrey and Joel Sobel (1987): "Equilibrium Selection in Signaling Games," *Econometrica*, 647-61.

Blouin, Max and Roberto Serrano (2001), "A Decentralized Market with Common Values Uncertainty: Non-Steady States," *Review of Economic Studies*, 323-346.

Cho, In-Koo and David Kreps (1987) "Signaling games and stable equilibria," *Quarterly Journal of Economics,* 179-221

Diamond, Peter (1971), "A model of price adjustment," *Journal of Economic Theory*, 156-168.

Duffie, Darrell and Gustavo Manso (2007): "Information Percolation in Large Markets," *American Economic Review* 97 (2), 203-209.

Duffie, Darrell, Semyon Malamud, and Gustavo Manso (2009): "Information Percolation With Equilibrium Search Dynamics," *Econometrica*, 1513–1574.

Duggan, John and Cesar Martinelli (2001), "A Bayesian Model of Voting in Juries," *Games and Economic Behavior*, 259-294.

Feddersen, Timothy and Wolfgang Pesendorfer (1997): "Voting Behavior and Information Aggregation in Elections with Private Information," *Econometrica*, 1029-1058.

Gale, Douglas (1987), "Limit Theorems for Markets with Sequential Bargaining," *Journal of Economic Theory*, 20-54.

Guerrieri, Veronica, Robert Shimer, and Randall Wright (2010): "Adverse Selection in Competitive Search Equilibrium," *Econometrica*, 1823-1862.

Hörner, Johannes and Nicolas Vieille, (2009): "Public vs. Private Offers in the Market for Lemons," *Econometrica*, 29–69.

Inderst, Roman (2005), “Matching markets with Adverse Selection,” *Journal of Economic Theory*, 145–166.

Lauermann, Stephan (2011), “Dynamic Matching and Bargaining Games: A General Approach,” Working paper University of Michigan.

Lauermann, Stephan and Asher Wolinsky (2011), “Common Values Procurement Auctions with Endogenous Biddership” in preparation.

Mailath, George, Okuno-Fujiwara, Masahiro and Andrew Postlewaite (1993), “On Belief Based Refinements in Signaling Games,” *Journal of Economic Theory*, 241-276.

Milgrom, Paul (1979), “A Convergence Theorem for Competitive Bidding with Differential Information,” *Econometrica*, 670-688.

Moreno, Diego and John Wooders (2010): “Decentralized Trade Mitigates the Lemons Problem,” *International Economic Review,* 383–399.

Myerson, Roger (1983), “Mechanism Design by an Informed Principal,” *Econometrica,* 1767–1797.

Pesendorfer, Wolfgang and Jeroen Swinkels (1997): “The Loser’s Curse and Information Aggregation in Common Value Auctions,” *Econometrica*, 1247–1281.

Smith, Lones and Peter Sorenson (2000), “Pathological Outcomes of Observational Learning,” *Econometrica*, 371-398.

Wilson, Robert (1977), “A Bidding Model of Perfect Competition,” *Review of Economic Studies,* 511-518.

Wolinsky, Asher (1990), “Information Revelation in a Market with Pairwise Meetings,” *Econometrica*, 1-23.