Second Fundamental Form of a Map (*).

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Summary. – *This paper is devoted to the study of the 2, fundamental form of a map, which generalizes this notion, well known for isometric immersions. We generalize results by Vilms, Yano, and Ishihara, and study in detail projective and umbilical maps.*

The notion of 2nd fundamental form of a mapping between manifolds endowed with connections, first constructed by J. EELLS ([Ee])—for the study of harmonic mappings—, generalizes the 2nd fundamental form of a submanifold isometrically immersed in a Riemannian manifold, and has been used by J. VILMS [Vi] to study totally geodesic mappings and Riemannian submersions. This author has proved the following theorems:

THEOREM A. – *Let $f\colon M \to M'$ be a totally geodesic mapping between Riemannian manifolds. Then*:

1) *f is the product of a totally geodesic Riemannian submersion, followed by a totally geodesic immersion,*

2) $\operatorname{Ker} f_*$ *has totally geodesic leaves.*

THEOREM B. – *Let $f\colon M \to M'$ be a Riemannian submersion with 2nd fundamental form σ. Then*:

1) *If X and Y are in* $\operatorname{Ker} f_*^\perp$, $\sigma(X, Y) = 0$,

2) $\sigma|_{\operatorname{Ker} f_* \times \operatorname{Ker} f_*} = 0$ *iff* $\operatorname{Ker} f_*$ *has totally geodesic leaves,*

3) $\sigma|_{\operatorname{Ker} f_* \times \operatorname{Ker} f_*^\perp} = 0$ *iff* $\operatorname{Ker} f_*^\perp$ *is integrable.*

ZVI HAR'EL [Ha] has used a similar method in order to study projective mappings.

In a slightly different approach,—computation in local coordinates—YANO and ISHIHARA [Ya & Is] define relatively affine mappings, the *2nd* fundamental form of which is orthogonal to $f(M)$, and prove:

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THEOREM C. – *Let $f: M \to M'$ be a relatively affine mapping between Riemannian manifolds.*

1) *If M is connected, then f is of constant rank,*

2) $\operatorname{Ker} f_*$ *is a parallel distribution.*

A Third and fruitful use of the 2nd fundamental form, in the light of some of its properties (nullity, umbilicity) is frequently made for the study of isometric immersions between Riemannian manifolds (e.g. [Ch]). Therefore we have studied mappings of constant rank by means of their 2nd fundamental form, from a more general viewpoint than those of the above mentionned authors.

In section I we introduce the various notions we shall need, about which more details can be found in [Ee] and [Do], and remark that:

PROPOSITION I.5.3. – *Let $f: M \to M'$ be a mapping of constant rank between manifolds endowed with symmetric connections, and let σ be its 2nd fundamental form*

$$\operatorname{Ker} f_* \quad \textit{is parallel iff it is included in} \quad \operatorname{Ker} \sigma .$$

PROPOSITION I.5.1. – $\operatorname{Ker} f_*$ *is totally geodesic iff σ is null on* $\operatorname{Ker} f_* \times \operatorname{Ker} f_*$ *and*

PROPOSITION I.5.4. – *Assume M is a Riemannian manifold.* $\operatorname{Ker} f_*^{\perp}$ *is integrable and totally geodesic iff* $\sigma|_{\operatorname{Ker} f_* \times \operatorname{Ker} f_*^{\perp}} = 0$.

In section II, we factorize a map between Riemannian manifolds into the product of a diffeomorphism followed by a Riemannian submersion and by an isometric immersion, which allows us to give the following results:

THEOREM II.3.1. – [Generalization of theorem C, 2)].

Let $f: M \to M'$ be a map of constant rank between Riemannian manifolds, and τ be the orthogonal projection of its 2nd fundamental form σ onto the tangent space of $f(M)$.

Then $\operatorname{Ker} f_*$ *is parallel iff it is included in* $\operatorname{Ker} \tau$.

THEOREM II.3.4. – *Which supplements the results of theorem C.*

Let $f: (M, g) \to (M', g')$ be a relatively affine mapping. Assume M is connected, simply connected and complete. Then

1) *M is isometric to a direct product $M_1 \times M_2$, where $TM_1 = \operatorname{Ker} f_*$, and M_2 is locally diffeomorphic to $f(M)$,*

2) *if M_2 admits the de Rham de composition, $M_2 = M_2^1 \times \ldots \times M_2^k$, then, for a fixed i, the distribution $f_* TM_2^i$ defines a foliation of $f(M)$, every leaf of which is irreducible and homothetic to M_2^i. Moreover the ratio of this homothecy is independent of the leaf so that all leaves are isometric.*

COROLLARY II.3.2. – *Assume $f: (M, g) \to (M', g')$ is a map of constant rank between Riemannian manifolds. $f(M)$ is a totally geodesic submanifold of M', iff the 2nd fundamental form of f is tangent to $f(M)$.*

THEOREM II.3.3. – *Let (M, g) be a Riemannian manifold and $f: M \to M'$ be a $\mathcal{C}^\infty$ mapping of constant rank. Then there exists a metric g_1 on M w.r.t. to which*

1) $\operatorname{Ker} f_*^\perp$ *is a totally geodesic plane field,*

2) *the integral foliation of* $\operatorname{Ker} f_*$ *is Riemannian.*

In section III, projective maps are investigated into: indeed we found that there was an underlying confusion in the proof of ZVI HAR'EL [Ha]. We must distinguish between projective—preserving piecewise geodesics—and strongly projective maps —which map any geodesic either into a geodesic, or a point—. As for geodesic preserving maps, they are necessarily immersions. (For the terminology, we refer to definitions III.1.1 and III.1.2.)

Strongly projective maps are the only ones which satisfy the following theorem, generalizing the characteristic property of projective diffeomorphisms:

THEOREM III.2.2. – *Let $f: (M, \nabla) \to (M', \nabla')$ be a mapping of constant rank between manifolds endowed with torsionless linear connections. We denote by σ its 2nd fundamental form. Then f is strongly projective iff it satisfies the following property:*

(*) *There exists a 1-form ω on M such that*

$$\forall X, Y \in TM\,, \quad \sigma(X, Y) = \omega(X) f_* Y + \omega(Y) f_* X\,.$$

A counterexample shows that this theorem cannot be generalized for just any projective map.

Moreover for a strongly projective mapping $f: M \to M'$, we have:

PROPOSITION III.2.1. – 1) *The foliation defined by* $\operatorname{Ker} f_*$ *is totally geodesic,*

2) *$f(M)$ is a totally geodesic submanifold of M'.*

If besides f is a strongly projective map between Riemannian manifolds, then we have:

THEOREM III.3. – 1) $\operatorname{Ker} f_*^\perp$ *is integrable and defines a totally umbilical foliation,*

2) *there exists a Riemannian metric g_1 on M for which f is totally geodesic.*

We give examples of projective maps which are not strongly projective, of strongly projective maps which do not satisfy $\operatorname{Ker} f_* \subset \operatorname{Ker} \sigma$—which disagrees with

ZVI HAR'EL'S assertions [Ha]—and show that strongly projective maps between euclidean spaces are necessarily affine.

In section IV we offer 4 definitions for the umbilicity of a mapping, which generalize the notion of umbilicity for isometric immersions. We display some examples, then we prove the results given in the following table:

M = connected, simply connected complete manifold M' = space of constant curvature $f: M \to M'$ mapping of constant rank.			
Hypothesis	Conclusions about		
	M	f	$f(M)$
f weakly g-umbilical of rank $\geqslant 2$		immersion	convex hyper-surface of a t.g. submf. of M'
M irreducible f strongly g-umbilical	isometric to a sphere	homothecy	sphere
f weakly g'-umbilical	$M = M_1 \times M_2$ M_2 diffeomorphic to a sphere	with parallel kernel	sphere
f strongly g-umbilical	$M = M_1 \times M_2$ M_2 isometric to a sphere	homothecy	sphere

g denotes the metric of M, and g' that of M'.

At last, starting from CHEN'S [Ch 1] definition of the extrinsic sphere, we define spheric mappings—the image of which is an extrinsic sphere in the special case of an isometric immersion—and we prove:

THEOREM IV.4.2.1. – *Let $f: (M, g) \to (M'^{2n'}, g')$ be a spheric map, into a Kähler manifold of real dimension $2n'$. Assume M is connected, simply connected and complete, and f is analytic of rank $2n' - 2$. Then one of the irreducible components of (M, g) is isometric to an even dimensional sphere.*

Which we can compare to Chen's following result:

THEOREM D. – *Let M^{2n} be a complete exinsic sphere in any Kähler manifold $\tilde{M}^{2m}$. If there exists $2m - 2n$ mutually orthogonal parallel unit vector fields along M^{2n}, then M^{2n} is isometric to sphere S^{2n}, the radius of which is the inverse of the length of the mean curvature vector.*

In section V, also devoted to maps between Riemannian manifolds, we display integral formulas relating the norms of the 2nd fundamental forms of f, $f(M)$, and of the leaves of $\operatorname{Ker} f_*$, in the case when there exists a function ϱ such that for every X orthogonal to $\operatorname{Ker} f_*$ we have:

$$\|f_* X\| = \varrho \|X\| .$$

In particular we obtain the:

COROLLARY V.3. – *Assume f is a mapping of constant rank from a compact Riemannian manifold (M, g) into a Riemannian manifold (M', g'), which induces a Riemannian submersion from M unto $(f(M), g')$. If the fibre F of f is compact, with the notations of* I.4, *we have*:

$$\int_M \|\sigma\|^2 \geqslant \int_M \|\sigma_0\|^2 + (\operatorname{vol} F) \int_M \|\sigma'\|^2 .$$

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We shall omit any proofs that are simple computations, or that can be found in the litterature.

1. – Second fundamental form of a map.

In this study, manifolds, mappings, vector fields, sections, and so on, will always be supposed of class C^∞.

f will be a mapping of constant rank, from a manifold M into a manifold M', the respective dimensions of which we denote by n and n'. $f(M)$ is an (immersed) submanifold of M'. We denote by f_* the differential of f.

In the case when M (resp. M') is Riemannian, its metric is denoted by g (resp. g') and connection ∇ (resp. ∇') will be its Levi-Civita's connection. The points of M are denoted by $m \ldots$ (resp. $m' \ldots$).

I.1. *Fiber bundles.*

We denote by: *TM the tangent bundle of M*, with fiber $T_m M$ over m.

$f^{-1}(TM')$ *the f-induced bundle*, with base-space M and fiber $T_{f(m)} M'$ over m.

—When (M', g') is Riemannian, g' induces a metric on $f^{-1}(TM')$, also denoted by g'.—

$f_* TM$ *the image bundle*, subbundle of $f^{-1}(TM')$ with fiber $f_* T_m M$ over m.

$\operatorname{Ker} f_*$ *the vertical distribution*, integrable subbundle of TM with fiber $(\operatorname{Ker} f_*)_m$ over m.

The maximal integral submanifolds of $\operatorname{Ker} f_*$ are called the leaves of the kernel. In the case when (M, g) is Riemannian, we denote by:

$\operatorname{Ker} f_*^{\perp}$ *the horizontal distribution*, subbundle of TM with fiber $(\operatorname{Ker} f_*)_m^{\perp}$ over m.

In the case when (M', g') is Riemannian, we denote by:

$f_* TM^{\perp}$ *the subbundle of* $f^{-1}(TM')$, *with fiber* $(f_* T_m M)^{\perp}$—orthogonal complement of $f_* T_m M$ for g'—over m.

1.2. *Fields along f.*

Sections of $f^{-1}(TM')$ are called (*vector*) *fields along f*. In particular every field X on M induces a vector field $f_* X$ along f, s.t. $(f_* X)_m = (f_*)_m X_m$.

Every field X' on M' induces a vector field $f^* X'$ along f, s.t. $(f^* X')_m = X'_{f(m)}$. For clearness, we shall sometimes write X' instead of $f^* X'$.

1.3. *Linear connections.*

Assume M and M' are endowed with linear connections ∇ and ∇'. We have:

DEFINITION AND PROPOSITION I.3.1. – *There exists one unique linear connection* $\overline{\nabla}'$ *on* $f^{-1}(TM')$ *such that*:

(1) *for every* $m \in M$, *every* $X \in T_m M$, *and every field* Y' *on* M':

$$\overline{\nabla}'_X \xi' = \nabla'_{f_* X} Y'|_{f(M)}$$

where we have put $\xi' = f^* Y'$ *and where* $|$ *denotes the restriction.* $\overline{\nabla}'$ *is called the f-induced connection on* $f^{-1}(TM')$.

PROOF. – Let $X \in T_m M$ and η' be a field along f.

In a neighborhood U' of $f(m)$ we can find n' fields (e'_α) which form a basis of $T_{m'} M'$ at every point $m' \in U'$.

Put $U = \eta'^{-1}(TU')$ and $e_\alpha = f^* e'_\alpha$.

We can write, on U:

$$\eta' = \varphi^\alpha e_\alpha \quad \text{where } \varphi^\alpha \text{ are functions on } U\,.$$

Then we must have:

$$\overline{\nabla}'_X \eta' = (X\varphi^\alpha)\, e_\alpha + \varphi^\alpha \nabla'_{f_* X} e'_\alpha\,.$$

Moreover if (ε'_β) is another moving frame on U'_1 we can write $e'_\alpha = P^\beta_\alpha \varepsilon'_\beta$, P^β_α being functions on $U' \cap U'_1$.

Then we have:

$$\eta' = \psi^\beta \varepsilon_\beta \quad \text{where} \quad \psi^\beta = (f^* P^\beta_\alpha)\varphi^\alpha \quad \text{and} \quad \varepsilon_\beta = f^* \varepsilon'_\beta$$

so that:

$$(X\varphi^\alpha)\, e_\alpha + \varphi^\alpha f^* \nabla'_{f_* X} e'_\alpha = (X\varphi^\alpha)(f^* P^\beta_\alpha)\, \varepsilon_\beta + \varphi^\alpha (f_* X)(P^\beta_\alpha)\, \varepsilon_\beta + \varphi^\alpha (f^* P^\beta_\alpha) \nabla'_{f_* X} \varepsilon'_\beta =$$
$$= (X\psi^\beta)\, \varepsilon_\beta + \psi^\beta \nabla'_{f_* X} \varepsilon'_\beta\,.$$

Thus, $\overline{\nabla}'$ is well defined, not depending on the choice of the frame. One can easily see that $\overline{\nabla}'$ is a linear connection.

EXAMPLE I.3.2. – Assume

$$\gamma: \begin{cases}]-\varepsilon, \varepsilon[\to M \\ \quad t \quad \mapsto \gamma(t) \end{cases}$$

is a regular curve.

The γ-induced connection on $\gamma^{-1}(TM)$ yields just what one denotes by $\nabla_{d/dt} V$ for every vector field V along γ.

1.3.3. *Properties of* $\overline{\nabla}'$.

For every X, Y, *fields on* M

Y', *field on* M'

ξ', η', *fields along* f, *we have*:

(1) *if* f *is an immersion*, $\overline{\nabla}'_X f_* Y = \nabla'_{f_* X} f_* Y$

(2) *if* ∇' *is torsion free*: $\overline{\nabla}'_X f_* Y - \overline{\nabla}'_Y f_* X = f_*[X, Y]$

(3) $\overline{\nabla}'_X \overline{\nabla}'_Y \xi' - \overline{\nabla}'_Y \overline{\nabla}'_X \xi' - \overline{\nabla}'_{[X,Y]} \xi' = K'(f_* X, f_* Y)\xi'$, *where* K' *denotes the curvature tensor of* ∇'

(4) *if* $X \in \operatorname{Ker} f_*$, $\overline{\nabla}'_X Y' = 0$

(5) *if* M' *is Riemannian, if* $X \in \operatorname{Ker} f_*$, *and if* ξ' *is* $(f_* TM)^\perp$ *valued*: $\overline{\nabla}'_X \xi' \in (f_* TM)^\perp$

(6) *if* M' *is Riemannian and* ∇' *its Levi-Civita's connection*, $\overline{\nabla}' g' = 0$.

We omit proofs. (2), (3), (6) are proved in [Do].

I.3.4. *Important remark.*

$f_* X = 0$ does not imply, for every ξ', $\overline{\nabla}'_X \xi' = 0$, though this equality does hold if $\xi' = f^* Y'$—Y' being a field on M'—.

I.3.5. *Connection* $\overline{\nabla}$ denotes the direct sum of ∇ and ∇' on $TM \oplus f^{-1}(TM')$ and its tensor algebra.

I.3.6. $\overline{\nabla}'^\perp$, *connection on* $f_* TM^\perp$.

In the case where M' is Riemannian, we have:

PROPOSITION AND DEFINITION I.3.6. – *For any field* X *on* M *and any section* ξ' *of* $f_* TM^\perp$, *we put*:

$$\overline{\nabla}'^\perp_X \xi' = \text{orthogonal projection of } \overline{\nabla}'_X \xi' \text{ on } f_* TM^\perp .$$

Thus defined, $\overline{\nabla}'^\perp$ is a linear connection on $f_* TM^\perp$ such that $\overline{\nabla}'^\perp g' = 0$.

$\overline{\nabla}'^\perp$ is called *connection associated to* f.

The proof, similar to the corresponding one for isometric immersions, is omitted.

1.4. *2nd fundamental forms.*

I.4.1. σ, *2nd fundamental form of* f.

THEOREM AND DEFINITION I.4.1.1. – *For every fields* X *and* Y *on* M, *we have*:

$$(\overline{\nabla} f_*)(X, Y) = \overline{\nabla}'_X f_* Y - f_* \nabla_X Y .$$

The bilinear mapping $\sigma: TM \times TM \to TM'$ *defined by*

$$\sigma(X, Y) = \overline{\nabla}'_X f_* Y - f_* \nabla_X Y$$

is called the 2nd fundamental form of f.

If moreover ∇ *and* ∇' *are torsion free*, σ *is symmetric.*

In the sequel, all connections are supposed symmetric.

PROOF. – Apply the definition of $\overline{\nabla} f_*$.

I.4.2. σ_0, *2nd fundamental form of* $\operatorname{Ker} f_*$.

DEFINITION I.4.2. – *Assume M is Riemannian. We denote by*

$$\sigma_{0_m}: (\operatorname{Ker} f_*)_m \times (\operatorname{Ker} f_*)_m \to (\operatorname{Ker} f_*)_m^\perp$$

the 2nd fundamental form of the leaf of the kernel at m.

I.4.3. σ', *2nd fundamental form of $f(M)$*. – Is defined whenever M' is a Riemannian manifold, $f(M)$ being an isometrically immersed submanifold.

I.4.4. σ_1, *2nd fundamental form of* $(\operatorname{Ker} f_*)^\perp$.

Recall that if M is a Riemannian manifold, ∇ its Levi-Civita's connection, P a plane field on M, and v the orthogonal projection on $P^\perp$, then the 2nd fundamental form θ of P is defined by:

$$\forall m \in M,\ \forall X,\ Y \in P_m, \qquad \theta(X, Y) = \tfrac{1}{2} v(\nabla_X Y + \nabla_Y X)$$

—cf. [Re]—.

P is integrable iff $\theta(X, Y) = v(\nabla_X Y)$, and θ is then the 2nd fundamental form of the leaves of P.

DEFINITION I.4.4.1. – *We denote by σ_1 the 2nd fundamental form of the distribution* $\operatorname{Ker} f_*^\perp$.

I.4.5. *Composition of maps.*

Assume M, M', M'' are 3 manifolds endowed with linear connections and $f: M \to M'$, $f': M' \to M''$ are mappings with respective 2nd fundamental forms σ and σ'.

If σ'' denotes the 2nd fundamental form of $f' \circ f$, we have:

$$\forall X,\ Y \in TM, \qquad \sigma''(X, Y) = f'_* \sigma(X, Y) + \sigma'(f_* X, f_* Y).$$

—cf. [Fe & Sa] e.g.—.

I.5. *Geometrical interpretation of σ.*

We shall prove the following results:

PROPOSITION I.5.1. – $\operatorname{Ker} f_*$ *is totally geodesic iff σ is null on* $\operatorname{Ker} f_* \times \operatorname{Ker} f_*$.

PROPOSITION I.5.2. – *Assume M is a Riemannian manifold. If X and $Y \in (\operatorname{Ker} f_*)_m$, we have*:

$$\sigma(X, Y) = -f_* \sigma_0(X, Y) .$$

PROPOSITION I.5.3. – $\operatorname{Ker} f_*$ *is parallel iff it is included in* $\operatorname{Ker} \sigma$.

PROPOSITION I.5.4. – *Assume M is a Riemannian manifold.* $\operatorname{Ker} f_*^\perp$ *is integrable and totally geodesic iff* $\sigma|_{\operatorname{Ker} f_* \times \operatorname{Ker} f_*^\perp} = 0$.

Those propositions generalize results by VILMS [Vi]. The 3rd proposition then implies:

THEOREM I.5.5. – *Assume (M, g) and (M', g') are Riemannian manifolds and $f\colon M \to M'$ is a $\mathcal{C}^\infty$ map of constant rank, the 2nd fundamental form of which we denote by σ.*

If $\operatorname{Ker} f_* \subset \operatorname{Ker} \sigma$.

Then M admits a local decomposition:

$$M = M_1 \times M_2 , \quad \textit{where} \quad TM_1 = \operatorname{Ker} f_* \quad \textit{and} \quad TM_2 = \operatorname{Ker} f_*^\perp .$$

This theorem is a generalization of VILM's [Vi] result about totally geodesic maps, and, as we shall see later on, of YANO and ISHIHARA's [Ya & Is] result about relatively affine maps.

REMARK. – If M is connected, simply connected and complete, this theorem is then global.

COROLLARY I.5.6. – *If M is locally irreducible, every mapping $f\colon M \to M'$ of constant non null rank, satisfying* $\operatorname{Ker} f_* \subset \operatorname{Ker} \sigma$, *is an immersion.*

PROOF OF PROPOSITIONS. – It is based on:

LEMMA I.5.7. – *If $Y_m \in (\operatorname{Ker} f_*)_m$, then $\sigma(X_m, Y_m) = -f_* \nabla_{X_m} Y$, for every $X_m \in T_m M$ and every section Y of* $\operatorname{Ker} f_*$ *taking the value Y_m at m.*

This lemma is an immediate consequence of the definition of σ. It implies propositions I.5.1, I.5.2 and I.5.3.

To prove proposition I.5.4, we first note that $\operatorname{Ker} f_*^\perp$ is integrable and totally geodesic iff for every $\operatorname{Ker} f_*^\perp$ valued fields X, Y and for every $\operatorname{Ker} f_*$ valued field U, we have:

$$g(\nabla_X Y, U) = 0 .$$

But, ∇ being metric: $g(\nabla_X Y, U) = -g(\nabla_X U, Y)$, so that $\operatorname{Ker} f_*^\perp$ is integrable and totally geodesic iff $\nabla_X U \in \operatorname{Ker} f_*$, that is, by lemma I.5.7, $\sigma(X, U) = 0$.

2. – Decomposition of a map in the Riemannian case.

II.1. *Metrics on M, $f(M)$, and factorization of f.*

II.1.1. *$(f(M), g')$ and σ'.*

The metric g' of M' induces a Riemannian structure on $f(M)$, also denoted by g'. The 2nd fundamental form σ' of $f(M)$—cf. I.4.3.—is then the 2nd fundamental form of the canonical injection:

$$j\colon \big(f(M), g'\big) \to (M', g') .$$

II.1.2. *Metrics g_1 on M, σ_2 and σ_3.*

We now construct a new metric g_1 on M such that $f_3\colon (M, g_1) \to \big(f(M), g'\big)$, defined by $\forall m \in M$, $f_3(m) = f(m)$ be a Riemannian submersion:

For $X, Y \in T_m M$, we put

$$(g_1)_m(X, Y) = g_m(X, Y) \quad \text{if} \quad X, Y \in \operatorname{Ker} f_*$$

$$g_1(X, Y) = 0 \quad \text{if} \quad X \in \operatorname{Ker} f_* \quad \text{and} \quad Y \in \operatorname{Ker} f_*^{\perp}$$

$$g_1(X, Y) = g'(f_* X, f_* Y) \quad \text{if} \quad X, Y \in \operatorname{Ker} f_*^{\perp}$$

and we extend g_1 into a bilinear symmetric form on $T_m M \times T_m M$.

PROPOSITION II.1.2. – *Tensor field g_1 endows M with a Riemannian structure, and f_3 is a Riemannian submersion.*

PROOF. – Omitted.

We shall denote by i the identity diffeomorphism: $(M, g) \to (M, g_1)$ and σ_2 its 2nd fundamental form.

by σ_3 the 2nd fundamental form of the Riemannian submersion f_3.

II.1.3. *Factorization.*

We can regard f as the product $f = j \circ f_3 \circ i$:

$$\begin{array}{ccc} (M, g) & \xrightarrow{f} & (M', g') \\ {\scriptstyle i}\downarrow & & \uparrow{\scriptstyle j} \\ (M, g_1) & \xrightarrow{f_3} & \big(f(M), g'\big) \end{array}$$

where j is an isometric immersion; f_3 is a Riemannian submersion; i is a diffeomorphism.

II.2. *Tensors τ and ν.*

DEFINITION II.2.1. – *We define 2 tensor fields*

$$\tau\colon TM\times TM \to f_*TM$$

$$\nu\colon TM\times TM \to (f_*TM)^\perp$$

by putting $\tau(X, Y)$ = orthogonal projection of $\sigma(X, Y)$ on f_*TM; $\nu(X, Y)$ = orthogonal projection of $\sigma(X, Y)$ on $(f_*TM)^\perp$.

Then we have:

PROPOSITION II.2.2. – $\nu = f^*\sigma'$. And its immediate consequence.

COROLLARY II.2.3. – $\operatorname{Ker} f_* \subset \operatorname{Ker} \nu$

$$\operatorname{Ker} f_* \cap \operatorname{Ker} \tau \subset \operatorname{Ker} \sigma = \operatorname{Ker} \tau \cap f_*^{-1}(\operatorname{Ker} \sigma') .$$

PROOF. – Applying I.4.5, we have for $X, Y \in TM$

$$\sigma(X, Y) = \sigma'(f_*X, f_*Y) + \sigma_3(X, Y) + f_*\circ\sigma_2(X, Y) .$$

As $\sigma'(f_*X, f_*Y) \in (f_*TM)^\perp$ and $\sigma_3(X, Y) + f_*\circ\sigma_2(X, Y) \in f_*TM$ we can see that

$$\nu(X, Y) = \sigma'(f_*X, f_*Y) ,$$

that is proposition II.2.2.

II.3. *Geometrical viewpoint.*

II.3.1. *Study of M.*

As $\operatorname{Ker} f_* \subset \operatorname{Ker} \nu$ and $\sigma = \tau + \nu$, one can reformulate propositions I.5.1 to I.5.4 and theorem I.5.5 by replacing σ by τ. In particular we have:

THEOREM II.3.1. – *Assume $f\colon (M, g) \to (M', g')$ is a C^∞ map of constant rank between Riemannian manifolds. The conditions:*

(i) $\operatorname{Ker} f_* \subset \operatorname{Ker} \sigma$

(ii) $\operatorname{Ker} f_* \subset \operatorname{Ker} \tau$

are equivalent.

If they hold, then M admits a local decomposition $M = M_1 \times M_2$, where $TM_1 = \operatorname{Ker} f_*$ and $TM_2 = \operatorname{Ker} f_*^\perp$.

II.3.2. *Study of* $f(M)$.

Proposition II.2.2 yields:

COROLLARY II.3.2. – *Assume* $f\colon (M, g) \to (M', g')$ *is a map of constant rank between Riemannian manifolds.* $f(M)$ *is a totally geodesic submanifold of* M' *iff the 2nd fundamental form of* f *is tangent to* $f(M)$.

For applications, see also § IV: umbilical maps.

II.3.3. *The integral foliation of* $\operatorname{Ker} f_*$ *and the distribution* $\operatorname{Ker} f_*^{\perp}$.

THEOREM II.3.3. – *Let* (M, g) *be a Riemannian manifold and* $f\colon M \to M'$ *be a* C^∞ *mapping of constant rank.*

Then there exists a metric g_1 *on* M *with respect to which*

1) $\operatorname{Ker} f_*^{\perp}$ *is a totally geodesic plane field,*

2) *the integral foliation of* $\operatorname{Ker} f_*$ *is Riemannian.*

PROOF. – It is based on lemma I.B.2 by CARRIÈRE [Ca] who gives the following characterization of Riemannian foliations: *a foliation* $\mathcal{F}$ *of a Riemannian manifold* M *is Riemannian with respect to the metric of* M *iff for every unitary field* ξ *orthogonal to* $\mathcal{F}$, $\nabla_\xi \xi$ *is orthogonal to* $\mathcal{F}$ (∇ being the Levi-Civita connection of M).

This lemma implies that the orthogonal complement of a totally geodesic plane field, whenever integrable, is Riemannian for the metric used.

Thus assertion 2) is an immediate consequence of 1). We shall now prove 1): let g_1 be the metric defined in II.1.2 and ∇^1 the associated Levi-Civita connection.

For X and Y $\operatorname{Ker} f_*^{\perp}$-valued vector fields on M, Z $\operatorname{Ker} f_*$-valued field on M, we can compute $g_1(\nabla^1_X Y, Z)$ and using properties (2) and (6) of $\overline{\nabla}$, we find:

$$2g_1(\nabla^1_X Y, Z) = g_1\big([X, Y], Z\big)\,.$$

So we see that the 2nd fundamental form σ_1 of $\operatorname{Ker} f_*^{\perp}$, defined in I.4.4 is null, q.e.d.

II.3.4. *Relatively affine maps: the case where* $\tau = 0$.

A relatively affine map is a map between Riemannian manifolds the 2nd fundamental form of which is orthogonal to $f(M)$ (cf. [Ya & Is]), i.e. such that $\tau = 0$. Yano and Ishihara have proved that every relatively affine map is of constant rank.

We supplement here the result obtained by these authors, proving:

THEOREM II.3.4. – *Assume* $f\colon (M, g) \to (M', g')$ *is a relatively affine map. Assume moreover that* M *is connected, simply connected and complete. Then,*

1) M *is isometric to a Riemannian product* $M_1 \times M_2$, *where* $TM_1 = \operatorname{Ker} f_*$ *and* M_2 *is locally diffeomorphic to* $f(M)$.

2) *If M_2 admits the de Rham decomposition $M_2 = M_2^1 \times \ldots \times M_2^k$, then for a fixed i, the distribution $f_* TM_2^i$ defines a foliation of $f(M)$, every leaf of which is irreducible and homothetic to M_2^i. Moreover the ratio of this homothecy is independent of the leaf so that all leaves are isometric.*

PROOF. – Assertion 1) is proved by [Ya & Is]. Here we just apply II.3.1. Assertion 2). f defines a local diffeomorphism from M_2 unto $f(M)$—see [Di] e.g.—.So we can define a metric $g^\# = f_* g$ on $f(M)$ such that $g^\#(f_*X, f_*Y) = g(X, Y)$ for X, $Y \in TM_2$. The Levi-Civita connection associated to $g^\#$ satisfies $\nabla^\#_{f_*X} f_* Y = f_* \nabla_X Y$.

On the other hand, by the definition of the 2nd fundamental form we have:

$$\begin{aligned}\overline{\nabla}_X f_* Y &= f_* \nabla_X Y + \nu(X, Y) = \nabla^\#_{f_*X} f_* Y + \sigma'(f_*X, f_*Y)\\ &= \nabla'_{f_*X} f_* Y \quad \text{by property (1) of } \overline{\nabla}\,.\end{aligned}$$

Thus, $\nabla^\#$ is the tangent component of ∇'. Hence $g^\#$ and g' induce on $f(M)$ the same Levi-Civita's connection, and f maps parallel distributions on (M_2, g) into parallel distributions on $(f(M), g')$.

If $M_2 = M_2^1 \times \ldots \times M_2^i \times \ldots \times M_2^k$ is the de Rham decomposition of M_2 we see that for a fixed i, $f_* TM_2^i$ defines a totally geodesic foliation of $f(M)$, with irreducible leaves. Let $M_2''^i$ be a leaf of the integral foliation of TM_2^i in M, and $M_2'^i = f(M_2''^i)$ its image by f. $M_2'^i$ is a leaf of $f_* TM_2^i$.

$M_2'^i$ and $M_2''^i$ being totally geodesic—in $f(M)$ and M respectively—, metrics $g^\#$ and g' induce the same Levi-Civita's connection on $M_2'^i$ and by lemma 1 in [Ko & No], p. 242, we see that $g^\#$ and g' are homothetic on $M_2'^i$: there exists $\lambda^i > 0$ s.t.

$$\forall X^i,\ Y^i \in TM_2^i\,, \qquad g(X^i, Y^i) = \lambda^i g'(f_*X^i, f_*Y^i)\,.$$

We must now prove that λ^i is constant (does not depend on the choice of the leaf $M_2'^i$). Therefore for $X \in TM$ we compute

$$\begin{aligned}Xg(X^i, Y^i) &= (X\lambda^i) g'(f_*X^i, f_*Y^i) + \lambda^i X g'(f_*X^i, f_*Y^i)\\ &= g(\nabla_X X^i, Y^i) + g(X^i, \nabla_X Y^i)\,.\end{aligned}$$

Making use of property (6) of $\overline{\nabla}$ we find:

$$\begin{aligned}&g(\nabla_X X^i, Y^i) + g(X^i, \nabla_X Y^i) =\\ &\qquad = (X\lambda^i) g'(f_*X^i, f_*Y^i) + \lambda^i g'(f_*\nabla_X X^i, f_*Y^i) + \lambda^i g'(f_*X^i, f_*\nabla_X Y^i)\end{aligned}$$

But TM_2^i being parallel, $\nabla_X X^i \in TM_2^i$.

Hence

$$\lambda^i g'(f_*\nabla_X X^i, f_*Y^i) = g(\nabla_X X^i, Y^i) \quad \text{and} \quad \lambda^i g'(f_*X^i, f_*\nabla_X Y^i) = g(X^i, \nabla_X Y^i)\,.$$

Thus $(X\lambda^i) = 0$ q.e.d.

3. – Projective maps.

III.1. *Definitions and remarks.*

III.1.1. *Geodesics.*

A C^∞ *map* $t \mapsto \gamma(t)$ *from an open interval* $I \subset \mathbb{R}$ *into a manifold* M *endowed with a linear connection is said to be a geodesic if it satisfiès a) et b);*

a) γ *is an immersion (i.e.* $\dot{\gamma} \neq 0$ *for every* t*);*

b) $\nabla_{d/dt}\dot{\gamma} = \lambda\dot{\gamma}$, *where* $\lambda \in C^\infty(I)$.

III.1.2. *Piecewise geodesics.*

A C^∞ *map* $t \mapsto \gamma(t)$ *from an open interval* $I \subset \mathbb{R}$ *into a manifold* M *endowed with a linear connection* ∇ *is said to be a piecewise geodesic if it satisfies b).*

III.1.3. *Projective maps.*

A C^∞ *map* $f: (M, \nabla) \to (M', \nabla')$ *between manifolds endowed with linear connections is said to be projective if for every piecewise geodesic* γ *on* M, $f\circ\gamma$ *is a piecewise geodesic on* M'.

III.1.4. *Strongly projective maps.*

A C^∞ *map* $f: (M, \nabla) \to (M', \nabla')$ *between manifolds endowed with linear connections is said to be strongly projective if for every geodesic* γ *on* M, *either* $f\circ\gamma$ *is a geodesic on* M', *or the image of* $f\circ\gamma$ *is a point.*

III.1.5. *Remark.*

Mappings f that map every geodesic γ into a geodesic are immersions because they map regular curves into regular curves.

III.1.6. *Remark.*

If $f: (M, \nabla) \to (M', \nabla')$ is a strongly projective map, a geodesic on M is either tangent of transverse to $\operatorname{Ker} f_*$ at every point.

III.2. *Study of strongly projective maps and projective maps between manifolds endowed with linear connections.*

We omit the proof of the following.

PROPOSITION III.2.1. – *Assume* $f: (M, \nabla) \to (M', \nabla')$ *is a strongly projective map of constant rank between manifolds endowed with linear connections. Then*

1) *the integral foliation of* $\operatorname{Ker} f_*$ *is totally geodesic;*

2) $f(M)$ *is a totally geodesic submanifold of* M'.

And we can now state

FUNDAMENTAL THEOREM III.2.2. – *Let $f\colon (M, \nabla) \to (M', \nabla')$ be a mapping of constant rank between manifolds endowed with torsionless linear connections. We denote by σ its 2nd fundamental form. Then f is strongly projective iff it satisfies the following property:*

(*) *There exists a 1-form ω on M such that:*

$$\forall X,\ Y \in TM\,, \qquad \sigma(X, Y) = \omega(X) f_* Y + \omega(Y) f_* X\,.$$

In the proof we shall use the following lemmas

LEMMA III.2.3. – *Assume $f\colon (M, \nabla) \to (M', \nabla')$ is a map of constant rank between manifolds endowed with linear connections, $\gamma\colon\,]-\varepsilon, \varepsilon[\, \to M$ is a regular curve on M, and V is a field along γ. Then*

1) *In the neighborhood of any $t \in\,]-\varepsilon, \varepsilon[$, V can be regarded as the restriction along γ of a field Y on M.*

2) *We have $(\nabla'_{d/dt} f_* V)_t = (\overline{\nabla}'_{\dot\gamma} f_* Y)_{\gamma(t)}$ for any $t \in\,]-\varepsilon, \varepsilon[$.*

LEMMA III.2.4. – *Assume $f\colon (M, \nabla) \to (M', \nabla')$ is a map of constant rank between manifolds endowed with linear connections, and $\gamma\colon s \mapsto \gamma(s)$ is a geodesic on M, with affine parameter s—i.e. such that $\nabla_{d/ds}\dot\gamma = 0$—.*

Then we have: $\sigma(\dot\gamma, \dot\gamma) = \nabla_{d/ds} f_* \dot\gamma$.

PROOF OF LEMMA III.2.3. – Assertion 1) follows from the fact that γ is an immersion.

For assertion 2), let $(e'_\alpha)_{\alpha=1,\dots,n'}$ denote a frame of TM' in the neighborhood of $m'_0 = f\circ\gamma(t_0)$.

We write $f_* Y = \varphi^\alpha e'_\alpha$, φ^α being functions on M.

At the point $m = \gamma(t)$ we have:

$$\overline{\nabla}'_{\dot\gamma} f_* Y = \dot\gamma(\varphi^\alpha)\, e'_\alpha + \varphi^\alpha \nabla'_{f_*\dot\gamma}\, e'_\alpha\,,$$

and, as $(f_* V)_t = \varphi^\alpha_{\gamma(t)} (e'_\alpha)_{f(m)}$

$$\nabla'_{d/dt} f_* V = \frac{d}{dt}(\varphi^\alpha \circ \gamma)\, e'_\alpha + (\varphi^\alpha\circ\gamma)\, \nabla'_{\dot{\widehat{f\circ\gamma}}}\, e'_\alpha\,.$$

The identities $\dot\gamma = \gamma_*(d/dt)$ and $\dot{\widehat{f\circ\gamma}} = f_*\dot\gamma$ give the result 2).

LEMMA III.2.4. – *Is an immediate consequence of lemma* III.2.3.

PROOF OF THE THEOREM. – *A) The condition is necessary.* The totally geodesic distribution $\operatorname{Ker} f_*$ admits a ($\mathcal{C}^\infty$) supplementary autoparallel distribution, N_1. We shall now study the colinearity of $\sigma(T, T)$ and $f_* T$ for $T \in T_m M$.

Let $t \mapsto \gamma(t)$ be the geodesic from m, s.t. $\dot\gamma(0) = T$, with affine parameter t.

If $T \in (\operatorname{Ker} f_*)_m$, by proposition III.2.1 and lemma III.2.4, $\sigma(\dot\gamma, \dot\gamma) = 0$.

If $T \notin (\operatorname{Ker} f_*)_m$, $f \circ \gamma$ is a geodesic on M', γ is transverse to $\operatorname{Ker} f_*$, and by lemma III.2.4, $\sigma(\dot\gamma, \dot\gamma)$ is colinear to $f_* \dot\gamma$. So if $T \in N_1$ there exists a function $\omega_1 \colon N_1 \to \mathbb{R}$ s.t. $\sigma(T, T) = 2\omega_1(T) f_* T$.

Using a proof by ZVI HAR'EL [Ha] we see that ω_1 is a linear map. f_* and σ being $\mathcal{C}^\infty$, so is ω_1.

Now for $T \notin \operatorname{Ker} f_*$ we write $T = T_0 + T_1$ with $T_0 \in \operatorname{Ker} f_*$, $T_1 \in N_1$ and we have:

$$\sigma(T, T) = \sigma(T_0, T_0) + 2\sigma(T_0, T_1) + 2\omega_1(T_1) f_* T = 2\sigma(T_0, T_1) + 2\omega_1(T_1) f_* T .$$

As $\sigma(T, T)$ is colinear to $f_* T$, there exists a mapping $\omega_0 \colon \operatorname{Ker} f_* \times N_1 \to \mathbb{R}$ such that:

$$\sigma(T_0, T_1) = \omega_0(T_0, T_1) f_* T = \omega_0(T_0, T_1) f_* T_1 .$$

As $f_* T_1$ is nowhere zero, σ and f_* being $\mathcal{C}^\infty$, we see that ω_0 is $\mathcal{C}^\infty$.

Now, using the bilinearity of σ we can see that ω_0 is linear w.r. to T_0 and does not depend on T_1.

Se we define $\omega \colon \operatorname{Ker} f_* \to \mathbb{R}$. By

$$\sigma(T_0, T_1) = \omega(T_0) f_* T$$

and this equality still holds when $T_1 = 0$.

Putting $\omega(T) = \omega(T_0) + \omega_1(T_1)$ we have:

$$\sigma(T, T) = 2\omega(T) f_* T .$$

Hence $\sigma(X, Y) = \frac{1}{4}[\sigma(X + Y, X + Y) - \sigma(X - Y, X - Y)] = \omega(X) f_* Y + \omega(Y) f_* Y$

q.e.d.

B) The condition is sufficient. If (*) holds, for two $\operatorname{Ker} f_*$-valued fields X and Y we have:

$$\sigma(X, Y) = 0 = - f_* \nabla_X Y$$

by lemma I.5.7, so that $\operatorname{Ker} f_*$ is totally geodesic.

Using lemma III.2.4, it is easy to see that any geodesic in M is mapped either into a geodesic on M', or into a point.

A piecewise geodesic γ being a geodesic on the open set where $\dot\gamma \neq 0$, lemma III.2.4 provides also the following characterization for projective maps:

THEOREM III.2.5. – *Let* $f\colon (M, \nabla) \to (M', \nabla')$ *be a mapping of constant rank between manifolds endowed with torsionless linear connections, and denote by* σ *its 2nd fundamental form. Then* f *is projective iff for every* $X \in TM$, $\sigma(X, X)$ *and* f_*X *are colinear.*

III.3. *The Riemannian case.*

By our definition III.1.1, a geodesic is an immersion $\sigma\colon I =]-\varepsilon, \varepsilon[\to M$ such that $\gamma(I)$ is a totally geodesic submanifold of M. Example I.3.2 shows that the 2nd fundamental form of γ satisfies $\sigma_\gamma(d/dt, d/dt) = \nabla_{d/dt}\dot{\gamma}$, so that γ is a totally geodesic map (i.e. $\sigma_\gamma = 0$) iff t is an affine parameter for γ. One knows that every geodesic admits affine parameters. In the case when rank $f > 1$ the notion of geodesic curve is naturally extended into the notion of strongly projective mapping (with characteristic property $(*)$). We shall show here that a change of metric can make any strongly projective map into a totally geodesic one. We have:

THEOREM III.3. – *Assume* $f\colon (M, g) \to (M', g')$ *is a strongly projective mapping between Riemannian manifolds. Then*

1) $\operatorname{Ker} f_*^\perp$ *is integrable and defines a totally umbilical foliation, which is totally geodesic iff* $\operatorname{Ker} f_* \subset \operatorname{Ker} \omega$.

2) *There exists a metric* g_1 *on* M *for which* f *is totally geodesic.*

PROOF OF 1). – Assume X, Y are $\operatorname{Ker} f_*^\perp$ valued fields and Z is a $\operatorname{Ker} f_*$ valued field on M. We have:

$$\sigma(X, Z) = -f_*\nabla_X Z = \omega(Z) f_* X . \tag{a}$$

Hence $g(\nabla_X Y, Z) = -g(Y, \nabla_X Z) = \omega(Z) g(X, Y)$ and by symmetry:

$$g([X, Y], Z) = 0$$

so that $\operatorname{Ker} f_*^\perp$ is integrable.

Moreover $g(\sigma_1(X, Y), Z) = \omega(Z) g(X, Y)$, so that $\operatorname{Ker} f_*^\perp$ is umbilical—totally geodesic iff $\operatorname{Ker} f_* \subset \operatorname{Ker} \omega$—.

PROOF OF 2). – Let g_1 be the metric defined in II.1.2. We shall prove that $\operatorname{Ker} f_*$ is totally geodesic w.r. to g_1.

Using the identity

$$2g_1(\nabla^1_X Y, Z) = Xg_1(Y, Z) + Yg_1(X, Z) - Zg_1(X, Y) + \\ + g_1([X, Y], Z) + g_1([Z, X], Y) + g_1(X, [Z, Y])$$

and the definition of g_1, we find, for X and $Y \in \operatorname{Ker} f_*$

$$2g_1(\nabla^1_X Y, Z) = 2g(\nabla_X Y, Z) .$$

$\operatorname{Ker} f_*$ being totally geodesic for g, taking Z in $\operatorname{Ker} f_*^\perp$ we see that $\operatorname{Ker} f_*$ is totally geodesic for g_1.

Applying theorem 3.3 by J. VILMS [Vi] we see that the 2nd fundamental form σ_3 of $f_3\colon (M, g_1) \to (f(M), g')$ is null.

The 2nd fundamental form of $j\circ f\colon (M, g_1) \to (M', g')$ being $\sigma_3 + \nu$ is null. So f is totally geodesic with respect to g_1.

III.4. *Examples.*

III.4.1. *First example of strongly projective map.*

Let $f\colon \mathbb{E}^2 \to \mathbb{E}^2$

$$(x^1, x^2) \mapsto (\sin^2 x^1, \cos^2 x^1)$$

f is of rank one whenever $x^1 \neq k\pi/2$.

It maps $\mathbb{E}^2$ into the totally geodesic submanifold:

$$\begin{cases} x^1 + x^2 = 1 \\ 0 \leqslant x^1 \leqslant 1 . \end{cases}$$

We have $\sigma(X, Y) = \omega(X) f_* Y + \omega(Y) f_* X$ with

$$\omega(X) = X^1 \operatorname{cotg} 2x^1$$

so that f is strongly projective without being totally geodesic. $\operatorname{Ker} f_*^\perp$ is a line so that we have $\operatorname{Ker} f_* \subset \operatorname{Ker} \sigma$.

III.4.2. *Second example of strongly projective map.*

Consider the VRANCEANU [Vr] surface $M^2 \subset \mathbb{E}^4$, the points of which satisfy:

$$\begin{cases} x^1 = r(u) \cos u \cos v \\ x^2 = r(u) \cos u \sin v \\ x^3 = r(u) \sin u \cos v \\ x^4 = r(u) \sin u \sin v \end{cases}$$

Put $M' = S^2 \setminus \{N, S\}$ parametrized by the latitude $\theta \in]-\pi/2, \pi/2[$, and the longitude $\varphi \in [0, 2\pi[$. Define $f\colon M \to M'$ by $f(u, v) = (0, v)$. We can see that $f_* X = X^2(\partial/\partial\varphi)$ and $\sigma(X, Y) = \omega(X) f_* Y + \omega(Y) f_* X$ with $\omega(X) = -X^1(\dot r/r)$ so that f is strongly projective and $\operatorname{Ker} f_* \not\subset \operatorname{Ker} \omega$.

III.4.3. *Example of projective, not strongly projective map.*

Let p be the orthogonal projection from the sphere $S^2 \setminus \{N, S\} \subset \mathbb{E}^3$ on its axis $]N, S[$. Being $\mathbb{R}$-valued, p is projective. But it maps a great circle (c) on S^2 into a twice covered segment: its image is a totally geodesic submanifold of $[N, S]$, but $p \circ c$ is not an immersion. The leaves of $\operatorname{Ker} f_*$ are the horizontal circles (not totally geodesic). On the other hand, a computation shows that the 2nd fundamental form of p does not satisfy $(*)$.

III.4.4. *Example of strongly projective map satisfying* $\operatorname{Ker} \sigma = \{0\}$.

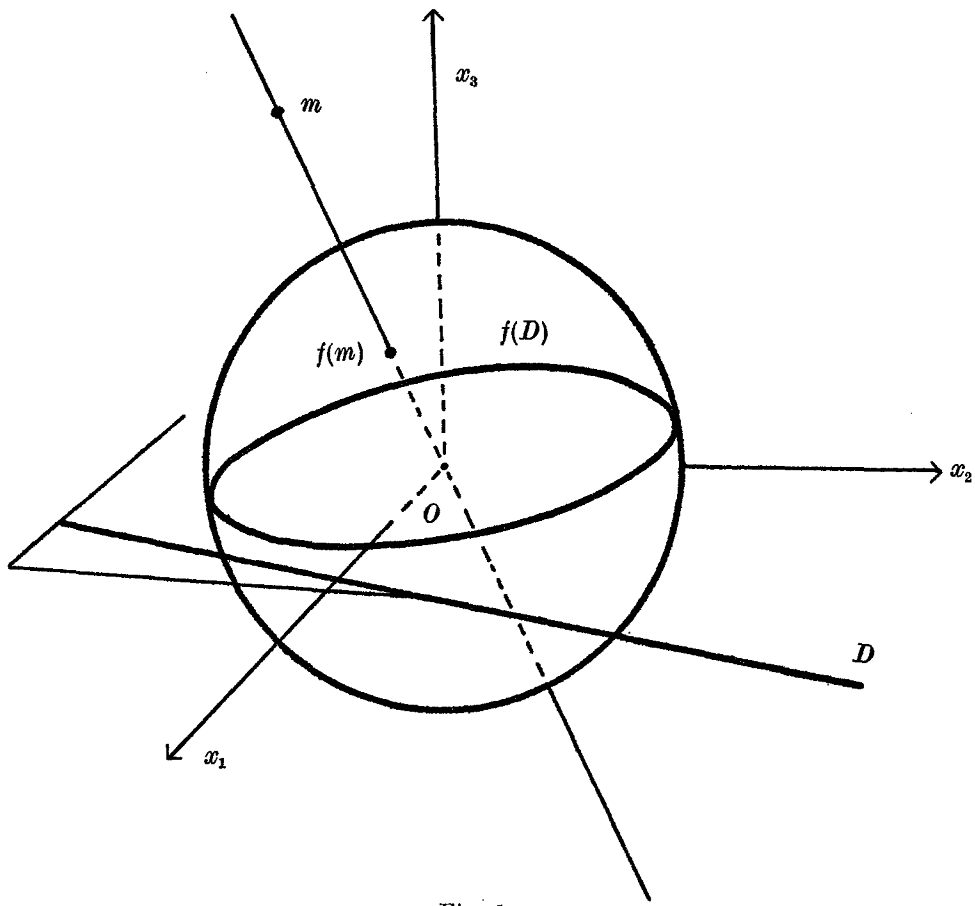

Fig. 1.

Consider the map $f \colon \mathbb{E}^3 \setminus \{0\} \to S^2$

$$(x^1, x^2, x^3) \mapsto \left(\frac{x^1}{r}, \frac{x^2}{r}, \frac{x^3}{r}\right)$$

where $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$.

Let D be a straight line in $\mathbb{E}^3$. If $0 \in D$, $f(D)$ is the point $D \cap S^2$. If $0 \notin D$, $f(D)$ is included in the intersection of the plane defined by (O, D) and S^2: it is an open subset of a great circle. Thus f is strongly projective.

The leaves of $\operatorname{Ker} f_*$ are the straight lines through O (O being excluded). The kernel of f_* at a point $m \in \mathbb{E}^3$ is generated by the position vector $\vec{m}$. Being non parallel, $\operatorname{Ker} f_*$ is not included in $\operatorname{Ker} \sigma$ (proposition I.5.3). In fact, using lemma I.5.7 one can easily compute the 2nd fundamental form σ of f. We have:

$$\sigma(X, Y) = \omega(X) f_* Y + \omega(Y) f_* X ,$$

where $\omega(\vec{m}) = -1$ and $\operatorname{Ker} \omega = \operatorname{Ker} f_*^{\perp}$.

Thus $\operatorname{Ker} \sigma = \{0\}$.

PROPOSITION III.4.5. *Strongly projective maps of rank $\geqslant 2$ between euclidean spaces are affine.*

IDEA OF THE PROOF. – We first establish that we need only investigate the case of immersions, then we show strongly projective immersions map straight lines into straight lines.

1) Consider $f\colon M = \mathbb{E}^n \to M' = \mathbb{E}^{n'}$ and suppose f is strongly projective, of constant rank k.

The leaves of $\operatorname{Ker} f_*$ (resp. $\operatorname{Ker} f_*^{\perp}$) are $n - k$ planes (resp. k-planes). If Π is a leaf of $\operatorname{Ker} f_*^{\perp}$ and π the orthogonal projection on Π—totally geodesic—consider the following factorization of f:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\pi"'] & M' \\ \Pi \arrow[ur, "f'"'] & \end{tikzcd}$$

where f', restriction of f to Π, is an immersion.

Being parallel, $\operatorname{Ker} f_*$ is included in $\operatorname{Ker} \sigma$ (Prop. I.5.3). Hence one can see that f is strongly projective iff so is f', and f is affine iff so is f'.

2) Suppose now moreover that f is an immersion of rank $k \geqslant 2$. $f(M)$ is a connected open subset of a k-plane in M'. The image of a straight line by f is included in a straight line.

Thus one can easily state that, if D' is a straight line in M:

1) $f^{-1}(D')$ is either $\emptyset$ or a straight line D in M;

2) we have $f(D) = D'$;

3) f satisfies the hypothesis of the fundamental theorem of affine geometry, and hence is affine.

4. – Umbilical maps.

IV.1. *Definitions.*

The following 4 definitions can be regarded as natural extensions of the usual notion of umbilicity for submanifolds.

Let $f\colon (M, g) \to (M', g')$ be a map of constant rank between Riemannian manifolds, with 2nd fundamental form σ.

IV.1.1. *g-umbilicity.*

f is said to be weakly g-umbilical if there exists

1) a field ξ along f, nowhere O, with values in $\operatorname{Ker} f_*^\perp$;

2) a field Z on M, such that for every X and Y in TM we have: $\sigma(X, Y) = g(X, Y)(\xi + f_* Z)$.

If moreover σ is orthogonal to $f_* TM$ (that is $Z = 0$) f is said to be *strongly g-umbilical.*

IV.1.2. *g'-umbilicity.*

f is said to be weakly g'-umbilical if there exists fields ξ and Z as in IV.1.1, such that for every X and Y in TM we have: $\sigma(X, Y) = g'(f_* X, f_* Y)(\xi + f_* Z)$. If moreover σ is orthogonal to $f_* TM$ ($Z = 0$) f is said to be *strongly g'-umbilical.*

REMARK. – No Riemannian submersion can be umbilical because for such maps, $\sigma|_{\operatorname{Ker} f_*^\perp \times \operatorname{Ker} f_*^\perp} = 0$—cf. [Vi]—, which would imply $\xi = 0$ in the umbilical case.

IV.2. *Examples.*

IV.2.1. *g-umbilicity.*

PROPOSITION IV.2.1. – *Let M be a convex hypersurface of the euclidean space $\mathbb{E}^{n+1}$. We denote by g the metric in $\mathbb{E}^{n+1}$ and $\tilde{\nabla}$ the associated Levi-Civita connection.*

1) *There exists one unique metric g_1 on M s.t. if $f_1\colon (M, g_1) \to (\mathbb{E}^{n+1}, g)$ denotes the canonical injection, and ν the orthogonal projection of its 2nd fundamental form on $TM^\perp$ we have:*

$$\forall X, Y \in TM, \quad \nu(X, Y) = g_1(X, Y)\xi,$$

ξ being a unitary vector field orthogonal to M.

2) *f_1 is weakly g_1-umbilical iff there exists a field U on M such that the 2nd fundamental form σ' of M isometrically immersed in $\mathbb{E}^{n+1}$ satisfy*:

$$\forall X,\, Y,\, Z \in TM\,, \qquad (\tilde{\nabla}_Z \sigma')(X, Y) = 2g\big(\sigma'(X, Y), \sigma'(U, Z)\big)\,.$$

3) *f_1 is strongly g_1-umbilical iff σ' is parallel* ($\tilde{\nabla}\sigma' = 0$).

PROOF. – *Assertion* 1) As M is convex, $TM^\perp$ is orientable. Let ξ be a unitary field in $TM^\perp$. Define a bilinear symmetric form g_1 on TM by:

$$g_1(X, Y) = \langle \sigma'(X, Y), \xi \rangle\,.$$

M being convex, we can chose ξ such that g_1 be positive definite at every point. Thus (M, g_1) is a Riemannian manifold. Factorize f_1 as in II.1.3.

$$\begin{tikzcd}
(M, g_1) \arrow[r, "f_1"] & (\mathbb{E}^{n+1}, g) \\
(M, g) \arrow[u, "i"] \arrow[ur, "j"'] &
\end{tikzcd}$$

By proposition II.2.2 and by the definition of g_1 we have $\nu(X, Y) = \sigma'(X, Y) = g_1(X, Y)\xi$, and 1) is satisfied. As any change of metric on M does not alter ν, g_1 is the only suitable metric.

Assertion 2) The tangent component τ of the 2nd fundamental form of f_1 is $\tau(X, Y) = \nabla_X Y - \nabla^1_X Y$.

We seek for a condition that it satisfy

$$\text{(i)} \qquad \tau(X, Y) = g_1(X, Y)\, U\,,$$

U being a field on M.

A computation, using the Codazzi equation for M immersed in $\mathbb{E}^{n+1}$ yields: $-2g\big(\sigma'(\tau(X, Y), Z), \xi\big) = g\big((\tilde{\nabla}_Z \sigma')(X, Y), \xi\big)$.

σ' being definite, we see that condition (i) is equivalent to $(\tilde{\nabla}_Z \sigma')(X, Y) = 2g\big(\sigma'(X, Y), \sigma'(U, Z)\big)$.

Assertion 3) Is an immediate consequence of the definition of strong umbilicity.

IV.2.2. *g'-umbilicity*: *Projection of $S^n \times \mathbb{R}$ into S^n.*

Consider the cylinder $M = S^n \times \mathbb{R}$ and the map from M into $\mathbb{E}^{n+1}$

$$f\colon M \to \mathbb{E}^{n+1}$$

$$(m, z) \mapsto m\,.$$

We can factorize f as follows:

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathbb{E}^{n+1} \\ & S^n \arrow[ur, "j"'] & \end{tikzcd}$$

where $p\colon (m, z) \mapsto m$ is the—totally geodesic—orthogonal projection on S^n, and where j is the—totally umbilical—canonical isometric immersion.

Denoting by H the mean curvature vector of S^n immersed in $\mathbb{E}^{n+1}$, using lemma I.5.7 and I.4.5 we find:

$$\sigma(X, Y) = \langle f_* X, f_* Y \rangle H$$

so that f is strongly g'-umbilical.

IV.3. *Theorems.*

IV.3.1. *g-umbilical maps.*

THEOREM IV.3.1.1. – *Let $f\colon (M, g) \to (M', g')$ be a weakly g-umbilical map. Then*

1) *f is an immersion.*

2) *If rank $f \geqslant 2$ and if M' is a space of constant curvature, $f(M)$ is a convex hypersurface of a totally geodesic submanifold of M'.*

THEOREM IV.3.1.2. – *Assume $f\colon (M, g) \to (M', g')$ is a strongly g-umbilical map from a simply connected irreducible manifold into a space of constant curvature. Then M is isometric to a sphere.*

PROOF. – 1*st theorem.* 1) If f is weakly g-umbilical we have $\operatorname{Ker} \nu = \{0\}$. Hence f_* is injective by corollary II.2.3 ($\operatorname{Ker} f_* \subset \operatorname{Ker} \nu$).

2) Assume now that M' is a space of constant curvature. Consider metric g_1 as in II.1.2: $g_1(X, Y) = g'(f_* X, f_* Y)$ and the factorization of II.1.3.

Using I.4.5, we see that the 2nd fundamental form σ_4 of the isometric immersion $j \circ f_3$ satisfies:

$$\sigma_4(X, Y) = \sigma'(f_* X, f_* Y)\,.$$

By proposition II.2.2 $\sigma_4(X, Y) = \nu(X, Y) = g(X, Y)\xi$. We shall now prove, using a method of GRIFONE and MORVAN [Gr & Mo], that ξ is parallel in the normal bundle $f_* TM^\perp$. Let us write the Codazzi equation for $j \circ f_3$—cf. [Ch] e.g.— for $X, Y, Z \in TM$:

$$\nabla'^{\perp}_X \sigma_4(Y, Z) - \nabla'^{\perp}_Y \sigma_4(X, Z) = \sigma_4(\nabla^1_X Y, Z) - \sigma_4(\nabla^1_Y X, Z) + \sigma_4(Y, \nabla^1_X Z) - \sigma_4(X, \nabla^1_Y Z)$$

where $\nabla'^{\perp}$ is the connection induced by ∇' on $f_*TM^{\perp}$. That is

$$Xg(Y,Z)\xi + g(Y,Z)\nabla'^{\perp}_X\xi - g(\nabla^{\perp}_X Y, Z)\xi - g(Y, \nabla^{\perp}_X Z)\xi =$$
$$= Yg(X,Z)\xi + g(X,Z)\nabla'^{\perp}_Y\xi - g(\nabla^{\perp}_Y X, Z)\xi - g(X, \nabla^{\perp}_Y Z)\xi\,.$$

If $\nabla'^{\perp}\xi$ is non zero on an open set, the projection of this equation on the distribution orthogonal to ξ in $f_*TM^{\perp}$ yields $g(Y,Z)\nabla'^{\perp}_X\xi = g(X,Z)\nabla'^{\perp}_Y\xi$.

Defining $L\colon TM \to f_*TM^{\perp}$ by $L(X) = \nabla'_X\xi$, one can easily see that L would be a rank 1 linear map, satisfying moreover $\operatorname{Ker} L = \operatorname{Ker} g$, which is impossible. Hence $\nabla'^{\perp}\xi = 0$ and the distribution M'' generated by f_*TM and ξ is integrable and totally geodesic. The 2nd fundamental form of M isometrically immersed in M'' is the definite form $\sigma_4\colon \forall X \neq 0$, $\sigma_4(X,X) \neq 0$. Thus M is a convex hypersurface of M''.

2nd theorem. – Being strongly g-umbilical, f is relatively affine and weakly g-umbilical. Thus by theorem II.3.4, f is an homothecy and one can easily see that $f(M)$ is a totally umbilical, closed submanifold of M', without boundary. Thus $f(M)$ is an hypersphere of a totally geodesic submanifold of M' ([Ch]).

IV.3.2. *g'-umbilical maps.*

THEOREM IV.3.2.1. – *The image of M by a weakly g'-umbilical map is a totally umbilical submanifold of M'.*

THEOREM IV.3.2.2. – *Assume f is a weakly g'-umbilical map from a simply connected complete manifold M into a space of constant curvature M'. Then*

1) *$f(M)$ is a sphere.*

2) *M admits a decomposition $M_1 \times M_2$ where $TM_1 = \operatorname{Ker} f_*$ and M_2 is diffeomorphic to the sphere $f(M) \subset M'$.*

THEOREM IV.3.2.3. – *Assume f is a strongly g'-umbilical map from a simply connected complete manifold M into a space of constant curvature. Then*

1) *$f(M)$ is a sphere.*

2) *M admits a decomposition $M_1 \times M_2$ where $TM_1 = \operatorname{Ker} f_*$ and M_2 is isometric to a sphere.*

PROOF. – *The 1st theorem.* Is an application of II.2.2.

The 2nd theorem. We have $\operatorname{Ker} f_* \subset \operatorname{Ker}\sigma$, thus by theorem I.5.5 we can write $M = M_1 \times M_2$, M_2 being diffeomorphic to $f(M)$, which is totally umbilical, hence included into a sphere. Moreover $f(M)$ is complete an has no boundary since f is of constant rank. $f(M)$ is then the whole sphere.

The 3rd theorem. By similar arguments as in II.3.4 we can see that M_2, as $f(M)$, is irreducible, and that f_* induces an homothecy: $TM_2 \to f_* TM$. Hence M_2 is isometric to a sphere.

IV.4. *Spherical maps.*

Generalizing the definition of « extrinsic sphere » by B. Y. CHEN [Ch 1] we set:

IV.4.1. *Definition.*

A map $f\colon (M, g) \to (M', g')$ is said to be spherical if it is strongly g'-umbilical and if its mean curvature vector ξ is parallel in the normal bundle (that is for $\overline{\nabla}'^{\perp}$: cf. I.3.6), and non null.

IV.4.2. *Spherical maps into a Kähler manifold.*

We shall now prove

THEOREM IV.4.2.1. – *Assume $f\colon (M, g) \to (M'^{2n'}, g')$ is a spherical map, with values in a Kähler manifold of real dimension $2n'$. If M is simply connected complete, and f analytical of rangk $2n' - 2$, then one of the irreducible components of (M, g) is isometric to an even dimensional sphere.*

This theorem is based on two lemmas.

LEMMA IV.4.2.2. – *Assume $X \in T_m M$ and ζ is a section of $f_* TM^{\perp}$. Denoting by A' the 2nd fundamental tensor of $f(M)$ isometrically immersed in M', we have:*

$$\overline{\nabla}'_X \zeta = - A'_\zeta f_* X + \overline{\nabla}'^{\perp}_X \zeta .$$

LEMMA IV.4.2.3. – *Assume $X \in T_m M$ and ζ is a field along f. Denoting by J the complex structure of M' we have:*

$$\overline{\nabla}'_X J\zeta = J \overline{\nabla}'_X \zeta .$$

PROOF OF THE LEMMAS. – 1*st lemma.* Property (5) of $\overline{\nabla}'$ shows that the tangent component of $\overline{\nabla}'_X \zeta$ depends only on $f_* X$.

On the other hand for $Y \in T_m M$ and for any section Y' of $f_* TM$ such that $Y'_m = f_* Y$, we have:

$$\begin{aligned} g'(\overline{\nabla}'_X \zeta, f_* Y) &= - g'(\zeta, \overline{\nabla}'_X Y') \quad \text{as } \overline{\nabla}' \text{ is metric} \\ &= - g'\big(\zeta, \sigma'(X, Y)\big) = - g'\big(\zeta, \sigma'(f_* X, f_* Y)\big) \end{aligned}$$

by prop. II.2.2,

$$= - g'(A'_\zeta f_* X, f_* Y) \quad \text{by the definition}$$

of A'_ζ.

Hence we get the lemma.

2nd lemma. We omit the proof, which is a computation in local coordinates.

PROOF OF THE THEOREM. – We shall use here the method of CHEN [Ch 1]. By our assumptions, f is strongly g'-umbilical:

$$\sigma(X, Y) = g'(f_* X, f_* Y)\xi\,, \qquad \frac{\xi}{\|\xi\|}\text{ is parallel for } \nabla'^{\perp}\,,$$

and $\|\xi\|$ is constant.

We can apply theorem II.3.4: $M = M_1 \times M_2$ where $TM_1 = \operatorname{Ker} f_*$ and if M_2 admits the de Rham decomposition $M_2 = M_2^1 \times \ldots \times M_2^k$, f induces an homothecy of a submanifold $M_2''^i$ in M—isomorphic to M_2^i—into a leaf $M_2'^i$ of $f_* TM_2^i$; We denote its ratio by λ^i.

We can chose a unitary section of $f_* TM^\perp$, η, orthogonal to ξ. $\overline{\nabla}'^\perp$ being metric, we have $\overline{\nabla}'^\perp \eta = 0$. Hence $\overline{\nabla}'_X \eta = -A'_\eta f_* X + \overline{\nabla}'^\perp_X \eta = 0$, η being orthogonal to the mean curvature vector of $f(M)$. We define a function φ on M_2^i by

$$\varphi(m) = g'_{f(m)}\left(J\frac{\xi}{\|\xi\|}\eta\right).$$

By a computation we can see that

$$\nabla_X d\varphi = -(\lambda^i\|\xi\|)^2 g(X,\ \ )\varphi\,.$$

Moreover there exists at least one i for which φ is non constant, for if the contrary held, one could see that $\{\eta, J_\eta\}$ would generate $f_* TM^\perp$, hence we would have $\|\xi\| = 0$.

The result of OBATA [Ob] then proves that M_2^i is isometric to the sphere of radius $1/\lambda^i\|\xi\|$ in E^{2r+1}, where $2r = \dim M_2^i$.

5. – Integral formulas.

We shall here state formulas relating the norms of the 2nd fundamental forms of f, of $f(M)$, and of $\operatorname{Ker} f_*$, in the case where f induces a conformal map from $\operatorname{Ker} f_*^\perp$ into $f_* TM$—e.g. when f is a Riemannian submersion or a mapping of rank 1—. In the sequel we denote by $\|\cdots\|$ the norm of any type of tensor, for either metric g, or metric g'.

V.1. *The conformal case.*

Our results will follow from the

PROPOSITION V.1.1. – *Assume $f\colon (M, g) \to (M', g')$ is a mapping of constant rank between Riemannian manifolds. Suppose there exists a function ϱ on M s.t. for any*

$X \in \operatorname{Ker} f_*^\perp$ *we have*: $\|f_* X\| = \varrho \|X\|$. *With the notations of* I.4, *we have at any point of* M

$$\|\sigma\|^2 \geqslant \varrho^2 \|\sigma_0\|^2 + \varrho^4 \|\sigma'\|^2 .$$

Hence we obtain:

COROLLARY V.1.2. – *Assume* $f\colon (M, g) \to (M', g')$ *is a mapping of constant rank between Riemannian manifolds, M being compact.*

Suppose there exists a function ϱ *on* M *s.t. for* $X \in \operatorname{Ker} f_*^\perp$ *we have*:

$$\|f_* X\| = \varrho \|X\| .$$

Then, with the notations of I.4 *we have*:

$$\int_M \|\sigma\|^2 \geqslant \varrho_0^2 \Big[\int_M \|\sigma_0\|^2 + \varrho_0^2 \int_M \|\sigma'\|^2 \Big]$$

where ϱ_0 *denotes the lower bound of* ϱ.

PROOF OF PROPOSITION V.1.1. – Is a direct computation of $\|\sigma\|$, using at m an orthonormal frame $\{e_1, \ldots, e_n\}$ s.t. $\{e_{n-r+1}, \ldots, e_n\}$ generates $\operatorname{Ker} f_*^\perp$.

We have $\|f_* e_i\| = \varrho \|e_i\|$ for $i > n - r$ and

$$\|\sigma\|^2 = \sum_{i,j=1}^{n} \|\sigma(e_i, e_j)\|^2 = \sum_{i,j=1}^{n} \|\tau(e_i, e_j)\|^2 + \sum_{i,j=1}^{n} \|\nu(e_i, e_j)\|^2 .$$

Using propositions I.5.2 and II.2.2, and the definitions of the norms, we find the required equality.

V.2. *The case of a fibration.*

Whenever f defines a fibration with compact fiber F, we obtain:

COROLLARY V.2.1. – *With the hypothesis of corollary* V.1.2, *if f defines a fibration with compact fiber F, we have*:

$$\int_M \|\sigma\|^2 \geqslant \varrho_0^2 \Big[\int_M \|\sigma_0\|^2 + \varrho_0^2 (\operatorname{vol} F) \int_{f(M)} \|\sigma'\|^2 \Big]$$

and

COROLLARY V.2.2. – *Assume f is a map of constant rank* 1 *from an orientable compact Riemannian manifold into a Riemannian manifold* (M', g'). *Suppose moreover that f defines a fibration with compact fiber F. With the notations of* I.4, *we have*:

$$\int_M \|\sigma\|^2 \geqslant \inf_M \|f_*\|^2 \Big[\int_M \|\sigma_0\|^2 + \inf_M \|f_*\|^2 (\operatorname{vol} F) \int_C k^2 \Big]$$

where C is the curve $f(M)$ and k its curvature.

V.3. *Case of a Riemannian submersion.*

Using a result by HERMANN [He] we obtain.

COROLLARY V.3. – *Assume f is a mapping of constant rank from a compact Riemannian manifold (M, g) into a Riemannian manifold (M', g'), which induces a Riemannian submersion from M unto $(f(M), g')$. If the fibre F or f is compact, with the notations of* I.4, *we have*:

$$\int_M \|\sigma\|^2 \geq \int_M \|\sigma_0\|^2 + (\operatorname{vol} F)\int_M \|\sigma'\|^2 .$$

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