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SECOND FUNDAMENTAL MEASURE OF GEOMETRIC SETS AND LOCAL APPROXIMATION OF CURVATURES

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Abstract

Using the theory of normal cycles, we associate with each geometric subset of a Riemannian manifold a —tensor-valued— curvature measure, which we call its second fundamental measure. This measure provides a finer description of the geometry of singular sets than the standard curvature measures. Moreover, we deal with approximation of curvature measures. We get a local quantitative estimate of the difference between curvature measures of two geometric subsets, when one of them is a smooth hypersurface.

1. Introduction

In his article *Euler characteristic and finitely additive Steiner measures* [**26**], John Milnor raises the following question: *"In what sense do two sets have to be close to each other, in order to guarantee that their curvature measures are close to each other?"* Before we address this question, let us briefly review the history of curvature measures.

Defining and studying the curvatures of singular spaces goes back to Steiner (1840) in the convex case (see [**30**] for instance). Given a convex body K of the Euclidean space $\mathbb{E}^n$, he showed that the volume of the parallel body of K at distance ε is a polynomial of degree n in ε. When the boundary of K is smooth, the coefficients of this polynomial are, up to a constant depending on n, the integrals of the k^{-th}*-mean curvatures* of the boundary of K, that is the symmetric functions of its principal curvatures. Thus, these coefficients, called *Quermassintegrale* by Minkowski, are good candidates to generalize curvatures to the case of convex hypersurfaces, without assuming any regularity condition. The tubes formula, proved by H. Weyl in 1939 [**32**], states that this interpretation of integrals of curvatures in terms of the volume of parallel bodies also holds if one drops the convexity assumption but assumes smoothness, provided ε is small enough.

In 1958, H. Federer made a breakthrough in two directions [**14**]:

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- He defined a large class of subsets, including smooth submanifolds and convex bodies, for which it is possible to define reasonable generalizations of curvature: the *subsets of positive reach.* His approach consists again in considering the volume of parallel bodies. Basically, he observed that the key point in the tubes formula for both the smooth and the convex case is that the orthogonal projection on the studied subset is defined in a neighborhood of it. Subsets of positive reach are defined to be the ones for which this holds.
- He showed that one could actually associate with each subset K of positive reach in $\mathbb{E}^n$ and each integer $k \leq n$ a measure on $\mathbb{E}^n$, called the k^{-th} curvature measure of K. When K is a smooth submanifold, its k^{-th} curvature measure evaluated on a set U is nothing but the integral of the k^{-th} mean curvature of K on U. Curvature measures thus give much finer information than the *Quermassintegrale* since they determine, in the smooth case, the k^{-th} mean curvatures at every point of the subset.

Unfortunately, Federer's approach could not handle some simple objects such as non-convex polyhedra. The next step has been accomplished by P. Wintgen and M. Zähle, [**33**], [**36**]. These authors noticed that in the smooth case, curvature measures arise as integrals over the unit normal bundle of the submanifold of $(n-1)-$differential forms on $ST\mathbb{E}^n$ that are invariant under rigid motion. The geometry of a submanifold is thus contained in the current determined by its unit normal bundle. Now as they showed, for certain union $\mathcal{A}$ of subsets of positive reach, a suitable generalization of this current can be defined, which they called the *normal cycle* $N(\mathcal{A})$ of $\mathcal{A}$. As a consequence, curvature measures can be defined for such subsets by integrating corresponding differential forms. One important property of the normal cycle is additivity: it satisfies

$$N(\mathcal{A} \cup \mathcal{A}') = N(\mathcal{A}) + N(\mathcal{A}') - N(\mathcal{A} \cap \mathcal{A}'), \tag{1}$$

whenever both sides are defined. In particular, the normal cycle of a non necessarily convex polyhedron can be computed from a triangulation of it by applying the inclusion-exclusion principle to the normal cycles of the simplices of the triangulation. Finally, J. Fu [**19**], [**16**], [**17**] showed that normal cycles could be defined for a very broad class of subsets which he called *geometric subsets.* In particular, semialgebraic sets, subanalytic sets and more generally definable sets are geometric (see [**20**] and [**5**], [**6**] for the last point).

The line of research proposed by J. Milnor has already received some attention in the past. The problem of *continuity* of curvature measures first appeared in the context of convex subsets of $\mathbb{E}^n$. It could be proved that if a sequence of convex bodies K_n has a Hausdorff limit K, then the Quermassintegrale of K_n converge to the ones of K. Using integral-geometric considerations, tight estimates can even be obtained for the difference between the total mean curvature of K_n and the one of K. But continuity with respect to the Hausdorff topology does not hold for smooth submanifolds (*resp.* subsets of positive reach), unless one assumes additionally that the curvatures of the sequence of submanifolds are uniformly bounded from above (*resp.* that the reach of the sequence of subsets with positive reach is bounded from below) [**14**]. Under these assumptions, estimates of differences of curvature measures are known. Also, a convergence theorem has been proved by J. Fu [**18**] for a sequence of triangulated polyhedra inscribed in a smooth surface and tending to it for the Hausdorff distance, under the assumption that the fatness of the triangulations is bounded from below. This theorem relies on the compactness theorem for integral currents; it does not give any quantitative information about the approximation.

The contribution of this article is twofold:

1) First of all, remark that the k^{th}-mean curvatures of a smooth hypersurface determine principal curvatures but not principal directions. In order to get a finer description of the geometry of singular sets, it is natural to look for a generalization of the second fundamental form of an immersion to the singular case. This is the goal of the first part of this article, which considers geometric compact subsets of a Riemannian manifold M and relies on normal cycle theory. Mimicking the construction of the invariant $(n-1)$-forms, we define a $(0,2)$-tensor valued $(n-1)$-form on the horizontal space of M, that we plug in the normal cycle of the considered geometric subset $\mathcal{K}$. In this way, we create a new curvature measure which we call *the second fundamental measure associated to* $\mathcal{K}$. Of course, when $\mathcal{K}$ is smooth, we get the integral of the second fundamental form.
2) The second part of this article deals with approximation of curvature measures, including the second fundamental measure defined in the first part. Apart from the case of convex subsets or the one of subsets with positive reach, no quantitative estimate of the difference between curvature measures of two "close" subsets seems to be known. In this article, under a certain condition, we bound the difference of the curvature measures of two geometric sets when one of them is a smooth hypersurface. In particular, we refine the result of J. Fu by giving a quantitative version of it. More precisely, we give an estimate of the flat norm of the difference of the normal cycle of a compact n-manifold K of $\mathbb{E}^n$ whose boundary is a smooth hypersurface and the normal cycle of a compact geometric subset $\mathcal{K}$ in terms of the mass of the normal cycle K, the Hausdorff distance between their boundary, the maximal

angle between the normal to K and the "normals" to $\mathcal{K}$, and an *a priori* upperbound on the norm of the second fundamental form of the boundary of K. We thereby give an answer to the question raised by J. Milnor in the special case where one of the two sets is smooth.

The paper is organized as follows: after a brief summary of the needed background and notations, we define in Section 3 the tensor $\mathbf{h}$ that generalizes the second fundamental form of a hypersurface. We give its explicit expression for polyhedra. Section 4 is devoted to the approximation theorem abovementioned. Finally, Section 5 gives corollaries in the case where a smooth surface of $\mathbb{E}^3$ is approximated by polyhedra.

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2. Background and notations

2.1. A brief survey on the geometry of a tangent manifold. We refer to [**22**] [**23**] for details on the construction and the main properties of the second tangent bundle of a manifold. In this article, M denotes a smooth n-dimensional oriented manifold, $(TM \xrightarrow{\pi} M)$ its tangent bundle and $(TTM \xrightarrow{\pi_{TM}} TM)$ its second tangent bundle. We shall deal with the following diagram:

$$\begin{array}{ccc} TTM & \xrightarrow{d\pi} & TM \\ \pi_{TM} \downarrow & & \downarrow \pi \\ TM & \xrightarrow{\pi} & M \end{array}$$

and the following exact sequence of vector bundles over TM:

$$0 \to TM \times_M TM \xrightarrow{i} TTM \xrightarrow{j} TM \times_M TM \to 0,$$

where i is the natural injection defined by

$$i(u_1, u_2) = \frac{d}{dt}(u_1 + tu_2)_{|t=0},$$

and $j = (\pi_{TM}, d\pi)$. $J = i \circ j$ is an almost tangent structure of TM, $(J^2 = 0)$. The *vertical bundle*, that is, the kernel of j is denoted by $V(M)$. If m is a point of M and $w, z \in T_mM$, $z^v = i_w(z)$ is the *vertical lift* of z at w. In the next paragraphs, we shall use the *canonical vertical vector field* C associated to the one parameter group of homotheties with a positive ratio acting on the fibers of TM.

Suppose now that M is endowed with a Riemannian metric $\langle , \rangle$, and let ∇ be its Levi-Civita connection. This (linear) connection can be considered as a right splitting of the exact sequence, that is, a bundle morphism

$$\gamma : TM \times_M TM \to TTM,$$

such that

$$j \circ \gamma = Id_{TM \times_M TM}.$$

If $w, z \in T_m M$, $z^h = \gamma_w(z)$ is the *horizontal lift* of z at w and $H_w(M) = \mathrm{Im}(\gamma(w, .))$ is the *horizontal bundle* at w. At every point z of TM, one has:

$$T_z TM = V_z(M) \oplus H_z(M).$$

We denote by $h : TTM \to TTM$ the *horizontal projection* on the horizontal bundle H, and by $v = \mathrm{Id} - h$ the *vertical projection* on the vertical bundle $V(M)$.

We also endow the bundle $(\pi_{TM} : TTM \to TM)$ with the (Riemannian) *Sasaki metric* (still denoted by $\langle , \rangle$), such that the vertical subbundle and the horizontal subbundle are orthogonal, and with the almost complex structure F, $(F^2 = -Id)$ defined by

$$\begin{array}{rcl} FJ & = & h, \\ Fh & = & -J \end{array}$$

compatible with the Sasaki metric. The 2-form $\mathbf{\Omega} = \langle ., F. \rangle$ on TM is the canonical symplectic structure. It is exact, $\mathbf{\Omega} = d\alpha$, where α is the *Liouville form*, that is the 1-form dual to FC, in the duality induced by the Sasaki metric.

In our context, it will be useful to deal with the connection forms, and to use the Maurer-Cartan formalism. We introduce the bundle $\mathcal{T}M = TM \backslash \{0\}$, that is, the tangent bundle without the 0-section. Let $z_m \neq 0$ be a point of $\mathcal{T}M$. Let $(e_1, \ldots, e_n)$ be any orthonormal frame $T_m M$, such that $e_n = \frac{z}{\|z\|}$. We denote by $(e_1^*, \ldots, e_n^*)$ the dual frame. If ω denotes the (Levi-Civita) connection form on TM (considered as a vector valued one form taking its values in the Lie algebra of $SO(n)$), the Maurer-Cartan structure equations can be written as:

$$de_i^* = \omega_i^j \wedge e_j^*, \;\; d\omega_i^j = \omega_i^k \wedge \omega_k^j + \Omega_i^j,$$

where Ω_i^j are the curvature forms of the connection, related to the curvature tensor R of M by:

$$\Omega_i^j(X, Y) = \langle R(X, Y)e_i, e_j \rangle, \forall X, Y \in TM.$$

Now we take the pullback of the 1-forms e_i^* by π. We get n covectors $(\theta_1, \ldots, \theta_n)$ on TM, null on the vertical bundle $V(M)$: $\theta_i = \pi^*(e_i^*), \forall i$. One has: $\forall X, Y \in TM$,

$$d\theta_i(X^v, Y^v) = 0,$$

$$d\theta_i(X^h, Y^h) = (\omega_i^j \wedge e_j^*)(X, Y), \tag{2}$$

$$d\theta_i(X^h, Y^v) = 0.$$

We define the 1-forms ϖ_i^j on $\mathcal{T}M$ by

$$\varpi_i^j(X^h) = \omega_i^j(X), \varpi_i^j(X^v) = 0, \forall X \in TM.$$

Finally, we have: $d\theta_i = \varpi_i^j \wedge \theta_j$.

Associated to this frame, we define the n 1-forms $\Theta_i = F^*(\theta_i), \forall i$. These forms are null on the horizontal bundle $H(M)$, and satisfy: $\Theta_i(e_j^v) = \delta_{ij}$. One has:

$$d\Theta_i = \varpi_i^k \wedge \Theta_k + \Re_i^n, \tag{3}$$

where $\Re_i^n$ are the 2-forms on $\mathcal{T}M$ defined at z by:

$$\Re_i^n(X, Y) = \langle R(d\pi(X), d\pi(Y))z, e_i\rangle.$$

2.2. The Lipschitz-Killing $(n-1)$-forms on $\mathcal{T}M$. With the notations of the previous section, consider the $(n-1)$-form

$$(\theta_1 + t\Theta_1) \wedge \cdots \wedge (\theta_{n-1} + t\Theta_{n-1})_z.$$

When z varies, this defines a differential $(n-1)$-form on $\mathcal{T}M$. This expression can be considered as a polynomial in the variable t; (remark that the coefficient ϕ_k of t^k is a differential form which does not depend on the orthonormal frame $(e_1, \dots, e_n = \frac{z}{\|z\|})$). Trivially one has:

$$\phi_k = \sum_\pi (-1)^{|\pi|} \theta_{\pi(1)} \wedge \cdots \wedge \theta_{\pi(k)} \wedge \Theta_{\pi(k+1)} \wedge \cdots \wedge \Theta_{\pi(n-1)}.$$

Classically, for $1 \leq k \leq n-1$, the $(n-1)$-form ϕ_k on $\mathcal{T}M$ is called the *k^{th}-Lipschitz-Killing curvature form.* Usually, these forms are only defined on the unit tangent bundle STM. With the Liouville 1-form α and the symplectic 2-form $\mathbf{\Omega}$ they generate the $C^\infty(M)$ exterior algebra of differential forms on STM invariant on each fiber by the group of rotations.

In the following, the bundle $\Lambda(TM)$ of exterior forms on TM will be endowed with its classical comass norm induced by the Riemannian structure on TM: if $\Phi \in \Lambda^k(TM)$,

$$\|\Phi\| = \sup_{z \in TM} \{\langle \Phi, Z\rangle, Z \text{ unit simple k-vector of } T_zTM\}.$$

The comass norms of the Lipschitz-Killing forms are bounded as follows:

Proposition 1. *Let M be a n-dimensional Riemannian manifold. Then,*

- *each invariant form ϕ_k satisfies $\|\phi_k\| = 1$;*

- *moreover, if the norm of the curvature tensor of M is bounded by a positive constant R, then*

$$\|d\phi_k\| \leq C(k, n, R),$$

where $C(k,n,R)$ is a positive constant depending on the dimension and on the bound R.

Proof of Proposition 1. The first item is trivial. For the second one, one has:

$$\begin{aligned} d\phi_k &= \sum \pm d[\theta_{\pi(1)} \wedge \cdots \wedge \theta_{\pi(k)} \wedge \Theta_{\pi(k+1)} \wedge \cdots \wedge \Theta_{\pi(n-1)}] \\ &= \sum \pm \theta_{\pi(1)} \wedge \cdots \wedge d\theta_{\pi(j)} \wedge \cdots \wedge \theta_{\pi(k)} \wedge \Theta_{\pi(k+1)} \wedge \cdots \wedge \Theta_{\pi(n-1)} \\ &\quad + \sum \pm \theta_{\pi(1)} \wedge \cdots \wedge \cdots \wedge \theta_{\pi(k)} \wedge \Theta_{\pi(k+1)} \wedge \cdots \\ &\qquad\qquad \wedge d\Theta_{\pi(k+l)} \wedge \cdots \wedge \Theta_{\pi(n-1)}. \end{aligned}$$

To bound $\|d\phi_k\|$, we use equations (2) and (3), replacing the terms $d\theta_i$ and $d\Theta_i$ by their values in terms of ϖ_i^j and $\Re_i^n$. We get a sum of indecomposable forms which are the wedge products of θ_i, Θ_i and $\Re_i^n$, (the terms involving $\varpi_i^j, 1 \leq i, j \leq n-1$ cancel). The conclusion follows.
q.e.d.

Remarks.

1) In particular, if M is flat, ($R = 0$), then we deduce that $d\phi_k$ has an expression of the type $\sum \pm\theta_{i_1} \wedge \cdots \wedge \omega^j_{i_{n-k-1}} \wedge \Theta_{i_{n-k}} \wedge \cdots \wedge \Theta_{n-1}$. The norm of each decomposable term of this sum is 1, each term appearing at most $k+1$ times, and the terms of type $\sum \pm\theta_{i_1} \wedge \cdots \wedge \Theta_{i_{n-k-1}} \wedge \Theta_{i_{n-k}} \wedge \cdots \wedge \Theta_{n-1}$ appearing $k+1$ times. We deduce that

$$\|d\phi_k\| = k+1, \forall k \leq n-2;$$
$$\|d\phi_{n-1}\| = 0.$$

2) In the case where the manifold is the three dimensional Euclidean space $\mathbb{E}^3$, we get three 2-forms on $\mathcal{T}\mathbb{E}^3$. We give here their explicit expressions in the standard frame $(x_1, x_2, x_3, y_1, y_2, y_3)$, at any point (m, y) such that $\|y\| = 1$:

$$\phi_{\mathcal{A}} = y_1 dx_2 \wedge dx_3 + y_2 dx_3 \wedge dx_1 + y_3 dx_1 \wedge dx_2;$$
$$\phi_{\mathcal{G}} = y_1 dy_2 \wedge dy_3 + y_2 dy_3 \wedge dy_1 + y_3 dy_1 \wedge dy_2;$$
$$\begin{aligned} \phi_{\mathcal{H}} &= y_1(dx_2 \wedge dy_3 + dx_2 \wedge dy_3) + y_2(dx_3 \wedge dy_1 + dy_3 \wedge dx_1) \\ &\quad + y_3(dx_1 \wedge dy_2 + dy_1 \wedge dx_2). \end{aligned}$$

The norm of these forms is attained on $ST\mathbb{E}^3$, from which we deduce that

$$\|\phi_{\mathcal{A}}\| = \|\phi_{\mathcal{G}}\| = \|\phi_{\mathcal{H}}\| = 1,$$
$$\|d\phi_{\mathcal{A}}\| = 1, \quad \|d\phi_{\mathcal{G}}\| = 0, \quad \|d\phi_{\mathcal{H}}\| = 2.$$

3) Suppose now that the manifold is the unit sphere S^3. In this case, the curvature form is non zero, and satisfies:

$$\Omega_i^j = \theta_i \wedge \theta_j.$$

We deduce by a direct computation:

$$\|\phi_{\mathcal{A}}\| = \|\phi_{\mathcal{G}}\| = \|\phi_{\mathcal{H}}\| = 1,$$
$$\|d\phi_{\mathcal{A}}\| = 1, \quad \|d\phi_{\mathcal{G}}\| = 2, \quad \|d\phi_{\mathcal{H}}\| = 2.$$

2.3. Normal cycles of geometric subsets. We denote by $\mathcal{D}^m$ the $\mathbb{R}$-vector space of C^∞ differential m-forms with compact support on M^n, endowed with the C^∞ topology. The topological dual of $\mathcal{D}^m$ is the $\mathbb{R}$-vector space $\mathcal{D}_m$ of *currents* on M^n. In the following, we can assume that all the currents we consider have a support included in a fixed compact subset of TM. The mass $\mathbf{M}$ and the flat semi-norm $\mathcal{F}$ are defined on $\mathcal{D}_m$ as follows: for every T in $\mathcal{D}_m$,

$$\mathbf{M}(T) = \sup\{T(\phi), \phi \in \mathcal{D}^m, \|\phi\| \leq 1\};$$
$$\mathcal{F}(T) = \sup\{T(\phi), \phi \in \mathcal{D}^m, \|\phi\| \leq 1, \|d\phi\| \leq 1\}.$$

Using Hahn-Banach theorem, it can be proved [**15**] that

$$\mathcal{F}(T) = \inf\{\mathbf{M}(T - \partial S) + \mathbf{M}(S), S \in \mathcal{D}_{m+1}\}. \tag{4}$$

The currents we deal with in the theory of *normal cycle* are closed *integral* m-currents, that is, *rectifiable* currents whose boundaries are rectifiable. They are supported in the unit tangent manifold STM, which is canonically endowed with the contact structure defined by the restriction of the Liouville form α. If T is a $(n-1)$-current supported in STM, T is said to be *Legendrian* if it cancels α and the symplectic form $\mathbf{\Omega}$, that is

$$\forall \phi \in \mathcal{D}^{n-2}(STM), \langle T, \phi \wedge \alpha \rangle = 0,$$
$$\forall \phi \in \mathcal{D}^{n-3}(STM), \langle T, \phi \wedge \mathbf{\Omega} \rangle = 0.$$

As explained in the introduction, the curvature measures of a smooth submanifold arise as integrals of the Lipschitz-Killing forms on its unit normal bundle. The normal cycle is a closed current generalizing the unit normal bundle for a certain class of subsets, called geometric subsets by J. Fu [**17**]. J. Fu's approach is rather indirect: he exhibits certain properties any reasonable definition of a normal cycle should satisfy, and then shows that there is at most one current satisfying the properties. Geometric subsets are the ones for which this current exists.

If the ambient space is a Euclidean space, the fundamental result of J. Fu can be stated as follows [**17**]:

Let $\mathcal{K}$ be a compact subset of $\mathbb{E}^n$. Consider the function

$$i_{\mathcal{C}} : ST\mathbb{E}^n \to \mathbb{R},$$

defined by

$$i_{\mathcal{K}}(m,\xi) = \lim_{r\to 0}\lim_{s\to 0}[\chi(\mathcal{K}\cap B(m,r)\cap\{p \text{ such that } (p-m).\xi \leq t\})|_{t=-s}^{t=+s}].$$

Remark that when $\mathcal{K}$ is a stratified set, $i_{\mathcal{K}}(m,\xi)$ is just the index of m as critical point of the height function defined by ξ. Moreover $i_{\mathcal{K}}$ may be not defined, but if $i_{\mathcal{K}}$ exists and $\sum_{m\in\mathbb{E}^n}\phi(m,\xi)i_{\mathcal{K}}(m,\xi)$ is finite for almost every ξ in S^{n-1}, J. Fu proved the following:

Theorem A. *Let $\mathcal{K}$ be a compact subset of $\mathbb{E}^n$. There exists at most one closed compactly supported integral $(n-1)$-current $N(\mathcal{K})$ of $ST\mathbb{E}^n$ such that*

- *$N(\mathcal{K})$ is Legendrian,*
- *for all smooth functions ϕ in $ST\mathbb{E}^n$,*

$$N(\mathcal{K})(\phi(m,\xi)dv_{S^{n-1}}) = \int_{S^{n-1}}\sum_{m\in\mathbb{E}^n}\phi(m,\xi)i_{\mathcal{K}}(m,\xi)dv_{S^{n-1}}.$$

Following [**20**], any compact subset $\mathcal{A}$ of M such that $N(\mathcal{A})$ exists is said to be *geometric*, and $N(\mathcal{A})$ is called its normal cycle.

The main property of the normal cycle is its additivity. More precisely, if $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{A}\cap\mathcal{A}'$ are geometric, then $\mathcal{A}\cup\mathcal{A}'$ is geometric and equation (1) is satisfied.

Let us examine classical examples:

- The normal cycle of a smooth submanifold of M is nothing but its unit normal bundle.
- If σ is a k-simplex of $\mathbb{E}^n$, $N(\sigma)$ is its normal cone, that is the integral closed current of $\mathbb{E}^n\times S^{n-1}$ defined by the (closed oriented) $(n-1)$-submanifold of $\mathbb{E}^n\times S^{n-1}$ given by
$$S = \{(m,\xi)\in\mathbb{E}^n\times S^{n-1} \text{ such that } \langle\vec{mp},\xi\rangle \leq 0, \forall p\in\sigma\}.$$
- To compute the normal cycle of a polyhedron P of $\mathbb{E}^n$, we decompose it in simplices, and we use (1). The resulting current is independent on the decomposition.

2.4. Lipschitz-Killing curvature measures of a geometric subset. The k^{th}-Lipschitz-Killing curvature $\mathcal{M}_k^{\mathcal{K}}$ associated to a geometric subset $\mathcal{K}$ of a Riemannian manifold M is defined by:

$$\mathcal{M}_k^{\mathcal{K}}(B) = \langle N(\mathcal{K}), \chi_{\pi^{-1}(B)}\phi_k\rangle,$$

for all Borel subsets B. From now on, and to simplify the notations, we will put

$$\langle N(\mathcal{K})_{\pi^{-1}(B)},\phi\rangle = \langle N(\mathcal{K}),\chi_{\pi^{-1}(B)}\phi\rangle.$$

In other words and roughly speaking, $N(\mathcal{K})_{\chi_{\pi^{-1}(B)}}$ denotes *"the part of the normal cycle which lies above B"*. Moreover, we put $\mathcal{M}_k^{\mathcal{K}}(\mathcal{K}) = \mathcal{M}_k(\mathcal{K})$. The explicit expression of these measures can be given in particular cases:

1) Suppose that the geometric subset is a compact domain K whose boundary is a smooth compact oriented hypersurface W:
$$x : W \hookrightarrow M,$$
is a codimension one (isometric) immersion of an (oriented) Riemannian manifold W into an (oriented) Riemannian manifold M. For simplicity, we identify as usual W and its image by x. Let ξ be the unit normal vector field compatible with the orientation. We will use the following notations:
$$\hbar : TW \times TW \to \mathcal{C}^{\infty}(W)$$
denotes the second fundamental form of the immersion x in the direction ξ, and A_ξ denotes the Weingarten endomorphism (in the direction ξ). One has with obvious notations, $\forall X, Y \in TW$,
$$\nabla_X Y = \nabla'_X Y + \hbar(X,Y)\xi, \quad \nabla_X \xi = -A_\xi X.$$
We denote by $\Xi_k = \{\lambda_{i_1}, \ldots, \lambda_{i_k}\}$ the k^{-th}-elementary symmetric function of the principal curvatures of W, and by dv_W the volume element of W. Then, if B is any Borel subset of M (see [**31**] for instance):
$$\mathcal{M}_k^K(B) = \int_{W \cap B} \Xi_k dv_W,$$
where dv_W denotes the volume form of W.
2) Suppose now that the geometric set is a polyhedron P of $\mathbb{E}^n$, (endowed with a triangulation —the results will not depend on the triangulation—), that is the underlying space of a linearly embedded simplicial complex, supposed finite. To evaluate the Lipschitz-Killing measures of P, we begin to evaluate them on the elementary simplices. If σ^l is a l-simplex of $\mathbb{E}^n$ then one gets, for every k,
$$\mathcal{M}_k(\sigma^l) = \sum_{\sigma^{n-k} \subset \sigma^l} \mathrm{Vol}_{n-k}(\sigma^{n-k}).(\sigma^{n-k}, \sigma^l)^*,$$
and for any Borel subset B of W,
$$\mathcal{M}_k^{\sigma^l}(B) = \sum_{\sigma^{n-k} \subset \sigma^l} \mathrm{Vol}_{n-k}(\sigma^{n-k} \cap B).(\sigma^{n-k}, \sigma^l)^*,$$
where $(\sigma^q, \sigma^l)^*$ denotes the exterior dihedral angle.

The Lipschitz-Killing curvatures of a polyhedron P are evaluated by considering P as a union of simplices, and applying formula (1). These results are consistent with [**9**].

3. The second fundamental measure

3.1. The fundamental $(n-1)$-form. In this section, we introduce a $(n-1)$-form on $\mathcal{T}M$ depending (bi)linearly on two horizontal vector fields. Let $\mathcal{H}$ be the $(n-1)$ dimensional subbundle of the horizontal bundle $H(M)$ orthogonal to FC:

$$\mathcal{H} = FC^{\perp_H}.$$

We build a tensor field of type $(0,2)$

$$\mathbf{h} : \mathcal{H} \times \mathcal{H} \to \Lambda^{n-1}\mathcal{T}M,$$

acting on $\mathcal{H}$ and taking its values in the space of differential $(n-1)$-forms on $\mathcal{T}M$, in the following way: let U be an open neighborhood of a point $z \in \mathcal{T}M$. One more time, let $(e_1, \dots, e_n)$ be a local orthonormal frame on U, such that $e_n = \frac{z}{\|z\|}$. Let $\theta_i = \pi^*(e_i^*), 1 \le i \le n$. For two fixed vectors $e_{i_0} \neq e_n, e_{j_0} \neq e_n$, we define the $(n-1)$-form

$$\mathbf{h}_{i_0,j_0} = (-1)^{n-i_0}\theta^1 \wedge \cdots \wedge \hat{\theta^{i_0}} \wedge \cdots \wedge \hat{\theta^n} \wedge \Theta_{j_0}.$$

When the two indices i_0, j_0 vary, the previous formula defines a tensor of type $(0,2)$, independent of the orthonormal local frame $(e_1, \dots, e_n)$: if $X, Y \in \mathcal{H}, X = \sum_{i=1}^{n-1} X^i e_i^h, Y = \sum_{i=1}^{n-1} Y^i e_i^h$, then

$$\mathbf{h}(X,Y) = \sum_{i,j} X^i Y^j \mathbf{h}_{i,j}.$$

To simplify the notations, we put $\mathbf{h}(X,Y) = \mathbf{h}^{X,Y}$.

Remark. Here is a global construction of the same tensor $\mathbf{h}$: let $X, Y \in \mathcal{H}$. Let τ_X be the $(n-2)$-form on $\mathcal{H}$ defined by

$$\tau_X(u_1, \dots, u_{n-2}) = \det(FC, X, u_1, \dots, u_{n-2}).$$

Let Y^* be the 1-form dual to FY in $V(M)$. Then,

$$\mathbf{h} : \mathcal{H} \times \mathcal{H} \to \Lambda^{n-1}\mathcal{T}M,$$

satisfies

$$\mathbf{h}(X,Y) = \tau_X \wedge Y^*.$$

We give

Definition 1. The form $\mathbf{h}^{X,Y}$ is called the **fundamental $(n-1)$-form** associated to the couple (X,Y).

The tensor $\mathbf{h}$ has an interesting property involving the symplectic structure $\mathbf{\Omega}$ on TM:

Proposition 2. *For every X, Y in $\mathcal{H}$, there exists a $(n-3)$-form $\Psi^{X,Y}$ on $\mathcal{T}M$ such that:*

$$\mathbf{h}^{X,Y} - \mathbf{h}^{Y,X} = \mathbf{\Omega} \wedge \Psi^{X,Y}.$$

Proof of Proposition 2. Using the linearity, we shall deal with an orthonormal frame θ^i, Θ^j. The only interesting case occurs when we consider two indices $\mathbf{h}_{i_0 j_0}$ with i_0 and j_0 different of n. In this case,

$$\begin{aligned}
\mathbf{h}_{i_0 j_0} - \mathbf{h}_{j_0 i_0} &= (-1)^{n-i_0}\theta^1 \wedge \cdots \wedge \hat{\theta^{i_0}} \wedge \cdots \wedge \theta^j \wedge \cdots \wedge \hat{\theta^n} \wedge \Theta_{j_0} \\
&\quad - (-1)^{n-j_0}\theta^1 \wedge \cdots \wedge \hat{\theta}^{j_0} \wedge \cdots \wedge \theta^j \wedge \cdots \wedge \hat{\theta^n} \wedge \Theta_{i_0} \\
&= \pm(\theta^{i_0} \wedge \Theta_{i_0} + \theta^{j_0} \wedge \Theta_{j_0}) \wedge (\Lambda_{k \neq i_0, j_0, n}\theta^k) \\
&= \pm \mathbf{\Omega} \wedge (\theta^i \wedge \cdots \wedge \theta^j).
\end{aligned}$$

q.e.d.

The following proposition gives a bound on the norm of $\mathbf{h}$ and $d\mathbf{h}$. The norm $\|.\|$ is the usual one, and the norm $\|.\|_1$ is defined by $\|X\|_1 = \sup(\|X\|, \|\nabla X\|)$, where ∇ is the covariant derivative in TM.

Proposition 3. *For all $X, Y \in \mathcal{H}$, one has:*

- $\|\mathbf{h}^{X,Y}\| \leq C(n,R)\|X\|\|Y\|$, *where $C(n,R)$ is a real constant depending on the dimension of M, and on the norm of its curvature tensor;*
- $\|d\mathbf{h}^{X,Y}\| \leq C_1(n,R)\|X\|_1\|Y\|_1$, *where $C_1(n,R)$ is a real constant depending on the dimension of M, and on the norm of its curvature tensor.*

Sketch of proof of Proposition 3. The map

$$\mathbf{h} : \mathcal{H} \times \mathcal{H} \to \Lambda(TM)$$

defined by $\mathbf{h}(X,Y) = \mathbf{h}^{X,Y}$ is bilinear and C^∞; the differential

$$d : (\Lambda(TM), \|.\|_1) \to (\Lambda(TM), \|.\|)$$

is linear and continuous. Using Maurer-Cartan equations, we see in the local expression of $\mathbf{h}$ that the differential of terms of type $d\Theta_i$ involves the curvature tensor of M. The conclusion follows by simple computations. q.e.d.

Remark. It must be noticed that this construction can be generalized as follows: instead of taking two indices i_0, j_0, it is possible to take an arbitrary number of indices $l_1, \ldots, l_p$ different to n and to consider the tensor $\mathbf{h}_{l_1,\ldots,l_p}$ obtained by taking out the 1-forms θ_{l_i} and adding the corresponding Θ_{l_i}. In such a way, one constructs new tensors

$$\mathbf{h}_{l_1,\ldots,l_p} = \theta^1 \wedge \cdots \wedge \hat{\theta^{l_1}} \wedge \cdots \wedge \hat{\theta^{l_p}} \wedge \cdots \wedge \hat{\theta^n} \wedge \Theta_{l_1} \wedge \cdots \wedge \Theta_{l_p}.$$

For instance, if we take four indices, and if we plug the resulting tensor on the unit normal bundle of a hypersurface, one gets an expression involving its curvature tensor, by Gauss equation. The computation is left to the reader. An extensive study of such tensors has been done by L. Bröcker and A. Bernig, [**6**], [**5**], [**4**].

3.2. Second fundamental measure of geometric set. In the same flavor as 2.4, we give

Definition 2. Let $\mathcal{K}$ be a compact geometric subset of M, and X, Y be any vector fields lying in $\mathcal{H}$. Then, the second fundamental measure $\mathbf{h}_{\mathcal{K}}^{X,Y}$ associated to $\mathcal{K}$ in the directions X, Y is defined by

$$\mathbf{h}_{\mathcal{K}}^{X,Y}(B) = \langle N(\mathcal{K}), \chi_{\pi^{-1}(B)}\mathbf{h}^{X,Y}\rangle,$$

for every Borel subset B of M.

Here is a remarkable symmetry property of $\mathbf{h}_{\mathcal{K}}$:

Proposition 4. *Let $\mathcal{K}$ be any geometric subset and B be any Borel subset of M. Then $\mathbf{h}_{\mathcal{K}}^{X,Y}(B)$ is symmetric in X, Y.*

Proof of Proposition 4. This is a direct consequence of the fact that normal cycles are Legendrian, and thus cancel the symplectic form $\mathbf{\Omega}$ restricted to STM. Using Proposition 2 and its notations,

$$\mathbf{h}_{\mathcal{K}}^{X,Y}(B) - \mathbf{h}_{\mathcal{K}}^{Y,X}(B) = \langle N(\mathcal{K}), \chi_{\pi^{-1}(B)}\mathbf{\Omega} \wedge \Psi^{X,Y}\rangle = 0.$$

q.e.d.

As in 2.4 1. and 2., we can give an explicit expression of $\mathbf{h}$ in particular cases:

1) First of all, suppose that K is a compact domain of M whose boundary is an oriented (smooth) hypersurface of W. We need to introduce the Gauss map G associated to the immersion of W:

$$G : W \hookrightarrow TM,$$

defined by

$$G(m) = (m, \xi_m).$$

Using the isomorphism $j_{\xi_m} \times \varsigma_{\xi_m}$ between $H_{\xi_m}(M) \times V_{\xi_m}(M)$ and $T_mM \times T_mM$, we get:

$$(j_{\xi_m} \times \varsigma_{\xi_m}) \circ dG(X_m) = ((m, X_m), (m, -A_{\xi_m}(X)).$$

Proposition 5. *One has: $\forall X, Y \in TW$,*

$$\hbar(X, Y)dv_W = G^*\mathbf{h}(X^h, Y^h),$$

where dv_W denotes the volume form of W.

Proof of Proposition 5. Let $e_1, \ldots, e_n$ be a local frame of M such that $e_1, \ldots, e_{n-1}$ are tangent to W and e_n is normal to M. Let (e_{i_0}, e_{j_0}) be two vectors of this frame, different to e_n. One has

$$\begin{aligned}
&G^*\mathbf{h}(e_{i_0}^h, e_{j_0}^h)(e_1, \ldots, e_{n-1}) \\
&= \mathbf{h}(e_{i_0}^h, e_{j_0}^h)(dG(e_1), \ldots, dG(e_{n-1})) \\
&= \Theta_{j_0}(dG(e_{i_0})) = \hbar(e_{i_0}, e_{j_0}).
\end{aligned}$$

q.e.d.

A direct consequence of Proposition 5 is that, for $X, Y \in \mathcal{H}$, one has

$$G^*(\mathbf{h}^{X,Y}) = \hbar(d\pi(X), d\pi(Y))dv_W.$$

An immediate corollary can be stated as follows: let B be a Borel subset of M. We have:

Corollary 1. *For all $X, Y \in \mathcal{H}$, one has:*

$$\mathbf{h}_K^{X,Y}(B) = \int_{B\cap W} \hbar(d\pi(X), d\pi(Y))dv_W.$$

This is why we call $\mathbf{h}_{\mathcal{K}}$ the second fundamental measure associated with $\mathcal{K}$.

Remarks.

- Proposition 5 implies the symmetry of the tensor $G^*\mathbf{h}(.^h, .^h)$, since the second fundamental form $\hbar$ is symmetric; (this last property can be seen directly by using the fact that the normal bundle of the hypersurface is Lagrangian in TTW). In some sense, it is the infinitesimal version of Proposition 4, which establishes the symmetry of the second fundamental measure in the general case.
- Suppose now that the ambient space is $\mathbb{E}^n$. The canonical parallelism of $\mathbb{E}^n$ and $T\mathbb{E}^n$ allows to identify at each point m, $T_m\mathbb{E}^n$ and $\mathbb{E}^n$, and at each vector $\xi_m \in T_m\mathbb{E}^n$, $H_{\xi_m}(\mathbb{E}^n)$, $V_{\xi_m}(\mathbb{E}^n)$ and $\mathbb{E}^n$. Let X, (*resp.* Y) be a parallel vector field on $\mathbb{E}^n$, considered as a horizontal vector field, and let X', (*resp.* Y') be its (orthogonal) projection on the $(n-1)$-subbundle $\mathcal{H}$ of the horizontal bundle. If W is a hypersurface of $\mathbb{E}^n$, remark that the restriction of X', (*resp.* Y') to the unit normal bundle of W satisfies (using the previous identifications) $d\pi(X') = \mathrm{pr}_{TW}X$, (*resp.* $d\pi(Y') = \mathrm{pr}_{TW}Y$). Consequently, Corollary 1 can be stated as follows:

$$\mathbf{h}_K^{X',Y'}(B) = \int_{B\cap W} \hbar(\mathrm{pr}_{TW}X, \mathrm{pr}_{TW}Y)dv_W.$$

Moreover, since Y is parallel, $\nabla_{\mathrm{pr}_{TW}X}Y = 0$ (where ∇ denotes the Levi-Civita connection on $\mathbb{E}^n$). If we decompose the restriction of Y to W in its tangent and normal component,

$$Y = \mathrm{pr}_{TW}Y + \alpha_Y\xi,$$

we get:

$$\hbar(\mathrm{pr}_{TW}X, \mathrm{pr}_{TW}Y) = -(\mathrm{pr}_{TW}X)(\alpha_Y).$$

In other words, $\mathbf{h}_K^{X,Y}(B)$ measures the integral on $W \cap B$ of the variation of α_Y in the direction $\mathrm{pr}_{TW}X$.

2) Now, we assume that P is a n-dimensional polyhedron of $\mathbb{E}^n$. We shall evaluate $\mathbf{h}_P^{X,Y}$ for any vector fields $X, Y \in \mathcal{H}$. Since the normal cycle $N(P)$ can be decomposed as a sum of elementary currents, the support of which lies above each simplex of dimension $i, 1 \leq i \leq n-1$, we shall evaluate $\mathbf{h}_P^{X,Y}$ above each simplex. If σ^k is any k-dimensional simplex of P, the support of $N(P)$ lying above σ^k is the product of σ^k by a portion of a vertical $(n-k-1)$-sphere. In particular, the support of $N(P)_{|\sigma^{n-2}}$ is the product $\sigma^{n-2} \times C_\sigma$, where C is a portion of circle. Let $(e_1, \ldots, e_{n-2})$ be an orthonormal frame field tangent to σ^{n-2}. Any point of $\sigma^{n-2} \times C_\sigma$ is a couple (m, e_{n-1}), where m is a point of σ^{n-2} and e_{n-1} is a unit vector orthogonal to σ^{n-2}. With these notations, we have the following:

Theorem 1. *For every Borel set $B \subset \mathbb{E}^n$:*

$$\mathbf{h}_P^{X,Y}(B) = \sum_{\sigma^{n-2} \subset \partial P} \int_{\sigma^{n-2} \cap B \times C} \langle X, e^h_{(n-1)} \rangle \langle Y, e^h_{(n-1)} \rangle,$$

where $e^h_{(n-1)}$ denotes the horizontal lift of $e_{(n-1)}$.

Sketch of proof of Theorem 1. At each point of $ST\mathbb{E}^n$, the form $\mathbf{h}^{X,Y}$ is the wedge product of an $(n-2)$-form tangent to the horizontal bundle, and a 1-form tangent to the vertical bundle. Consequently, when we plug it in the normal cycle of P, the only non null contribution is given by the $(n-2)-$simplices of ∂P. The explicit computation trivially gives Theorem 1. q.e.d.

4. An approximation result

In this section, we shall compare the second fundamental measure (*resp.* curvature measures) of a compact domain K of a n-dimensional Riemannian manifold M whose boundary is a (smooth oriented) hypersurface W and the second fundamental measure (*resp.* curvature measures) of a geometric compact subset $\mathcal{K}$ with boundary $\mathcal{W}$ *close to it*. The result we obtain can be considered as a quantitative version of the *convergence theorem* of J. Fu, proved for sequences of triangulations, [**18**]. Remark that we shall not use the compactness theorem for currents, which is a crucial tool in [**18**]. We are not able to prove our result in whole generality: for a technical reason, we need a restrictive condition on a *"small part"* of the boundary $\mathcal{W}$ of $\mathcal{K}$. We introduce

Definition 3. Let $\mathcal{K}$ be a subset of the n-dimensional manifold M. $\mathcal{K}$ is said to be **weakly regular** if there exists a point of $\partial\mathcal{K}$ having a neighborhood in $\mathcal{K}$ diffeomorphic to a n-dimensional half space.

For instance, a codimension one polyhedron is weakly regular. This definition may seem somewhat artificial, but we will need it in the proof of Theorem 2, when we apply the constancy theorem for integral currents.

4.1. Fine tubular neighborhood of a hypersurface. Since W is smooth, there exists a tubular neighborhood U of W on which the orthogonal projection pr_W on W (or simply pr if there is no possible confusion) is well defined. Such a neighborhood U will be called a **fine** tubular neighborhood of W in M. For every point p in U, there exists a unique point $\mathrm{pr(p)} \in \mathrm{W}$ that realizes the distance from p to W. If $\mathcal{F}$ is the geodesic foliation defined on U orthogonal to W, the point $\mathrm{pr}(p)$ is the intersection point of W with the geodesic $\gamma^p : [0,1] \to U$ tangent to $\mathcal{F}$ and such that $\gamma^p(0) = p, \gamma^p(1) = \mathrm{pr}(p)$. Since γ^p hits W orthogonally, one has $\dot{\gamma}^p(1) = \pm d(p,W)\xi_{\mathrm{pr(p)}}$ (where ξ is the outward unit normal to W, and $d(p,W)$ denotes the distance from p to W). For any $t > 0$ (*resp.* $t < 0$), we denote by W_t the hypersurface parallel to W at distance t of K (*resp.* the complement of K). If t is small enough, W_t is smooth and included in U. We denote by $\hbar_t$ its second fundamental form. If B is any Borel subset of U, we shall consider the domain $\tilde{B} \subset U$ spanned by the geodesics of $\mathcal{F}$ begining at B and ending on W. We need the following geometric quantities. We denote by δ_B the Hausdorff distance between B and W, by $\|\hbar_{\mathrm{pr_{W_t}}(B)}\|$ the supremum of the (operator) norm of the second fundamental form of W_t on $\mathrm{pr}_{W_t}(B)$.

Definition 4. A subset $\mathcal{A}$ of M is said to be **strongly close** to W if

1) $\mathcal{A}$ lies in a fine tubular neighborhood of W,
2) the orthogonal projection pr defines an homeomorphism from $\mathcal{A}$ onto W.

From now on, we assume that $\mathcal{K}$ and K are such that $\mathcal{W}$ is strongly close to W. The main quantity involved in the study of the couple K and $\mathcal{K}$ is the *angular deviation*. We give now a precise definition. We need some notations: If (p, n_p) is a point of spt $\mathrm{N}(\mathcal{K})$, we denote by $\mathbf{n}$ the vector field parallel along the geodesic γ^p and whose initial value is n_p.

Definition 5. Let $p \in \mathcal{W}$. The **angular deviation** between p and $\mathrm{pr}(p)$ is the maximal angle α_p between $\mathbf{n}_{\mathrm{pr}(p)}$ and $\xi_{\mathrm{pr}(p)}$, when $(p, n_p) \in \mathrm{spt}N(\mathcal{K})$. If B is any Borel subset of $\mathcal{W}$, the **angular deviation** between B and $\mathrm{pr}(B)$ is the real number $\alpha_B = \sup_{p\in B} \alpha_p$.

4.2. A homotopy between normal cycles. We keep the notations and definitions of the introduction of Section 4 and Subsection 4.1. By rescaling the metric, one can always assume that *the norm of the sectional curvatures of M is bounded by* 1. To simplify our computations, we shall use this convention in the following theorem and its corollaries.

Theorem 2. *If $\mathcal{W}$ is weakly regular and strongly close to W, then for any (connected) Borel subset $B \subset \mathcal{W}$,*

$$\begin{aligned}&\mathcal{F}(N(\mathcal{K})_{|T_B M} - N(K)_{|T_{\mathrm{pr}_W(B)}M})\\ &\leq c(n)(\delta_B + \alpha_B)[e^{2\delta_B}(1 + \hbar_{\mathrm{pr}_W(B)})^4]^{n-1}\\ &\qquad \cdot (\mathbf{M}(N(\mathcal{K})_{|T_B M}) + \mathbf{M}(\partial N(\mathcal{K}_{|T_B M}))),\end{aligned}$$

where $c(n)$ is a constant depending on the dimension n of M.

In particular, if the ambient space M is the Euclidean space $\mathbb{E}^n$ then $\delta_B \|\hbar_{\mathrm{pr}_W(B)}\| < 1$ and

$$\begin{aligned}&\mathcal{F}(N(\mathcal{K})_{|T_B M} - N(K)_{|T_{\mathrm{pr}_W(B)}M})\\ &\leq (\delta_B + \alpha_B)\left[\frac{2(1 + \|\hbar_{\mathrm{pr}_W(B)}\|)}{1 - \delta_B\|\hbar_{\mathrm{pr}_W(B)}\|}\right]^{n-1}\\ &\qquad \cdot (\mathbf{M}(N(\mathcal{K})_{|T_B M}) + \mathbf{M}(\partial N(\mathcal{K}_{|T_B M}))).\end{aligned}$$

The rest of this section is devoted to the proof of Theorem 2. Let U be a fine tubular neighborhood of W in which $\mathcal{W}$ lies. Consider the map f defined by the following diagram:

$$\begin{array}{ccc} T_U M & \xrightarrow{f} & \mathrm{spt}\ N(K) \\ \pi \downarrow & & \uparrow G \\ U & \xrightarrow{\mathrm{pr}} & W \end{array}$$

To simplify the notations, we define the $(n-1)$-current D by $D = N(\mathcal{K})_{|T_B M}$ and E by $E = N(K)_{|T_{\mathrm{pr}(B)}M}$.

Lemma 1. *One has:*

$$f_\sharp(D) = E.$$

Proof of Lemma 1. We apply the constancy theorem ([**15**]) to f, $N(\mathcal{K})$ and $N(K)$: since the support of the image by f of $N(\mathcal{K})$ is included in the support of $N(K)$, there exists an integer c such that

$$f_\sharp(N(\mathcal{K})) = cN(K).$$

We need to prove that $c = 1$. First of all, by a classical property of (proper) smooth maps between currents, (see [**15**], p. 359), one has:

$$f_\sharp(N(\mathcal{K})_{|f^{-1}(A)}) = f_\sharp(N(\mathcal{K}))_{|A},$$

for every Borel subset A of $T_U M$. Thus

$$f_\sharp(N(\mathcal{K})_{|f^{-1}(A)}) = cN(K)_{|A}. \tag{5}$$

We need to prove that c=1. By assumption, $\mathcal{W}$ contains a point having a neighborhood (in $\mathcal{K}$) diffeomorphic to a half space. In this neighborhood there exists a neighborhood $\mathcal{U}$ whose closure is diffeomorphic to a half ball. The subset $\mathcal{K}$ is the union of $\mathcal{U}$ and the closure of $\mathcal{K}\backslash\mathcal{U}$. Since $\mathcal{U}$, K and $\overline{\mathcal{K}\backslash\mathcal{U}} \cap \mathcal{U}$ are geometric, one can apply the additivity property of the normal cycle ([**17**]), and deduce that the normal cycle of $\mathcal{K}$ over $\mathcal{U}\cap\mathcal{W}$ is the current associated to the unit normal bundle (of dimension 1) of $\mathcal{U}\cap\mathcal{W}$. Since the restriction of pr_W to $\mathcal{U}$ is one-one onto $\mathrm{pr}_W(\mathcal{U})$, the restriction of f to the support of $N(\mathcal{K})_{|T_{\mathcal{U}}M}$ is (smooth and) one-one onto $N(K)_{|T_{\mathrm{pr}_W\mathcal{U}}M}$, and

$$f_\sharp((N(\mathcal{K})_{|T_{\mathcal{U}}M}) = N(K)_{|T_{\mathrm{pr}_W(\mathcal{U})}M}.$$

Taking $A = T_{\mathrm{pr_W}(\mathcal{U})}M$ in (5) we deduce that $c = 1$. q.e.d.

Now, we define a homotopy g between f and the identity. We put

$$g : T_{\tilde{B}}M \times [0,1] \to TM,$$

with

$$g(X_x, t) = (1-t)X_{\gamma^x(t)} + t\xi_{\gamma^x(t)},$$

where ξ is the unit vector field tangent to γ^x which extends the outward normal to W at $\gamma^x(1)$, and X is the vector field over γ^x obtained by parallel transport of X_x. We now use g to bound the flat norm of $D-E$.

Proposition 6.

$$\begin{aligned}&\mathcal{F}(D-E)\\ &\le (\mathbf{M}(D) + \mathbf{M}(\partial D)) \sup_t \sup_{\mathrm{spt}\, D} \left[\left(\left\| \frac{dg}{dt} \right\| \right) (\|dg_t\|^{n-2}, \|dg_t\|^{n-1}) \right].\end{aligned}$$

Proof of Proposition 6.

Let C be the n-current defined by:

$$C = g_\sharp(D \times [0,1]).$$

The homotopy formula for currents (cf. [**15**], 4.1.9.) gives immediately

$$\partial C = f_\sharp(D) - D - g_\sharp(\partial D \times [0,1]).$$

Thus by lemma 1:

$$D - E = \partial C - g_\sharp(\partial D \times [0,1]).$$

By definition of the flat norm,

$$\mathcal{F}(D-E) \le \mathbf{M}(C) + \mathbf{M}(g_\sharp(\partial D \times [0,1]).$$

To evaluate $\mathbf{M}(C)$ and $\mathbf{M}(g_\sharp(\partial D \times [0,1]))$, we use the fact that D is representable by integration. By a computation similar to ([**14**] 4.1.9.), we have:

$$\begin{aligned}\mathbf{M}(C) &= \mathbf{M}(g_\sharp(D \times [0,1]))\\ &\le \int_{[0,1]} \mathbf{M}(D) \sup_{\mathrm{spt}\, D\times\{t\}} \left(\left\| \frac{dg}{dt} \right\| \right) \sup_{\mathrm{spt}\, D\times\{t\}} (\|dg_t\|^{n-1}) dt,\end{aligned}$$

and

$$\mathbf{M}(g_\sharp(\partial D \times [0,1])) \leq \int_{[0,1]} \mathbf{M}(\partial D) \sup_{\operatorname{spt} D \times \{t\}} \left\| \frac{dg}{dt} \right\| \sup_{\operatorname{spt} D \times \{t\}} (\|dg_t\|^{n-2}) dt.$$

Proposition 6 follows. q.e.d.

Now we need to evaluate each term of the right hand side of Proposition 6. To evaluate $\|\frac{dg}{dt}\|_{(X_x,t)}$, we remark that for a fixed $x \in U$, $\frac{dX}{dt}$ and $\frac{d\xi}{dt}$ lie in $H(M) \subset TTM$, since they are parallel vector fields along the geodesic γ^x. Using the Sasaki metric and the isometry between $H(M)_{\dot\gamma(t)}$ and $T_{\gamma(t)}M$, we deduce that at each point t,

$$\left\| \frac{dg}{dt} \right\|^2_{(X_x,t)} \leq l_x^2 + \| - X_{\gamma^x(t)} + \xi_{\gamma^x(t)}\|^2,$$

where $l_x = d(x, W)$. Hence:

Lemma 2. $\sup_{\operatorname{spt} D} \|\frac{dg}{dt}\| \leq \delta_B + \alpha_B$.

To evaluate $\|dg_t\|$, (for a fixed t), we write

$$g_t = g_{1t} + g_{2t},$$

with $g_{1t}(X_x) = t\xi_{\gamma^x(t)}$, and $g_{2t}(X_x) = (1-t)X_{\gamma^x(t)}$.

One has $g_{1t}(X_x) = G_{((1-t)l,t)} \circ \exp^{\perp W_l}_{|tl|}(x)$, where l is the (signed) distance from x to W, $\exp^{\perp W_l}_t$ is the normal exponential map with respect to W_l (see below), and for all $t_1, t_2 \in \mathbb{R}$, $G_{(t_1,t_2)} : W_{t_1} \to TM$ is the Gauss map, defined by $G_{(t_1,t_2)}(m) = t_2\xi_m$, with the same notations as before. The norm of its differential satisfies

$$\|dG_{(t_1,t_2)}\|^2 = 1 + t_2^2 \|\hbar_{W_{t_1}(B)}\|^2.$$

On the other hand, for $u \in \mathbb{R}$, $\exp^{\perp W_t}_u : W_t \to M$ is the normal exponential map associated to W_t defined by

$$\exp^{\perp W_t}_u(m) = \exp_m(u\tilde\xi_m), \forall m \in W_t,$$

where $\tilde\xi_m = \frac{\dot\gamma^m(0)}{\|\dot\gamma^m(0)\|}$ is the normal of W_t at m towards W. Assume that u is small enough so that $\exp^{\perp W_t}_u$ takes its values into $\tilde B$. Using classical comparison theorems, (see [**24**] for instance), one can bound the norm of the differential of $\exp^{\perp W_t}_u$ (restricted to $\operatorname{pr}_{W_t}(B)$). Since we have bounded the sectional curvatures of M by 1, one has:

$$\|d\exp^{\perp W_t}_u\| \leq |\cosh u + \|\hbar_{\operatorname{pr}_{W_t}(B)}\| \sinh u|.$$

Letting

$$\mathcal{E}_B = \sup_{|t| \leq \delta_B} \sup_{0 \leq u \leq t} (|\cosh u + \|\hbar_{\operatorname{pr}_{W_t}(B)}\| \sinh u|),$$

we have:

$$\|d\exp_u^{\perp W_t}\| \leq \mathcal{E}_B,$$

at each point of $\mathrm{pr}(B)$ if $u \leq |t|$. Remark that

$$\mathcal{E}_B \leq \left(1 + \sup_{|t|\leq\delta_B} \|\hbar_{\mathrm{pr}_{W_t}(B)}\|\right) e^{\delta_B}.$$

Moreover, when δ_B is close to 0, $\mathcal{E}_B$ is close to 1. Since $\|dg_{1t}\| \leq \|dG_{(1-t)l,t)}\|\|d\exp_{|tl|}^{\perp W_l}\|$ we deduce

Lemma 3.

$$\|dg_{1t}\| \leq (1+\delta_B^2\|\hbar_{\mathrm{pr}_{W_t}(B)}\|^2)^{\frac{1}{2}}\mathcal{E}_B \leq (1+\delta_B\|\hbar_{\mathrm{pr}_{W_t}(B)}\|)\mathcal{E}_B.$$

On the other hand, g_{2t} is proportional to the parallel transport along the geodesic foliation $\mathcal{F}$. If γ^p is a geodesic of $\mathcal{F}$ parametrized by the arc length, we denote by $\tau_0^t(\gamma^p)$ (or simply τ^t) the parallel transport along γ^p between 0 and t. With these notations, $g_{2t}(X_x) = (1-t)\tau^{tl_x}$ where l_x denotes the distance from x to W. Putting $\gamma_t(p) = \gamma^p(t)$, one has:

Lemma 4.

$$\|dg_{2t}\| \leq c(n)\mathcal{E}_B\delta_B(1+\delta_B),$$

where $c(n)$ is a constant depending only on the dimension n of M.

Sketch of proof of Lemma 4. Let $v \in T_pM$, $\tau^t(v) = \tau_0^t(\gamma^p)(v) \in T_{\gamma^p(t)}M$. We need to bound

$$(d\tau^t)_v : T_v(TM) \to T_{\tau^t(v)}(TM).$$

Consider the decomposition $T_v(TM) = H_v(M) \oplus V_v(M)$. Let $\zeta \in T_vTM$. Write

$$\zeta = U_p^h + V_p^v,$$

where U_p^h denotes the horizontal lift of a vector $U_p \in T_pM$, and V_p^v the vertical lift of a vector $V_p \in T_pM$. If $v(s)$ is a curve on TM such that $v(0) = v$ and $v'(0) = \zeta$, then $(\nabla_{\frac{\partial}{\partial s}} v)_{s=0} = V_p$.

Now consider the curve w on TM defined by:

$$w(s) = \tau_0^t(\gamma^{(\pi\circ v)(s)})(v(s)).$$

Remark that $(d\tau^t)_v(\zeta) = w'(0)$. We introduce the vector field:

$$C_p^\zeta(t) = (\nabla_{\frac{\partial}{\partial s}} w)_{s=0} - \tau^t((\nabla_{\frac{\partial}{\partial s}} v)_{s=0}).$$

One has

$$(d\tau^t)_v(\zeta) = ((d\gamma_t)_p(U_p))^h + (\tau^t(V_p) + C_p^\zeta(t))^v;$$

C_p^ζ satisfies the following differential equation:

$$\nabla_{\frac{\partial}{\partial t}} C_p^\zeta(t) = -R(\xi_{\sigma_p(t)}, (d\gamma_t)_p(U_p))\tau^t(v).$$

Since $\gamma_t(x) = \gamma^x(t) = \exp_t^{\perp W_l}$ where l is the distance between x and W, and since the norm of the curvature tensor R is bounded by $\frac{4}{3}$, we deduce a bound $k(B,\zeta)$ on $\|\nabla_{\frac{\partial}{\partial t}} C_p^\zeta(t)\|$ in terms of $\mathcal{E}_B$, δ_B and $\|\zeta\|$. Hence

$$\frac{d}{dt}\|C_p^\zeta(t)\|^2 = 2\langle C_p^\zeta(t), \nabla_{\frac{\partial}{\partial t}} C_p^\zeta(t)\rangle \leq 2k(B,\zeta)\|C_p^\zeta(t)\|.$$

So by integration:

$$\|C_p^\zeta(t)\|^2 \leq 2tk(B,\zeta)\sup_{s\leq t}\|C_p^\zeta(s)\|,$$

from which we deduce the bound given by Lemma 4. q.e.d.

Since $g = g_1 + g_2$, we obtain the (rough) bound:

Proposition 7.

$$\|dg_t\| \leq c(n)(1 + \|\hbar_{\mathrm{pr}_{W_t}(B)}\|)e^{\delta_B}(1 + \delta_B\|\hbar_{\mathrm{pr}_{W_t}(B)}\| + \delta_B(1+\delta_B)).$$

Using the work of M. Zambon [**34**], we obtain immediately an upper-bound of the norm of the second fundamental form of W_t in terms of the second fundamental form of W:

$$\|\hbar_{\mathrm{pr}_{W_t}(B)}\| \leq \|\hbar_{\mathrm{pr}_W(B)}\| + (22 + 2\|\hbar_{\mathrm{pr}_W(B)}\|^2)t,$$

from which we deduce:

Proposition 8.

$$\|dg_t\| \leq c(n)e^{2\delta_B}(1 + \hbar_{\mathrm{pr}_W(B)})^4,$$

where $c(n)$ is a constant depending only on the dimension n of M.

Combining this proposition with Proposition 6 and Lemma 2, we deduce the first part of Theorem 2. Suppose now that M is the Euclidean space $\mathbb{E}^n$. The computations can be simplified since $\exp_u^{\perp W}(m) = m + u\xi_m$, $\|d\exp_u^{\perp W}\| \leq 1 + u\|\hbar_{\mathrm{pr}_W(B)}\|$ and $\mathcal{E}_B \leq 1 + \delta_B\|\hbar_{\mathrm{pr}_W(B)}\|)$. Under the same assumptions, B satisfies $\delta_B\|\hbar_{\mathrm{pr}_W(B)}\| < 1$, $\|d\mathrm{pr}_{|\tilde{B}}\| \leq \frac{1}{1-\delta_B\|\hbar_{\mathrm{pr}(B)}\|}$, and $\|dg_t\|^{n-1} \leq ((1-t) + t(\frac{1+\|\hbar_{\mathrm{pr}_W(B)}\|}{1-\delta_B\|\hbar_{\mathrm{pr}_W(B)}\|})^{n-1} \leq (\frac{2(1+\|\hbar_{\mathrm{pr}_W(B)}\|)}{1-\delta_B\|\hbar_{\mathrm{pr}_W(B)}\|})^{n-1})$, from which we deduce the second part of Theorem 2. q.e.d.

4.3. Approximations of curvatures. Once we have bounded the flat norm between normal cycles of K and $\mathcal{K}$, and the norm of the Lipschitz-Killing forms, the fundamental $(n-1)$-form and their exterior derivatives, we deduce the following approximation result *(the norm of the sectional curvatures of the ambient manifold M is still bounded by* 1):

Corollary 2. *Under the assumptions of Theorem 2, for every $k, 0 \leq k \leq n-1$, we have*

$$\begin{aligned}
&|\mathcal{M}_k^{\mathcal{K}}(B) - \mathcal{M}_k^{K}(pr(B))| \\
&\leq C(n,k)(\delta_B + \alpha_B)[e^{2\delta_B}(1 + \hbar_{\mathrm{pr}_W(B)})^4]^{n-1} \\
&\qquad \cdot (\mathbf{M}(N(\mathcal{K})_{|T_B M}) + \mathbf{M}(\partial(N(\mathcal{K})_{|T_B M}))).
\end{aligned}$$

Moreover, if X, Y are vector fields of $\mathcal{H}$,

$$\begin{aligned}
&|\langle \mathbf{h}_{\mathcal{K}}^{X,Y}, B\rangle - \langle \mathbf{h}_{K}^{X,Y}, \mathrm{pr}_W(B)\rangle| \\
&\leq C_1(n)\|X\|_1\|Y\|_1(\delta_B + \alpha_B)[e^{2\delta_B}(1 + h_{\mathrm{pr}_W(B)})^4]^{n-1} \\
&\qquad \cdot (\mathbf{M}(N(\mathcal{K})_{|T_B} M) + \mathbf{M}(\partial(N(\mathcal{K})_{|T_B M}))),
\end{aligned}$$

where $C(n,k), C_1(n)$ are constant depending on the dimension of the ambient manifold.

Roughly speaking, this corollary can be interpreted as follows: if $\mathcal{W}$ is strongly close to W and

- W and $\mathcal{W}$ are close, (that is $\delta_{\mathcal{W}}$ is *small*);
- W and $\mathcal{W}$ have close normals, (that is α_B is *small*);
- the norm of the second fundamental form of W is *not too big*, (that is $\|\hbar_W\|$ is *not too big*);
- the "total curvature" of $\mathcal{W}$ is *not too big*;

  then, W and $\mathcal{W}$ have close curvature measures.

Indeed, the mass of the normal cycle of $\mathcal{K}$ is in a certain sense a measure of the total curvature of $\mathcal{W}$. For instance, if $\mathcal{W}$ is a smooth surface in $\mathbb{E}^3$, then

$$\mathbf{M}(N(\mathcal{K})) = \int_{\mathcal{W}} (1 + k_1^2 + k_2^2 + k_1^2 k_2^2)^{1/2} dv_{\mathcal{W}},$$

where k_1, k_2 are the principal curvatures of $\mathcal{W}$. The requirement that the mass of the normal cycle of $\mathcal{K}$ is not too big cannot be removed as the following example shows:

Take $W \subset \mathbb{E}^3$ be a smooth surface containing a (flat) square. Let $\mathcal{W}$ be the surface obtained from W by adding conic spikes with height μ, slope θ, and centered on the vertices of a grid of size η contained in the square. We assume that $2\mu < \theta\eta$, so that the spikes do not overlap. The surface $\mathcal{W}$ is closely inscribed in W and when μ, θ, and η go to 0, δ_B and α_B also go to 0. However, one can tune the decay of these parameters in such a way that the difference H between the total mean curvature of $\mathcal{W}$ and the one of W goes to infinity. H is simply the sum of the total mean curvatures of all the spikes of $\mathcal{W}$. The total mean curvature of each spike is a function of μ and θ that is linear with respect to μ by homogeneity. Thus $H \simeq \mu/\eta^2 \phi(\theta)$ for some function ϕ. Calculations show that $\phi(\theta) = \Omega(\theta)$ when θ goes to 0. Thus if one chooses $\theta = \eta^{1/3}$ and μ such that $2\mu \leq \theta\eta$ holds, e.g. $\mu = \eta^{4/3}/3$, then one has $H = \Omega(\eta^{-1/3})$. In this example, the total mean curvature does not converge because the mass of the normal cycle of the volume enclosed by $\mathcal{W}$ is unbounded.

Proof of Corollary 2. As a consequence of the definition of the flat norm (4), one has for any (smooth) $(n-1)$-form ω on TM,

$$|\langle N(\mathcal{K}) - N(K), \omega\rangle| \leq \mathcal{F}(N(\mathcal{K}) - N(K)) \sup\{\|\omega\|, \|d\omega\|\}.$$

However, there is a technical problem: the Lipschitz-Killing curvature forms ω_k are not defined on the null section of the tangent bundle of the ambient space, and the homotopy can pass through this null section since it leaves the unit tangent bundle. So we introduce a fixed smooth function $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi = 1$ on a neighborhood of 1, and whose support lies in $[1/2, 3/2]$. Then we define on TM the smooth forms $\breve{\omega}_k(m, \xi) = \phi(\|\xi\|)\omega_k(m, \frac{\xi}{\|\xi\|})$. Remarking that $\breve{\omega}_k$ and their exterior derivatives are bounded by a constant $\check{C}(n, k)$, we have

$$\begin{aligned}
&|\langle N(\mathcal{K})_{T_B M} - N(K)_{T_{\mathrm{pr}_W(B)} M}, \omega_k\rangle| \\
&= |\langle N(\mathcal{K})_{T_B M} - N(K)_{T_{\mathrm{pr}_W(B)} M}, \breve{\omega}_k\rangle| \\
&\leq \tilde{C}(n, k)(\mathcal{F}(N(\mathcal{K})_{|T_B M} - N(K)_{T_{\mathrm{pr}_W(B)} M}),
\end{aligned}$$

from which we deduce the first part of Corollary 2. A similar argument can be given for the approximation of the fundamental $(n-1)$-form $\mathbf{h}$. q.e.d.

5. Application: triangulated mesh inscribed in a smooth surface

This section applies Corollary 2 in one of the most simple and most common situations in practice: A triangulated mesh that approximates a smooth surface in the three dimensional Euclidean space. By *triangulated mesh* (or simply *mesh*) we mean a compact oriented piecewise linear surface without boundary linearly embedded in $\mathbb{E}^3$. In the following, T denotes a mesh and $\mathcal{K}$ denotes the closure of the bounded component of $\mathbb{E}^3 \setminus T$.

5.1. A relative of the second fundamental measure. For practical purposes, we consider the measures $\mathbf{h}_{\mathcal{K}}^{U,V}$ where U and V lie in a finite-dimensional space of vector fields. Though other possibilities may be of interest, we restrict ourselves to the 3-dimensional space of vector fields obtained by projecting on $\mathcal{H}$ horizontal lifts of constant vector fields on $\mathbb{E}^3$. If X is a constant vector field on $\mathbb{E}^3$, let X' be the vector field obtained by this process. The resulting measures $\mathbf{h}_{\mathcal{K}}^{X',Y'}$ define a $(2,0)-$tensor valued measure $\mathbf{h}_{\mathcal{K}}$ by setting $\mathbf{h}_{\mathcal{K}}(B)(X,Y) = \mathbf{h}_{\mathcal{K}}^{X',Y'}(B)$ for all Borel sets $B \subset \mathbb{E}^3$ and all couples of vectors X and Y in $\mathbb{E}^3$

(here we identify $T\mathbb{E}^3$ with $\mathbb{E}^3 \times \mathbb{E}^3$). The expression of this measure for meshes is the following:

$$\mathbf{h}_{\mathcal{K}}(B)$$
$$= \sum_{e \,\text{edge of}\, T} \frac{l(e \cap B)}{2} [(\beta(e) - \sin \beta(e)) e^+ \otimes e^+ + (\beta(e) + \sin \beta(e)) e^- \otimes e^-]$$

where e^+(*resp.* e^-) denote the normalized sum (*resp.* difference) of the unit outward normal vector to the triangles incident on e, and $\beta(e)$ is the angle between these normal vectors. The sign of $\beta(e)$ is chosen positive if and only if the edge e is convex.

In practice, we actually prefer a related tensor-valued measure, which we denote by $\tilde{\mathbf{h}}_{\mathcal{K}}$, because its expression for meshes is simpler. Given two vectors X and Y in $\mathbb{E}^3$ and a point $(p, n) \in \mathbb{E}^3 \times \mathbb{E}^3$, we set

$$\tilde{\mathbf{h}}^{X,Y}_{(p,n)} = X \wedge (n \times Y)$$

and for each Borel set $B \subset \mathbb{E}^3$

$$\tilde{\mathbf{h}}_{\mathcal{K}}(B)(X, Y) = \langle N(\mathcal{K})_{|B}, \tilde{\mathbf{h}}^{X,Y} \rangle.$$

The form $\tilde{\mathbf{h}}^{X,Y}$ is actually similar to the form $\mathbf{h}^{X',Y'}$, since when the ambient space is $\mathbb{E}^3$ one has

$$\mathbf{h}^{X',Y'}_{(p,n)} = (n \times X) \wedge Y$$

For a compact 3-manifold K with smooth boundary W, the measure $\tilde{\mathbf{h}}_K$ is related to the tensor field obtained from the second fundamental form of W by exchanging its eigenvalues. More precisely:

$$\tilde{\mathbf{h}}_K(B)(X, Y) = \int_B \hbar(j\mathrm{pr}_{TW} X, j\mathrm{pr}_{TW} Y) dv_W$$

where j is the almost complex structure of W compatible with the metric. As remarked by A. Bernig, the tensor involved in the integral is nothing but $(\mathrm{trace}\, \hbar)\mathrm{Id} - \hbar$. The computation of $\tilde{\mathbf{h}}$ in the piecewise linear case yields

$$\tilde{\mathbf{h}}_{\mathcal{K}}(B) = \sum_{e \text{ edge of} P} \beta(e) l(e \cap B) \overrightarrow{e} \otimes \overrightarrow{e}$$

where $\overrightarrow{e}$ denotes the unit 3-vector with the same direction as the edge e. The proofs of these results are left to the reader.

5.2. Approximation results. Following [**18**], we say that a mesh T is *strongly inscribed* to W if it is strongly close to W and all its vertices lie in W. Note that in this case, fine tubular neighborhoods U_r of W are nothing but tubular neighborhoods of W of radius smaller than the reach of W. We denote by $r(t)$ the circumradius of a triangle t.

Corollary 3. *Let W be a smooth oriented hypersurface of $\mathbb{E}^3$ bounding a compact subset K of $\mathbb{E}^3$. Let T be a mesh bounding a closed compact $\mathcal{K}$ and strongly inscribed in W. Let B be a union of triangles of T. Then,*

$$|\mathcal{M}_H^{\mathcal{K}}(B) - \mathcal{M}_H^K(\mathrm{pr}(B))| \leq C_W \mathbf{K}\epsilon;$$

$$|\mathcal{M}_G^{\mathcal{K}}(B) - \mathcal{M}_G^K(\mathrm{pr}(B))| \leq C_W \mathbf{K}\epsilon;$$

$$\|\mathbf{h}_{\mathcal{K}}(B) - \mathbf{h}_K((\mathrm{pr}(B))\| \leq C_W \mathbf{K}\epsilon;$$

$$\|\tilde{\mathbf{h}}_{\mathcal{K}}(B) - \tilde{\mathbf{h}}_K((\mathrm{pr}(B))\| \leq C_W \mathbf{K}\epsilon;$$

where C_W is a real number depending only on the norm of the second fundamental form of W,

$$\mathbf{K} = \sum_{t\in T, t\subset \overline{B}} r(t)^2 + \sum_{t\in T, t\subset \overline{B}, t\cap\partial B\neq\emptyset} r(t), \ \textit{and } \epsilon = \max\{r(t), t \subset B\}.$$

The proof of this corollary essentially amounts to giving explicit bounds for the various quantities involved in Corollary 2. In particular, it results from [**28**] that the angular deviation between T and W is bounded by a constant times the norm of the second fundamental form of W times ϵ. The Hausdorff distance between T and W is bounded by 2ϵ. The term $\mathbf{K}$ (times C_W) is a bound on the mass of the normal cycle of T lying above B; it is obtained by bounding the mass lying above each simplex of T, using the bound on the angular deviation.

Note that $\mathbf{K}$ is bounded by a function of the minimal angle in the triangles of T times the area of B plus the length of ∂B. We deduce that if all angles in the triangles of T are bounded from below (this is the fatness condition used in [**18**]), then the bounds on the differences between curvature measures given in Corollary 3 are $O(\mathrm{area}(B) + \mathrm{length}(\partial B))C_W\epsilon$.

We conclude this section by some experimental results. For any vertex v, we compute the tensor $\tilde{\mathbf{h}}_{\mathcal{K}}(B)$ obtained by taking as B the union of all triangles whose vertices can be joined to v by at most 2 edges of the mesh. In the case of the ellipsoid (Figure 1), we found that the eigendirection associated with the eigenvalue of smallest magnitude was close to the normal to the surface at v, whereas the two others were close to the principal directions, which is what one could expect. In Figure 3, we displayed the direction associated with the smallest principal curvature estimated on a mesh of Michelangelo's David. This example is more delicate to evaluate: as we are dealing with a real world object, the smooth surface being approximated is hard to define. Still, the fact that one can recognize Michelangelo's masterpiece from the displayed directions shows in some sense the validity of the result.

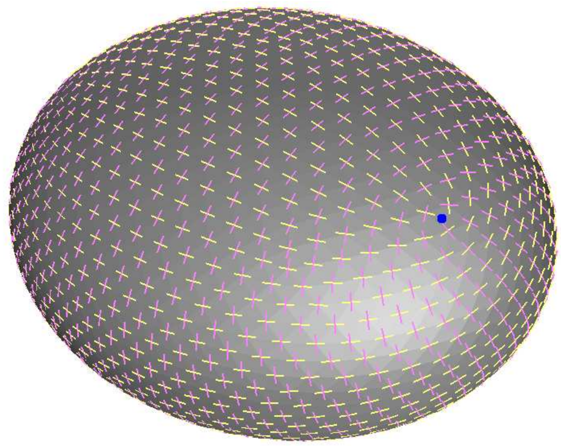

Figure 1. The principal directions estimated on an ellipsoid meshed with 1442 vertices, are very similar to the actual ones, whose integral lines are shown in Figure 2.

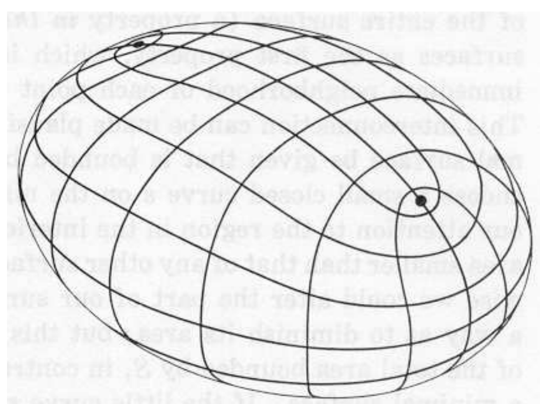

Figure 2. Lines of curvature of an ellipsoid (from [**25**]).

5.3. Geometry from samples. In practice, real world objects are not known through a mesh, but rather by a set of points measured on their surface, for instance using a laser scanner. A possible way of estimating the curvature of the object is to build an approximating mesh of it based on the sampled points, and then compute the second fundamental measure of the obtained mesh.

The latest algorithms devoted to mesh reconstruction from sampled points aim at building a particular mesh associated with the original surface and the point cloud. This mesh, called the restricted Delaunay triangulation, turns out to be particularly well-suited for purposes of curvature estimation through curvature measures.

Figure 3. Directions of minimal curvature estimated on a mesh of Michelangelo's David.

Let us first recall some standard definitions. If $P \subset \mathbb{R}^3$ is a finite point set and $p \in P$, the *Voronoi polytope* of p is the set of points x in $\mathbb{R}^3$ such that the distance from x to p is less or equal than the distance from x to q for all $q \in P$. The *Voronoi diagram* of P is the cell structure on $\mathbb{R}^3$ induced by all Voronoi polytopes. The *Delaunay triangulation* is the triangulation of the convex hull of P dual to the Voronoi diagram.

Definition 6. If W is a surface embbedded in $\mathbb{R}^3$ and P is a finite subset of W, the Delaunay triangulation of P restricted to W is the union of all Delaunay simplices whose dual Voronoi cell meets W (Figure 4).

Restricted Delaunay triangulations are good approximations provided the sampling density is sufficient. The precise formulation of this condition involves the notion of *local feature size*. The local feature size $\mathrm{lfs}(p)$ of a point $p \in W$ is its distance to the subset of $\mathbb{R}^3$ where the projection on W is not defined. The local feature size is in some sense

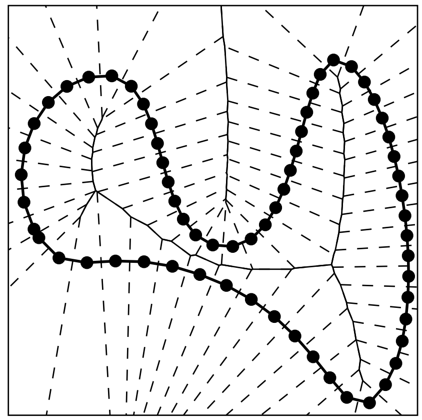

Figure 4. Restricted Delaunay triangulation of a sampled curve (in bold). Voronoi edges that meet the curve are dashed, the other ones are solid.

a local version of the reach as defined by Federer, since the reach of a surface coincides with the minimum of lfs over the whole surface.

Definition 7. *P is said to be an ε-sample of W if for all $p \in W$, the ball centered at p and with radius $\varepsilon \mathrm{lfs}(p)$ contains at least a point of P (Figure 5).*

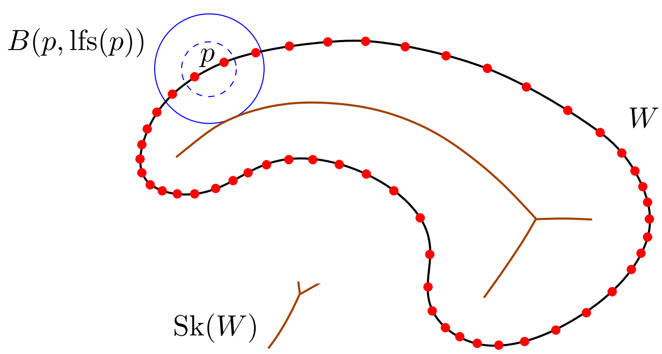

Figure 5. A 1/2-sample of a curve W. $\mathrm{Sk}(W)$ denotes the discontinuity locus of the projection on W.

N. Amenta *et al.* [**2**] proved that if P is an ε-sample of W for $\varepsilon < 0.06$, then its Delaunay triangulation restricted to W is strongly inscribed in W. Moreover, they showed that the angular deviation between both was bounded by a constant times ε, as well as their Hausdorff distance. Unfortunately, the bound $\mathbf{K}$ on the mass of the normal cycle of the triangulation in Corollary 3 is more difficult to bound, even in the case of restricted Delaunay triangulations of ε-samples for small ε. One can actually build examples where not only K is arbitrarily large, but also is the mass of the normal cycle of the triangulation, no matter how small ε is (figure 6).

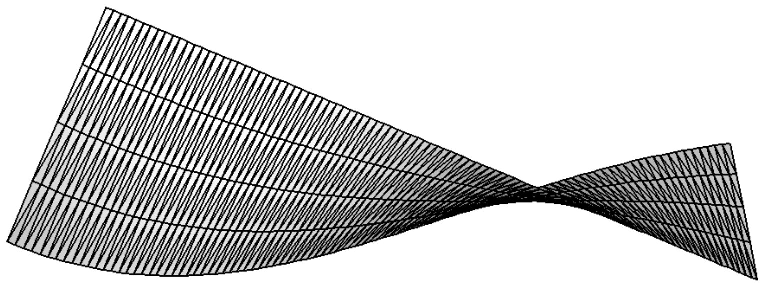

Figure 6. A sequence of restricted Delaunay triangulations whose normal cycles have unbounded mass.

The triangulation depicted in Figure 6 is the restricted Delaunay triangulation of a particular sampling of a helicoid. This sampling is such that the density d_a of samples in the direction of the axis of the helicoid is much larger than the density d_f of samples along the fibers. If one chooses, for instance, $d_a = d_f^2$ and lets d_f go to infinity, then the normal cycles of these triangulations have masses tending to infinity. Corollary 3 thus gives a poor bound on the difference between the curvature measure of the helicoid and the one of its triangulation in this case.

A way to circumvent the problem is to require that the sampling is locally uniform in the sense of [**29**], which is a reasonable assumption from a practical point of view. In this case, it can be shown that the triangles of the restricted Delaunay triangulation have their smallest angle larger than a given constant. The conclusions of the previous section thus apply:

Corollary 4. *Let W be a closed smooth hypersurface of $\mathbb{E}^3$. Let T be the restricted Delaunay triangulation of a locally uniform ε-sample of W with $\varepsilon < 0.06$. Let B be a union of triangles of T. Then, the bounds on the differences between the curvature measures* (*resp. second fundamental measures*) *given in Corollary* 3 *are* $O(\mathrm{area}(B) + \mathrm{length}(\partial B))C_W\epsilon$.

Computing the second fundamental measure of a mesh reconstructed from a sampling of an object is thus a reliable way to estimate the curvature tensor of the original object.

6. Conclusion

We have given conditions under which the curvature measures of a hypersurface are close to the ones of a given smooth hypersurface. Unfortunately, our approach breaks down as soon as both hypersurfaces are singular. We leave to the reader the following question which, if the answer were positive, would settle the problem of approximation of Lipschitz-Killing curvatures in the general case:

Open problem. Let V and W be geometric subsets of M and let f be a homeomorphism between V and W. Let:

1) $\delta = \sup_{m\in M} d(m, f(m))$,
2) α be the maximum over $m \in M$ of the Hausdorff distance between the support of $N(V) \llcorner T_m M$ and the one of $N(W) \llcorner T_{f(m)} M$.

Can one bound the difference between the Lipschitz-Killing curvatures of V and W by a function of δ, α, the curvature of M, and the masses of $N(V)$ and $N(W)$ that tend to 0 with δ and α?

We note that for curves in Euclidean space, an even stronger conjecture is true. Indeed, it can be shown that the length difference between two curves V and W is bounded by δ times the masses of $N(V)$ and $N(W)$, times a dimension dependent constant [**11**].

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