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Second Hankel determinant for certain subclasses of bi-univalent functions

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Abstract: In the present paper, we obtain the upper bounds for the second Hankel determinant for certain subclasses of analytic and bi-univalent functions. Moreover, several interesting applications of the results presented here are also discussed.

Key words: Analytic functions, univalent functions, bi-univalent functions, subordination between analytic functions, Hankel determinant

1. Introduction and definitions

Let $\mathcal{A}$ denote the family of functions f analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ that are univalent in $\mathbb{U}$. The Koebe one-quarter theorem (see, for example, [9]) ensures that the image of $\mathbb{U}$ under every $f \in \mathcal{S}$ contains a disk of radius $1/4$. Clearly, every $f \in \mathcal{S}$ has an inverse function f^{-1} satisfying $f^{-1}(f(z)) = z \ (z \in \mathbb{U})$ and $f(f^{-1}(w)) = w \ (|w| < r_0(f);\ r_0(f) \geqq 1/4)$, where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots .$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1).

In 1967, Lewin [21] showed that, for every function $f \in \sigma$ of the form (1.1), the second coefficient of f satisfies the estimate $|a_2| < 1.51$. In 1967, Brannan and Clunie [2] conjectured that $|a_2| \leqq \sqrt{2}$ for $f \in \sigma$. Later, Netanyahu [22] proved that $\max\limits_{f\in\sigma} |a_2| = \frac{4}{3}$. In 1985, Kedzierawski [17] proved the Brannan–Clunie conjecture for bi-starlike functions. In 1985, Tan [31] obtained the bound for a_2, namely that $|a_2| < 1.485$, which is the best

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known estimate for functions in the class σ. Brannan and Taha [3] obtained estimates on the initial coefficients $|a_2|$ and $|a_3|$ for functions in the classes of bi-starlike functions of order β and bi-convex functions of order β.

The study of bi-univalent functions was revived in recent years by Srivastava et al. [30] and a considerably large number of sequels to the work of Srivastava et al. [30] have appeared in the literature since then. In particular, several results on coefficient estimates for the initial coefficients $|a_2|$, $|a_3|$, and $|a_4|$ were proved for various subclasses of σ (see, for example, [1, 4, 5, 10, 12, 14, 16, 25, 28, 29, 32, 33]).

Recently, Deniz [7] and Kumar et al. [19] both extended and improved the results of Brannan and Taha [3] by generalizing their classes by means of the principle of subordination between analytic functions. The problem of estimating the coefficients $|a_n|$ $(n \geqq 2)$ is still open (see also [29] in this connection).

Among the important tools in the theory of univalent functions are Hankel determinants, which are used, for example, in showing that a function of bounded characteristic in $\mathbb{U}$, that is, a function that is a ratio of two bounded analytic functions, with its Laurent series around the origin having integral coefficients, is rational [6]. The Hankel determinants $H_q(n)$ $(n = 1, 2, 3, \cdots, \; q = 1, 2, 3, \cdots)$ of the function f are defined by (see [23])

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \qquad (a_1 = 1).$$

This determinant was discussed by several authors with $q = 2$. For example, we know that the functional $H_2(1) = a_3 - a_2^2$ is known as the Fekete–Szegö functional and one usually considers the further generalized functional $a_3 - \mu a_2^2$ where μ is some real number (see [11]). Estimating for the upper bound of $\left|a_3 - \mu a_2^2\right|$ is known as the Fekete–Szegö problem. In 1969, Keogh and Merkes [18] solved the Fekete–Szegö problem for the classes of starlike and convex functions. One can see the Fekete–Szegö problem for the classes of starlike functions of order β and convex functions of order β in special cases in the paper of Orhan et al. [24]. On the other hand, quite recently, Zaprawa (see [34, 35]) studied the Fekete–Szegö problem for some classes of bi-univalent functions. In special cases, he gave the Fekete–Szegö problem for the classes of bi-starlike functions of order β and bi-convex functions of order β.

The second Hankel determinant $H_2(2)$ is given by $H_2(2) = a_2a_4 - a_3^2$. The bounds for the second Hankel determinant $H_2(2)$ were obtained for the classes of starlike and convex functions in [15]. Lee et al. [20] established the sharp bound for $|H_2(2)|$ by generalizing their classes by means of the principle of subordination between analytic functionds. In their paper [20], one can find the sharp bound for $|H_2(2)|$ for the functions in the classes of starlike functions of order β and convex functions of order β. Recently, Deniz et al. [8] and Orhan et al. [26] found the upper bound for the functional $H_2(2) = a_2a_4 - a_3^2$ for the subclasses of bi-univalent functions.

The object of the present paper is to seek the upper bound for the functional $\left|a_2a_4 - a_3^2\right|$ for $f \in \mathcal{N}_\sigma(\beta)$ and $f \in \mathcal{N}_\sigma^\alpha$, which are defined as follows.

Definition 1 *(see [30]) A function $f(z)$ given by (1.1) is said to be in the class $f \in \mathcal{N}_\sigma(\beta)$ $(0 \leqq \beta < 1)$ if the following conditions are satisfied:*

$$f \in \sigma \quad and \quad \Re\left(f'(z)\right) > \beta \qquad (z \in \mathbb{U};\; 0 \leqq \beta < 1) \tag{1.2}$$

and

$$\Re\left(g'(w)\right) > \beta \qquad \left(w \in \mathbb{U};\ 0 \leqq \beta < 1\right), \tag{1.3}$$

where the function g *is given by*

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{1.4}$$

Definition 2 *(see [30]) A function* $f(z)$ *given by* (1.1) *is said to be in the class* $f \in \mathcal{N}_\sigma^\alpha$ $(0 < \alpha \leqq 1)$ *if the following conditions are satisfied*:

$$f \in \sigma \quad \text{and} \quad \left|\arg\left(f'(z)\right)\right| \leqq \frac{\alpha\pi}{2} \qquad \left(z \in \mathbb{U};\ 0 < \alpha \leqq 1\right) \tag{1.5}$$

and

$$\left|\arg\left(g'(w)\right)\right| < \frac{\alpha\pi}{2} \qquad \left(w \in \mathbb{U};\ 0 < \alpha \leqq 1\right), \tag{1.6}$$

where the function g *is defined by* (1.4).

For special values of the parameters α and β, we have

$$\mathcal{N}_\sigma(0) = \mathcal{N}_\sigma^1 = \mathcal{N}_\sigma.$$

Let $\mathcal{P}$ be the class of functions with positive real part consisting of all analytic functions $\mathcal{P} : \mathbb{U} \to \mathbb{C}$ satisfying $p(0) = 1$ and $\Re\left(p(z)\right) > 0$.

To establish our main results, we shall require the following lemmas.

Lemma 1 *(see, for example, [27]) If the function* $p \in \mathcal{P}$ *is given by the following series*:

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots , \tag{1.7}$$

then the sharp estimate given by

$$|c_k| \leqq 2 \qquad (k = 1, 2, 3, \cdots)$$

holds true.

Lemma 2 *(see [13]) If the function* $p \in \mathcal{P}$ *is given by the series* (1.7), *then*

$$2c_2 = c_1^2 + x(4 - c_1^2), \tag{1.8}$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)\left(1 - |x|^2\right)z \tag{1.9}$$

for some x *and* z *with* $|x| \leqq 1$ *and* $|z| \leqq 1$.

2. Main results

Our first main result for the class $f \in \mathcal{N}_\sigma(\beta)$ is stated as follows:

Theorem 1 *Let* $f(z)$ *given by* (1.1) *be in the class* $\mathcal{N}_\sigma(\beta)$ $(0 \leqq \beta < 1)$. *Then*

$$\left|a_2 a_4 - a_3^2\right| \leqq \begin{cases} \frac{(1-\beta)^2}{2}\left(2\left(1-\beta\right)^2 + 1\right) & \left(\beta \in \left[0, \frac{11-\sqrt{37}}{12}\right]\right), \\ \frac{(1-\beta)^2}{16}\left(\frac{60\beta^2 - 84\beta - 25}{9\beta^2 - 15\beta + 1}\right) & \left(\beta \in \left(\frac{11-\sqrt{37}}{12}, 1\right)\right). \end{cases} \tag{2.1}$$

Proof Let $f \in \mathcal{N}_\sigma(\beta)$ and $g = f^{-1}$. Then

$$f'(z) = \beta + (1-\beta)p(z) \text{ and } g'(w) = \beta + (1-\beta)q(w) \tag{2.2}$$

where the functions $p(z)$ and $q(z)$ given by

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots$$

and

$$q(w) = 1 + d_1 w + d_2 w^2 + \cdots$$

are in class $\mathcal{P}$.

Comparing the coefficients in (2.2), we have

$$2a_2 = (1-\beta)c_1, \tag{2.3}$$

$$3a_3 = (1-\beta)c_2, \tag{2.4}$$

$$4a_4 = (1-\beta)c_3, \tag{2.5}$$

and

$$-2a_2 = (1-\beta)d_1, \tag{2.6}$$

$$3\left(2a_2^2 - a_3\right) = (1-\beta)d_2, \tag{2.7}$$

$$-4\left(5a_2^3 - 5a_3a_2 + a_4\right) = (1-\beta)d_3. \tag{2.8}$$

From (2.3) and (2.6), we find that

$$c_1 = -d_1 \tag{2.9}$$

and

$$a_2 = \frac{(1-\beta)}{2}c_1. \tag{2.10}$$

Now, from (2.4), (2.7) and (2.10), we get

$$a_3 = \frac{(1-\beta)^2}{4}c_1^2 + \frac{(1-\beta)}{6}\left(c_2 - d_2\right). \tag{2.11}$$

Also, from (2.5) and (2.8), we find that

$$a_4 = \frac{5(1-\beta)^2}{24}c_1\left(c_2 - d_2\right) + \frac{(1-\beta)}{8}\left(c_3 - d_3\right). \tag{2.12}$$

Thus, we can easily establish that

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| = \Bigg| &-\frac{(1-\beta)^4}{16}c_1^4 + \frac{(1-\beta)^3}{48}c_1^2\left(c_2 - d_2\right) \\ &+\frac{(1-\beta)^2}{16}c_1\left(c_3 - d_3\right) - \frac{(1-\beta)^2}{36}\left(c_2 - d_2\right)^2\Bigg|. \end{aligned} \tag{2.13}$$

According to Lemma 2 and (2.9), we write

$$\left.\begin{array}{l} 2c_2 = c_1^2 + x(4-c_1^2) \\ 2d_2 = d_1^2 + y(4-d_1^2) \end{array}\right\} \Longrightarrow c_2 - d_2 = \frac{4-c_1^2}{2}(x-y) \tag{2.14}$$

and

$$4c_3 = c_1^3 + 2(4-c_1^2)c_1 x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)\left(1-|x|^2\right)z,$$

$$4d_3 = d_1^3 + 2(4-d_1^2)d_1 y - d_1(4-d_1^2)y^2 + 2(4-d_1^2)\left(1-|y|^2\right)w.$$

Moreover, we have

$$\begin{aligned} c_3 - d_3 &= \frac{c_1^3}{2} + \frac{c_1\left(4-c_1^2\right)}{2}(x+y) - \frac{c_1\left(4-c_1^2\right)}{4}(x^2+y^2) \\ &\quad + \frac{\left(4-c_1^2\right)}{2}\left(\left(1-|x|^2\right)z - \left(1-|y|^2\right)w\right), \end{aligned} \tag{2.15}$$

$$c_2 + d_2 = c_1^2 + \frac{\left(4-c_1^2\right)}{2}(x+y) \tag{2.16}$$

for some x, y and z, w with $|x| \leqq 1, |y| \leqq 1, |z| \leqq 1$ and $|w| \leqq 1$. Using (2.14) and (2.15) in (2.13), and applying the triangle inequality, we have

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| &= \Bigg| -\frac{(1-\beta)^4}{16}c_1^4 + \frac{(1-\beta)^3}{96}c_1^2(4-c_1^2)(x-y) \\ &\quad + \frac{(1-\beta)^2}{16}c_1\left[\frac{c_1^3}{2} + \frac{(4-c_1^2)c_1}{2}(x+y) - \frac{(4-c_1^2)c_1}{4}(x^2+y^2) + \frac{(4-c_1^2)}{2}\left((1-|x|^2)z - (1-|y|^2)w\right)\right] \\ &\quad - \frac{(1-\beta)^2}{144}(4-c_1^2)^2(x-y)^2 \Bigg| \\ &\leqq \frac{(1-\beta)^4}{16}c_1^4 + \frac{(1-\beta)^2}{32}c_1^4 + \frac{(1-\beta)^2}{16}c_1(4-c_1^2) \\ &\quad + \left[\frac{(1-\beta)^3}{96}c_1^2(4-c_1^2) + \frac{(1-\beta)^2}{32}c_1^2(4-c_1^2)\right](|x|+|y|) \\ &\quad + \left[\frac{(1-\beta)^2}{64}c_1^2(4-c_1^2) - \frac{(1-\beta)^2}{32}c_1(4-c_1^2)\right](|x|^2+|y|^2) + \frac{(1-\beta)^2}{144}(4-c_1^2)^2(|x|+|y|)^2. \end{aligned}$$

Since $p \in \mathcal{P}$, we have $|c_1| \leqq 2$. Letting $c_1 = c$, we may assume without loss of generality that $c \in [0,2]$. Thus, for $\lambda = |x| \leqq 1$ and $\mu = |y| \leqq 1$, we obtain

$$\left|a_2a_4 - a_3^2\right| \leqq T_1 + T_2(\lambda+\mu) + T_3(\lambda^2+\mu^2) + T_4(\lambda+\mu)^2 = F(\lambda,\mu),$$

where

$$T_1 = T_1(c) = \frac{(1-\beta)^2}{32} c \left[\left(1 + 2(1-\beta)^2 \right) c^3 + 2(4-c^2) \right] \geqq 0,$$

$$T_2 = T_2(c) = \frac{(1-\beta)^2}{96} c^2 (4-c^2)(4-\beta) \geqq 0,$$

$$T_3 = T_3(c) = \frac{(1-\beta)^2}{64} c(4-c^2)(c-2) \leqq 0,$$

$$T_4 = T_4(c) = \frac{(1-\beta)^2}{144} (4-c^2)^2 \geqq 0.$$

Now we need to maximize $F(\lambda,\mu)$ in the closed square $\mathbb{S} = \{(\lambda,\mu) : 0 \leqq \lambda \leqq 1, 0 \leqq \mu \leqq 1\}$ for $c \in [0,2]$. We must investigate the maximum of $F(\lambda,\mu)$ according to $c = (0,2)$, $c = 0$ and $c = 2$, keeping in view the sign of $F_{\lambda\lambda}F_{\mu\mu} - (F_{\lambda\mu})^2$.

First, let $c \in (0,2)$. Since $T_3 < 0$ and $T_3 + 2T_4 > 0$ for $c \in (0,2)$, we conclude that

$$F_{\lambda\lambda}F_{\mu\mu} - (F_{\lambda\mu})^2 < 0.$$

Thus, the function F cannot have a local maximum in the interior of the square $\mathbb{S}$. Now we investigate the maximum of F on the boundary of the square $\mathbb{S}$.

For $\lambda = 0$ and $0 \leqq \mu \leqq 1$, we obtain

$$F(0,\mu) = G(\mu) = (T_3 + T_4)\mu^2 + T_2\mu + T_1.$$

We consider the following two cases separately.

Case 1. Let $T_3 + T_4 \geqq 0$. In this case, for $0 < \mu < 1$ and for any fixed c with $0 < c < 2$, it is clear that

$$G'(\mu) = 2(T_3 + T_4)\mu + T_2 > 0 \qquad (0 < \mu < 1),$$

that is, that $G(\mu)$ is an increasing function. Hence, for fixed $c \in (0,2)$, the maximum of $G(\mu)$ occurs at $\mu = 1$, and

$$\max\{G(\mu)\} = G(1) = T_1 + T_2 + T_3 + T_4.$$

Case 2. Let $T_3 + T_4 < 0$. Since

$$T_2 + 2(T_3 + T_4) \geqq 0$$

for any fixed c with $0 < c < 2$, it is clear (in this case) that

$$T_2 + 2(T_3 + T_4) < 2(T_3 + T_4)\mu + T_2 < T_2 \qquad (0 < \mu < 1),$$

which shows that $G'(\mu) > 0$. Hence, for fixed $c \in (0,2)$, the maximum of $G(\mu)$ occurs at $\mu = 1$. Similarly, for $\mu = 0$ and $0 \leqq \lambda \leqq 1$, we get

$$\max\{F(\lambda,0)\} = \max\{G(\lambda)\} = G(1) = T_1 + T_2 + T_3 + T_4.$$

For $\lambda = 1$ and $0 \leqq \mu \leqq 1$, we obtain

$$F(1,\mu) = H(\mu) = (T_3 + T_4)\mu^2 + (T_2 + 2T_4)\mu + T_1 + T_2 + T_3 + T_4.$$

Thus, from the above Case 1 and Case 2 for $T_3 + T_4$, we get

$$\max\{H(\mu)\} = H(1) = T_1 + 2T_2 + 2T_3 + 4T_4.$$

Similarly, for $\mu = 1$ and $0 \leqq \lambda \leqq 1$, we have

$$\max\{F(\lambda, 1)\} = \max\{H(\lambda)\} = H(1) = T_1 + 2T_2 + 2T_3 + 4T_4.$$

Since $G(1) \leqq H(1)$ for $c \in (0,2)$, we have

$$\max\{F(\lambda, \mu)\} = F(1,1)$$

on the boundary of the square $\mathbb{S}$. Thus, clearly, the maximum of the function $F(\lambda, \mu)$ occurs when $\lambda = 1$ and $\mu = 1$ in the closed square $\mathbb{S}$ and for $c \in (0,2)$.

Let $K : (0,2) \to \mathbb{R}$ be given by

$$K(c) = \max\{F(\lambda, \mu)\} = F(1,1) = T_1 + 2T_2 + 2T_3 + 4T_4. \tag{2.17}$$

Substituting the values of T_1, T_2, T_3, and T_4 into the function $K(c)$ defined by (2.17) yields

$$K(c) = \frac{(1-\beta)^2}{144}\left[\left(9\beta^2 - 15\beta + 1\right)c^4 + (34 - 12\beta)c^2 + 64\right].$$

We now investigate the maximum value of $K(c)$ in the interval $(0,2)$. By elementary calculation, we find that

$$K'(c) = \frac{(1-\beta)^2}{36}\left[\left(9\beta^2 - 15\beta + 1\right)c^3 + (17 - 6\beta)c\right]. \tag{2.18}$$

As a result of some calculations, we can accomplish the following results.

Result 1. Let

$$9\beta^2 - 15\beta + 1 \geqq 0,$$

that is,

$$\beta \in \left[0, \frac{5 - \sqrt{21}}{6}\right].$$

Then $K'(c) > 0$ for every $c \in (0,2)$. Furthermore, since $K(c)$ is an increasing function in the interval $(0,2)$, it has no maximum value in this interval.

Result 2. Let

$$9\beta^2 - 15\beta + 1 < 0,$$

that is,

$$\beta \in \left(\frac{5 - \sqrt{21}}{6}, 1\right).$$

Then $K'(c) = 0$ implies the real critical point given by

$$c_{0_1} = \sqrt{\frac{6\beta - 17}{9\beta^2 - 15\beta + 1}}.$$

In the case when

$$\beta \in \left(\frac{5 - \sqrt{21}}{6}, \frac{11 - \sqrt{37}}{12} \right],$$

then $c_{0_1} \geqq 2$, that is, c_{0_1} lies outside of the interval $(0, 2)$. In the case when

$$\beta \in \left(\frac{11 - \sqrt{37}}{12}, 1 \right),$$

then $c_{0_1} < 2$, that is, c_{0_1} is in the interior of the interval $[0, 2]$. Furthermore, since $K''(c_{0_1}) < 0$, the maximum value of $K(c)$ occurs at $c = c_{0_1}$ for

$$\beta \in \left(\frac{11 - \sqrt{37}}{12}, 1 \right).$$

Thus, clearly, it is observed that

$$\max_{0<c<2} \{K(c)\} = K(c_{0_1}) = K\left(\sqrt{\frac{6\beta - 17}{9\beta^2 - 15\beta + 1}} \right) = \frac{(1-\beta)^2}{2} \left(\frac{15\beta^2 - 21\beta - \frac{25}{4}}{18\beta^2 - 30\beta + 2} \right) \tag{2.19}$$

for

$$\beta \in \left(\frac{11 - \sqrt{37}}{12}, 1 \right).$$

Secondly, let $c = 2$ and $(\lambda, \mu) \in \mathbb{S}$. We then obtain a constant function of the dependent variables λ and μ as follows:

$$F(\lambda, \mu) = \frac{(1-\beta)^2}{2} (2\beta^2 - 4\beta + 3) \tag{2.20}$$

for every $0 \leqq \beta < 1$.

Finally, let $c = 0$ and $(\lambda, \mu) \in \mathbb{S}$. We then find that

$$F(\lambda, \mu) = \frac{(1-\beta)^2}{9} (\lambda + \mu)^2.$$

We can easily see that the maximum of $F(\lambda, \mu)$ occurs at $\lambda = \mu = 1$ and we have

$$\max\{F(\lambda, \mu)\} = F(1, 1) = \frac{4\,(1-\beta)^2}{9} \tag{2.21}$$

for every β $(0 \leqq \beta < 1)$.

From (2.19), (2.20), and (2.21), it is easily seen that

$$\frac{4\,(1-\beta)^2}{9} < \frac{(1-\beta)^2}{2}(2\beta^2-4\beta+3) < \frac{(1-\beta)^2}{2}\left(\frac{15\beta^2-21\beta-\frac{25}{4}}{18\beta^2-30\beta+2}\right)$$

for

$$\beta \in \left(\frac{11-\sqrt{37}}{12}, 1\right).$$

We thus obtain the second inequality of (2.1) for

$$\beta \in \left(\frac{11-\sqrt{37}}{12}, 1\right).$$

On the other hand, since the following inequality:

$$\frac{4\,(1-\beta)^2}{9} < \frac{(1-\beta)^2}{2}(2\beta^2-4\beta+3)$$

is satisfied for every β $(0 \leqq \beta < 1)$, we obtain the first inequality of (2.1) for

$$\beta \in \left[0, \frac{11-\sqrt{37}}{12}\right].$$

This completes the proof of Theorem 1. □

Our second main result for the class $\mathcal{N}_\sigma^\alpha$ is given by Theorem 2 below.

Theorem 2 *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{N}_\sigma^\alpha$ $(0 < \alpha \leqq 1)$. Then*

$$\left|a_2a_4 - a_3^2\right| \leqq \begin{cases} \frac{4\alpha^2}{9} & \left(0 < \alpha \leqq \frac{7}{24}\right), \\ \frac{\alpha^2}{48}\left(\frac{64\alpha^2-144\alpha+5}{12\alpha^2-12\alpha+1}\right) & \left(\frac{7}{24} \leqq \alpha \leqq \frac{1+\sqrt{2}}{4}\right), \\ \frac{\alpha^2\left(8\alpha^2+1\right)}{6} & \left(\frac{1+\sqrt{2}}{4} \leqq \alpha \leqq 1\right). \end{cases} \tag{2.22}$$

Proof Let $f \in \mathcal{N}_\sigma^\alpha$, $0 < \alpha \leqq 1$, and $g = f^{-1}$. Then

$$f'(z) = [p(z)]^\alpha \qquad \text{and} \qquad g'(w) = [q(w)]^\alpha, \tag{2.23}$$

where the functions $p(z)$ and $q(z)$ given by

$$p(z) = 1 + c_1z + c_2z^2 + \cdots \qquad \text{and} \qquad q(w) = 1 + d_1w + d_2w^2 + \cdots$$

are in class $\mathcal{P}$.

Now, upon equating the coefficients in (2.23), we have

$$2a_2 = \alpha c_1, \tag{2.24}$$

$$3a_3 = \alpha c_2 + \frac{\alpha(\alpha-1)}{2}c_1^2, \tag{2.25}$$

$$4a_4 = \alpha c_3 + \alpha(\alpha-1)c_1c_2 + \frac{\alpha(\alpha-1)(\alpha-2)c_1^3}{6}, \tag{2.26}$$

and

$$-2a_2 = \alpha d_1, \tag{2.27}$$

$$3\left(2a_2^2 - a_3\right) = \alpha d_2 + \frac{\alpha(\alpha-1)}{2}d_1^2, \tag{2.28}$$

$$-4\left(5a_2^3 - 5a_2a_3 + a_4\right) = \alpha d_3 + \alpha(\alpha-1)d_1d_2 + \frac{\alpha(\alpha-1)(\alpha-2)d_1^3}{6}. \tag{2.29}$$

From (2.24) and (2.27), we obtain

$$c_1 = -d_1 \tag{2.30}$$

and

$$a_2 = \frac{\alpha c_1}{2}. \tag{2.31}$$

Now, from (2.25), (2.28), and (2.31), we find that

$$a_3 = \frac{\alpha^2 c_1^2}{4} + \frac{\alpha(c_2 - d_2)}{6}. \tag{2.32}$$

Also, from (2.26) and (2.29), we get

$$a_4 = \frac{\alpha(\alpha-1)(\alpha-2)c_1^3}{24} + \frac{5\alpha^2 c_1(c_2-d_2)}{24} + \frac{\alpha(c_3-d_3)}{8} + \frac{\alpha(\alpha-1)c_1(c_2+d_2)}{8}. \tag{2.33}$$

We can thus easily establish that

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| = \Bigg| &\frac{\alpha^2(\alpha-1)(\alpha-2)c_1^4}{48} - \frac{\alpha^4 c_1^4}{16} + \frac{\alpha^3 c_1^2(c_2-d_2)}{48} \\ &+ \frac{\alpha^2 c_1(c_3-d_3)}{16} - \frac{\alpha^2(c_2-d_2)^2}{36} + \frac{\alpha^2(\alpha-1)c_1^2(c_2+d_2)}{16}\Bigg|. \end{aligned} \tag{2.34}$$

Using (2.14), (2.15), and (2.16) in (2.34), we have

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| \leqq &\frac{\alpha^2(\alpha-1)(\alpha-2)c_1^4}{48} + \frac{\alpha^4 c_1^4}{16} + \frac{\alpha^2 c_1^4}{32} + \frac{\alpha^2(\alpha-1)c_1^4}{16} + \frac{\alpha^2 c_1(4-c_1^2)}{16} \\ &+ \frac{\alpha^3 c_1^2(4-c_1^2)}{24}(|x|+|y|) + \frac{\alpha^2 c_1(4-c_1^2)(c_1-2)}{64}(|x|^2+|y|^2) + \frac{\alpha^2(4-c_1^2)^2}{144}(|x|+|y|)^2. \end{aligned}$$

Since $p(z) \in \mathcal{P}$, we obtain $|c_1| \leqq 2$. Taking $c_1 = c$, we may assume without any loss of generality that $c \in [0,2]$. Thus, for

$$\lambda = |x| \leqq 1 \qquad \text{and} \qquad \mu = |y| \leqq 1,$$

we obtain

$$\left|a_2a_4 - a_3^2\right| \leqq M_1 + M_2(\lambda+\mu) + M_3(\lambda^2+\mu^2) + M_4(\lambda+\mu)^2 = \Psi(\lambda,\mu),$$

where

$$M_1 = M_1(c) = \frac{\alpha^2}{96}\left[\left(8\alpha^2+1\right)c^4 - 6c^3 + 24c\right] \geqq 0,$$

$$M_2 = M_2(c) = \frac{\alpha^3}{24}c^2(4-c^2) \geqq 0,$$

$$M_3 = M_3(c) = \frac{\alpha^2}{64}c(4-c^2)(c-2) \leqq 0,$$

$$M_4 = M_4(c) = \frac{\alpha^2}{144}(4-c^2)^2 \geqq 0.$$

Therefore, we need to maximize $\Psi(\lambda,\mu)$ in the closed square $\mathbb{S}$ given by

$$\mathbb{S} = \{(\lambda,\mu) : 0 \leqq \lambda \leqq 1 \quad \text{and} \quad 0 \leqq \mu \leqq 1\}.$$

In order to determine the maximum of $\Psi(\lambda,\mu)$, we can analogously follow the derivation of the maximum of $F(\lambda,\mu)$ in Theorem 1. Thus, clearly, the maximum of $\Psi(\lambda,\mu)$ occurs at $\lambda = 1$ and $\mu = 1$ in the closed square $\mathbb{S}$. Let $\Phi : (0,2) \to \mathbb{R}$ defined by

$$\Phi(c) = \max\{\Psi(\lambda,\mu)\} = \Psi(1,1) = M_1 + 2\left(M_2+M_3\right) + 4M_4. \tag{2.35}$$

Substituting the values of M_1, M_2, M_3, and M_4 into the function $\Phi(c)$ given by (2.35), we get

$$\Phi(c) = \frac{\alpha^2}{144}\left[\left(12\alpha^2 - 12\alpha + 1\right)c^4 + (48\alpha-14)c^2 + 64\right].$$

Let

$$P = 12\alpha^2 - 12\alpha + 1, \quad Q = 48\alpha - 14, \qquad \text{and} \qquad R = 64. \tag{2.36}$$

Then, since

$$\max_{0\leqq t\leqq 4}\left\{\left(Pt^2+Qt+R\right)\right\} = \begin{cases} R & \left(Q \leqq 0;\ P \leqq -\frac{Q}{4}\right), \\ 16P+4Q+R & \left(Q \geqq 0 \ \text{and} \ P \geqq -\frac{Q}{8} \ \text{or} \ Q \leqq 0 \ \text{and} \ P \geqq -\frac{Q}{4}\right), \\ \frac{4PR-Q^2}{4P} & \left(Q > 0;\ P \leqq -\frac{Q}{8}\right), \end{cases} \tag{2.37}$$

we have

$$\left|a_2a_4 - a_3^2\right| \leqq \frac{\alpha^2}{144}\begin{cases} R & \left(Q \leqq 0;\ P \leqq -\frac{Q}{4}\right), \\ 16P+4Q+R & \left(Q \geqq 0 \ \text{and} \ P \geqq -\frac{Q}{8} \ \text{or} \ Q \leqq 0 \ \text{and} \ P \geqq -\frac{Q}{4}\right), \\ \frac{4PR-Q^2}{4P} & \left(Q > 0;\ P \leqq -\frac{Q}{8}\right), \end{cases}$$

where P, Q, and R are given by (2.36).

This completes the proof of Theorem 2. □

For $\beta = 0$ in Theorem 1 or for $\alpha = 1$ in Theorem 2, we obtain the coefficient estimate given by the corollary below.

Corollary. *Let* $f(z)$ *given by* (1.1) *be in the class* $\mathcal{N}_\sigma$*. Then*

$$\left|a_2a_4 - a_3^2\right| \leqq \frac{3}{2}.$$

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