Second Modules over Noncommutative Rings

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Joint Work With: S.CEKEN- P.F. SMITH

Introduction

Throughout all rings have identity elements and all modules are unital.

Definitions

i) A right R-module M is called prime in case $M \neq 0$ and $\operatorname{ann}_R(M) = \operatorname{ann}_R(N)$ for every non-zero submodule N of M.
ii) A right R-module M will be called a second module provided $M \neq 0$ and $\operatorname{ann}_R(M) = \operatorname{ann}_R(M/N)$ for every proper submodule N of M. .

- By a prime submodule of M, we mean a submodule P such that the module M/P is prime.
- By a second submodule of M, we mean a submodule which is also a second module.
- In [S.Annin Attached primes over noncommutative rings, J. Pure Appl. Algebra 212 (2008), 510-521.] second modules are called coprime.

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Prime submodules

Prime modules and prime submodules of modules have been studied by various authors over the past 30 years

•J. Dauns, Prime modules, J. Reine Angew Math. 298 (1978), 156-181.

•C.-P. Lu, Prime submodules of modules, Comm. Math. Univ. Sancti Pauli 33 (1984), 61-69.

•R. L. McCasland and P. F. Smith, Prime submodules of Noetherian modules, Rocky Mtn. J. 23 (1993), 1041-1062.

•Y. Tiras, A, Harmanci and P. F. Smith, A characterization of prime submodules, J. Algebra 212 (1999), 743-752.

Second modules

The study of second modules and second submodules of modules have been instigated by

•S. Yassemi, The dual notion of prime submodules. Arch. Math. Brno, 37 (2001), 273-278.

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•S. Ebrahimi-Atani, On secondary modules over Dedekind domains, Southeast Asian Bull. Math. 25 (1) (2001), 1-6.

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Let R be a commutative ring and let M be a non-zero R-module. Given any element $r \in R$, let $\mu_r : M \to M$ denote the endomorphism of M defined by $\mu_r(m) = rm\,(m \in M)$.

- M is prime if and only if for each $r \in R$ either μ_r is zero or a monomorphism.
- M is prime if and only if for any r in R and m in M, $rm = 0$ implies that $m = 0$ or $rM = 0$.
- M is second if and only if for each $r \in R$ either μ_r is zero or an epimorphism.
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The fundamental concepts of the second modules

If R is any ring and M is a second R-module then $P = \text{ann}_R(M)$ is a prime ideal of R
because if $MAB = 0$, for some ideals A and B of R, and $0 \neq MA$ then we get that $M = MA$ and so $MB = 0$.

In this case, M is called a P-second module.
Clearly a simple modules are both prime and second modules.

More generally, a homogeneous semisimple modules are both prime and second.

If R is a simple ring then every non-zero module is a prime second module.

Conversely, every ring R such that the right R-module R is a second module is simple.

Clearly every non-zero submodule of a prime module is prime and every non-zero homomorphic image of a second module is second.

Lemma

Let R be a ring such that every prime ideal is maximal. Then a right R-module M is prime if and only if M is second.
Moreover, if R is commutative then the module M is second if and only if M is homogeneous semisimple.

Proof.

Suppose first that M is prime. Then $M \neq 0$ and $P = \text{ann}_R(M)$ is a prime, and hence maximal ideal of R.Let N be any proper submodule of M. Then $P \subseteq \text{ann}_R(M/N) \subset R$, so that $P = \text{ann}_R(M/N)$. It follows that M is a second module.
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The Results

Corollary

Let R be either a commutative von Neumann regular ring or a right perfect ring . Then a non-zero module M is second if and only if M is homogeneous semisimple.

Lemma

Let R be a ring such that R/P is right Artinian for every right primitive ideal P. Then the following statements are equivalent for a module M.

1. *M is a prime module which contains a simple submodule.*
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2. *For every ideal A of R either $MA = 0$ or $M = MA$.*
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$(i) \Rightarrow (ii)$ Suppose that $M \neq MA$ for any ideal A of R. Then MA is a proper submodule. If $B = \text{ann}_R(M/MA)$ then (i) gives that $MB = 0$. But we know that $A \subseteq B$ and hence $MA = 0$.
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$(iv) \Rightarrow (i)$Let N be a proper submodule and let $C = \text{ann}_R(M/N)$. Then $\text{ann}_R(M) \subseteq C$ and $MC \subseteq N \neq M$ so that $C = \text{ann}_R(M)$ and $MC = 0$. Thus $\text{ann}_R(M) = \text{ann}_R(M/N)$ and hence$M$ is second. □

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Consequences

- Let P be a prime ideal of a ring R and let N be a submodule of a module M such that the modules N and M/N are both P-second. Then M is P-second if and only if $MP = 0$.
- Let M be a P-second module for some prime ideal P of R. Then every non-zero pure submodule of M is P-second.
- Let A be an ideal of a ring R and let M be a R-module such that $MA = 0$. Then the R-module M is a second module if and only if the (R/A)-module M is a second module.
- Let P be a prime ideal of a commutative ring R. Then the sum of any non-empty collection of P-second submodules of a R-module X is also a P-second submodule of X.

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Lemma

Let R be a prime right Goldie ring. Then

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The Theorems

- Let R be a ring such that R/P is a left bounded left Goldie ring for every prime ideal P of R. Then
  1. a module M is a second module if and only if $Q = \text{ann}_R(M)$ is a prime ideal of R and M is a divisible right (R/Q)-module.
  2. a module M is a prime second module if and only if $Q = \text{ann}_R(M)$ is a prime ideal of R and M is a torsion-free injective right (R/Q)-module.
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For an arbitrary ring R, let M be a Bass R-module,(i.e, every proper submodule is contained in a maximal submodule)
Let P be an attached prime of M. There exists a proper submodule N of M such that M/N is P-second.
Let L be a maximal submodule of M such that $N \subseteq L$. Then $P = \text{ann}_R(M/N) = \text{ann}_R(M/L)$ and hence P is a right primitive ideal of R. Thus every attached prime ideal of a Bass module is right primitive.

Propositions

- Let R be a semilocal ring. Then every Bass R-module has a finite number of attached prime ideals.
- Let M be a non-zero R-module such that there exists an ideal P of R maximal in the collection of ideals A of R such that $M \neq MA$. Then P is an attached prime ideal of M and M/MP is a P-second module.
- Let M be a non-zero R-module. Then there exists a proper submodule N of M such that M/N is a second module if and only if there exist a proper submodule L of M and a prime ideal P of R such that P is maximal in the collection of ideals A of R such that $M \neq MA + L$.

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A second submodule L of a module M is called a maximal second submodule if L is not contained in another second submodule of M.

- Let N_i $(i \in I)$ be chain of second submodules of a right modules M. Then $N = \cup_{i \in I} N_i$ is a second submodule of M.
- Then every second submodule of a nonzero module M is contained in a maximal second submodule of M.
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Theorem

Every non-zero Artinian module contains only a finite number of maximal second submodules.

Proof.

Suppose the result is false.
Let M be a non-zero Artinian right R-module such that M does not contain a finite number of maximal second submodules.
Let N be a non-zero submodule of M minimal with respect to the property that N does not contain a finite number of maximal second submodules.
Clearly N is not a second module.
Then there exists an ideal A of R such that $NA \neq 0$ and $N \neq NA$.
Let $L = \{x \in N : xA = 0\}$. Then L is a submodule of N such that $LA = 0$ and hence $L \neq N$. □

Proof.

Suppose that $L \neq 0$. By the choice of N, L contains only a finite number of maximal second submodules L_i $(1 \leq i \leq n)$, for some positive integer n, and NA contains only a finite number of maximal second submodules K_j $(1 \leq j \leq t)$, for some positive integer t.
Let H be a maximal second submodule of N. Then we get that either $HA = 0$ or $H = HA$.
If $HA = 0$ then $H \subseteq L$ and hence $H \subseteq L_i$ for some $1 \leq i \leq n$ and it follows that $H = L_i$.
If $H = HA$ then $H \subseteq NA$ so that $H \subseteq K_j$ for some $1 \leq j \leq t$. In this case, $H = K_j$.
Thus every maximal second submodule of N belongs to the list $L_1, \ldots, L_n, K_1, \ldots, K_t$ of submodules of N.
Thus N has at most $n + t$ maximal second submodules, a contradiction.
Now suppose that $L = 0$. In this case, $H = K_j$ for some $1 \leq j \leq t$ and again N has at most a finite number of maximal second submodules.
The result follows. □

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Thank you for your attentions