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# SECOND-ORDER BSDES WITH JUMPS: FORMULATION AND UNIQUENESS[1]

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In this paper, we define a notion of second-order backward stochastic differential equations with jumps (2BSDEJs for short), which generalizes the continuous case considered by Soner, Touzi and Zhang [*Probab. Theory Related Fields* **153** (2012) 149–190]. However, on the contrary to their formulation, where they can define pathwise the density of quadratic variation of the canonical process, in our setting, the compensator of the jump measure associated to the jumps of the canonical process, which is the counterpart of the density in the continuous case, depends on the underlying probability measures. Then in our formulation of 2BSDEJs, the generator of the 2BSDEJs depends also on the underlying probability measures through the compensator. But the solution to the 2BSDEJs can still be defined universally. Moreover, we obtain a representation of the $Y$ component of a solution of a 2BSDEJ as a supremum of solutions of standard backward SDEs with jumps, which ensures the uniqueness of the solution.

**1. Introduction.** Motivated by duality methods and maximum principles for optimal stochastic control, Bismut [7] studied a linear backward stochastic differential equation (BSDE). In their seminal paper [30], Pardoux and Peng generalized such equations to the nonlinear Lipschitz case and proved existence and uniqueness results in a Brownian framework. Since then, a lot of attention has been given to BSDEs and their applications, not only in stochastic control, but also in theoretical economics, stochastic differential games and financial mathematics.

Given a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P})$ generated by an $\mathbb{R}^d$-valued Brownian motion $B$, solving a BSDE with generator $f$ and terminal condition $\xi$ consists of finding a pair of progressively measurable processes $(Y, Z)$ such that

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s)\,ds - \int_t^T Z_s\,dB_s, \qquad \mathbb{P}\text{-a.s.}, t \in [0, T]. \tag{1.1}$$

The process $Y$ thus defined is a possible generalization of the conditional expectation of $\xi$, since when $f$ is the null function, we have $Y_t = \mathbb{E}_t^{\mathbb{P}}[\xi]$, and in this case,

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$Z$ is the process appearing in the $(\mathcal{F}_t)$-martingale representation of $(\mathbb{E}_t^{\mathbb{P}}[\xi])_{t\geq 0}$. In the case of a filtered probability space generated by both a Brownian motion $B$ and a Poisson random measure $\mu$ with compensator $\nu$, the martingale representation for $(\mathbb{E}_t^{\mathbb{P}}[\xi])_{t\geq 0}$ becomes

$$\mathbb{E}_t^{\mathbb{P}}[\xi] = \mathbb{E}^{\mathbb{P}}[\xi] + \int_0^t Z_s\,dB_s + \int_0^t \int_{\mathbb{R}^d\setminus\{0\}} \psi_s(x)(\mu-\nu)(dx,ds), \qquad \mathbb{P}\text{-a.s.},$$

where $\psi$ is a predictable function.

This leads to the following natural generalization of equation (1.1) to the case with jumps. We will say that $(Y, Z, U)$ is a solution to the BSDE with jumps (BSDEJ in the sequel) with generator $f$ and terminal condition $\xi$ if for all $t \in [0, T]$, we have $\mathbb{P}$-a.s.

$$\begin{aligned} Y_t = \xi &+ \int_t^T f_s(Y_s, Z_s, U_s)\,ds - \int_t^T Z_s\,dB_s \\ &- \int_t^T \int_{\mathbb{R}^d\setminus\{0\}} U_s(x)(\mu-\nu)(dx,ds). \end{aligned} \tag{1.2}$$

Tang and Li [37] were the first to prove existence and uniqueness of a solution for (1.2) in the case where $f$ is Lipschitz in $(y, z, u)$. Our aim in this paper is to generalize (1.2) to the second order, as introduced recently by Soner, Touzi and Zhang [36]. Their key idea in the definition of the second-order BSDEs (2BSDEs) is that the equation defining the solution has to hold $\mathbb{P}$-almost surely, for every $\mathbb{P}$ in a class of nondominated probability measures. They then manage to prove a uniqueness result using a representation of the solution of a 2BSDE as an essential supremum of solutions of standard BSDEs. This representation finds its origin in the deep link that 2BSDEs share with stochastic control theory and PDEs. In order to shed more light on this aspect, let us give the intuition behind this representation in the continuous case.

Suppose that we want to study the following fully nonlinear PDE:

$$-\frac{\partial u}{\partial t} - h\bigl(t, x, u(t,x), Du(t,x), D^2u(t,x)\bigr) = 0, \qquad u(T,x) = g(x). \tag{1.3}$$

If the function $\gamma \mapsto h(t, x, r, p, \gamma)$ is assumed to be convex, then it is equal to its double Fenchel–Legendre transform, and if we denote its Fenchel–Legendre transform by $f$, we have

$$h(t, r, p, \gamma) = \sup_{a\geq 0}\left\{\frac{1}{2}a\gamma - f(t, x, r, p, a)\right\}. \tag{1.4}$$

From (1.4), we expect, at least formally, that the solution $u$ of (1.3) is going to verify

$$u(t,x) = \sup_{a\geq 0} u^a(t,x),$$

where $u^a$ is defined as the solution of the following semi-linear PDE:

$$(1.5)\qquad -\frac{\partial u^a}{\partial t} - \frac{1}{2} a D^2 u^a(t,x) + f\bigl(t,x,u^a(t,x), Du^a(t,x), a\bigr) = 0, \qquad u^a(T,x) = g(x).$$

Since $u^a$ is linked to a classical BSDE, the 2BSDE associated to $u$ should correspond (in some sense) to the supremum of the family of BSDEs indexed by $a$. Furthermore, changing the process $a$ can be achieved by changing the probability measure under which the BSDE is written. We also emphasize that the generator of the BSDEs depends explicitly on $a$, which is actually the density of the quadratic variation of the martingale driving the BSDE.

For the sake of clarity, we will now briefly outline the main differences and difficulties due on the one hand to second-order framework and on the other hand to our jump setting.

(i) We remind the reader that our aim is to introduce an equation similar to (1.2). But as shown above in the continuous case, the generator $f$ will have to depend on the density of $[B, B]^c$, the pathwise continuous part of the quadratic variation $[B, B]$ of the canonical process $B$, but since we are in a jump setting, it will also have to depend on the compensator of the random jump measure $\mu_B$ associated to $B$. Exactly as in the continuous case, we can always give a pathwise definition of the density of $[B, B]^c$, which gives us directly an aggregator. However, it is generally impossible to find an aggregator for the compensator of the jump measure; see Section 2.3 for more details. This forces us to consider in our jump setting 2BSDEs whose generator depends explicitly on the underlying probability measure. This is an important difference with the framework considered in [36]. However, in spite of this, the solution to the 2BSDEJs is still-defined independently of the probability measures considered.

(ii) A second major difference with (1.2) in the second-order case, is, as we recalled earlier, that the BSDE has to hold $\mathbb{P}$-almost surely for every probability measure $\mathbb{P}$ lying in a wide family of probability measures. Under each $\mathbb{P}$, $[B, B]^c$ and $\mu_B$ have, respectively, a prescribed density and a prescribed jump measure compensator. This is why we can intuitively understand the 2BSDEJ (3.3) as a BSDEJ with model uncertainty, where the uncertainty affects both the quadratic variation and the jump measure of the process driving the equation.

(iii) The last major difference with (1.2) in the second-order case is the presence of an additional nondecreasing process $K$ in the equation. To have an intuition for $K$, one has to have in mind representation (4.1) that we prove in Theorem 4.1, stating that the $Y$ part of a solution of a 2BSDEJ is an essential supremum of solutions of standard BSDEJs. The process $K$ maintains $Y$ above any solution $y^{\mathbb{P}}$ of a BSDEJ, with given quadratic variation and jump measure under $\mathbb{P}$. The process $K$ is then formally analogous to the nondecreasing process appearing in reflected BSDEs (as defined in [15], e.g.).

There are many other possible approaches in the literature to handle volatility and/or jump measure uncertainty in stochastic models [1, 6, 11, 12]. Among them, Peng [31] introduced a notion of Brownian motion with uncertain variance structure, called $G$-Brownian motion. This process is defined without making reference to a given probability measure. It refers instead to the $G$-Gaussian law, defined by a partial differential equation; see [32] for a detailed exposition and references. We also would like to mention the very recent works by Neufeld and Nutz [25, 26], which appeared during the revision of this paper, which provide an elegant and very important extension to the work of Peng, by allowing a very general type of uncertainties for the whole triplet of characteristics of a given Lévy process, very in much in the spirit of the approach we follow in this paper. We emphasize that their approach is very general, which is why they have to deal with delicate measurability issues, and that, roughly speaking, their results could be used to define the solution to a 2BSDEJ with a generator equal to 0, without having to impose any continuity assumptions on the terminal condition.

Finally, recall that Pardoux and Peng [30] proved that if the randomness in $f$ and $\xi$ is induced by the current value of a state process defined by a forward stochastic differential equation, the solution to the BSDE (1.1) could be linked to the solution of a semilinear PDE by means of a generalized Feynman–Kac formula. Similarly, Soner, Touzi and Zhang [36] showed that 2BSDEs generalized the point of view of Pardoux and Peng, in the sense that they are connected to the larger class of fully nonlinear PDEs. In this context, the 2BSDEJs are the natural candidates for a probabilistic solution of fully nonlinear integro-differential equations. This is the purpose of our accompanying paper [21].

The rest of this paper is organized as follows. In Section 2, we introduce the set of probability measures on the Skorohod space $\mathbb{D}$ that we will work with. Using the notion of martingale problems on $\mathbb{D}$, we construct probability measures under which the canonical process has given characteristics. In Section 3, we define the notion of 2BSDEJs and show how it is linked to standard BSDEJs. Section 4 is devoted to our uniqueness result and some a priori estimates. The Appendix is dedicated to the proof of some important technical results needed throughout the paper.

## 2. Preliminaries.

2.1. *A primer on* 2*BSDEJs and main difficulties*. Before giving all notation in detail and a precise definition of 2BSDEJs, we would like to start by presenting the main object of interest in this paper, as well as the main difficulties we need to address in our framework.

First, as mentioned in the Introduction, we shall consider the following 2BSDEJ, for $0 \le t \le T$ and $\mathbb{P}$-a.s.:

$$\begin{aligned}Y_t = \xi + \int_t^T \widehat{F}_s^{\mathbb{P}}(Y_s, Z_s, U_s)\,ds - \int_t^T Z_s\,dB_s^{\mathbb{P},c}\\ - \int_t^T \int_E U_s(x)\tilde{\mu}_B^{\mathbb{P}}(dx, ds) + K_T^{\mathbb{P}} - K_t^{\mathbb{P}},\end{aligned}$$

for every $\mathbb{P} \in \mathcal{P}_H^\kappa$, which is a family, not necessarily dominated, of local martingale probability measures. These different probability measures represent the model uncertainty. $B^{\mathbb{P},c}$ and $\tilde{\mu}_B^{\mathbb{P}}$ denote, respectively, the continuous local martingale part and the compensated jump measure associated to the purely discontinuous local martingale part of the canonical process $B$ under any local martingale measure $\mathbb{P}$. We reiterate that in contrast to (1.2), we have to add a nondecreasing process $K^{\mathbb{P}}$ to account for the fact that solutions to 2BSDEJs have to be understood as suprema of families of classical BSDEJs.

Let us now highlight the new difficulties in our framework compared to the continuous 2BSDEs as considered in [36]. While a crucial issue in their definition of the 2BSDEs is the aggregation of the quadratic variation of the canonical process $B$ under a wide family of probability measures, here, in general, the aggregation of the jump compensators associated to $B$ is not possible; see Section 2.3 for more details. This is the reason why the generator $\widehat{F}^{\mathbb{P}}$ and the compensated jump measure $\widetilde{\mu}^{\mathbb{P}}$ above depend explicitly on the probability measure, through the jump compensator defined under each $\mathbb{P}$. This is an important difference which may lead one to think that it might not be possible to define the solution $(Y, Z, U)$ of a 2BSDEJ in a universal way (i.e., to say so that it does not depend explicitly on the measure $\mathbb{P}$). This would be very unfortunate from the point of view of applications, since, if we look, for instance, at classical problems of portfolio optimization in finance, the process $Z$ is usually related to the corresponding optimal investment strategy. Therefore, in a context of uncertainty, one will definitely need an optimal strategy which works for every possible model, that is to say for every measure $\mathbb{P}$. Nonetheless, we prove that the solution of a 2BSDEJ, $(Y, Z, U)$, can still be constructed in such a way that it is defined for all $\omega$, independently of probability measures; we refer the reader to our companion paper [21] for more details.

Another crucial point in the definition of 2BSDEs in [36], is that they work under a set of measure corresponding to the so-called strong formulation of stochastic control. Roughly speaking, this corresponds to considering the laws under the Wiener measure of stochastic integrals with respect to the canonical process $B$, with the constraint that these integrands have to take values in the space of symmetric definite positive matrices. Such a choice has several extremely important advantages: first of all, it allows them to define their measures through a unique reference measure (i.e., the Wiener measure), and even more importantly, they showed that all the measures thus constructed satisfy the martingale representation property and the Blumenthal 0–1 law, which are known to be fundamental properties for the wellposedness of classical BSDEs (which, as recalled in the Introduction are a kind of nonlinear martingales). Therefore, in our framework, we have to be able to retrieve the strong formulation. However, there is no longer any clear choice for a reference measure as soon as jumps are added into the mix. This will therefore lead us to consider a whole family of reference measures, which makes in turn the problem more complicated. Moreover, even though the only assumption needed in [36] to retrieve the martingale representation property is that the admissible volatilities are symmetric definite positive (and therefore invertible), in a framework with jumps, we need to consider special jumps compensators with some restrictions, but which still are flexible enough to be able to model as many types of jump measure uncertainty as possible. Overcoming these main difficulties is the most important contribution of this paper.

2.2. *The stochastic basis*. We first introduce the notations used in the paper. Let $\Omega := \mathbb{D}([0,T],\mathbb{R}^d)$ be the space of càdlàg paths defined on $[0,T]$ with values in $\mathbb{R}^d$ and such that $w(0)=0$, equipped with the Skorohod topology, so that it is a complete, separable metric space; see [5], for instance.

We denote $B$ the canonical process, $\mathbb{F} := \{\mathcal{F}_t\}_{0\le t\le T}$ the filtration generated by $B$, $\mathbb{F}^+ := \{\mathcal{F}_t^+\}_{0\le t\le T}$ the right limit of $\mathbb{F}$ and for any $\mathbb{P}$, $\overline{\mathcal{F}}_t^{\mathbb{P}} := \mathcal{F}_t^+ \vee \mathcal{N}^{\mathbb{P}}(\mathcal{F}_t^+)$ where

$$\mathcal{N}^{\mathbb{P}}(\mathcal{G}) := \{E \in \Omega,\ \text{there exists } \widetilde{E} \in \mathcal{G} \text{ such that } E \subset \widetilde{E} \text{ and } \mathbb{P}(\widetilde{E}) = 0\}.$$

We then define as in [36] a local martingale measure $\mathbb{P}$ as a probability measure such that $B$ is a $\mathbb{P}$-local martingale. We then associate to the jumps of $B$ a counting measure $\mu_B$, which is a random measure on $\mathbb{R}^+ \times E$ equipped with its Borel $\sigma$-field $\mathcal{B}(\mathbb{R}^+) \times \mathcal{B}(E)$ (where $E := \mathbb{R}^d \setminus \{0\}$), defined pathwise by

$$\mu_B(A,[0,t]) := \sum_{0<s\le t} \mathbf{1}_{\{\Delta B_s \in A\}} \qquad \forall t \ge 0, \forall A \in \mathcal{B}(E). \tag{2.1}$$

We recall that (see, e.g., Theorem I.4.18 in [19]) under any local martingale measure $\mathbb{P}$, we can decompose $B$ uniquely into the sum of a continuous local martingale, denoted by $B^{\mathbb{P},c}$, and a purely discontinuous local martingale, denoted by $B^{\mathbb{P},d}$. We emphasize that such a decomposition depends on the underlying probability measure. Then we define $\overline{\mathcal{P}}_W$ as the set of all local martingale measures $\mathbb{P}$, such that $\mathbb{P}$-a.s.:

(i) The quadratic variation of $B^{\mathbb{P},c}$ is absolutely continuous with respect to the Lebesgue measure $dt$, and its density takes values in $\mathbb{S}_d^{>0}$, which is the space of all $d \times d$ real valued positive definite matrices.

(ii) The compensator $\lambda_t^{\mathbb{P}}(dx,dt)$ of the jump measure $\mu_B$ exists under $\mathbb{P}$ and can be decomposed, for some $\mathbb{F}$-predictable random measure $\nu^{\mathbb{P}}$ on $E$, as follows:

$$\lambda_t^{\mathbb{P}}(dx,dt) = \nu_t^{\mathbb{P}}(dx)\,dt.$$

We will denote by $\widetilde{\mu}_B^{\mathbb{P}}(dx,dt)$ the corresponding compensated measure, and for simplicity, we will often call $\nu^{\mathbb{P}}$ the compensator of the jump measure associated to $B$.

REMARK 2.1. In this paper, we always assume that under the probability measures that we consider, the canonical process is a local martingale, whose quadratic variation and jump compensator change depending on the measure considered.

Formally, it means that we do not consider drift uncertainty. Hence, the reader may wonder why we do not consider more generally probability measures under which the canonical process is a semimartingale with a triplet of characteristics which can all vary. In a nutshell, the framework considered here is completely sufficient for us in order to give wellposedness results for 2BSDEs with jumps, and we did not want to make our presentation confusing. However, we emphasize that all the above results can be easily extended to the more general case of drift, volatility and jump uncertainty. For related results, the reader may consult [29], and the very recent preprint [25].

In this discontinuous setting, we will say that a probability measure $\mathbb{P} \in \overline{\mathcal{P}}_W$ satisfies the martingale representation property if for any $(\overline{\mathbb{F}}^{\mathbb{P}}, \mathbb{P})$-local martingale $M$, there exists a unique $\overline{\mathbb{F}}^{\mathbb{P}}$-predictable processes $H$ and a unique $\overline{\mathbb{F}}^{\mathbb{P}}$-predictable function $U$ such that $(H, U) \in \mathbb{H}^2_{\mathrm{loc}}(\mathbb{P}) \times \mathbb{J}^2_{\mathrm{loc}}(\mathbb{P})$ (these spaces are defined later in Section 3.2) and

$$M_t = M_0 + \int_0^t H_s \, dB_s^{\mathbb{P},c} + \int_0^t \int_E U_s(x) \widetilde{\mu}_B^{\mathbb{P}}(dx, ds), \qquad \mathbb{P}\text{-a.s.}$$

We now follow [35] and introduce the so-called universal filtration. For this we let $\mathcal{P}$ be a given subset of $\overline{\mathcal{P}}_W$ and define the following:

DEFINITION 2.1. A property is said to hold $\mathcal{P}$-quasi-surely ($\mathcal{P}$-q.s. for short), if it holds $\mathbb{P}$-a.s. for all $\mathbb{P} \in \mathcal{P}$.

2.3. *Aggregation* (*or the absence of it*). In this section, we discuss issues related to aggregation of the quadratic variation of the canonical process $B$ and the absence of aggregation of the jump compensators associated to the jumps of $B$.

Let $\mathcal{P} \subset \overline{\mathcal{P}}_W$ be a set of nonnecessarily dominated probability measures, and let $\{X^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}\}$ be a family of random variables indexed by $\mathcal{P}$. One can think, for example, of the stochastic integrals $X_t^{\mathbb{P}} := {}^{(\mathbb{P})}\int_0^t H_s \, dB_s$, where $\{H_t, t \geq 0\}$ is a predictable process.

DEFINITION 2.2. An *aggregator* of the family $\{X^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}\}$ is a r.v. $\widehat{X}$ such that

$$\widehat{X} = X^{\mathbb{P}}, \qquad \mathbb{P}\text{-a.s., for every } \mathbb{P} \in \mathcal{P}.$$

Bichteler [4], Karandikar [20], and more recently Nutz [28] all showed in different contexts and under different assumptions, that it is possible to find an aggregator for the Itô stochastic integrals ${}^{(\mathbb{P})}\int_0^t H_s \, dB_s$. A direct consequence of this result is the possibility to define the quadratic variation process $\{[B, B]_t, t \geq 0\}$

pathwise.[3] Indeed, using Itô's formula, we can write for any local martingale measure $\mathbb{P}$,

$$[B, B]_t = |B_t|^2 - 2\int_0^t B_{s-}\, dB_s, \qquad \mathbb{P}\text{-a.s.},$$

and the aggregation of the stochastic integrals automatically yields the aggregation of the bracket $\{[B, B]_t, t \geq 0\}$.

Next, since $[B, B]$ has finite variation, we can define its path-by-path continuous part $[B, B]^c$ (by subtracting the sum of the jumps) and finally the corresponding density

$$\hat{a}_t := \overline{\lim_{\varepsilon \downarrow 0}} \frac{[B, B]^c_t - [B, B]^c_{t-\varepsilon}}{\varepsilon}.$$

Notice that since for any local martingale measure $\mathbb{P}$,

$$[B, B]^c = \langle B^{\mathbb{P},c} \rangle, \qquad \mathbb{P}\text{-a.s.},$$

then $\hat{a}$ coincides with the density of quadratic variation of $B^{\mathbb{P},c}$, $\mathbb{P}$-a.s. Therefore $\hat{a}$ takes values in $\mathbb{S}_d^{>0}$, $dt \times d\mathbb{P}$-a.e., and

$$\hat{a}_t = \frac{d\langle B^{\mathbb{P},c}\rangle_t}{dt}, \qquad \mathbb{P}\text{-a.s.}$$

More generally than the above examples, Soner, Touzi and Zhang [35], motivated by the study of stochastic target problems under volatility uncertainty, obtained an aggregation result for a family of probability measures corresponding to the laws of some continuous martingales on the canonical space $\Omega = \mathcal{C}(\mathbb{R}^+, \mathbb{R}^d)$, under a *separability* assumption on the quadratic variations (see their Definition 4.8) and under an additional *consistency* condition (which is usually only necessary) for the family to aggregate. A related result, not limited to the case of volatility uncertainty was then obtained by Cohen [9]. In our setting, this naturally leads to the question of whether it is possible or not to find an aggregator for the family of jump compensators $\nu^{\mathbb{P}}$.

However, unlike with the quadratic variation which can be either obtained through the Doob–Meyer decomposition of the local submartingale $\langle B \rangle$ or through the use of Itô's formula, the predictable compensator can only be obtained thanks to the Doob–Meyer decomposition of the nondecreasing process $[B, B]$. It is therefore obvious that this compensator depends explicitly on the underlying probability measure, and it is not clear at all whether an aggregator always exists or not.

This actually goes deeper, and in any reasonable setting of jump uncertainty, it is actually not possible to define such an aggregator, as showed in the following simple examples.

[3]The following construction was proposed to us by Marcel Nutz, whom we thank warmly. It is also used in a more general context in the recent preprint [25], where the absence of aggregation in a jump setting is also made clear. We urge the reader to consult their very interesting results.

EXAMPLE 2.1. Consider two probability measures $\mathbb{P}_1$ and $\mathbb{P}_2$ such that under $\mathbb{P}_1$ the canonical process $B$ is a Lévy process with characteristics $(0, 1, \lambda_1 \delta_{\{1\}})$ where the intensity of jumps $\lambda_1$ is a constant, and under $\mathbb{P}_2$ the canonical process $B$ is a Lévy process with characteristics $(0, 1, \lambda_2 \delta_{\{1\}})$ where $\lambda_2$ is a constant different from $\lambda_1$ (it is a classical result that these probabilities are uniquely defined on the Skorohod space $\mathbb{D}$). Since only the jump intensities are different, from the classical theory of change of measures, we know that $\mathbb{P}_1$ and $\mathbb{P}_2$ are actually equivalent, so their null-sets are the same, and we cannot find an aggregator which is simultaneously equal to $\lambda_1$ and $\lambda_2$ on the same support of $\mathbb{P}_1$ and $\mathbb{P}_2$.

EXAMPLE 2.2. Even in the case of pure jump martingale measures, we can still have a simple counterexample. Actually, we consider probability measures such that the canonical process $B$ is a pure jump local martingale. Under $\mathbb{P}_1$, $B$ is a Lévy process with characteristics $(0, 0, 2\delta_{\{1\}} + 4\delta_{\{-1\}})$, and under $\mathbb{P}_2$, $B$ is a Lévy process with characteristics $(0, 0, 3\delta_{\{1\}} + 5\delta_{\{-1\}})$. Then $\mathbb{P}_1$ and $\mathbb{P}_2$ are equivalent, and they are both martingale measures.

Therefore, we will not try to aggregate the family of compensators of jump measure in our formulation of 2BSDEJs. We emphasize that this feature is also shared by the drift, which can obviously be changed by using Girsanov's theorem. Hence, among the three elements of the characteristic triplet of a semimartingale, as defined in [19], for instance, the quadratic variation plays a peculiar role, in the sense that this is the only one which can be aggregated when uncertainty about this triplet is considered.

Notwithstanding this unavoidable fact, as proved in the following sections, the solution to the 2BSDEJs, which is our object of interest, can still be aggregated. To begin, we will use in the following subsection the notion of martingale problem for semimartingales with general characteristics (as defined in the book by Jacod and Shiryaev [19] to which we refer), in order to construct a probability measure under which the canonical process has a given quadratic variation and a given jump measure.

2.4. *Characterization by martingale problems.* In this section, we extend the connection between diffusion processes and probability measures established in [35] thanks to weak solutions of SDEs, to our general jump case with the more general notion of martingale problems.

Let $\mathcal{N}$ be the set of $\mathbb{F}$-predictable random measures $\nu$ on $\mathcal{B}(E)$ satisfying

$$\begin{aligned} &\int_0^t \int_E (1 \wedge |x|^2) \nu_s(\omega, dx)\, ds < +\infty \quad \text{and} \\ &\qquad \int_0^t \int_{|x|>1} x \nu_s(\omega, dx)\, ds < +\infty \qquad \text{for all } \omega \in \Omega, \end{aligned} \tag{2.2}$$

and let $\mathcal{D}$ be the set of $\mathbb{F}$-predictable processes $\alpha$ taking values in $\mathbb{S}_d^{>0}$ with

$$\int_0^T |\alpha_t(\omega)|\,dt < +\infty \qquad \text{for all } \omega \in \Omega.$$

We define a martingale problem as follows:

DEFINITION 2.3. For $\mathbb{F}$-stopping times $\tau_1 \le \tau_2$, for $(\alpha, \nu) \in \mathcal{D} \times \mathcal{N}$ and for a probability measure $\mathbb{P}_1$ on $\mathcal{F}_{\tau_1}$, we say that $\mathbb{P}$ is a solution of the *martingale problem* $(\mathbb{P}_1, \tau_1, \tau_2, \alpha, \nu)$ if:

(i) $\mathbb{P} = \mathbb{P}_1$ on $\mathcal{F}_{\tau_1}$.

(ii) The canonical process $B$ on $[\tau_1, \tau_2]$ is a semimartingale under $\mathbb{P}$ with characteristics

$$\left(-\int_{\tau_1}^{\cdot}\int_E x\mathbf{1}_{|x|>1}\nu_s(dx)\,ds, \int_{\tau_1}^{\cdot}\alpha_s\,ds, \nu_s(dx)\,ds\right).$$

REMARK 2.2. We refer to Theorem II.2.21 in [19] for the fact that $\mathbb{P}$ is a solution of the martingale problem $(\mathbb{P}_1, \tau_1, \tau_2, \alpha, \nu)$ if and only if the following properties hold:

(i) $\mathbb{P} = \mathbb{P}_1$ on $\mathcal{F}_{\tau_1}$.

(ii) The processes $M$, $J$ and $Q$ defined below are $\mathbb{P}$-local martingales on $[\tau_1, \tau_2]$:

$$M_t := B_t - \sum_{\tau_1 \le s \le t} \mathbf{1}_{|\Delta B_s|>1}\Delta B_s + \int_{\tau_1}^t \int_E x\mathbf{1}_{|x|>1}\nu_s(dx)\,ds, \qquad \tau_1 \le t \le \tau_2,$$

$$J_t := M_t^2 - \int_{\tau_1}^t \alpha_s\,ds - \int_{\tau_1}^t \int_E x^2 \nu_s(dx)\,ds, \qquad \tau_1 \le t \le \tau_2,$$

$$Q_t := \int_{\tau_1}^t \int_E g(x)\mu_B(dx, ds) - \int_{\tau_1}^t \int_E g(x)\nu_s(dx)\,ds, \qquad \tau_1 \le t \le \tau_2, \forall g \in \mathcal{C}^+(\mathbb{R}^r),$$

where $\mathcal{C}^+(\mathbb{R}^r)$ is a discriminating family of bounded Borel functions; see Remark II.2.20 in [19] for more details.

We say that the martingale problem associated to $(\alpha, \nu)$ has a unique solution if, for every stopping times $\tau_1, \tau_2$ and for every probability measure $\mathbb{P}_1$, the martingale problem $(\mathbb{P}_1, \tau_1, \tau_2, \alpha, \nu)$ has a unique solution.

Let now $\overline{\mathcal{A}}_W$ be the set of $(\alpha, \nu) \in \mathcal{D} \times \mathcal{N}$, such that there exists a solution to the martingale problem $(\mathbb{P}^0, 0, +\infty, \alpha, \nu)$, where $\mathbb{P}^0$ is such that $\mathbb{P}^0(B_0 = 0) = 1$.

We also denote by $\mathcal{A}_W$ the set of $(\alpha, \nu) \in \overline{\mathcal{A}}_W$ such that there exists a unique solution to the martingale problem $(\mathbb{P}^0, 0, +\infty, \alpha, \nu)$. We denote $\mathbb{P}_\nu^\alpha$ this unique solution and finally set

$$\mathcal{P}_W := \{\mathbb{P}_\nu^\alpha, (\alpha, \nu) \in \mathcal{A}_W\}.$$

REMARK 2.3. We take here as an initial condition that $B_0 = 0$. There is actually no loss of generality, since at the end of the day, the probability measures under which we are going to work will all satisfy the Blumenthal 0–1 law. Hence, $B_0$ will have to be a constant, and we choose 0 for simplicity.

2.5. *The strong formulation.* We now face the following problem. As reminded in the Introduction, the predictable martingale representation property is a crucial ingredient for the whole BSDE theory, as well as the Blumenthal 0–1 law. Hence the set $\mathcal{P}_W$ defined above is far too large for our purpose. In this section, we will concentrate on a subset of $\mathcal{P}_W$ which will only contain probability measures that do satisfy the predictable representation property and the Blumenthal 0–1 law. For this purpose, let us first consider any so-called Lévy measure, that is to say any deterministic (i.e., which does not depend on $\omega$) measure $F \in \mathcal{N}$. It is a well-known result that for any such measure $F$, $(I_d, F) \in \mathcal{A}_W$, and that the corresponding unique solution $\mathbb{P}_{0,F} := \mathbb{P}_F^{I_d}$ satisfies the predictable martingale representation property as well as the Blumenthal 0–1 law. Let us then define

$$\mathcal{A}_{\text{det}} := \{(I_d, F), F \in \mathcal{N} \text{ and } F \text{ is deterministic}\}.$$

We would like to use this set as a base to build a class of probability measures under which the canonical process has, formally, the following dynamics:

$$dB_t = \alpha_s^{1/2}\, dW_s + \int_E \beta_s(x)\big(\mu(dx, ds) - F(dx)\, ds\big), \tag{2.3}$$

for some given processes $\alpha$ and $\beta$, and where $W$ is a Brownian motion and $\mu$ is a Poisson random measure with compensator $F$. This can usually be done by considering the law under $\mathbb{P}_{0,F}$ of a well chosen stochastic process; see (2.4) below. There are then two questions one should ask:

(i) How large can one choose the corresponding family of compensators $F$ while ensuring that the predictable martingale representation property and the Blumenthal 0–1 law hold?

(ii) Since, on a fundamental level, the notion of a 2BSDEJ that we want to define corresponds to a stochastic control problem where the objective function is a family of BSDEJs indexed by the family of probability measures considered, the chosen class of controls (i.e., here the family of compensators $F$) has to be rich enough for the dynamic programming property to hold. In particular, the family of compensators has to be stable by concatenation and bifurcation; see, for instance, [8].

Since the family $\mathcal{A}_{\text{det}}$ is clearly not stable by concatenation and bifurcation (recall that the compensators in $\mathcal{A}_{\text{det}}$ are deterministic), it has to be enlarged, but in such a way that we do not lose either the predictable martingale representation property or the Blumenthal 0–1 law. Such a result can be achieved by a classical construction, detailed in Section A.1, by considering the so-called separable class of coefficients generated by $\mathcal{A}_{\text{det}}$ (see Definition A.2), which we denote $\widetilde{\mathcal{A}}_{\text{det}}$. We also designate by $\mathcal{P}_{\widetilde{\mathcal{A}}_{\text{det}}}$ the set of measures corresponding to this separable class of coefficients. Then, in virtue of Proposition A.1, all the measures in $\mathcal{P}_{\widetilde{\mathcal{A}}_{\text{det}}}$ do satisfy the predictable martingale representation property and the Blumenthal 0–1 law.

For simplicity, we let $\mathcal{V}$ designate the measure $F \in \mathcal{N}$ such that $(I_d, F) \in \widetilde{\mathcal{A}}_{\text{det}}$. Moreover, we will still denote $\mathbb{P}_{0,F} := \mathbb{P}_F^{I_d}$, for any $F \in \mathcal{V}$.

Let us now detail what kind of processes $\alpha$ and $\beta$ we can choose in (2.3). For $\alpha$, we can basically take any process in $\mathcal{D}$. For $\beta$ however, the situation is a bit more complicated, since the admissible $\beta$ will necessarily have to depend on the measure $F \in \mathcal{V}$ chosen. First of all, we introduce the following set $\mathcal{R}_F$ of $\mathbb{F}$-predictable functions $\beta : E \longmapsto \mathbb{R}$ such that for Lebesgue almost every $s \in [0, T]$,

$$|\beta_s|(\omega, x) \leq C\big(1 \wedge |x|\big), \qquad F_s(dx)\text{-a.e., for every } \omega \in \Omega,$$

and for every $\omega \in \Omega$,

$x \longmapsto \beta_s(\omega, x)$ is strictly monotone on the support of the law of $\Delta B_s$ under $\mathbb{P}_{0,F}$.

We will then denote by $\beta_s^{(-1)}(\omega, \cdot)$ the corresponding inverse function. While the first condition is common, since it implies in particular that for every $\beta \in \mathcal{R}_F$, we have

$$\int_0^T \int_E |\beta_s|^2(x) F_s(dx)\, ds < +\infty, \qquad \mathbb{P}_{0,F}\text{-a.s.},$$

the second one may seem surprising. Nonetheless, it is a natural condition in our context, since, as we will see, it will be linked to problems of stochastic flow inversion for SDEs with jumps; see below for more details.

Next, for each $F \in \mathcal{V}$ and for each $(\alpha, \beta) \in \mathcal{D} \times \mathcal{R}_F$, we define

$$\mathbb{P}_F^{\alpha,\beta} := \mathbb{P}_{0,F} \circ \big(X_\cdot^{\alpha,\beta}\big)^{-1},$$

where

$$\begin{aligned} X_t^{\alpha,\beta} &:= \int_0^t \alpha_s^{1/2}\, dB_s^{\mathbb{P}_{0,F},c} \\ &\quad + \int_0^t \int_E \beta_s(x)\big(\mu_B(dx, ds) - F_s(dx)\, ds\big), \qquad \mathbb{P}_{0,F}\text{-a.s.} \end{aligned} \tag{2.4}$$

We then define

$$\overline{\mathcal{P}}_S := \bigcup_{F \in \mathcal{V}} \big\{\mathbb{P}_F^{\alpha,\beta}, (\alpha, \beta) \in \mathcal{D} \times \mathcal{R}_F\big\}.$$

REMARK 2.4. Let us discuss a bit the kind of measures that are in the set $\overline{\mathcal{P}}_S$. First of all, there are almost no restrictions (except mild integrability conditions) on the admissible $\alpha$. This means that basically, our framework covers all types of volatility uncertainty. However, when it comes to the jump compensators which are allowed, the situation is more complicated. Indeed, according to a result of Jacod (see [18], Theorem 14.53, page 471), if we take one measure $F \in \mathcal{A}_{\mathrm{det}}$ which is nonatomic and with infinite mass, then every $\nu \in \mathcal{N}$ can be written as the image of $F$ by some $\mathbb{F}$-predictable function $\beta$. Therefore, it would appear that there was no need for us to consider more than one $F$. However, the strong formulation that we consider is tailor-made so that we can recover the predictable martingale representation property, and as we will see below, this puts restrictions on the possible $\beta$ we can consider (namely they have to be invertible). Hence considering only one $F$ could seriously limit the range of compensators we can reach. This is the reason why we chose to consider a whole family of measures $F$. However, it is a difficult problem to know how large the set of compensators we consider is when compared to $\mathcal{N}$. Nonetheless, from the point of view of applications, we think that it does not induce any important restrictions, since the set $\mathcal{V}$ by itself contains already more than all the possible compensators of additive processes.

Notice then that $\alpha$ is the density of the quadratic variation of the continuous part of $X^{\alpha,\beta}$ and

$$dB_s^{\mathbb{P}_{0,F},c} = \alpha_s^{-1/2}\, dX_s^{\alpha,c},$$

under $\mathbb{P}_{0,F}$. Moreover, the compensator of the measure associated to the jumps of $X^{\alpha,\beta}$ is the measure $\nu^{F,\beta}(dx)\,dt$ where

$$\nu_t^{F,\beta}(\omega, A) := \int_E \mathbf{1}_{\beta_t(\omega,x)\in A} F_t(\omega, dx) \qquad \text{for any } A \in \mathcal{B}(E),$$

that is to say the image of the measure $F$ by $\beta$. Besides, we have $\Delta X_s^{\alpha,\beta} = \beta_s(\Delta B_s)$ under $\mathbb{P}_{0,F}$.

Before pursuing, we would like to be able to define for any $\mathbb{P} \in \mathcal{P}_W$ and any $F \in \mathcal{V}$ a process $L^{\mathbb{P},F}$, whose law under $\mathbb{P}$ is the same as the law of $B$ under $\mathbb{P}_{0,F}$. If $F$ were deterministic, then this would amount to constructing an additive process which would be the sum of $\mathbb{P}$-Brownian motion and a pure jump $\mathbb{P}$-martingale with compensator $F$, which is classical result. When $F \in \mathcal{V}$, it is indeed random, but it has the special structure (A.1). Hence, the previous construction can easily be carried out recursively in this case. If in addition, the probability measure $\mathbb{P} =: \mathbb{P}_F^{\alpha,\beta}$ is actually in $\overline{\mathcal{P}}_S$, then we can instead define

$$L_\cdot^{\mathbb{P}_F^{\alpha,\beta},F} := W_t^{\mathbb{P}_F^{\alpha,\beta}} + \int_0^\cdot \int_E \beta_s^{(-1)}(x)\big(\mu_B(dx, ds) - \nu_s^{\mathbb{P}_F^{\alpha,\beta}}(dx)\, ds\big), \qquad \mathbb{P}_F^{\alpha,\beta}\text{-a.s.,} \tag{2.5}$$

where $W^{\mathbb{P}_F^{\alpha,\beta}}$ is a $\mathbb{P}_F^{\alpha,\beta}$ Brownian motion defined by

$$W_t^{\mathbb{P}_F^{\alpha,\beta}} := \int_0^\cdot \hat{a}_s^{-1/2}\, dB_s^{\mathbb{P}_F^{\alpha,\beta},c},$$

and where we remind the reader that since the law of $B$ under $\mathbb{P}_F^{\alpha,\beta}$ is the same as the law of $X^{\alpha,\beta}$ under $\mathbb{P}_{0,F}$, the support of the law of the jumps of $L^{\mathbb{P}_F^{\alpha,\beta},F}$ under $\mathbb{P}_F^{\alpha,\beta}$ is the image by $\beta$ of the support of the law of the jumps of $B$ under $\mathbb{P}_{0,F}$, so that $\beta^{(-1)}$ is indeed well-defined in the above expression.

Then, $\overline{\mathcal{P}}_S$ is a subset of $\mathcal{P}_W$, and we have by definition for any $F \in \mathcal{V}$,

$$\begin{aligned}&\text{the } \mathbb{P}_F^{\alpha,\beta}\text{-distribution of } \big(B, \hat{a}, \nu^{\mathbb{P}_F^{\alpha,\beta}}, L^{\mathbb{P}_F^{\alpha,\beta},F}\big)\\ &\quad = \text{the } \mathbb{P}_{0,F}\text{-distribution of } \big(X^{\alpha,\beta}, \alpha, \nu^{F,\beta}, B\big).\end{aligned} \tag{2.6}$$

Let us note immediately that the above implies that $B$ has the following characteristics under $\mathbb{P}_F^{\alpha,\beta}$:

$$\begin{aligned}&\hat{a}_t(B.) = \alpha_t\big(L^{\mathbb{P}_F^{\alpha,\beta},F}(B.)\big) \quad \text{and}\\ &\nu_t^{\mathbb{P}_F^{\alpha,\beta}}(B., dx) = \nu_t^{F,\beta}\big(L^{\mathbb{P}_F^{\alpha,\beta},F}(B.), dx\big), \qquad \mathbb{P}_F^{\alpha,\beta}\text{-a.s.}\end{aligned} \tag{2.7}$$

Now we want to recover the predictable martingale representation property. One possible solution would be to have a characterization of $\overline{\mathcal{P}}_S$ in terms of completed filtrations, exactly as in Lemma 8.1 of [35] in the continuous case. Roughly speaking, their result uses crucially a fact, which translated in our notation, reads

$$\overline{\mathbb{F}^B}^{\mathbb{P}_{0,F}} \subset \overline{\mathbb{F}^{X^{\alpha,\beta}}}^{\mathbb{P}_{0,F}}.$$

When there are no jump terms, this result is actually trivial, as soon as the matrix $\alpha$ is invertible (notice that also in our case, the reverse inclusion is immediate). However in our setting, because the jumps of $X^{\alpha,\beta}$ and $B$ are related by

$$\Delta X_s^{\alpha,\beta} = \beta_s(\Delta B_s), \qquad \mathbb{P}_{0,F}\text{-a.s.},$$

even though we know that $X^{\alpha,\beta}$ and $B$ jump at the same times, if the function $\beta$ is not invertible on the support of the law of the jumps of $B$ under $\mathbb{P}_{0,F}$, we cannot identify the size of a jump of $B$ by only observing a jump of $X^{\alpha,\beta}$. This is a well-known problem in the literature of SDEs in a jump setting; see, for example, Fujiwara and Kunita [16] or Protter [33]. This is exactly the reason why we assumed that the invertibility of the maps $\beta \in \mathcal{R}_F$.

We then have the following characterization of $\overline{\mathcal{P}}_S$, which is similar to Lemma 8.1 in [35]:

LEMMA 2.1. $\overline{\mathcal{P}}_S = \{\mathbb{P} \in \mathcal{P}_W, \exists F \in \mathcal{V}, \overline{\mathbb{F}^{L^{\mathbb{P},F}}}^{\mathbb{P}} = \overline{\mathbb{F}^B}^{\mathbb{P}}\}$.

PROOF. First of all, let $\mathbb{P}_F^{\alpha,\beta} \in \overline{\mathcal{P}}_S$. Then by definition, we have

$$\overline{\mathbb{F}^B}^{\mathbb{P}_{0,F}} = \overline{\mathbb{F}^{X^{\alpha,\beta}}}^{\mathbb{P}_{0,F}}.$$

Now we can use (2.6) to obtain that $\overline{\mathbb{F}^{L^{\mathbb{P}_F^{\alpha,\beta},F}}}^{\mathbb{P}_F^{\alpha,\beta}} = \overline{\mathbb{F}^B}^{\mathbb{P}_F^{\alpha,\beta}}$.

Conversely, let $\mathbb{P} \in \mathcal{P}_W$ be such that there exists some $F \in \mathcal{V}$ and $\overline{\mathbb{F}^{L^{\mathbb{P},F}}}^{\mathbb{P}} = \overline{\mathbb{F}^{B}}^{\mathbb{P}}$. Then, there exists some measurable function $\zeta$ such that $B_\cdot = \zeta(L_\cdot^{\mathbb{P},F})$, $\mathbb{P}$-a.s.

Now notice that by definition, the law of $\zeta(B_\cdot)$ under $\mathbb{P}_{0,F}$ is the same as the law of $\zeta(L_\cdot^{\mathbb{P},F})$ under $\mathbb{P}$; that is, this is the same as the law of $B$ under $\mathbb{P}$. Therefore, since $B$ is a $(\mathbb{P}, \mathbb{F})$-local martingale by definition, $\zeta(B_\cdot)$ is a $(\mathbb{P}_{0,F}, \mathbb{F})$-local martingale. However, since, as recalled above, $\mathbb{P}_{0,F}$ has the predictable martingale representation property, there exist a $\overline{\mathbb{F}}^{\mathbb{P}_{0,F}}$-predictable process $\alpha$ and a $\overline{\mathbb{F}}^{\mathbb{P}_{0,F}}$-predictable function $\beta$ such that

$$\zeta(B)_t = \int_0^t \alpha_s^{1/2}\, dB_s^{\mathbb{P}_{0,F},c} + \int_0^t \int_E \beta_s(x)\big(\mu_B(dx, ds) - F_s(dx)\, ds\big), \qquad \mathbb{P}_{0,F}\text{-a.s.}$$

Notice also that we can always take a $\mathbb{P}_{0,F}$ version of $\alpha$ and $\beta$ which is $\mathbb{F}$-predictable. Then, we actually have $\zeta(B_\cdot) = X_F^{\alpha,\beta}$. Fix now any measurable and bounded function $\varphi$, and we have

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}[\varphi(B_\cdot)] &= \mathbb{E}^{\mathbb{P}}[\varphi(\zeta(L_\cdot^{\mathbb{P},F}))] = \mathbb{E}^{\mathbb{P}_{0,F}}[\varphi(\zeta(B_\cdot))] \\ &= \mathbb{E}^{\mathbb{P}_{0,F}}[\varphi(X_\cdot^{\alpha,\beta})] = \mathbb{E}^{\mathbb{P}_F^{\alpha,\beta}}[\varphi(B_\cdot)],\end{aligned}$$

which means that $\mathbb{P} = \mathbb{P}_F^{\alpha,\beta}$. $\square$

As an immediate consequence, we deduce the following, since for any $F \in \mathcal{V}$, we have the martingale representation property for any $(\overline{\mathbb{F}^{L^{\mathbb{P},F}}}^{\mathbb{P}}, \mathbb{P})$-local martingale, and the Blumenthal 0–1 law holds for the filtration $\overline{\mathbb{F}^{L^{\mathbb{P},F}}}^{\mathbb{P}}$.

LEMMA 2.2. *Every probability measure in $\overline{\mathcal{P}}_S$ satisfies the predictable martingale representation property and the Blumenthal* 0–1 *law.*

PROOF. Fix some $\mathbb{P}_F^{\alpha,\beta} \in \overline{\mathcal{P}}_S$. Let us start with the predictable martingale representation property. We start by denoting for simplicity,

$$\overline{\mathbb{F}}^{\alpha,\beta} := \overline{\mathbb{F}^{L^{\mathbb{P}_F^{\alpha,\beta},F}}}^{\mathbb{P}_F^{\alpha,\beta}}.$$

Let $M$ be a $(\overline{\mathbb{F}}^{\mathbb{P}_F^{\alpha,\beta}}, \mathbb{P}_F^{\alpha,\beta})$-local martingale; then it is also a $(\overline{\mathbb{F}}^{\alpha,\beta}, \mathbb{P}_F^{\alpha,\beta})$-local martingale. Then by the standard predictable martingale representation theorem, we know that there exist a unique pair $(\widetilde{H}, \widetilde{U})$ of $\overline{\mathbb{F}}^{\alpha,\beta}$-predictable process and function such that, $\mathbb{P}_F^{\alpha,\beta}$-a.s.

$$\begin{aligned}M_t = M_0 &+ \int_0^t \widetilde{H}_s\, dW_s^{\mathbb{P}_F^{\alpha,\beta}} \\ &+ \int_0^t \int_E \widetilde{U}_s(x)\big(\mu_{L^{\mathbb{P}_F^{\alpha,\beta}},F}(dx, ds) - F_s\big(L^{\mathbb{P}_F^{\alpha,\beta},F}(B_\cdot), dx\big)\, ds\big).\end{aligned}$$

Define

$$H := \hat{a}^{-1/2}\widetilde{H} \quad \text{and} \quad U(x) := \widetilde{U}\big(\beta^{(-1)}(x)\big).$$

Then, using (2.7) and (2.5), we obtain directly that

$$M_t = M_0 + \int_0^t H_s\, dB_s^{\mathbb{P}_F^{\alpha,\beta},c} + \int_0^t \int_E U_s(x)\big(\mu_B(dx,ds) - \nu_s^{\mathbb{P}_F^{\alpha,\beta}}(dx)\,ds\big),$$

$$\mathbb{P}_F^{\alpha,\beta}\text{-a.s.}$$

The Blumenthal 0–1 law can then be directly deduced; see the proof of Lemma 8.2 in [35] for details. □

## 3. Preliminaries on 2BSDEJs.

3.1. *The nonlinear generator.* In this subsection we will introduce the function which will serve as the generator of our 2BSDEJ. Let us define the following spaces for $p \geq 1$:

$$\widehat{L}^p := \big\{\xi,\ \mathcal{F}_T\text{-measurable, s.t. } \xi \in L^p(\nu),\ \text{for every } \nu \in \mathcal{N}\big\}.$$

We then consider a map

$$H_t(\omega, y, z, u, \gamma, \widetilde{v}) : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times \widehat{L}^2 \times D_1 \times D_2 \to \mathbb{R},$$

where $D_1 \subset \mathbb{R}^{d\times d}$ is a given subset containing 0 and $D_2 \subset \widehat{L}^1$ is the domain of $H$ in the variable $\widetilde{v}$.

Define the following conjugate of $H$ with respect to $\gamma$ and $\widetilde{v}$ by

$$\begin{aligned} &F_t(\omega, y, z, u, a, \nu) \\ &\quad := \sup_{\{\gamma,\widetilde{v}\}\in D_1\times D_2} \left\{\frac{1}{2}\operatorname{Tr}(a\gamma) + \int_E \widetilde{v}(e)\nu(de) - H_t(\omega, y, z, u, \gamma, \widetilde{v})\right\}, \end{aligned} \tag{3.1}$$

for $a \in \mathbb{S}_d^{>0}$ and $\nu \in \mathcal{N}$.

In the remainder of this paper, we formulate the needed hypothesis for the generator directly on the function $F$, and the BSDEs we consider also include the case where $F$ does not take the form (3.1). Nonetheless, this particular form allows us to retrieve easily the framework of the standard BSDEs or of the $G$-stochastic analysis on the one hand (see Sections 3.4 and 3.5), and to establish the link with the associated PDEs on the other hand. In the latter cases, $H$ is evaluated at $\widetilde{v}(\cdot) = Av(\cdot)$, where $A$ is the following nonlocal operator, defined for any $\mathcal{C}^1$ function $v$ on $\mathbb{R}^d$ with bounded gradient, and $y \in \mathbb{R}^d$ by:

$$(Av)(t,y)(e) := v(t, e+y) - v(t,y) - \mathbf{1}_{\{|e|\leq 1\}} e.(\nabla v)(t,y)$$

$$\text{for } e \in E \text{ and } t \in [0,T].$$

The assumptions on $v$ ensure that $(Av)(t,y)(\cdot)$ is an element of $\widehat{L}^1$.

The operator $A$ applied to $v$ will not appear again in the paper, but this particular nonlocal form comes from the intuition that the 2BSDEJs is an essential supremum of standard BSDEJs. Indeed, solutions to Markovian BSDEJs provide viscosity solutions to some parabolic partial integro-differential equations with similar nonlocal operators; see [2] for more details.

We define for any $\mathbb{P} \in \overline{\mathcal{P}}_S$

$$\widehat{F}_t^{\mathbb{P}}(y,z,u) := F_t\big(y,z,u,\hat{a}_t,\nu_t^{\mathbb{P}}\big) \quad \text{and} \quad \widehat{F}_t^{\mathbb{P},0} := \widehat{F}_t^{\mathbb{P}}(0,0,0). \tag{3.2}$$

We denote by $D^1_{F_t(y,z,u)}$ the domain of $F$ in $a$ and by $D^2_{F_t(y,z,u)}$ the domain of $F$ in $\nu$, for a fixed $(t,\omega,y,z,u)$. As in [36] we fix a constant $\kappa \in (1,2]$ and restrict the probability measures in $\mathcal{P}_H^\kappa \subset \overline{\mathcal{P}}_S$.

DEFINITION 3.1. $\mathcal{P}_H^\kappa$ consists of all $\mathbb{P} \in \overline{\mathcal{P}}_S$ such that:

(i) $\mathbb{E}^{\mathbb{P}}[\int_0^T \int_E |x|^2 \nu_t^{\mathbb{P}}(dx)\,dt] < +\infty$;

(ii) $\underline{a}^{\mathbb{P}} \le \hat{a} \le \overline{a}^{\mathbb{P}}$, $dt \times d\mathbb{P}$-a.s. for some $\underline{a}^{\mathbb{P}}, \overline{a}^{\mathbb{P}} \in \mathbb{S}_d^{>0}$ and

$$\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T \big|\widehat{F}_t^{\mathbb{P},0}\big|^\kappa\,dt\right)^{2/\kappa}\right] < +\infty.$$

REMARK 3.1. The above conditions assumed on the probability measures in $\mathcal{P}_H^\kappa$ ensure that under any $\mathbb{P} \in \mathcal{P}_H^\kappa$, the canonical process $B$ is actually a true càdlàg martingale. This will be important when we will define standard BSDEJs under each of these probability measures.

We now state our assumptions on the function $F$ which will be our main interest in the sequel.

ASSUMPTION 3.1. (i) The domains $D^1_{F_t(y,z,u)} = D^1_{F_t}$ and $D^2_{F_t(y,z,u)} = D^2_{F_t}$ are independent of $(\omega,y,z,u)$.

(ii) For fixed $(y,z,a,\nu)$, $F$ is $\mathbb{F}$-progressively measurable in $D^1_{F_t} \times D^2_{F_t}$.

(iii) The following uniform Lipschitz-type property holds. For all $(y,y',z,z',u,t,a,\nu,\omega)$,

$$\big|F_t(\omega,y,z,u,a,\nu) - F_t(\omega,y',z',u,a,\nu)\big| \le C\big(|y-y'| + \big|a^{1/2}(z-z')\big|\big).$$

(iv) For all $(t,\omega,y,z,u^1,u^2,a,\nu)$, there exist two processes $\gamma$ and $\gamma'$ such that

$$\int_E \delta^{1,2}u(x)\gamma'_t(x)\nu(dx) \le F_t\big(\omega,y,z,u^1,a,\nu\big) - F_t\big(\omega,y,z,u^2,a,\nu\big)$$
$$\le \int_E \delta^{1,2}u(x)\gamma_t(x)\nu(dx),$$

where $\delta^{1,2}u := u^1 - u^2$ and $c_1(1 \wedge |x|) \le \gamma_t(x) \le c_2(1 \wedge |x|)$ with $-1+\delta \le c_1 \le 0$, $c_2 \ge 0$, and $c'_1(1 \wedge |x|) \le \gamma'_t(x) \le c'_2(1 \wedge |x|)$ with $-1+\delta \le c'_1 \le 0$, $c'_2 \ge 0$, for some $\delta > 0$.

(v) $F$ is uniformly continuous in $\omega$ for the Skorohod topology, that is to say that there exists some modulus of continuity $\rho$ such that for all $(t, \omega, \omega', y, z, u, a, \nu)$,

$$\bigl|F_t(\omega, y, z, u, a, \nu) - F_t\bigl(\omega', y, z, u, a, \nu\bigr)\bigr| \le \rho\bigl(d_S\bigl(\omega_{\cdot\wedge t}, \omega'_{\cdot\wedge t}\bigr)\bigr),$$

where $d_S$ is the Skorohod metric and where $\omega_{\cdot\wedge t}(s) := \omega(s \wedge t)$.

REMARK 3.2. Assumptions (i) and (ii) are classic in the second-order framework; see [36]. Lipschitz assumption (iii) is standard in the BSDE theory due to the paper [30]. Hypothesis (iv) allows us to have a comparison theorem in the framework with jumps; it was introduced in [34] and is also present in [2] in the form of an equality. The last hypothesis (v) of uniform continuity in $\omega$ is also proper to the second-order framework; it is linked to our intensive use of regular conditional probability distributions in [21] to construct our solutions in a pathwise manner, thus avoiding complex issues related to negligible sets. Moreover, we emphasize that unlike [36], we consider here the Skorohod topology instead of the topology induced by the uniform norm. This is linked to the fact that we need our space $\Omega$ to be separable. Furthermore, notice that if we restrict ourselves to the Wiener space as in [36], we recover their assumption since the topologies induced by the uniform norm and the Skorohod metric are then equivalent. Nonetheless, this property will only be useful for us in our accompanying paper [21].

REMARK 3.3. (i) For $\kappa_1 < \kappa_2$, applying Hölder's inequality gives us

$$\mathbb{E}^{\mathbb{P}}\biggl[\biggl(\int_0^T |\widehat{F}_t^{\mathbb{P},0}|^{\kappa_1}\,dt\biggr)^{2/\kappa_1}\biggr] \le C\mathbb{E}^{\mathbb{P}}\biggl[\biggl(\int_0^T |\widehat{F}_t^{\mathbb{P},0}|^{\kappa_2}\,dt\biggr)^{2/\kappa_2}\biggr],$$

where $C$ is a constant. Then it is clear that $\mathcal{P}_H^{\kappa}$ is decreasing in $\kappa$.

(ii) Assumption 3.1, together with the fact that $\widehat{F}_t^{\mathbb{P},0} < +\infty$, $\mathbb{P}$-a.s. for every $\mathbb{P} \in \mathcal{P}_H^{\kappa}$, implies that $\hat{a}_t \in D^1_{F_t}$ and $\nu_t^{\mathbb{P}} \in D^2_{F_t}$ $dt \times \mathbb{P}$-a.s., for all $\mathbb{P} \in \mathcal{P}_H^{\kappa}$.

3.2. *The spaces and norms.* We now define as in [36], the spaces and norms which will be needed for the formulation of the 2BSDEJs.

For $p \ge 1$, $L_H^{p,\kappa}$ denotes the space of all $\mathcal{F}_T$-measurable scalar r.v. $\xi$ with

$$\|\xi\|^p_{L_H^{p,\kappa}} := \sup_{\mathbb{P}\in\mathcal{P}_H^{\kappa}} \mathbb{E}^{\mathbb{P}}\bigl[|\xi|^p\bigr] < +\infty.$$

$\mathbb{H}_H^{p,\kappa}$ denotes the space of all $\mathbb{F}^+$-predictable $\mathbb{R}^d$-valued processes $Z$ with

$$\|Z\|^p_{\mathbb{H}_H^{p,\kappa}} := \sup_{\mathbb{P}\in\mathcal{P}_H^{\kappa}} \mathbb{E}^{\mathbb{P}}\biggl[\biggl(\int_0^T \bigl|\hat{a}_t^{1/2} Z_t\bigr|^2\,dt\biggr)^{p/2}\biggr] < +\infty.$$

$\mathbb{D}_H^{p,\kappa}$ denotes the space of all $\mathbb{F}^+$-progressively measurable $\mathbb{R}$-valued processes $Y$ with

$$\mathcal{P}_H^{\kappa}\text{-q.s. càdlàg paths, and } \|Y\|^p_{\mathbb{D}_H^{p,\kappa}} := \sup_{\mathbb{P}\in\mathcal{P}_H^{\kappa}} \mathbb{E}^{\mathbb{P}}\Bigl[\sup_{0\le t\le T} |Y_t|^p\Bigr] < +\infty.$$

$\mathbb{J}_H^{p,\kappa}$ denotes the space of all $\mathbb{F}^+$-predictable functions $U$ with

$$\|U\|^p_{\mathbb{J}_H^{p,\kappa}} := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\bigg[\bigg(\int_0^T\int_E \big|U_t(x)\big|^2 \nu_t^{\mathbb{P}}(dx)\,dt\bigg)^{p/2}\bigg] < +\infty.$$

For each $\xi \in L_H^{1,\kappa}$, $\mathbb{P}\in\mathcal{P}_H^\kappa$ and $t\in[0,T]$, denote

$$\mathbb{E}_t^{H,\mathbb{P}}[\xi] := \operatorname*{ess\,sup^{\mathbb{P}}}_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})} \mathbb{E}_t^{\mathbb{P}'}[\xi],$$

where

$$\mathcal{P}_H^\kappa(t^+,\mathbb{P}) := \{\mathbb{P}'\in\mathcal{P}_H^\kappa : \mathbb{P}' = \mathbb{P} \text{ on } \mathcal{F}_t^+\}.$$

Then we define for each $p\geq\kappa$,

$$\mathbb{L}_H^{p,\kappa} := \{\xi \in L_H^{p,\kappa} : \|\xi\|_{\mathbb{L}_H^{p,\kappa}} < +\infty\},$$

where

$$\|\xi\|^p_{\mathbb{L}_H^{p,\kappa}} := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\Big[\operatorname*{ess\,sup^{\mathbb{P}}}_{0\leq t\leq T} \big(\mathbb{E}_t^{H,\mathbb{P}}[|\xi|^\kappa]\big)^{p/\kappa}\Big].$$

REMARK 3.4. Except for $\mathbb{L}_H^{p,\kappa}$, the definitions of the previous spaces are classic, but the second-order framework induces the presence of an essential supremum over our family of probability measures. As for $\mathbb{L}_H^{p,\kappa}$, it appears naturally in the a priori estimates; we refer to [36] for more details.

Finally, we denote by $\mathrm{UC}_b(\Omega)$ the collection of all bounded and uniformly continuous maps $\xi:\Omega\to\mathbb{R}$ with respect to the Skorohod distance $d_S$, and we let

$$\mathcal{L}_H^{p,\kappa} := \text{the closure of } \mathrm{UC}_b(\Omega) \text{ under the norm } \|\cdot\|_{\mathbb{L}_H^{p,\kappa}} \quad \text{for every } 1<\kappa\leq p.$$

REMARK 3.5. In our accompanying paper [21], we will prove existence for 2BSDEJs for terminal conditions belonging to the space $\mathcal{L}_H^{2,\kappa}$. We therefore think that it is important to give a few examples of terminal conditions belonging to it. First of all, with applications of 2BSDEJs to fully nonlinear PIDEs, we at least would like functions of the form $f(B_T)$ to be in $\mathcal{L}_H^{2,\kappa}$. But it is a well-known result that the application $\omega\mapsto B_t(\omega)$ is continuous for the Skorohod topology for Lebesgue almost every $t\in[0,T]$, including $t=0$ and $t=T$. Hence, it is easy to see that for a Lipschitz function $f$, $f(B_t)\in\mathcal{L}_H^{2,\kappa}$ for a.e. $t\in[0,T]$, including $t=0$ and $t=T$. We also refer the reader to our accompanying paper [21] for more explanations and intuitions about this problem. Finally we would like to mention that the recent results of [25, 26], which appeared during the revision of this paper, could be used to obtain existence of a solution when $F=0$, but with a terminal condition $\xi\in\mathbb{L}_H^{2,\kappa}$. It is a very interesting and difficult problem to see whether their approach could be extended to general generators $F$.

For a given probability measure $\mathbb{P} \in \mathcal{P}_H^\kappa$, the spaces $L^p(\mathbb{P})$, $\mathbb{D}^p(\mathbb{P})$, $\mathbb{H}^p(\mathbb{P})$ and $\mathbb{J}^p(\mathbb{P})$ correspond to the above spaces when the set of probability measures is only the singleton $\{\mathbb{P}\}$. Finally, we have $\mathbb{H}^p_{\mathrm{loc}}(\mathbb{P})$ denotes the space of all $\mathbb{F}^+$-predictable $\mathbb{R}^d$-valued processes $Z$ with

$$\left(\int_0^T |\hat{a}_t^{1/2} Z_t|^2\,dt\right)^{p/2} < +\infty, \qquad \mathbb{P}\text{-a.s.}$$

$\mathbb{J}^p_{\mathrm{loc}}(\mathbb{P})$ denotes the space of all $\mathbb{F}^+$-predictable functions $U$ with

$$\left(\int_0^T \int_E |U_t(x)|^2 \nu_t^{\mathbb{P}}(dx)\,dt\right)^{p/2} < +\infty, \qquad \mathbb{P}\text{-a.s.}$$

3.3. *Formulation.* We shall consider the following 2BSDEJ, for $0 \le t \le T$ and $\mathcal{P}_H^\kappa$-q.s.:

$$\begin{aligned} Y_t = \xi &+ \int_t^T \widehat{F}_s^{\mathbb{P}}(Y_s, Z_s, U_s)\,ds - \int_t^T Z_s\,dB_s^{\mathbb{P},c} \\ &- \int_t^T \int_E U_s(x)\tilde{\mu}_B^{\mathbb{P}}(dx, ds) + K_T^{\mathbb{P}} - K_t^{\mathbb{P}}. \end{aligned} \tag{3.3}$$

DEFINITION 3.2. We say $(Y, Z, U) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa} \times \mathbb{J}_H^{2,\kappa}$ is a solution to 2BSDEJ (3.3) if:

- $Y_T = \xi$, $\mathcal{P}_H^\kappa$-q.s.
- For all $\mathbb{P} \in \mathcal{P}_H^\kappa$ and $0 \le t \le T$, the process $K^{\mathbb{P}}$ defined below is predictable and has nondecreasing paths $\mathbb{P}$-a.s.

$$\begin{aligned} K_t^{\mathbb{P}} := Y_0 - Y_t &- \int_0^t \widehat{F}_s^{\mathbb{P}}(Y_s, Z_s, U_s)\,ds + \int_0^t Z_s\,dB_s^{\mathbb{P},c} \\ &+ \int_0^t \int_E U_s(x)\tilde{\mu}_B^{\mathbb{P}}(dx, ds). \end{aligned} \tag{3.4}$$

- The family $\{K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_H^\kappa\}$ satisfies the minimum condition

$$K_t^{\mathbb{P}} = \underset{\mathbb{P}' \in \mathcal{P}_H(t^+, \mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}} \mathbb{E}_t^{\mathbb{P}'}\big[K_T^{\mathbb{P}'}\big], \qquad 0 \le t \le T, \mathbb{P}\text{-a.s.}, \forall \mathbb{P} \in \mathcal{P}_H^\kappa. \tag{3.5}$$

Moreover if the family $\{K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_H^\kappa\}$ can be aggregated into a universal process $K$, we call $(Y, Z, U, K)$ a solution of the 2BSDEJ (3.3).

Following [36], in addition to Assumption 3.1, we will always assume the following:

ASSUMPTION 3.2. (i) $\mathcal{P}_H^\kappa$ is not empty.

(ii) The process $F$ satisfies the following integrability condition:

$$\phi_H^{2,\kappa} := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\operatorname*{ess\,sup}^{\mathbb{P}}_{0\le t\le T}\left(\mathbb{E}_t^{H,\mathbb{P}}\left[\int_0^T |\widehat{F}_s^{\mathbb{P},0}|^\kappa\, ds\right]\right)^{2/\kappa}\right] < +\infty. \tag{3.6}$$

3.4. *Connection with standard BSDEJs*. Let us assume that $H$ is linear in $\gamma$ and $\widetilde{v}$, in the following sense:

$$H_t(y,z,u,\gamma,\widetilde{v}) := \frac{1}{2}\operatorname{Tr}[I_d\gamma] + \int_E \widetilde{v}(e)\nu^*(de) - f_t(y,z,u), \tag{3.7}$$

where $\nu^* \in \mathcal{N}$. We then have the following result:

LEMMA 3.1. *If $H$ is of the form* (3.7), *then $D^1_{F_t} = \{I_d\}$, $D^2_{F_t} = \{\nu^*\}$ and*

$$F_t(\omega,y,z,u,a,\nu) = F_t(\omega,y,z,u,Id,\nu^*) = f_t(y,z,u).$$

PROOF. First notice that

$$\begin{aligned} &H_t(\omega,y,z,u,\gamma,\widetilde{v}) \\ &\quad= \sup_{(a,\nu)\in\mathbb{S}_d^{>0}\times\mathcal{N}}\left\{\frac{1}{2}\operatorname{Tr}(a\gamma) + \int_0^T\int_E \widetilde{v}(e)\nu_s(\omega)(ds,de) - \delta_{Id}(a) - \delta_{\nu^*}(\nu)\right\} \\ &\qquad - f_t(y,z,u), \end{aligned}$$

where $\delta_A$ denotes the characteristic function of a subset $A$ in the convex analysis sense.

By definition of $F$, we get

$$F_t(\omega,y,z,u,a,\nu) = f_t(y,z,u) + H^{**}(a,\nu),$$

where $H^{**}$ is the double Fenchel–Legendre transform of the function

$$(a,\nu) \mapsto \delta_{Id}(a) + \delta_{\nu^*}(\nu),$$

which is convex and lower-semicontinuous.

This then implies that

$$F_t(\omega,y,z,u,a,\nu) = f_t(y,z,u) + \delta_{Id}(a) + \delta_{\nu^*}(\nu),$$

which is the desired result. $\square$

If we further assume that $\mathbb{E}^{\mathbb{P}_{\nu^*}}[\int_0^T |f_t(0,0,0)|^2\,dt] < +\infty$, then $\mathcal{P}_H^\kappa = \{\mathbb{P}_{\nu^*}\}$ and the minimum condition on $K = K^{\mathbb{P}_{\nu^*}}$ implies that $0 = \mathbb{E}^{\mathbb{P}_{\nu^*}}[K_T]$, which means that $K \equiv 0$, $\mathbb{P}_{\nu^*}$-a.s., and the 2BSDEJ is reduced to a classical BSDEJ.

3.5. *Connection with $G$-expectations and $G$-Lévy processes*.

3.5.1. *Reminder on $G$-Lévy processes*. In their recent paper, Hu and Peng [17] introduced a new class of processes with independent and stationary increments, called $G$-Lévy processes. These processes are defined intrinsically, that is, without making reference to any probability measure.

Let $\widetilde{\Omega}$ be a given set, and let $\mathcal{H}$ be a linear space of real valued functions defined on $\widetilde{\Omega}$, containing the constants and such that $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$. A sublinear expectation is a functional $\widehat{\mathbb{E}} : \mathcal{H} \to \mathbb{R}$ which is monotone increasing, constant preserving, sub-additive and positively homogeneous. We refer to Definition 1.1 of [32] for more details. The triple $(\widetilde{\Omega}, \mathcal{H}, \widehat{\mathbb{E}})$ is called a sublinear expectation space.

DEFINITION 3.3. A $d$-dimensional càdlàg process $\{X_t, t \geq 0\}$ defined on a sublinear expectation space $(\widetilde{\Omega}, \mathcal{H}, \widehat{\mathbb{E}})$ is called a $G$-Lévy process if:

(i) $X_0 = 0$.

(ii) $X$ has independent increments: $\forall s, t > 0$, the random variable $(X_{t+s} - X_t)$ is independent from $(X_{t_1}, \dots, X_{t_n})$, for each $n \in \mathbb{N}$ and $0 \leq t_1 < \cdots < t_n \leq t$. The notion of independence used here corresponds to Definition 3.10 in [32].

(iii) $X$ has stationary increments: $\forall s, t > 0$, the distribution of $(X_{t+s} - X_t)$ does not depend on $t$. The notion of distribution used here corresponds to the definition given in Section 3 of [32].

(iv) For each $t \geq 0$, there exists a decomposition $X_t = X_t^c + X_t^d$, where $\{X_t^c, t \geq 0\}$ is a continuous process and $\{X_t^d, t \geq 0\}$ is a pure jump process.

(v) $(X_t^c, X_t^d)$ is a $2d$-dimensional process satisfying conditions (i), (ii) and (iii) of this definition and

$$\lim_{t \to 0^+} \frac{1}{t} \widehat{\mathbb{E}}\left(|X_t^c|^3\right) = 0, \qquad \widehat{\mathbb{E}}\left(|X_t^d|\right) \leq Ct, t \geq 0$$

for a real constant $C$.

In [17], Hu and Peng proved the following Lévy–Khintchine representation for $G$-Lévy processes:

THEOREM 3.1 (Hu and Peng [17]). *Let $\{X_t, t \geq 0\}$ be a $G$-Lévy process. Then for each Lipschitz and bounded function $\varphi$, the function $u$ defined by $u(t, x) := \widehat{\mathbb{E}}(\varphi[x + X_t])$ is the unique viscosity solution of the following partial integro-differential equation*:

$$\partial_t u(t, x) - \sup_{(b, \alpha, \nu) \in \mathcal{U}} \left\{ \int_E \left[u(t, x+z) - u(t, x)\right] \nu(dz) + \langle Du(t, x), b\rangle + \frac{1}{2} \operatorname{Tr}\left[D^2 u(t, x) \alpha \alpha^T\right] \right\} = 0,$$

*where $\mathcal{U}$ is a subset of $\mathbb{R}^d \times \mathbb{R}^{d \times d} \times \mathcal{M}_R^+$ satisfying*

$$\sup_{(b, \alpha, \nu) \in \mathcal{U}} \left\{ \int_{\mathbb{R}^d} |z| \nu(dz) + |b| + \operatorname{Tr}\left[\alpha \alpha^T\right] \right\} < +\infty,$$

*and where $\mathcal{M}_R^+$ denotes the set of positive Radon measures on $E$.*

Notice that Hu and Peng study the case of $G$-Lévy processes with a discontinuous part that is of finite variation.

3.5.2. *A connection with a particular 2BSDEJ.* In our framework, we know that $B^d$ is a purely discontinuous semimartingale of finite variation under $\mathbb{P}_{0,F}$ if

$$\int_0^T \int_{|x|\leq 1} |x| F_s(dx)\, ds < +\infty, \qquad \mathbb{P}_{0,F}\text{-a.s.}$$

We give a function $H$ below, which is the natural candidate to retrieve the example of $G$-Lévy processes in our context. This link will be made clear in our accompanying paper [21].

Let $\widetilde{\mathcal{N}}$ be any subset of $\mathcal{V}$ that is convex and closed for the weak topology on $\mathcal{M}_R^+$. We define

$$\begin{aligned} &H_t(\omega, \gamma, \widetilde{v}) \\ &\quad := \sup_{(a,\nu)\in\mathbb{S}_d^{>0}\times\mathcal{N}} \left\{\frac{1}{2}\operatorname{Tr}(a\gamma) + \int_0^T \int_E \widetilde{v}(e)\nu_s(de)\, ds - \delta_{[a_1,a_2]}(a) - \delta_{\widetilde{\mathcal{N}}}(\nu)\right\}. \end{aligned}$$

Since $[a_1, a_2]$ and $\widetilde{\mathcal{N}}$ are closed convex spaces, $F_t(\omega, a, \nu)$ is the double Fenchel–Legendre transform in $(a, \nu)$ of the convex and lower semi-continuous function $(a, \nu) \mapsto \delta_{[a_1,a_2]}(a) + \delta_{\widetilde{\mathcal{N}}}(\nu)$ and then

$$F_t(\omega, a, \nu) = \delta_{[a_1,a_2]}(a) + \delta_{\widetilde{\mathcal{N}}}(\nu).$$

In [21], we prove that the 2BSDEJs are connected to a class of fully nonlinear partial integro-differential equations. With this particular function $H$ and its transform $F$, the PIDE we find is the one given in Theorem 3.1. If moreover $H_t(\omega, \gamma, \widetilde{v}) = H_t(\omega, \tilde{v})$ is independent of $\gamma$, and $\widetilde{\mathcal{N}} = \{\lambda\delta_{\{1\}}, \lambda_1 \leq \lambda \leq \lambda_2\}$ (which is convex and closed and where $\delta_{\{1\}}$ is a Dirac mass at the point 1), then $F_t$ is independent of $a$, and we obtain a 2BSDEJ giving a representation of the $G$-Poisson process.

**4. Uniqueness result.** In this section, we address the question of uniqueness of a solution to a 2BSDEJ. We follow the intuition provided in the Introduction and write the solution to a 2BSDEJ as a supremum in some sense of solutions to classical BSDEJs.

4.1. *Representation of the solution.* We have the following, which is similar to Theorem 4.4 of [36]:

THEOREM 4.1. *Let Assumptions 3.1 and 3.2 hold. Assume $\xi \in \mathbb{L}_H^{2,\kappa}$ and that $(Y, Z, U)$ is a solution to the 2BSDEJ (3.3). Then, for any $\mathbb{P} \in \mathcal{P}_H^\kappa$ and $0 \leq t_1 < t_2 \leq T$,*

$$Y_{t_1} = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t_1^+,\mathbb{P})} y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2}), \qquad \mathbb{P}\text{-}a.s., \tag{4.1}$$

*where, for any* $\mathbb{P} \in \mathcal{P}_H^\kappa$, $\mathbb{F}^+$*-stopping time* $\tau$, *and* $\mathcal{F}_\tau^+$*-measurable random variable* $\xi \in \mathbb{L}^2(\mathbb{P})$, $(y^{\mathbb{P}}(\tau,\xi), z^{\mathbb{P}}(\tau,\xi))$ *denotes the solution to the following standard BSDEJ on* $0 \le t \le \tau$

$$
\begin{aligned}
y_t^{\mathbb{P}} = \xi + \int_t^\tau \widehat{F}_s^{\mathbb{P}}(y_s^{\mathbb{P}}, z_s^{\mathbb{P}}, u_s^{\mathbb{P}})\,ds - \int_t^\tau z_s^{\mathbb{P}}\,dB_s^{\mathbb{P},c} \\
- \int_t^\tau \int_E u_s^{\mathbb{P}}(x) \tilde{\mu}_B^{\mathbb{P}}(dx,ds), \qquad \mathbb{P}\text{-}a.s.
\end{aligned}
\tag{4.2}
$$

*Consequently, the* 2*BSDEJ* (3.3) *has at most one solution in* $\mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa} \times \mathbb{J}_H^{2,\kappa}$.

REMARK 4.1. We first emphasize that existence and uniqueness results for the standard BSDEJs (4.2) are not given directly by the existing literature, since the compensator of the counting measure associated to the jumps of $B$ is not deterministic. However, since all the probability measures we consider satisfy the martingale representation property and the Blumenthal 0–1 law, it is clear that we can straightforwardly generalize the proof of existence and uniqueness of Tang and Li [37]; see also [3] and [10] for related results. Furthermore, the usual a priori estimates and comparison theorems will also hold.

Before giving the proof of the above theorem, we first state the following lemma which is a generalization of the usual comparison theorem proved by Royer; see Theorem 2.5 in [34]. Its proof is a straightforward generalization so we omit it.

LEMMA 4.1. *Let* $\mathbb{P} \in \mathcal{P}_H^\kappa$. *We consider two generators* $f^1$ *and* $f^2$ *satisfying Assumption* $H_{comp}$ *in* [34] (*which is a consequence of our more restrictive assumptions*). *Given two nondecreasing processes* $k^1$ *and* $k^2$, *let* $\xi^1$ *and* $\xi^2$ *be two terminal conditions for the following BSDEJs for* $i = 1, 2$,

$$
\begin{aligned}
y_t^i = \xi^i + \int_t^T f_s^i(y_s^i, z_s^i, u_s^i)\,ds - \int_t^T z_s^i\,dB_s \\
- \int_t^T \int_E u_s^i(x) \widetilde{\mu}^{\mathbb{P}}(dx,ds) + k_T^i - k_t^i, \qquad \mathbb{P}\text{-}a.s.
\end{aligned}
$$

*Denote by* $(y^1, z^1, u^1)$ *and* $(y^2, z^2, u^2)$ *the respective solutions. If* $\xi^1 \le \xi^2$, $k^1 - k^2$ *is nonincreasing and* $f^1(t, y_t^1, z_t^1, u_t^1) \le f^2(t, y_t^1, z_t^1, u_t^1)$, *then* $\forall t \in [0,T]$, $Y_t^1 \le Y_t^2$.

PROOF OF THEOREM 4.1. The proof follows the lines of the proof of Theorem 4.4 in [36]. First of all, if representation (4.1) holds, then $Y$ is uniquely defined. Moreover, since we have that

$$
d[Y,B]_t^c = Z_t d[B,B]_t^c = \hat{a}_t Z_t\,dt, \qquad \mathcal{P}_H^\kappa\text{-q.s.},
$$

$Z$ is also uniquely defined.

Then, since for any $\mathbb{P} \in \mathcal{P}_H^\kappa$, $B$ only has $\mathbb{P}$-totally inaccessible jump times, we know that $B$ and $K^{\mathbb{P}}$ never jump at the same time, $\mathbb{P}$-a.s. We deduce that

$$\Delta[Y, B]_t = U_t(\Delta B_t)\Delta B_t \mathbf{1}_{\Delta B_t \neq 0}, \qquad \mathcal{P}_H^\kappa\text{-q.s.} \tag{4.3}$$

We can then define

$$\widetilde{U}_t(x) := U_t(x)\mathbf{1}_{\Delta B_t = x}.$$

Then $\widetilde{U}$ is actually equal to $U$, $dt \times \nu_t^{\mathbb{P}}(dx)$, for any $\mathbb{P} \in \mathcal{P}_H^\kappa$. Using this version instead, and still denoting it $U$ for simplicity, we deduce that $U$ is uniquely defined by (4.3). Then the uniqueness of the process $K^{\mathbb{P}}$ is immediate. Let us now proceed with the proof of (4.1):

(i) Fix $0 \le t_1 < t_2 \le T$ and $\mathbb{P} \in \mathcal{P}_H^\kappa$. For any $\mathbb{P}' \in \mathcal{P}_H^\kappa(t_1^+, \mathbb{P})$ and $t_1 \le t \le t_2$, we have

$$\begin{aligned} Y_t = Y_{t_2} &+ \int_t^{t_2} \widehat{F}_s^{\mathbb{P}'}(Y_s, Z_s, U_s)\,ds - \int_t^{t_2} Z_s\,dB_s^{\mathbb{P}',c} \\ &- \int_t^{t_2}\int_E U_s(x)\widetilde{\mu}_B^{\mathbb{P}'}(dx, ds) + K_{t_2}^{\mathbb{P}'} - K_t^{\mathbb{P}'}, \qquad \mathbb{P}'\text{-a.s.} \end{aligned}$$

With Assumption 3.1, we can apply the above Lemma 4.1 under $\mathbb{P}'$ to obtain that $Y_{t_1} \ge y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2})$, $\mathbb{P}'$-a.s. Since $\mathbb{P}' = \mathbb{P}$ on $\mathcal{F}_{t_1}^+$, we get $Y_{t_1} \ge y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2})$, $\mathbb{P}$-a.s. and thus

$$Y_{t_1} \ge \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t_1^+, \mathbb{P})} y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2}), \qquad \mathbb{P}\text{-a.s.}$$

(ii) We now prove the reverse inequality. Fix $\mathbb{P} \in \mathcal{P}_H^\kappa$. We will show in (iii) below that

$$C_{t_1}^{\mathbb{P}} := \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t_1^+, \mathbb{P})} \mathbb{E}_{t_1}^{\mathbb{P}'}\big[\big(K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'}\big)^2\big] < +\infty, \qquad \mathbb{P}\text{-a.s.}$$

For every $\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})$, denote

$$\begin{aligned} &\delta Y := Y - y^{\mathbb{P}'}(t_2, Y_{t_2}), \qquad \delta Z := Z - z^{\mathbb{P}'}(t_2, Y_{t_2}) \quad \text{and} \\ &\delta U := U - u^{\mathbb{P}'}(t_2, Y_{t_2}). \end{aligned}$$

By the Lipschitz Assumption 3.1(iii), there exist two bounded processes $\lambda$ and $\eta$ such that for all $t_1 \le t \le t_2$,

$$\begin{aligned} \delta Y_t = &\int_t^{t_2} \big(\lambda_s \delta Y_s + \eta_s \hat{a}_s^{1/2}\delta Z_s\big)\,ds \\ &+ \int_t^{t_2} \big(\widehat{F}_s^{\mathbb{P}'}\big(y_s^{\mathbb{P}'}, z_s^{\mathbb{P}'}, U_s\big) - \widehat{F}_s^{\mathbb{P}'}\big(y_s^{\mathbb{P}'}, z_s^{\mathbb{P}'}, u_s^{\mathbb{P}'}\big)\big)\,ds \\ &- \int_t^{t_2} \delta Z_s\,dB_s^{\mathbb{P}',c} - \int_t^{t_2}\int_E \delta U_s(x)\widetilde{\mu}_B^{\mathbb{P}'}(dx, ds) + K_{t_2}^{\mathbb{P}'} - K_t^{\mathbb{P}'}, \qquad \mathbb{P}'\text{-a.s.} \end{aligned}$$

Define for $t_1 \le t \le t_2$ the following processes:

$$N_t^{\mathbb{P}'} := \int_{t_1}^t \eta_s \hat{a}_s^{-1/2} \, dB_s^{\mathbb{P}',c} + \int_{t_1}^t \int_E \gamma_s(x) \widetilde{\mu}_B^{\mathbb{P}'}(ds, dx)$$

and

$$M_t^{\mathbb{P}'} := \exp\biggl(\int_{t_1}^t \lambda_s \, ds\biggr) \mathcal{E}(N^{\mathbb{P}'})_t,$$

where $\mathcal{E}(N^{\mathbb{P}'})_t$ denotes the Doléans–Dade exponential martingale of $N_t^{\mathbb{P}'}$.

By the boundedness of $\lambda$ and $\eta$ and the assumption on $\gamma$ in Assumption 3.1(iv), we know that $M$ has moments (positive or negative) of any order; see [23] for the positive moments and Lemma A.6 in the Appendix for the negative ones. Thus we have for $p \ge 1$,

$$\mathbb{E}_{t_1}^{\mathbb{P}'}\Bigl[\sup_{t_1 \le t \le t_2} (M_t^{\mathbb{P}'})^p + \sup_{t_1 \le t \le t_2} (M_t^{\mathbb{P}'})^{-p}\Bigr] \le C_p, \qquad \mathbb{P}'\text{-a.s.} \tag{4.4}$$

Then, by Itô's formula, we obtain

$$\begin{aligned}
&d(M_t^{\mathbb{P}'} \delta Y_t) \\
&\quad = M_{t-}^{\mathbb{P}'} \, d(\delta Y_t) + \delta Y_{t-} \, dM_t^{\mathbb{P}'} + d[M^{\mathbb{P}'}, \delta Y]_t \\
&\quad = M_{t-}^{\mathbb{P}'} \Bigl[ \bigl(-\lambda_t \delta Y_t - \eta_t \hat{a}_t^{1/2} \delta Z_t - \widehat{F}_t^{\mathbb{P}'}(y_t^{\mathbb{P}'}, z_t^{\mathbb{P}'}, U_t) + \widehat{F}_t^{\mathbb{P}'}(y_t^{\mathbb{P}'}, z_t^{\mathbb{P}'}, u_t^{\mathbb{P}'})\bigr) \, dt \\
&\qquad\qquad + \delta Z_t \, dB_t^{\mathbb{P}',c} + \int_E \bigl(\delta U_t(x) + \gamma_t(x) \delta U_t(x)\bigr) \widetilde{\mu}_B^{\mathbb{P}',c}(dx, dt) \Bigr] \\
&\qquad + \delta Y_{t-} M_{t-}^{\mathbb{P}'} \biggl( \lambda_t \, dt + \eta_t \hat{a}_t^{-1/2} \, dB_t^{\mathbb{P}',c} + \int_E \gamma_t(x) \widetilde{\mu}_B^{\mathbb{P}'}(dx, dt) \biggr) \\
&\qquad + M_t^{\mathbb{P}'} \biggl( \eta_t \hat{a}_t^{1/2} \delta Z_t \, dt + \int_E \gamma_t(x) \delta U_t(x) \nu_t^{\mathbb{P}'}(dx) \, dt \biggr) - M_{t-}^{\mathbb{P}'} \, dK_t^{\mathbb{P}'}.
\end{aligned}$$

Thus, by Assumption 3.1(iv), we have

$$\begin{aligned}
\delta Y_{t_1} \le {}& - \int_{t_1}^{t_2} M_s^{\mathbb{P}'} \bigl(\delta Z_s + \delta Y_s \eta_s \hat{a}_s^{-1/2}\bigr) \, dB_s^{\mathbb{P}',c} + \int_{t_1}^{t_2} M_{s-}^{\mathbb{P}'} \, dK_s^{\mathbb{P}'} \\
& - \int_{t_1}^{t_2} M_{s-}^{\mathbb{P}'} \int_E \bigl(\delta U_s(x) + \delta Y_s \gamma_s(x) + \gamma_s(x) \delta U_s(x)\bigr) \widetilde{\mu}_B^{\mathbb{P}'}(dx, ds).
\end{aligned}$$

By taking conditional expectation, we obtain

$$\delta Y_{t_1} \le \mathbb{E}_{t_1}^{\mathbb{P}'}\biggl[\int_{t_1}^{t_2} M_{t-}^{\mathbb{P}'} \, dK_t^{\mathbb{P}'}\biggr]. \tag{4.5}$$

Applying the Hölder inequality, we can now write

$$
\begin{aligned}
\delta Y_{t_1} &\le \mathbb{E}_{t_1}^{\mathbb{P}'}\Big[\sup_{t_1\le t\le t_2}(M_t^{\mathbb{P}'})(K_{t_2}^{\mathbb{P}'}-K_{t_1}^{\mathbb{P}'})\Big] \\
&\le \Big(\mathbb{E}_{t_1}^{\mathbb{P}'}\Big[\sup_{t_1\le t\le t_2}(M_t^{\mathbb{P}'})^3\Big]\Big)^{1/3}\big(\mathbb{E}_{t_1}^{\mathbb{P}'}\big[(K_{t_2}^{\mathbb{P}'}-K_{t_1}^{\mathbb{P}'})^{3/2}\big]\big)^{2/3} \\
&\le C(C_{t_1}^{\mathbb{P}})^{1/3}\big(\mathbb{E}_{t_1}^{\mathbb{P}'}\big[K_{t_2}^{\mathbb{P}'}-K_{t_1}^{\mathbb{P}'}\big]\big)^{1/3}, \qquad \mathbb{P}\text{-a.s.}
\end{aligned}
$$

Taking the essential infimum on both sides completes the proof.

(iii) It remains to show that the estimate for $C_{t_1}^{\mathbb{P}}$ holds. But by definition, and the Lipschitz assumption on $F$, we clearly have

$$
\begin{aligned}
&\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\big[(K_{t_2}^{\mathbb{P}}-K_{t_1}^{\mathbb{P}})^2\big] \\
&\qquad\le C\big(\|Y\|_{\mathbb{D}_H^{2,\kappa}}^2+\|Z\|_{\mathbb{H}_H^{2,\kappa}}^2+\|U\|_{\mathbb{J}_H^{2,\kappa}}^2+\phi_H^{2,\kappa}\big)<+\infty,
\end{aligned}
\tag{4.6}
$$

since the last term on the right-hand side is finite thanks to the integrability assumed on $\xi$ and $F$. We then use the definition of the essential supremum (see Neveu [27], e.g.) to have the following equality:

$$
\operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t_1^+,\mathbb{P})}\mathbb{E}_{t_1}^{\mathbb{P}'}\big[(K_{t_2}^{\mathbb{P}'}-K_{t_1}^{\mathbb{P}'})^2\big]=\sup_{n\ge 1}\mathbb{E}_{t_1}^{\mathbb{P}_n}\big[(K_{t_2}^{\mathbb{P}_n}-K_{t_1}^{\mathbb{P}_n})^2\big], \qquad \mathbb{P}\text{-a.s.} \tag{4.7}
$$

for some sequence $(\mathbb{P}_n)_{n\ge1}\subset\mathcal{P}_H^\kappa(t_1^+,\mathbb{P})$. Moreover, in Lemma A.3 of the Appendix, it is proved that the set $\mathcal{P}_H^\kappa(t_1^+,\mathbb{P})$ is upward directed which means that for any $\mathbb{P}_1',\mathbb{P}_2'\in\mathcal{P}_H^\kappa(t_1^+,\mathbb{P})$, there exists $\mathbb{P}'\in\mathcal{P}_H^\kappa(t_1^+,\mathbb{P})$ such that

$$
\mathbb{E}_{t_1}^{\mathbb{P}'}\big[(K_{t_2}^{\mathbb{P}'}-K_{t_1}^{\mathbb{P}'})^2\big]=\max\big\{\mathbb{E}_{t_1}^{\mathbb{P}_1'}\big[(K_{t_2}^{\mathbb{P}_1'}-K_{t_1}^{\mathbb{P}_1'})^2\big],\mathbb{E}_{t_1}^{\mathbb{P}_2'}\big[(K_{t_2}^{\mathbb{P}_2'}-K_{t_1}^{\mathbb{P}_2'})^2\big]\big\}.
$$

Hence, by using a subsequence if necessary, we can rewrite (4.7) as

$$
\operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t_1^+,\mathbb{P})}\mathbb{E}_{t_1}^{\mathbb{P}'}\big[(K_{t_2}^{\mathbb{P}'}-K_{t_1}^{\mathbb{P}'})^2\big]=\lim_{n\to\infty}\uparrow\mathbb{E}_{t_1}^{\mathbb{P}_n}\big[(K_{t_2}^{\mathbb{P}_n}-K_{t_1}^{\mathbb{P}_n})^2\big], \qquad \mathbb{P}\text{-a.s.}
$$

With (4.6), we can then complete the proof exactly as in the proof of Theorem 4.4 in [36]. □

Finally, the comparison theorem below follows easily from the classical one for BSDEJs (see, e.g., Theorem 2.5 in [34]) and the representation (4.1).

THEOREM 4.2. *Let $(Y,Z,U)$ and $(Y',Z',U')$ be the solutions of 2BSDEJs with terminal conditions $\xi$ and $\xi'$, generators $\widehat{F}$ and $\widehat{F}'$, respectively (with the corresponding function $H$ and $H'$), and let $(y^{\mathbb{P}},z^{\mathbb{P}},u^{\mathbb{P}})$ and $(y'^{\mathbb{P}},z'^{\mathbb{P}},u'^{\mathbb{P}})$ the solutions of the associated BSDEJs. Assume that they both verify our Assumptions 3.1 and 3.2 and that we have:*

- $\mathcal{P}_H^\kappa \subset \mathcal{P}_{H'}^\kappa$;
- $\xi \leq \xi'$, $\mathcal{P}_H^\kappa$-*q.s.*;
- $\widehat{F}_t^{\mathbb{P}}(y_t'^{\mathbb{P}}, z_t'^{\mathbb{P}}, u_t'^{\mathbb{P}}) \leq \widehat{F}_t'^{\mathbb{P}}(y_t'^{\mathbb{P}}, z_t'^{\mathbb{P}}, u_t'^{\mathbb{P}})$, $\mathbb{P}$-*a.s., for all* $\mathbb{P} \in \mathcal{P}_H^\kappa$.

*Then* $Y \leq Y'$, $\mathcal{P}_H^\kappa$-*q.s.*

4.2. *A priori estimates*. We conclude this section by showing some a priori estimates which will be useful to obtain the existence of a solution in [21].

THEOREM 4.3. *Let Assumptions 3.1 and 3.2 hold. Assume* $\xi \in \mathbb{L}_H^{2,\kappa}$ *and* $(Y, Z, U) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa} \times \mathbb{J}_H^{2,\kappa}$ *is a solution to the 2BSDEJ (3.3). Let* $\{(y^{\mathbb{P}}, z^{\mathbb{P}}, u^{\mathbb{P}})\}_{\mathbb{P}\in\mathcal{P}_H^\kappa}$ *be the solutions of the corresponding BSDEJs (4.2). Then there exists a constant* $C_\kappa$ *such that*

$$\|Y\|^2_{\mathbb{D}_H^{2,\kappa}} + \|Z\|^2_{\mathbb{H}_H^{2,\kappa}} + \|U\|^2_{\mathbb{J}_H^{2,\kappa}} + \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\big[\big|K_T^{\mathbb{P}}\big|^2\big] \leq C_\kappa\big(\|\xi\|^2_{\mathbb{L}_H^{2,\kappa}} + \phi_H^{2,\kappa}\big),$$

$$\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \big\{\big\|y^{\mathbb{P}}\big\|^2_{\mathbb{D}^2(\mathbb{P})} + \big\|z^{\mathbb{P}}\big\|^2_{\mathbb{H}^2(\mathbb{P})} + \big\|u^{\mathbb{P}}\big\|^2_{\mathbb{J}^2(\mathbb{P})}\big\} \leq C_\kappa\big(\|\xi\|^2_{\mathbb{L}_H^{2,\kappa}} + \phi_H^{2,\kappa}\big).$$

PROOF. As in the proof of the representation formula in Theorem 4.1, the Lipschitz Assumption 3.1(iii) implies that there exist two bounded processes $\lambda$ and $\eta$ such that for all $t$, and $\mathbb{P}$-a.s.,

$$\begin{aligned} y_t^{\mathbb{P}} &= \xi + \int_t^T \big(\lambda_s y_s^{\mathbb{P}} + \eta_s \hat{a}_s^{1/2} z_s^{\mathbb{P}} + \widehat{F}_s^{\mathbb{P}}\big(0, 0, u_s^{\mathbb{P}}\big)\big)\, ds \\ &\quad - \int_t^T z_s^{\mathbb{P}}\, dB_s^{\mathbb{P},c} - \int_t^T \int_E u_s^{\mathbb{P}}(x) \widetilde{\mu}_B^{\mathbb{P}}(dx, ds). \end{aligned}$$

Define the following processes:

$$N_t^{\mathbb{P}} := \int_t^T \eta_s \hat{a}_s^{-1/2}\, dB_s^{\mathbb{P},c} + \int_t^T \int_E \gamma_s(x) \widetilde{\mu}_B^{\mathbb{P}}(dx, ds)$$

and

$$M_t := \exp\bigg(\int_t^T \lambda_s\, ds\bigg) \mathcal{E}\big(N^{\mathbb{P}}\big)_t,$$

where $\mathcal{E}(N^{\mathbb{P}})_t$ denotes the Doléans–Dade exponential martingale of $N_t^{\mathbb{P}}$. Then by applying Itô's formula to $M_t^{\mathbb{P}} y_t^{\mathbb{P}}$, we obtain

$$y_t^{\mathbb{P}} = \mathbb{E}_t^{\mathbb{P}}\bigg[M_T^{\mathbb{P}}\xi + \int_t^T M_s^{\mathbb{P}} \widehat{F}_s^{\mathbb{P}}\big(0, 0, u_s^{\mathbb{P}}\big)\, ds - \int_t^T \int_E M_s^{\mathbb{P}} \gamma_s(x) u_s^{\mathbb{P}}(x) \nu_s^{\mathbb{P}}(dx)\, ds\bigg].$$

Finally with Assumption 3.1(iv), the Hölder inequality and the inequality (4.4), we conclude that there exists a constant $C_\kappa$ depending only on $\kappa$, $T$ and the Lipschitz constant of $F$, such that for all $\mathbb{P}$,

$$\big|y_t^{\mathbb{P}}\big| \leq C_\kappa \mathbb{E}_t^{\mathbb{P}}\bigg[|\xi|^\kappa + \int_t^T \big|\widehat{F}_s^{\mathbb{P},0}\big|^\kappa\, ds\bigg]^{1/\kappa}. \tag{4.8}$$

This immediately provides the estimate for $y^{\mathbb{P}}$. Now by definition of our norms, we get from (4.8) and representation formula (4.1) that

$$\|Y\|^2_{\mathbb{D}_H^{2,\kappa}} \le C_\kappa\big(\|\xi\|^2_{\mathbb{L}_H^{2,\kappa}} + \phi_H^{2,\kappa}\big). \tag{4.9}$$

Now apply Itô's formula to $|Y|^2$ under each $\mathbb{P} \in \mathcal{P}_H^\kappa$. We get as usual for every $\epsilon > 0$

$$\begin{aligned}
|Y_0|^2 &+ \int_0^T \big|\hat{a}_t^{1/2} Z_t\big|^2\,dt + \int_0^T\int_E \big|U_t(x)\big|^2 \nu_t^{\mathbb{P}}(dx)\,dt \\
&= |\xi|^2 + 2\int_0^T Y_t \widehat{F}_t^{\mathbb{P}}(Y_t, Z_t, U_t)\,dt + 2\int_0^T Y_{t^-}\,dK_t^{\mathbb{P}} \\
&\quad - 2\int_0^T Y_t Z_t\,dB_t^{\mathbb{P},c} - \int_0^T\int_E \big(\big|U_t(x)\big|^2 + 2Y_{t^-}U_t(x)\big)\widetilde{\mu}_B^{\mathbb{P}}(dx,dt) \\
&\le 2\int_0^T |Y_t|\big|\widehat{F}_t^{\mathbb{P}}(Y_t, Z_t, U_t)\big|\,dt + 2\sup_{0\le t\le T}|Y_t| K_T^{\mathbb{P}} \\
&\quad - 2\int_0^T Y_t Z_t\,dB_t^{\mathbb{P},c} - \int_0^T\int_E \big(\big|U_t(x)\big|^2 + 2Y_{t^-}U_t(x)\big)\widetilde{\mu}_B^{\mathbb{P}}(dx,dt).
\end{aligned}$$

By our assumptions on $F$, we have

$$\begin{aligned}
&\big|\widehat{F}_t^{\mathbb{P}}(Y_t, Z_t, U_t)\big| \\
&\quad \le C\bigg(|Y_t| + \big|\hat{a}_t^{1/2} Z_t\big| + \big|\widehat{F}_t^{\mathbb{P},0}\big| + \bigg(\int_E \big|U_t(x)\big|^2 \nu_t^{\mathbb{P}}(dx)\bigg)^{1/2}\bigg).
\end{aligned}$$

With the usual inequality $2ab \le \frac{1}{\epsilon}a^2 + \epsilon b^2$, $\forall \epsilon > 0$, we obtain

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\bigg[\int_0^T \big|\hat{a}_t^{1/2} Z_t\big|^2\,dt + \int_0^T\int_E \big|U_t(x)\big|^2 \nu_t^{\mathbb{P}}(dx)\,dt\bigg] \\
&\quad \le C\mathbb{E}^{\mathbb{P}}\bigg[|\xi|^2 + \int_0^T |Y_t|\bigg(\big|\widehat{F}_t^{\mathbb{P},0}\big| + |Y_t| + \big|\hat{a}_t^{1/2} Z_t\big| \\
&\qquad\qquad + \bigg(\int_E \big|U_t(x)\big|^2 \nu_t^{\mathbb{P}}(dx)\bigg)^{1/2}\bigg)\,dt\bigg] \\
&\qquad + \mathbb{E}^{\mathbb{P}}\bigg[\int_0^T |Y_{t^-}|\,dK_t^{\mathbb{P}}\bigg] \\
&\quad \le C\bigg(\|\xi\|_{\mathbb{L}_H^{2,\kappa}} + \mathbb{E}^{\mathbb{P}}\bigg[\bigg(1 + \frac{C}{\varepsilon}\bigg)\sup_{0\le t\le T}|Y_t|^2 + \bigg(\int_0^T \big|\widehat{F}_t^{\mathbb{P},0}\big|\,dt\bigg)^2\bigg]\bigg) \\
&\qquad + \epsilon\mathbb{E}^{\mathbb{P}}\bigg[\int_0^T \big|\hat{a}_t^{1/2} Z_t\big|^2\,dt + \int_0^T\int_E \big|U_t(x)\big|^2 \nu_t^{\mathbb{P}}(dx)\,dt + \big|K_T^{\mathbb{P}}\big|^2\bigg].
\end{aligned} \tag{4.10}$$

Then by definition of our 2BSDEJ, we easily have

$$\mathbb{E}^{\mathbb{P}}[|K_T^{\mathbb{P}}|^2] \le C_0\mathbb{E}^{\mathbb{P}}\bigg[|\xi|^2 + \sup_{0\le t\le T}|Y_t|^2 + \int_0^T |\hat{a}_t^{1/2}Z_t|^2\,dt + \int_0^T\int_E |U_t(x)|^2\nu_t^{\mathbb{P}}(dx)\,dt + \bigg(\int_0^T |\widehat{F}_t^{\mathbb{P},0}|\,dt\bigg)^2\bigg], \tag{4.11}$$

for some constant $C_0$, independent of $\epsilon$. Now set $\epsilon := (2(1 + C_0))^{-1}$ and plug (4.11) into (4.10). One then gets

$$\mathbb{E}^{\mathbb{P}}\bigg[\int_0^T |\hat{a}_t^{1/2}Z_t|^2\,dt + \int_0^T\int_E U_t^2(x)\nu_t^{\mathbb{P}}(dx)\,dt\bigg] \le C\mathbb{E}^{\mathbb{P}}\bigg[|\xi|^2 + \sup_{0\le t\le T}|Y_t|^2 + \bigg(\int_0^T |\widehat{F}_t^{\mathbb{P},0}|\,dt\bigg)^2\bigg].$$

From this and the estimate for $Y$, we immediately obtain

$$\|Z\|_{\mathbb{H}_H^{2,\kappa}} + \|U\|_{\mathbb{J}_H^{2,\kappa}} \le C\big(\|\xi\|^2_{\mathbb{L}_H^{2,\kappa}} + \phi_H^{2,\kappa}\big).$$

Then the estimate for $K^{\mathbb{P}}$ follows from (4.11). The estimates for $z^{\mathbb{P}}$ and $u^{\mathbb{P}}$ can be proved similarly. □

THEOREM 4.4. *Let Assumptions 3.1 and 3.2 hold. For $i = 1, 2$, let us consider the solutions $(Y^i, Z^i, U^i, K^{\mathbb{P},i})$ of the 2BSDEJ (3.3) with terminal condition $\xi^i$. Then, there exists a constant $C_\kappa$ depending only on $\kappa$, $T$ and the Lipschitz constant of $F$ such that*

$$\|Y^1 - Y^2\|_{\mathbb{D}_H^{2,\kappa}} \le C_\kappa\|\xi^1 - \xi^2\|_{\mathbb{L}_H^{2,\kappa}},$$

$$\|Z^1 - Z^2\|^2_{\mathbb{H}_H^{2,\kappa}} + \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\Big[\sup_{0\le t\le T}|K_t^{\mathbb{P},1} - K_t^{\mathbb{P},2}|^2\Big] + \|U^1 - U^2\|^2_{\mathbb{J}_H^{2,\kappa}} \le C_\kappa\|\xi^1 - \xi^2\|_{\mathbb{L}_H^{2,\kappa}}\big(\|\xi^1\|_{\mathbb{L}_H^{2,\kappa}} + \|\xi^2\|_{\mathbb{L}_H^{2,\kappa}} + (\phi_H^{2,\kappa})^{1/2}\big).$$

PROOF. As in the previous theorem, we can obtain that there exists a constant $C_\kappa$ depending only on $\kappa$, $T$ and the Lipschitz constant of $\widehat{F}$, such that for all $\mathbb{P}$,

$$\big|y_t^{\mathbb{P},1} - y_t^{\mathbb{P},2}\big| \le C_\kappa\mathbb{E}_t^{\mathbb{P}}\big[|\xi^1 - \xi^2|^\kappa\big]^{1/\kappa}. \tag{4.12}$$

Now by definition, we get from (4.12) and representation formula (4.1) that

$$\|Y^1 - Y^2\|^2_{\mathbb{D}_H^{2,\kappa}} \le C_\kappa\|\xi^1 - \xi^2\|^2_{\mathbb{L}_H^{2,\kappa}}. \tag{4.13}$$

Applying Itô's formula to $|Y^1 - Y^2|^2$, under each $\mathbb{P} \in \mathcal{P}_H^\kappa$, leads to

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\left[\int_0^T |\hat{a}_t^{1/2}(Z_t^1 - Z_t^2)|^2\,dt + \int_0^T \int_E |U_t^1(x) - U_t^2(x)|^2 \nu_t^{\mathbb{P}}(dx)\,dt\right] \\
&\quad \le C\mathbb{E}^{\mathbb{P}}[|\xi^1 - \xi^2|^2] + \mathbb{E}^{\mathbb{P}}\left[\int_0^T |Y_t^1 - Y_t^2|\,d(K_t^{\mathbb{P},1} - K_t^{\mathbb{P},2})\right] \\
&\qquad + C\mathbb{E}^{\mathbb{P}}\left[\int_0^T |Y_t^1 - Y_t^2|\Big(|Y_t^1 - Y_t^2| + |\hat{a}_t^{1/2}(Z_t^1 - Z_t^2)| \right. \\
&\qquad\qquad \left. + \left(\int_E |U_t^1(x) - U_t^2(x)|^2 \nu_t^{\mathbb{P}}(dx)\,dt\right)^{1/2}\Big)\,dt\right] \\
&\quad \le C(\|\xi^1 - \xi^2\|^2_{\mathbb{L}_H^{2,\kappa}} + \|Y^1 - Y^2\|^2_{\mathbb{D}_H^{2,\kappa}}) \\
&\qquad + \frac{1}{2}\mathbb{E}^{\mathbb{P}}\left[\int_0^T |\hat{a}_t^{1/2}(Z_t^1 - Z_t^2)|^2\,dt + \int_0^T \int_E |U_t^1(x) - U_t^2(x)|^2 \nu_t^{\mathbb{P}}(dx)\,dt\right] \\
&\qquad + C\|Y^1 - Y^2\|_{\mathbb{D}_H^{2,\kappa}}\left(\mathbb{E}^{\mathbb{P}}\left[\sum_{i=1}^2 (K_T^{\mathbb{P},i})^2\right]\right)^{1/2}.
\end{aligned}$$

The estimates for $(Z^1 - Z^2)$ and $(U^1 - U^2)$ are now obvious from the above inequality and the estimates of Theorem 4.3. Finally the estimate for the difference of the nondecreasing processes is obvious by definition. $\square$

## APPENDIX

**A.1. Generating and separable class of coefficients.** We introduce the following notions inspired by [35]:

DEFINITION A.1. $\mathcal{A}_0 \subset \mathcal{A}_W$ is a generating class of coefficients if $\mathcal{A}_0$ is stable for the concatenation operation; that is, if $(a, \nu), (b, \beta) \in \mathcal{A}_0 \times \mathcal{A}_0$, then for each $t$,

$$(a\mathbf{1}_{[0,t]} + b\mathbf{1}_{[t,+\infty)}, \nu\mathbf{1}_{[0,t]} + \beta\mathbf{1}_{[t,+\infty)}) \in \mathcal{A}_0.$$

Notice that unlike [35], we do not impose their so-called "constant disagreement time property," as it is only useful for them to obtain their aggregation result, which, as mentioned before, is an hopeless goal in our framework.

DEFINITION A.2. We say that $\mathcal{A}$ is a separable class of coefficients generated by $\mathcal{A}_0$ if $\mathcal{A}_0$ is a generating class of coefficients and if $\mathcal{A}$ consists of all processes

$a$ and random measures $\nu$ of the form

$$(A.1)\qquad \begin{aligned} a_t(\omega) &= \sum_{n=0}^{+\infty}\sum_{i=1}^{+\infty} a_t^{n,i}(\omega)\mathbf{1}_{E_n^i}(\omega)\mathbf{1}_{[\tau_n(\omega),\tau_{n+1}(\omega))}(t), \\ \nu_t(\omega) &= \sum_{n=0}^{+\infty}\sum_{i=1}^{+\infty} \nu_t^{n,i}(\omega)\mathbf{1}_{\tilde{E}_n^i}(\omega)\mathbf{1}_{[\tilde{\tau}_n(\omega),\tilde{\tau}_{n+1}(\omega))}(t), \end{aligned}$$

where for each $i$ and for each $n$, $(a^{n,i}, \nu^{n,i}) \subset \mathcal{A}_0$, $\tau_n$ and $\tilde{\tau}_n$ are $\mathbb{F}$-stopping times with $\tau_0 = 0$, such that:

(i) $\tau_n < \tau_{n+1}$ on $\{\tau_n < +\infty\}$ and $\widetilde{\tau}_n < \widetilde{\tau}_{n+1}$ on $\{\widetilde{\tau}_n < +\infty\}$.
(ii) $\inf\{n \geq 0, \tau_n = +\infty\} + \inf\{n \geq 0, \widetilde{\tau}_n = +\infty\} < \infty$.
(iii) $\tau_n$ and $\widetilde{\tau}_n$ take countably many values in some fixed $I_0 \subset [0, T]$ which is countable and dense in $[0, T]$.
(iii) For each $n$, $(E_i^n)_{i\geq 1} \subset \mathcal{F}_{\tau_n}$ and $(\widetilde{E}_i^n)_{i\geq 1} \subset \mathcal{F}_{\tilde{\tau}_n}$ form a partition of $\Omega$.

REMARK A.1. If we refine the subdivisions, we can always take a common sequence of stopping times $(\tau_n)_{n\geq 0}$ and common sets $(E_i^n)_{i\geq 1,n\geq 0}$ for $a$ and for $\nu$. This will be used throughout this section.

The form for $a$ and $\nu$ in Definition A.2 is directly inspired by the so-called property of stability by concatenation and by bifurcation in the theory of stochastic control. As shown, for instance, in [13, 14] or [8] (see Remark 3.1), this property of control processes is tailor-made to be able to retrieve the dynamic programming principle, and is somehow the minimal stability property that must be verified. In our case, 2BSDEJs can be seen formally as a weak version of a stochastic control problem for which the controls are $a$ and $\nu$, and we will see below that the set of probably measures we will consider will have this stability property. This will be important for us in Proposition 4.2 of our accompanying paper [21], where we recover the dynamic programming principle.

The following proposition generalizes Proposition 4.11 of [35] and shows that a separable class of coefficients inherits the "good" properties of its generating class.

PROPOSITION A.1. *Let $\mathcal{A}$ be a separable class of coefficients generated by $\mathcal{A}_0$. Then*:

(i) *If $\mathcal{A}_0 \subset \mathcal{A}_W$, then $\mathcal{A} \subset \mathcal{A}_W$.*
(ii) *$\mathcal{A}$-quasi-surely is equivalent to $\mathcal{A}_0$-quasi-surely, where for any $\tilde{\mathcal{A}} \subset \mathcal{A}_W$, $\widetilde{\mathcal{A}}$-q.s. means $\mathbb{P}$-a.s. for every $\mathbb{P} \in \{\mathbb{P}_\nu^\alpha, (\alpha, \nu) \in \tilde{\mathcal{A}}\}$.*
(iii) *If every $\mathbb{P} \in \{\mathbb{P}_\nu^\alpha, (\alpha, \nu) \in \mathcal{A}_0\}$ satisfies the martingale representation property, then every $\mathbb{P} \in \{\mathbb{P}_\nu^\alpha, (\alpha, \nu) \in \mathcal{A}\}$ also satisfies the martingale representation property.*
(iv) *If every $\mathbb{P} \in \{\mathbb{P}_\nu^\alpha, (\alpha, \nu) \in \mathcal{A}_0\}$ satisfies the Blumenthal* 0–1 *law, then every probability measure $\mathbb{P} \in \{\mathbb{P}_\nu^\alpha, (\alpha, \nu) \in \mathcal{A}\}$ also satisfies the Blumenthal* 0–1 *law.*

As in [35], to prove this result, we need the following two lemmas. The first one is a straightforward generalization of Lemma 4.12 in [35], so we omit the proof. The second one is analogous to Lemma 4.13 in [35].

LEMMA A.1. *Let $\mathcal{A}$ be a separable class of coefficients generated by $\mathcal{A}_0$. For any $(a,\nu)\in\mathcal{A}$, and any $\mathbb{F}$-stopping time $\tau\in\mathcal{T}$, there exist $\tilde{\tau}\in\mathcal{T}$ with $\tilde{\tau}\geq\tau$, a sequence $(a^n,\nu^n)_{n\geq 1}\subset\mathcal{A}_0$ and a partition $(E_n)_{n\geq 1}\subset\mathcal{F}_\tau$ of $\Omega$ such that $\tilde{\tau}>\tau$ on $\{\tau<+\infty\}$ and*

$$a_t(\omega)=\sum_{n\geq 1}a_t^n(\omega)\mathbf{1}_{E_n}(\omega)\quad\text{and}\quad \nu_t(\omega)=\sum_{n\geq 1}\nu_t^n(\omega)\mathbf{1}_{E_n}(\omega),\qquad t<\tilde{\tau}. \tag{A.2}$$

*Finally, if $a$ and $\nu$ take the form (A.1) and $\tau\geq\tau_n$, then we can choose $\tilde{\tau}\geq\tau_{n+1}$.*

PROOF. We refer to the proof of Lemma 4.12 in [35]. □

LEMMA A.2. *Let $\tau_1,\tau_2\in\mathcal{T}$ be two stopping times such that $\tau_1\leq\tau_2$, and $(a^i,\nu^i)_{i\geq 1}\subset\overline{\mathcal{A}}_W$ and let $\{E_i, i\geq 1\}\subset\mathcal{F}_{\tau_1}$ be a partition of $\Omega$. Finally let $\mathbb{P}^0$ be a probability measure on $\mathcal{F}_{\tau_1}$, and let $\{\mathbb{P}^i, i\geq 1\}$ be a sequence of probability measures such that for each $i$, $\mathbb{P}^i$ is a solution of the martingale problem $(\mathbb{P}^0,\tau_1,\tau_2,a^i,\nu^i)$. Define*

$$\mathbb{P}(E):=\sum_{i\geq 1}\mathbb{P}^i(E\cap E_i)\qquad\textit{for all } E\in\mathcal{F}_{\tau_2},$$

$$a_t:=\sum_{i\geq 1}a_t^i\mathbf{1}_{E_i}\quad\textit{and}\quad \nu_t:=\sum_{i\geq 1}\nu_t^i\mathbf{1}_{E_i},\qquad t\in[\tau_1,\tau_2].$$

*Then $\mathbb{P}$ is a solution of the martingale problem $(\mathbb{P}^0,\tau_1,\tau_2,a,\nu)$.*

PROOF. By definition, $\mathbb{P}=\mathbb{P}^0$ on $\mathcal{F}_{\tau_1}$. In view of Remark 2.2, it is enough to prove that $M$, $J$ and $Q$ are $\mathbb{P}$-local martingales on $[\tau_1,\tau_2]$. By localizing if necessary, we may assume as usual that all these processes are actually bounded. For any stopping times $\tau_1\leq R\leq S\leq\tau_2$, and any bounded $\mathcal{F}_R$-measurable random variable $\eta$, we have

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}[(M_S-M_R)\eta]&=\sum_{i\geq 1}\mathbb{E}^{\mathbb{P}^i}[(M_S-M_R)\eta\mathbf{1}_{E_i}]\\&=\sum_{i\geq 1}\mathbb{E}^{\mathbb{P}^i}[\mathbb{E}^{\mathbb{P}^i}[(M_S-M_R)|\mathcal{F}_R]\eta\mathbf{1}_{E_i}]\\&=0.\end{aligned}$$

Thus $M$ is a $\mathbb{P}$-local martingale on $[\tau_1,\tau_2]$. We can prove similarly that $J$ and $Q$ are also $\mathbb{P}$-local martingales on $[\tau_1,\tau_2]$. □

PROOF OF PROPOSITION A.1. The proof follows closely the proof of Proposition 4.11 in [35], and we provide it for the convenience of the reader.

(i) We take $(a,\nu)\in\mathcal{A}$. Let us prove that $(a,\nu)\in\mathcal{A}_W$. We fix two stopping times $\theta_1,\theta_2$ in $\mathcal{T}$. We define a sequence $(\tilde{\tau}_n)_{n\geq 0}$ as follows:

$$\tilde{\tau}_0:=\theta_1 \quad \text{and} \quad \tilde{\tau}_n:=(\tau_n\vee\theta_1)\wedge\theta_2, \qquad n\geq 1.$$

To prove that the martingale problem $(\mathbb{P}^0,\theta_1,\theta_2,a,\nu)$ has a unique solution, we prove by induction on $n$ that the martingale problem $(\mathbb{P}^0,\tilde{\tau}_0,\tilde{\tau}_n,a,\nu)$ has a unique solution.

*Step* 1 *of the induction*. Let $n=1$, and let us first construct a solution to the martingale problem $(\mathbb{P}^0,\tilde{\tau}_0,\tilde{\tau}_1,a,\nu)$. For this purpose, we apply Lemma A.1 with $\tau=\tilde{\tau}_0$ and $\tilde{\tau}=\tilde{\tau}_1$, which leads to $a_t=\sum_{i\geq 1}a_t^i\mathbf{1}_{E_i}$ and $\nu_t=\sum_{i\geq 1}\nu_t^i\mathbf{1}_{E_i}$ for all $t<\tilde{\tau}_1$, where $(a^i,\nu^i)\in\mathcal{A}_0$ and $\{E_i, i\geq 1\}\subset\mathcal{F}_{\tilde{\tau}_0}$ forms a partition of $\Omega$. For $i\geq 1$, let $\mathbb{P}^{0,i}$ be the unique solution of the martingale problem $(\mathbb{P}^0,\tilde{\tau}_0,\tilde{\tau}_1,a_i,\nu_i)$ and define

$$\mathbb{P}^{0,a}(E):=\sum_{i\geq 1}\mathbb{P}^{0,i}(E\cap E_i) \qquad \text{for all } E\in\mathcal{F}_{\tilde{\tau}_1}.$$

Then Lemma A.2 tells us that $\mathbb{P}^{0,a}$ solves the martingale problem $(\mathbb{P}^0,\tilde{\tau}_0,\tilde{\tau}_1,a,\nu)$. Now let $\mathbb{P}$ be an arbitrary solution of the martingale problem $(\mathbb{P}^0,\tilde{\tau}_0,\tilde{\tau}_1,a,\nu)$, and let us prove that $\mathbb{P}=\mathbb{P}^{0,a}$. We first define

$$\mathbb{P}^i(E):=\mathbb{P}(E\cap E_i)+\mathbb{P}^{0,i}\big(E\cap E_i^c\big) \qquad \forall E\in\mathcal{F}_{\tilde{\tau}_1}.$$

Using Lemma A.2, and the facts that $a^i=a\mathbf{1}_{E_i}+a^i\mathbf{1}_{E_i^c}$ and $\nu^i=\nu\mathbf{1}_{E_i}+\nu^i\mathbf{1}_{E_i^c}$, we conclude that $\mathbb{P}^i$ solves the martingale problem $(\mathbb{P}^0,\tilde{\tau}_0,\tilde{\tau}_1,a^i,\nu^i)$. Since this problem has a unique solution, we thus have $\mathbb{P}^i=\mathbb{P}^{0,i}$ on $\mathcal{F}_{\tilde{\tau}_1}$. This implies that for each $i\geq 1$ and for each $E\in\mathcal{F}_{\tilde{\tau}_1}$, $\mathbb{P}^i(E\cap E_i)=\mathbb{P}^{0,i}(E\cap E_i)$, and finally

$$\mathbb{P}^{0,a}(E)=\sum_{i\geq 1}\mathbb{P}^{0,i}(E\cap E_i)=\sum_{i\geq 1}\mathbb{P}^i(E\cap E_i)=\mathbb{P}(E) \qquad \forall E\in\mathcal{F}_{\tilde{\tau}_1}.$$

*Step* 2 *of the induction*. We assume that the martingale problem $(\mathbb{P}^0,\tilde{\tau}_0,\tilde{\tau}_n,a,\nu)$ has a unique solution denoted by $\mathbb{P}^n$. Using the same reasoning as above, we see that the martingale problem $(\mathbb{P}^n,\tilde{\tau}_n,\tilde{\tau}_{n+1},a,\nu)$ has a unique solution, denoted by $\mathbb{P}^{n+1}$. Then the processes $M$, $J$ and $Q$ defined in Remark 2.2 are $\mathbb{P}^{n+1}$-local martingales on $[\tilde{\tau}_n,\tilde{\tau}_{n+1}]$, and since $\mathbb{P}^{n+1}$ coincides with $\mathbb{P}^n$ on $\mathcal{F}_{\tilde{\tau}_n}$, $M$, $J$ and $Q$ are also $\mathbb{P}^{n+1}$-local martingales on $[\tilde{\tau}_0,\tilde{\tau}_n]$. Hence $\mathbb{P}^{n+1}$ solves the martingale problem $(\mathbb{P}^0,\tilde{\tau}_0,\tilde{\tau}_{n+1},a,\nu)$. We suppose now that $\mathbb{P}$ is another arbitrary solution to the martingale problem $(\mathbb{P}^0,\tilde{\tau}_0,\tilde{\tau}_{n+1},a,\nu)$. By the induction assumption, $\mathbb{P}^n=\mathbb{P}$ on $\mathcal{F}_{\tilde{\tau}_n}$. Then $\mathbb{P}$ solves the martingale problem $(\mathbb{P}^n,\tilde{\tau}_n,\tilde{\tau}_{n+1},a,\nu)$, and by uniqueness $\mathbb{P}=\mathbb{P}^{n+1}$ on $\mathcal{F}_{\tilde{\tau}_{n+1}}$. The induction is now complete.

Note that $\mathcal{F}_{\theta_2}=\bigvee_{n\geq 1}\mathcal{F}_{\tilde{\tau}_n}$. Indeed, since $\inf\{n\geq 1,\tau_n=+\infty\}<+\infty$, then $\inf\{n\geq 1,\tilde{\tau}_n=\theta_2\}<+\infty$. This allows us to define $\mathbb{P}^\infty(E):=\mathbb{P}^n(E)$ for $E\in\mathcal{F}_{\tilde{\tau}_n}$ and to extend it uniquely to $\mathcal{F}_{\theta_2}$. Now using again Remark 2.2, we conclude that $\mathbb{P}^\infty$ solves $(\mathbb{P}^0,\theta_1,\theta_2,a,\nu)$ and is unique.

(ii) We now prove that $\mathcal{A}$-quasi-surely is equivalent to $\mathcal{A}_0$-quasi-surely.

We take $(a,\nu)\in\mathcal{A}$ and we apply Lemma A.1 with $\tau=+\infty$ to write $a_t=\sum_{i\geq 1}a_t^i\mathbf{1}_{E_i}$ and $\nu_t=\sum_{i\geq 1}\nu_t^i\mathbf{1}_{E_i}$ for all $t\geq 0$, where $(a^i,\nu^i)\in\mathcal{A}_0$ and $\{E_i,i\geq 1\}\subset\mathcal{F}_\infty$ forms a partition of $\Omega$. Take a set $E$ such that $\mathbb{P}_{\tilde{\nu}}^{\tilde{a}}(E)=0$ for every $(\tilde{a},\tilde{\nu})\in\mathcal{A}_0$, then

$$\mathbb{P}_\nu^a(E)=\sum_{i\geq 1}\mathbb{P}_\nu^a(E\cap E_i)=\sum_{i\geq 1}\mathbb{P}_{\nu^i}^{a^i}(E\cap E_i)=0.$$

(iii) By (i), since $\mathcal{A}\subset\mathcal{A}_W$, for any $(a,\nu)\in\mathcal{A}$, the corresponding martingale problem has a unique solution, which is therefore an extremal point in the set of solutions. Hence, we can apply Theorem III.4.29 in [19] to obtain immediately the predictable representation property.

(iv) Take $(a,\nu)\in\mathcal{A}$ of the form (A.1), in which we can take $\tau_0=0$ without loss of generality. $\mathbb{P}_\nu^a$ is the law on $[0,\tau_1]$ of a semimartingale with characteristics

$$\left(-\int_0^t\int_E x\mathbf{1}_{|x|>1}\tilde{\nu}_s(dx)\,ds,\int_0^t\tilde{a}_s\,ds,\tilde{\nu}_s(dx)\,ds\right),$$

where

$$\tilde{a}_t:=\sum_{i\geq 1}a^{0,i}\mathbf{1}_{E_0^i}\quad\text{and}\quad\tilde{\nu}_t:=\sum_{i\geq 1}\nu^{0,i}\mathbf{1}_{E_0^i},$$

where $\{E_0^i,i\geq 1\}\subset\mathcal{F}_0$ is a partition of $\Omega$. Since $\mathcal{F}_0$ is trivial, the partition is only composed of $\Omega$ and $\varnothing$, and then

$$\tilde{a}_t:=a_t^{0,1}\quad\text{and}\quad\tilde{\nu}_t=\nu_t^{0,1}.$$

Then for $E\in\mathcal{F}_{0+}$, $\mathbb{P}_\nu^a(E)=\mathbb{P}_{\tilde{\nu}}^{\tilde{a}}(E)=0$, because $\mathbb{P}_{\tilde{\nu}}^{\tilde{a}}$ satisfies the Blumenthal 0–1 law by hypothesis. □

### A.2. The measures $\mathbb{P}_F^{\alpha,\beta}$.

LEMMA A.3. *Fix an arbitrary measure $\mathbb{P}=\mathbb{P}_F^{\alpha,\beta}$ in $\mathcal{P}_H^\kappa$. The set $\mathcal{P}_H^\kappa(t^+,\mathbb{P})$ is upward directed; that is, for each $\mathbb{P}_1:=\mathbb{P}_{F_1}^{\alpha_1,\beta_1}$ and $\mathbb{P}_2:=\mathbb{P}_{F_2}^{\alpha_2,\beta_2}$ in $\mathcal{P}_H^\kappa(t^+,\mathbb{P})$, there exists $\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})$ such that $\forall u>t$,*

$$\mathbb{E}_t^{\mathbb{P}'}\big[\big(K_u^{\mathbb{P}'}-K_t^{\mathbb{P}'}\big)^2\big]=\max\big\{\mathbb{E}_t^{\mathbb{P}_1}\big[\big(K_u^{\mathbb{P}_1}-K_t^{\mathbb{P}_1}\big)^2\big],\mathbb{E}_t^{\mathbb{P}_2}\big[\big(K_u^{\mathbb{P}_2}-K_t^{\mathbb{P}_2}\big)^2\big]\big\}.\tag{A.3}$$

PROOF. We define the following $\mathcal{F}_{t^+}$-measurable sets:

$$E_1:=\big\{\omega\in\Omega:\mathbb{E}_t^{\mathbb{P}_2}\big[\big(K_u^{\mathbb{P}_2}-K_t^{\mathbb{P}_2}\big)^2\big](\omega)\leq\mathbb{E}_t^{\mathbb{P}_1}\big[\big(K_u^{\mathbb{P}_1}-K_t^{\mathbb{P}_1}\big)^2\big](\omega)\big\},$$

and $E_2:=\Omega\setminus E_1$. Then for all $A\in\mathcal{F}_T$, we define the probability measure $\mathbb{P}'$ by

$$\mathbb{P}'(A):=\mathbb{P}_1(A\cap E_1)+\mathbb{P}_2(A\cap E_2).$$

By definition, $\mathbb{P}'$ satisfies (A.3). Let us prove now that $\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})$. As in the proof of claim (4.17) in [36], for $s \in [0, T]$, we define the processes $\alpha^*$, $\beta^*$ and the measure $F^*$ as follows:

$$\begin{aligned}
\alpha_s^*(\omega) &:= \alpha_s(\omega)\mathbf{1}_{[0,t)}(s) \\
&\qquad + \big(\alpha_s^1(\omega)\mathbf{1}_{\{X^{\alpha,\beta}\in E_1\}}(\omega) + \alpha_s^2(\omega)\mathbf{1}_{\{X^{\alpha,\beta}\in E_2\}}(\omega)\big)\mathbf{1}_{[t,T]}(s), \\
\beta_s^*(\omega,x) &:= \beta_s(\omega,x)\mathbf{1}_{[0,t)}(s) \\
&\qquad + \big(\beta_s^1(\omega,x)\mathbf{1}_{\{X^{\alpha,\beta}\in E_1\}}(\omega) + \beta_s^2(\omega,x)\mathbf{1}_{\{X^{\alpha,\beta}\in E_2\}}(\omega)\big)\mathbf{1}_{[t,T]}(s), \\
F_s^*(\omega) &:= F_s(\omega)\mathbf{1}_{[0,t)}(s) \\
&\qquad + \big(F_s^1(\omega)\mathbf{1}_{\{X^{\alpha,\beta}\in E_1\}}(\omega) + F_s^2(\omega)\mathbf{1}_{\{X^{\alpha,\beta}\in E_2\}}(\omega)\big)\mathbf{1}_{[t,T]}(s),
\end{aligned}$$

where $X^{\alpha,\beta}$ is defined in (2.4).

First of all, we clearly have $F^* \in \mathcal{V}$, since this set is stable by concatenation and bifurcation by definition. We can therefore define the probability measure $\mathbb{P}_{0,F^*}$.[4] Moreover, we have

$$0 < \underline{\alpha} \wedge \underline{\alpha}^1 \wedge \underline{\alpha}^2 \le \alpha^* \le \overline{\alpha} \vee \overline{\alpha}^1 \vee \overline{\alpha}^2,$$

where $\underline{\alpha}$, $\overline{\alpha}$, $\underline{\alpha}^i$, $\overline{\alpha}^i$ are the lower and upper bounds of the processes $\alpha$ and $\alpha^i$. Next, we have to check that $\beta^* \in \mathcal{R}_{F^*}$. It is clear that for every $\omega \in \Omega$, $F^*(dx)$-a.e.,

$$\begin{aligned}
|\beta_s^*|(\omega,x) &\le \big(C\mathbf{1}_{0\le s<t} + \big(C_1\mathbf{1}_{X^{\alpha,\beta}\in E_1}(\omega) + C_2\mathbf{1}_{X^{\alpha,\beta}\in E_2}(\omega)\big)\mathbf{1}_{t\le s\le T}\big)(1\wedge|x|) \\
&\le C^*(1\wedge|x|),
\end{aligned}$$

since $F^*$ coincides with $F$ before $t$ and with either $F^1$ or $F^2$ after $t$.

The strict monotony of $x \longmapsto \beta_s^*(\omega,x)$ for Lebesgue almost every $s \in [0,T]$ and $\mathbb{P}_{0,F^*}$-a.e. $\omega \in \Omega$ follows similarly from the corresponding properties of $\beta$, $\beta^1$ and $\beta^2$ and the fact that the support of the law of the jumps of $B$ at time $s$ under $\mathbb{P}_{0,F^*}$ coincides with the support of the same law under $\mathbb{P}_{0,F}$ for $s < t$ and under either $\mathbb{P}_{0,F^1}$ or $\mathbb{P}_{0,F^2}$ for $s \ge t$.

We can check similarly that

$$\int_0^T \int_{\{|x|>1\}} x\nu_s^{F^*,\beta^*}(dx,ds) < +\infty$$

and

$$\mathbb{E}^{\mathbb{P}}\bigg[\int_0^T \int_E |x|^2 \nu_s^{F^*,\beta^*}(dx)\,ds\bigg] < +\infty.$$

[4] The attentive reader may have remarked that $F^*$ is not defined for every $\omega$, but only for those such that their path up to time $t^+$ is in the support of $\mathbb{P}$ restricted to $\mathcal{F}_{t^+}$. This may appear as a problem, however, since we know that the measure $\mathbb{P}^{\alpha^*,\beta^*}$ has to agree with $\mathbb{P}$ on $\mathcal{F}_{t^+}$, we actually only need to solve the martingale problem in the definition of $\mathbb{P}_{0,F^*}$ starting from time $t$.

Therefore, we have proved that $\mathbb{P}_{F^*}^{\alpha^*,\beta^*} \in \overline{\mathcal{P}}_S$. Moreover, using the same arguments as in the step 3 of the proof of Lemma A.3 in [21], we can easily show that $\mathbb{P}' = \mathbb{P}_{F^*}^{\alpha^*,\beta^*}$. Finally, we compute

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}'}\left[\int_0^T |\widehat{F}_s^{\mathbb{P}',0}|^2\,ds\right] \\
&\quad = \mathbb{E}^{\mathbb{P}}\left[\int_0^t |\widehat{F}_s^{\mathbb{P},0}|^2\,ds\right] + \mathbb{E}^{\mathbb{P}_1}\left[\int_t^T |\widehat{F}_s^{\mathbb{P}_1,0}|^2\,ds\,\mathbf{1}_{E_1}\right] \\
&\qquad + \mathbb{E}^{\mathbb{P}_2}\left[\int_t^T |\widehat{F}_s^{\mathbb{P}_2,0}|^2\,ds\,\mathbf{1}_{E_2}\right] \\
&\quad \le \mathbb{E}^{\mathbb{P}}\left[\int_0^T |\widehat{F}_s^{\mathbb{P},0}|^2\,ds\right] + \mathbb{E}^{\mathbb{P}_1}\left[\int_0^T |\widehat{F}_s^{\mathbb{P}_1,0}|^2\,ds\,\mathbf{1}_{E_1}\right] \\
&\qquad + \mathbb{E}^{\mathbb{P}_2}\left[\int_0^T |\widehat{F}_s^{\mathbb{P}_2,0}|^2\,ds\,\mathbf{1}_{E_2}\right]. \\
&\quad < +\infty.
\end{aligned}$$

Since by construction $\mathbb{P}'$ coincides with $\mathbb{P}$ on $\mathcal{F}_{t^+}$, we have indeed shown that $\mathbb{P}' \in \mathcal{P}_H^{\kappa}(t^+, \mathbb{P})$. $\square$

### A.3. $L^r$-integrability of exponential martingales.

LEMMA A.4. *Let $\delta > 0$ and $n \in \mathbb{N}^*$. Then there exists a constant $C_{n,\delta}$ depending only on $\delta$ and $n$ such that*

$$(1+x)^{-n} - 1 + nx \le C_{n,\delta} x^2 \qquad \text{for all } x \in [-1+\delta, +\infty).$$

PROOF. The inequality is clear for $x$ large enough; let us say $x \ge M$ for some $M > 0$. Then a simple Taylor expansion shows that this also holds in a neighborhood of 0, that is to say for $x \in [-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$. Finally, for $x \in (-1+\delta, -\varepsilon) \cup (\varepsilon, M)$, it is clear that we can choose $C$ large enough such that the inequality also holds. $\square$

Mémin [24] and then Lépingle and Mémin [23] proved some useful multiplicative decompositions of exponential semimartingales. We give here one of these representations that we will use in the proof of Lemma A.5.

PROPOSITION A.2 (Proposition II.1 of [22]). *Let $N$ be a local martingale and let $A$ be a predictable process with finite variation such that $\Delta A \neq -1$. We assume $N_0 = A_0 = 0$. Then there exists a local martingale $\widetilde{N}$ with $\widetilde{N}_0 = 0$ and such that*

$$\mathcal{E}(N+A) = \mathcal{E}(\widetilde{N})\mathcal{E}(A).$$

LEMMA A.5. *Let* $\lambda > 0$ *and* $M$ *be a local martingale with bounded jumps, such that* $\Delta M \geq -1+\delta$, *for a fixed* $\delta > 0$. *Let* $V^{-\lambda}$ *be the predictable compensator of*

$$\left\{W_t^{-\lambda} = \sum_{s\leq t}[(1+\Delta M_s)^{-\lambda} - 1 + \lambda\Delta M_s], t \geq 0\right\}.$$

*We have*:

(i) $\mathcal{E}^{-\lambda}(M) = \mathcal{E}(N^{-\lambda} + A^{-\lambda})$ *where*

$$A^{-\lambda} = \frac{\lambda(\lambda+1)}{2}\langle M^c, M^c\rangle^T + V^{-\lambda}, \qquad N^{-\lambda} = -\lambda M^T + W^{-\lambda} - V^{-\lambda}.$$

(ii) *There exist a local martingale* $\widetilde{N}^{-\lambda}$ *such that*

$$\mathcal{E}^{-\lambda}(M) = \mathcal{E}(\widetilde{N}^{-\lambda})\mathcal{E}(A^{-\lambda}).$$

PROOF. First note that thanks to Lemma A.4, for $\lambda > 0$, $(1+x)^{-\lambda} - 1 + \lambda x \leq Cx^2$, and thus $W^{-\lambda}$ is integrable. We set

$$T_n = \inf\left\{t \geq 0 : \mathcal{E}(M)_t \leq \frac{1}{n}\right\} \quad \text{and} \quad M_t^n = M_{t\wedge T_n}.$$

Then $M^n$ and $\mathcal{E}(M^n)$ are local martingales, $\mathcal{E}(M^n) \geq \frac{1}{n}$ and $\mathcal{E}(M^n)_t = \mathcal{E}(M)_t$ if $t < T_n$. The assumption $\Delta M > -1$ shows that $T_n$ tends to infinity when $n$ tends to infinity. For each $n \geq 1$, we apply Itô's formula to a $\mathcal{C}^2$ function $f_n$ that coincides with $x^{-\lambda}$ on $[\frac{1}{n}, +\infty)$,

$$\begin{aligned}
\mathcal{E}^{-\lambda}(M^n)_t &= 1 - \lambda\int_0^t \mathcal{E}^{-\lambda-1}(M^n)_{s^-}\,d\mathcal{E}(M^n)_s \\
&\quad + \frac{\lambda(\lambda+1)}{2}\int_0^t \mathcal{E}^{-\lambda-2}(M^n)_{s^-}\,d\langle(\mathcal{E}(M^n))^c\rangle_s \\
&\quad + \sum_{s\leq t}[\mathcal{E}^{-\lambda}(M^n)_s - \mathcal{E}^{-\lambda}(M^n)_{s^-} + \lambda\mathcal{E}^{-\lambda-1}(M^n)_{s^-}\Delta\mathcal{E}^{-\lambda}(M^n)_s] \\
&= 1 + \int_0^t \mathcal{E}(M^n)_{s^-}\,dX_s^n,
\end{aligned}$$

where

$$X_t^n := -\lambda M_t^n + \frac{\lambda(\lambda+1)}{2}\langle(M^n)^c, (M^n)^c\rangle_t + \sum_{s\leq t}[(1+\Delta M_s)^{-\lambda} - 1 + \lambda\Delta M_s],$$

and then $\mathcal{E}^{-\lambda}(M^n) = \mathcal{E}(X^n)$. Let us define the nontruncated counterpart $X$ of $X^n$:

$$X = -\lambda M + \frac{\lambda(\lambda+1)}{2}\langle M^c, M^c\rangle + W^{-\lambda}.$$

On the interval $[0, T_n[$, we have $X^n = X$ and $\mathcal{E}^{-\lambda}(M) = \mathcal{E}(X)$, now letting $n$ tend to infinity, we obtain that $\mathcal{E}^{-\lambda}(M)$ and $\mathcal{E}(X)$ coincide on $[0, +\infty[$, which is the point (i) of the lemma.

We want to use Proposition A.2 to prove the point (ii), so we need to show that $\Delta A > -1$. We set

$$S = \inf\{t \geq 0 : \Delta A_t^{-\lambda} \leq -1\}.$$

It is a predictable stopping time. Using this, and the fact that $M$ and $(W^{-\lambda} - V^{-\lambda})$ are local martingales, we have

$$\Delta A_S^{-\lambda} = \mathbb{E}[\Delta A_S^{-\lambda} | \mathcal{F}_{S-}] = \mathbb{E}[\Delta X_S | \mathcal{F}_{S-}] = \mathbb{E}[(1 + \Delta M_S)^{-\lambda} | \mathcal{F}_{S-}],$$

and since $\{S < +\infty\} \in \mathcal{F}_{S-}$,

$$0 \geq \mathbb{E}[\mathbf{1}_{\{S<+\infty\}}(1 + \Delta A_S^{-\lambda})] = \mathbb{E}[\mathbf{1}_{\{S<+\infty\}}(1 + \Delta M_S)^{-\lambda}].$$

Then $\Delta M_S \leq -1$ on $\{S < +\infty\}$, which means that $S = +\infty$ and $\Delta A > -1$ a.s. The proof is now complete. $\square$

We are finally in a position to state the lemma on $L^r$ integrability of exponential martingales for a negative exponent $r$.

LEMMA A.6. *Let $\lambda > 0$ and let $M$ be a local martingale with bounded jumps, such that $\Delta M \geq -1 + \delta$, for a fixed $\delta > 0$, and $\langle M, M \rangle_t$ is bounded $dt \times \mathbb{P}$-a.s. Then*

$$\mathbb{E}^{\mathbb{P}}[\mathcal{E}(M)_t^{-\lambda}] < +\infty \qquad dt \times \mathbb{P}\text{-a.s.}$$

PROOF. Let $n \geq 1$ be an integer. We will denote $\tilde{\mu}_M = \mu_M - \nu_M$ the compensated jump measure of $M$. Thanks to Lemma A.5, we write the decomposition

$$\mathcal{E}(M)^{-n} = \mathcal{E}(\widetilde{N}^{-n})\mathcal{E}\big(\tfrac{1}{2}n(n+1)\langle M^c, M^c\rangle + V^{-n}\big),$$

where $\widetilde{N}^{-n}$ is a local martingale, and $V^{-n}$ is defined as $V^{-\lambda}$. Using Lemma A.4, we have the inequality

$$V_t^{-n} \leq \int_0^t \int_E Cx^2 \nu_M(dx, ds),$$

and using the previous representation we obtain

$$\begin{aligned}
\mathcal{E}(M)_t^{-n} &\leq \mathcal{E}(\widetilde{N}^{-n})_t \mathcal{E}\left(\frac{1}{2}n(n+1)\langle M^c, M^c\rangle + \int_0^\cdot \int_E Cx^2 \nu_M(dx, ds)\right)_t \\
&\leq \mathcal{E}(\widetilde{N}^{-n})_t \exp\left(\left(\frac{1}{2}n(n+1) + C\right)\langle M, M\rangle_t\right) \\
&\leq C\mathcal{E}(\widetilde{N}^{-n})_t \qquad \text{since } \langle M, M\rangle_t \text{ is bounded.}
\end{aligned}$$

Let us prove now that the jumps of $\widetilde{N}^{-n}$ are strictly bigger than $-1$. We compute

$$\Delta\widetilde{N}^{-n} = \frac{\Delta N^{-n}}{1+\Delta A^{-n}} \qquad \text{where } A^{-n} \text{ is defined as in Lemma A.6}$$
$$= \frac{(1+\Delta M)^{-n}}{1+\Delta V^{-n}} - 1 > -1 \qquad \text{since } -1 < \Delta M \leq B \text{ and } \Delta V^{-n} > -1.$$

This implies that $\mathcal{E}(\widetilde{N}^{-n})$ is a positive supermartingale which equals 1 at $t=0$. We deduce

$$\mathbb{E}\big[\mathcal{E}(M)_t^{-n}\big] \leq C\mathbb{E}\big[\mathcal{E}\big(\widetilde{N}^{-n}\big)_t\big] \leq C.$$

We have the desired integrability for negative integers. We extend the property to any negative real number by Hölder's inequality. □

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