

SECOND-ORDER CONDITIONS FOR OPTIMIZATION PROBLEMS WITH CONSTRAINTS*

JEAN-PAUL PENOT†

Abstract. Using a projective approach, new necessary conditions and new sufficient conditions for optimization problems with explicit or implicit constraints are examined. They are compared to previous ones. A particular emphasis is given to mathematical programming problems with non-polyhedral constraints. This case occurs in particular when the constraints are defined in functional spaces.

Key words. Lagrangian, mathematical programming, multiplier, optimality conditions, projective tangent set, second-order conditions

AMS subject classifications. 49K27, 90C30, 46A20, 46N10, 52A05, 52A40

PII. S0363012996311095

1. Introduction. Devising efficient optimality conditions is an important objective when dealing with optimization problems. Therefore the literature on the subject is rich. (See [11], [15], [25], [29], [30], [41] for recent contributions.) Structured problems, such as mathematical programming problems, optimal control problems, continuous time problems, and semi-infinite programming problems, require a particular attention because the constraints are not necessarily defined by a finite number of scalar functions. This lack of polyhedrality causes a gap between necessary conditions and sufficient conditions, (see, for instance, [24], [28]). Moreover, the conditions cannot be given the simple and aesthetic form of the cases in which the constraints are polyhedral, as in [3], [4], [7], [17]–[19], and [23], for instance.

In [37] we reduced this gap to an acceptable extent: when the decision space is finite dimensional, the sufficient condition differs from the necessary condition by the replacement of an inequality by a strict inequality. As the unconstrained case shows, this difference is unavoidable. However, the second-order conditions of [37] are complex, and so are the conditions of [20], [27], and [31]. It is the purpose of the present work to present more handy conditions inspired by [11] and to compare them with recent proposals. It appears that the new conditions are not as selective as the previous ones: the sufficient (resp., the necessary) condition is a consequence of the sufficient (resp., necessary) condition of [37]. However, the new necessary condition is close to the sufficient condition, and such a fact is rather satisfactory.

For simplicity, we limit our study to the second-order case and we do not insist on the projective aspect of the tangent sets we deal with, which is just pointed out in section 2, although it is probably the main novelty here.

The optimality conditions are presented in section 3 along with a comparison with the results of [37]. Mathematical programming problems are considered in section 4. We devote section 5 to comparisons with recent works which came to our attention after the original version of the present paper was submitted. We are especially indebted to the referees for references [12] and [26]. We hope the clarifications we give

*Received by the editors October 21, 1996; accepted for publication (in revised form) December 29, 1997; published electronically November 17, 1998.

<http://www.siam.org/journals/sicon/37-1/31109.html>

†Université de Pau et des Pays de l'Adour, Laboratoire de Mathématiques appliquées, CNRS UPRES A 5033 64000 Pau, France (jean-paul.penot@univ-pau.fr).

will provide hints for obtaining concrete and convenient conditions in the specially structured cases mentioned above.

2. Projective tangent sets. In what follows, we denote by \mathbb{P} (resp., \mathbb{R}_+) the set of positive (resp., nonnegative) real numbers. The closed ball with center x and radius r in a normed vector space (n.v.s.) X is denoted by $B(x, r)$. The closure of a subset F of X is denoted by $\text{cl}F$. Recall that the projective space $P(X)$ associated with a vector space X is the set of equivalence classes of pairs $(v, r) \in X \times \mathbb{R}_+$ for the relation

$$(v, r) \sim (v', r') \text{ if } (v', r') = (tv, tr) \text{ for some } t > 0.$$

Obviously, $P(X)$ can be identified with the union

$$P(X) = X_1 \cup X_0,$$

where X_1 (resp., X_0) is the image of $X \times \{1\}$ (resp., $X \times \{0\}$) under the canonical mapping $p : X \times \mathbb{R}_+ \rightarrow P(X)$. We write $[v, r]$ to denote $p(v, r)$ and we call p the projective projection. If Y is another vector space and if $h : X \rightarrow Y$ is a positively homogeneous mapping, then h induces a mapping $h^P : P(X) \rightarrow P(Y)$ satisfying $h^P(p(x, r)) = p(h(x), r)$ for each $(x, r) \in X \times \mathbb{R}_+$, hence $h^P(X_1) \subset Y_1, h^P(X_0) \subset Y_0$, and if $h^P([x, 1]) = [y, 1]$, then $h^P([x, 0]) = [y, 0]$. Conversely, any mapping $\widehat{h} : P(X) \rightarrow P(Y)$ satisfying these conditions is the mapping h^P associated with some positively homogeneous map $h : X \rightarrow Y$.

DEFINITION 1. Given an integer $k \geq 2$, a subset F of an n.v.s. X , $x \in \text{cl} F$, $v_1, \dots, v_{k-1} \in X$, the projective tangent set of order k to F at (x, v_1, \dots, v_{k-1}) is the image $PT^k(F, x, v_1, \dots, v_{k-1})$ by the projective projection p of the set $\widehat{T}^k(F, x, v_1, \dots, v_{k-1})$ of pairs $(w, r) \in X \times \mathbb{R}_+$ such that there exist sequences $(t_n), (r_n)$ in \mathbb{P} with limits 0 and r , resp., $(w_n) \xrightarrow{\sigma} w$ (weak convergence) such that $(r_n^{-1}t_n) \rightarrow 0$ and

$$x_n := x + t_n v_1 + \frac{t_n^2}{2} v_2 + \dots + \frac{t_n^{k-1}}{(k-1)!} v_{k-1} + \frac{t_n^k}{k!} \frac{w_n}{r_n} \in F$$

for each n .

The preceding definition has been inspired by a notion presented in [11]; it is closely related to two notions given in [26]. A precise comparison will be given in the last section of the paper. Several variants are possible. For instance, one can take strong convergence instead of weak convergence in what precedes, or weak* convergence if X is a dual space. One could also use nets (or, rather, bounded nets). Also for some purposes, it would be possible to replace the condition $(r_n^{-1}t_n) \rightarrow 0$ by the weaker condition $(r_n^{-1}t_n w_n) \rightarrow 0$. Clearly, by its very definition, the weak tangent set of order k to F at (x, v_1, \dots, v_{k-1}) (also denoted by $F^k(x, v_1, \dots, v_{k-1})$),

$$T^k(F, x, v_1, \dots, v_{k-1}) = \limsup_{t \downarrow 0} k! t^{-k} \left(F - x - tv_1 \dots - \frac{t^{k-1}}{(k-1)!} v_{k-1} \right)$$

coincides with the set $F_1^k(x, v_1, \dots, v_{k-1})$, where

$$F_r^k(x, v_1, \dots, v_{k-1}) := \left\{ w \in X : (w, r) \in \widehat{T}^k(F, x, v_1, \dots, v_{k-1}) \right\}.$$

Here the limit sup is the sequential limit sup with respect to the weak topology.

It may be useful to split the set $PT^2(F, x, v)$ into two parts.. We observe that this set is the union of $p(F^2(x, v) \times \{1\})$ and $p(F_0^2(x, v) \times \{0\})$, where

$$F^2(x, v) = \left\{ w \in X : \exists (t_n) \searrow 0, \exists (w_n) \xrightarrow{\sigma} w, x + t_n v + \frac{1}{2} t_n^2 w_n \in F \ \forall n \in \mathbb{N} \right\}$$

is the familiar (weak upper) *second-order tangent set* to F at (x, v) and

$$F_0^2(x, v) = \left\{ w \in X : \exists (t_n) \downarrow 0, \exists (r_n) \downarrow 0, \exists (w_n) \xrightarrow{\sigma} w, (r_n^{-1} t_n) \rightarrow 0, \right. \\ \left. x + t_n v + \frac{1}{2} r_n^{-1} t_n^2 w_n \in F \ \forall n \right\}$$

is what will be called the *asymptotic second-order tangent cone* to F at (x, v) .

Similar decompositions hold for higher-order projective tangent sets. For the sake of simplicity, in what follows we focus our attention on the second-order case only.

Although the second-order tangent set to a smooth subset may be empty, as the example of

$$F := \{(r, s) \in \mathbb{R}^2 : r^2 = s^3\}, \quad x = (0, 0), \quad v := (1, 0)$$

shows, the following result asserts that the projective tangent set of order two in the reflexive case is always nonempty.

PROPOSITION 2.1. *Let $v \in F'(x) := T(F, x)$, where F is an arbitrary subset of the reflexive Banach space X and $x \in \text{cl } F$. Then either $F^2(x, v)$ or $F_0^2(x, v)$ is nonempty.*

Proof. By assumption, there exists a sequence $(t_n) \searrow 0$ such that the sequence (s_n) given by $s_n := t_n^{-1} d(x + t_n v, F)$ converges to 0. Since $0 \in F^2(x, v)$ if $s_n = 0$ for infinitely many n , we may assume $s_n > 0$ and set $r_n = \frac{1}{2} s_n^{-1} t_n$, $w_n = s_n^{-1} t_n^{-1} (z_n - x - t_n v)$, where $z_n \in F$ is such that $\|x + t_n v - z_n\| \leq 2s_n t_n$. Then $(r_n^{-1} t_n) = (2s_n) \rightarrow 0$ and

$$x + t_n v + \frac{1}{2} t_n^2 r_n^{-1} w_n = z_n \in F.$$

Taking a subsequence if necessary, we may suppose $(r_n) \rightarrow r$ for some $r \in [0, \infty]$ and (w_n) has a weak limit w in $2B_X$. If $r = \infty$, setting $w'_n = r_n^{-1} w_n$, we get $(w'_n) \rightarrow 0$ and $0 \in F^2(x, v)$ (strong). If $r \in \mathbb{P}$ the same choice of (w'_n) shows that $r^{-1} w \in F^2(x, v)$ (weak). Finally, if $r = 0$ we have $w \in F_0^2(x, v)$. \square

EXAMPLE 2.1. *Suppose F is the graph of a twice differentiable mapping $g : U \rightarrow V$ in $X = U \times V$, where U and V are n.v.s. Then, for $(u_0, v_0) \in F$, $(u, v) \in F'(u_0, v_0)$, $(w, z) \in X$, $r > 0$, one has*

$$((w, z), r) \in \widehat{T}^2 F((u_0, v_0), (u, v))$$

iff $z = g'(u_0)w + rg''(u_0)uu$, as an easy calculation shows. Since any submanifold of a normed vector space can be represented locally as a graph, this example applies in a variety of situations.

The following proposition shows the concept of projective tangent set is invariant under C^k -diffeomorphisms and thus can be extended to subsets of C^k -manifolds. We take $k = 2$ for simplicity.

PROPOSITION 2.2. *Let $g : X \rightarrow Y$ be a mapping of class C^2 on an open subset X_0 of X , let B be a subset of X , $x \in X_0 \cap \text{cl } B$, and let C be a subset of Y with $g(B) \subset C$. Then for each $v \in X$, $(w, r) \in \widehat{T}^2(B, x, v)$ one has*

$$(g'(x)w + rg''(x)vv, r) \in \widehat{T}^2(C, g(x), g'(x)v).$$

Proof. By definition there exist sequences $(t_n) \searrow 0$, $(w_n, r_n) \rightarrow (w, r)$ such that $\{r_n\} \subset \mathbb{P}$, $(r_n^{-1}t_n) \rightarrow 0$, and $x_n := x + t_nv + 2^{-1}r_n^{-1}t_n^2w_n \in B$ for each n . Then $g(x_n) \in C$, and since $(t_nv_n) \rightarrow 0$ strongly if $v_n := v + 2^{-1}r_n^{-1}t_nw_n$, for some $(y_n) \rightarrow 0$, one has

$$g(x_n) = g(x) + t_n g'(x)v + \frac{1}{2}r_n^{-1}t_n^2(g'(x)w_n + r_n g''(x)vv) + t_n^2 y_n$$

in view of Taylor's expansion. Setting $z_n := g'(x)w_n + r_n g''(x)vv + 2r_n y_n$ and observing that $(z_n) \xrightarrow{\sigma} z := g'(x)w + r g''(x)vv$ and that $(r_n^{-1}t_n) \rightarrow 0$, the result follows. \square

The following property will be useful. For $r = 1$ it corresponds to a property similar to the one observed in [34], [13, Proposition 3.1].

PROPOSITION 2.3. *Let C be a convex subset of X , and let $x \in C$, $v \in T(C, x)$. Then, for any $z \in T(T(C, x), v)$, $(w, r) \in \widehat{T}^2(C, x, v)$, one has*

$$(w + z, r) \in \widehat{T}^2(C, x, v).$$

Proof. By definition, $w \in F_r^2(x, v) := \left\{ w \in X : (w, r) \in \widehat{T}^2(C, x, v) \right\}$, so that there exist sequences $(r_n) \rightarrow r$, $(t_n) \rightarrow 0_+$, $(w_n) \rightarrow w$ such that $(r_n^{-1}t_n) \rightarrow 0$, $r_n > 0$, and

$$x_n = x + t_nv + \frac{1}{2}r_n^{-1}t_n^2w_n \in C$$

for each n . For any $y \in C$ and any $p, q \geq 0$ we have

$$\left(1 - \frac{1}{2}pr_n^{-1}t_n \right) x_n + \frac{1}{2}pr_n^{-1}t_n(x + qt_n(y - x)) \in C;$$

as $(r_n^{-1}t_n) \rightarrow 0$ we obtain $w + p(q(y - x) - v) \in F_r(x, v)$. As this set is closed, and as $\mathbb{R}_+(C - x)$ is dense in $T(C, x)$, we also have $w + p(T(C, x) - v) \subset F_r(x, v)$. Since $T(C, x)$ is convex, $\mathbb{R}_+(T(C, x) - v)$ is dense in $T(T(C, x), v)$ and we get $w + T(T(C, x), v) \subset F_r(x, v)$. \square

Among the variants of Definition 1, the following one seems to be noticeable. We will see in section 5 that this variant is closely related to Definition 2.2 of [26].

DEFINITION 2. *The second-order projective incident set to a subset F of X at (x, v) with $x \in F, v \in T(F, x)$ is the image by the projective projection of the set $\widehat{T}^{ii}(F, x, v)$ of $(w, r) \in X \times \mathbb{R}_+$ such that for any sequences $(t_n) \rightarrow 0_+$, $(r_n) \rightarrow r$ with $r_n > 0$, $(r_n^{-1}t_n) \rightarrow 0$, there exists a sequence $(w_n) \rightarrow w$ such that $x + t_nv + 2^{-1}r_n^{-1}t_n^2w_n \in F$ for each n .*

Given $r \in \mathbb{R}_+$ we denote by $\widehat{T}^{ii}(F, x, v, r)$ the set of $w \in X$ such that $(w, r) \in \widehat{T}^{ii}(F, x, v)$, and we use the similar notation $\widehat{T}^2(F, x, v, r)$ when $\widehat{T}^{ii}(F, x, v)$ is replaced with $\widehat{T}^2(F, x, v)$.

Part of the interest of this notion stems from the following property, the proof of which follows easily from the definition.

PROPOSITION 2.4. (a) *If F is convex, then $\widehat{T}^{ii}(F, x, v, r)$ is convex for each (x, v, r) ;*

(b) *if F is convex, then $(1 - \lambda)\widehat{T}^{ii}(F, x, v, r) + \lambda\widehat{T}^2(F, x, v, r) \subset \widehat{T}^2(F, x, v, r)$ for any $\lambda \in [0, 1]$;*

(c) $\widehat{T}^{ii}(F \times G, (x, y), (u, v), r) = \widehat{T}^{ii}(F, x, u, r) \times \widehat{T}^{ii}(G, y, v, r)$;

(d) $\widehat{T}^{ii}(F, x, u, r) \times \widehat{T}^2(G, y, v, r) \subset \widehat{T}^2(F \times G, (x, y), (u, v), r) \subset \widehat{T}^2(F, x, u, r) \times \widehat{T}^2(G, y, v, r)$.

Sequential concepts as in [21], [22], [37], and [38] can be devised for similar aims.

3. Optimality conditions. The following necessary optimality condition justifies the introduction of the projective tangent set of order two. The proof we present has been devised independently of the one in [12, Theorem 2]; however, results of this kind had been announced earlier by A. Cambini at a lecture in Marseille (see [11] for a partial account and section 5 for a comparison).

PROPOSITION 3.1. *Suppose $f : X \rightarrow \mathbb{R}$ is twice differentiable at $x \in F$ and attains a (local) minimum on $F \subset X$ at x . Then*

$$f'(x)v \geq 0 \text{ for each } v \in F'(x) = T(F, x),$$

and whenever $v \in F'(x) \cap \ker f'(x)$, one has

$$f'(x)w + rf''(x)vv \geq 0 \text{ for each } (w, r) \in \widehat{T}^2(F, x, v).$$

Clearly this last condition can be formulated on the second projective tangent space $PT^2(F, x, v)$.

Proof. Since for $g(\cdot) := f(\cdot) - f(x)$ we have $g(F) \subset \mathbb{R}_+$, the result follows from Proposition 2.2 and from the fact that for $y = 0, v = 0$ one has

$$\widehat{T}^2(\mathbb{R}_+, y, v) = \{(z, r) \in \mathbb{R} \times \mathbb{R}_+ : z \geq 0\}. \quad \square$$

EXAMPLE 3.1. *Let $F = \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R}$ in $X = \mathbb{R}^2$. Then $F'(0) = F$, and, as for $x = 0, v \in F$, the set $\widehat{T}^2(F, x, v)$ contains $(w, 1)$, with $w = 0$, a necessary optimality condition for f on F at 0 is $f'(0) = 0, f''(0)vv \geq 0$ for each $v \in F$. \square*

It may be useful to split the condition of Proposition 3.1 into two parts, using the decomposition of $PT^2(F, x, v)$ we described above.

COROLLARY 3.2. *If $f : X \rightarrow \mathbb{R}$ is twice differentiable at $x \in F$ and attains a local minimum on F at x , then $f'(x)v \geq 0$ for each $v \in F'(x)$, and when $v \in F'(x) \cap \ker f'(x)$, one has*

$$\begin{aligned} f'(x)w + f''(x)vv &\geq 0 \text{ for each } w \in F^2(x, v), \\ f'(x)w &\geq 0 \text{ for each } w \in F_0^2(x, v). \end{aligned}$$

The first condition is well known but the second one is new.

EXAMPLE 3.2. *Let $F = \{(r, s) \in \mathbb{R}^2 : s = |r|^\alpha\}$, where $\alpha \in]1, 2[$. Then for $x = (0, 0), v = (1, 0)$, the set $F^2(x, v)$ is empty but $F_0^2(x, v)$ contains $w = (0, 1)$. Thus a necessary condition for $(0, 0)$ to be a minimizer of f on F is $f'(0, 0) = 0$. Such a condition can also be obtained from [37, Theorem 1.2] via a computation similar to the one in [37, Example 1.4].*

EXAMPLE 3.3. *Given a subset F of the space $X, x \in F, v \in X \setminus \{0\}$, given $0 < p < q$, let us denote by $T^{q/p}(F, x, v)$ the set of vectors w , such that for some sequences $(s_n) \rightarrow 0_+, (w_n) \rightarrow w$ one has $x + s_n^p v + s_n^q w_n \in F$ for each n . Then if $q > 2p$, one has $0 \in F^2(x, v)$ whenever $T^{q/p}(F, x, v)$ is nonempty, while for $q = 2p$ one has $F^2(x, v) = T^{q/p}(F, x, v)$; for $q < 2p$ and for $w \in T^{q/p}(F, x, v)$ one has $(w, 0) \in \widehat{T}^2(F, x, v)$, as one can see by taking $(t_n) := (s_n^p), (r_n) := (s_n^{2p-q})$. In the last case, a necessary condition for f to attain a local minimum on F at x is $f'(x)w \geq 0$ whenever $f'(x)v = 0$ and $w \in T^{q/p}(F, x, v)$. The relationships with the higher-order optimality conditions of [14] and [15] will be considered elsewhere. \square*

The preceding examples prompt us to clarify the relationships between Corollary 3.2 (which is equivalent to Proposition 3.1) and [37, Theorem 1.2].

PROPOSITION 3.3. *The necessary optimality condition of [37]:*

$$\frac{1}{2}f''(x)vv + \liminf_{(t,u) \rightarrow (0,v), t>0, x+tu \in F} f'(x)t^{-1}(u-v) \geq 0 \quad \forall v \in F'(x) \cap \ker f'(x)$$

implies the necessary condition of Corollary 3.2.

Proof. Let $v \in F'(x) \cap \ker f'(x)$. The condition $f''(x)vv + f'(x)w \geq 0$ for each $w \in F^2(x, v)$ is a consequence of [37, Theorem 1.2] by [37, Corollary 1.3]. Let us derive the condition $f'(x)w \geq 0$ for $w \in F_0^2(x, v)$. Suppose on the contrary that $f'(x)w < 0$ for some $w \in F_0^2(x, u)$. Then there exist sequences $(t_n) \downarrow 0$, $(r_n) \downarrow 0$, $(w_n) \xrightarrow{\sigma} w$ such that $(r_n^{-1}t_n) \downarrow 0$, $x_n := x + t_nv + (2r_n)^{-1}t_n^2w_n \in F$ for each n . Setting $v_n := v + (2r_n)^{-1}t_nw_n$ we see that $(v_n) \rightarrow v$, $x + t_nv_n = x_n \in F$ and $f'(x)t_n^{-1}(v_n - v) = (2r_n)^{-1}f'(x)(w_n) \rightarrow -\infty$, which is a contradiction with our assumption. \square

Although the necessary condition of Proposition 3.1 is not as strong as [37, Theorem 1.2], one can still associate to it a sufficient condition of the same type (see also [11] and [12, Theorem 2] for a closely related result).

PROPOSITION 3.4. *If X is finite dimensional, if f is twice differentiable at $x \in F$, and if the following conditions hold, then x is a local strict minimizer of f on F :*

- (a) $f'(x)v \geq 0$ for each $v \in F'(x)$;
- (b) if $v \in F'(x) \cap \ker f'(x)$, $v \neq 0$, then $f'(x)w + rf''(x)vv > 0$ for each $(w, r) \in \widehat{T}^2(F, x, v) \setminus \{(0, 0)\}$.

Proof. Suppose on the contrary there exists a sequence (x_n) of $F \setminus \{x\}$ with limit x such that $f(x_n) \leq f(x)$ for each $n \in \mathbb{N}$. Let $t_n := \|x_n - x\|$, $v_n := t_n^{-1}(x_n - x)$. Taking a subsequence if necessary, we may suppose (v_n) has a limit v with norm 1. Let $s_n := \|v_n - v\|$. When $s_n = 0$ for infinitely many n we get $0 \in F^2(x, v)$ and $f'(x)v = 0$ (by (a) and the inequality $t_n^{-1}(f(x + t_nv_n) - f(x)) \leq 0$), and $f''(x)vv \leq 0$, a contradiction, as we can take $(w, r) = (0, 1)$ in (b). Thus we may suppose $s_n > 0$ for each n and assume that the sequence (r_n) given by $r_n := (2s_n)^{-1}t_n$ has a limit r in $\mathbb{R}_+ \cup \{\infty\}$, and the sequence $(w_n) := (s_n^{-1}(v_n - v))$ has a limit w with norm 1. Then $(r_n^{-1}t_n) = (2s_n) \rightarrow 0$ and

$$x + t_nv + \frac{1}{2}r_n^{-1}t_n^2w_n = x_n \in F \quad \forall n.$$

Thus, when r is finite, we have $(w, r) \in \widehat{T}^2(F, x, v)$, and, since $f(A) \subset f(x) - \mathbb{R}_+$ for $A = \{x_n : n \in \mathbb{N}\}$, we obtain $f'(x)v \leq 0$, hence $f'(x)v = 0$, and

$$f'(x)w + rf''(x)vv \leq 0$$

by a computation similar to the one in Proposition 2.2. This is a contradiction, as $w \neq 0$.

When $r = \infty$, setting $r' = 1$, $r'_n = 1$, $w'_n = r_n^{-1}w_n$, we observe that $(w'_n) \rightarrow 0$ and $x + t_nv + \frac{1}{2}t_n^2w'_n = x_n \in F$ for each n , so that $0 \in F^2(x, v)$ or $(w', r') \in \widehat{T}^2(F, x, v)$, and we get, as above, $f''(x)vv \leq 0$, which is a contradiction, since $(0, r') \in \widehat{T}^2(F, x, v)$, $v \neq 0$. \square

The preceding sufficient condition is in fact a consequence of the sufficient condition of [37, Theorem 1.7], as the following result shows.

PROPOSITION 3.5. *Suppose X is finite dimensional. If f is twice differentiable at $x \in F$ and $v \in F'(x) \cap \ker f'(x)$, $v \neq 0$, the condition*

$$(1) \quad f'(x)w + rf''(x)vv > 0 \text{ for each } (w, r) \in \widehat{T}^2(F, x, v) \setminus \{(0, 0)\}$$

implies the condition

$$(2) \quad \frac{1}{2}f''(x)vv + \liminf_{(t,u) \rightarrow (0,v), t>0, x+tu \in F} f'(x)t^{-1}(u - v) > 0 \quad \forall v \in F'(0) \cap \ker f'(x).$$

Proof. Suppose on the contrary that the first condition holds and there exist sequences $(t_n) \rightarrow 0_+$, $(v_n) \rightarrow v$ such that $x + t_n v_n \in F$ for each n and

$$\frac{1}{2}f''(x)vv + f'(x)t_n^{-1}(v_n - v) \rightarrow c \leq 0.$$

Let $s_n = \|v_n - v\|$. If $s_n = 0$ for infinitely many n , we have $w := 0 \in F^2(x, v)$ and $\frac{1}{2}f''(x)vv = c \leq 0$, so that for $r = \frac{1}{2}$, $w = 0$, we get a contradiction with (1).

Thus, we may assume $s_n > 0$ for each n and that $(r_n) := (2^{-1}s_n^{-1}t_n)$ has a limit r in $\mathbb{R}_+ \cup \{\infty\}$ and $(w_n) := (s_n^{-1}(v_n - v))$ has a limit $w \neq 0$. If $r = \infty$, setting $w'_n = 2s_n t_n^{-1}w_n$, we see that $(w'_n) \rightarrow 0$, $x_n := x + t_n v + \frac{1}{2}t_n^2 w'_n = x + t_n v_n \in F$ for each n , hence $w' := 0 \in F^2(x, v)$, and, as $t_n^{-1}(v_n - v) = \frac{1}{2}w'_n$, we get $f'(x)0 + \frac{1}{2}f''(x)vv = c \leq 0$, a contradiction with $(0, 1) \in \widehat{T}^2(F, x, v)$. If $r < \infty$ we have

$$x_n := x + t_n v_n = x + t_n v + \frac{1}{2}r_n^{-1}t_n^2 w_n \in F$$

and $(r_n^{-1}t_n) = (2s_n) \rightarrow 0$, so that $(w, r) \in \widehat{T}^2(F, x, v)$. Since

$$f'(x)w = \lim s_n^{-1}t_n f'(x)t_n^{-1}(v_n - v) = 2r \left(c - \frac{1}{2}f''(x)vv \right)$$

we get

$$f'(x)w + r f''(x)vv = 2rc \leq 0,$$

a contradiction. \square

However, the implication shown in the preceding condition can be partly reversed.

PROPOSITION 3.6. *Suppose that f is twice differentiable at $x \in F$. Then, for each $v \in (F'(x) \setminus \{0\}) \cap \ker f'(x)$, the condition*

$$(3) \quad \liminf_{(t,u) \rightarrow (0,v), t>0, x+tu \in F} f'(x)t^{-1}(u - v) + \frac{1}{2}f''(x)vv > 0$$

implies the condition

$$(4) \quad f'(x)w + r f''(x)vv > 0 \text{ for each } (w, r) \in \widehat{T}^2(F, x, v) \text{ with } r \neq 0.$$

Proof. Let $(w, r) \in \widehat{T}^2(F, x, v)$ with $r > 0$: there exist positive sequences $(t_n) \rightarrow 0$, $(r_n) \rightarrow r$, and a sequence $(w_n) \xrightarrow{\sigma} w$ such that $(r_n^{-1}t_n) \rightarrow 0$ and

$$x_n := x + t_n v + \frac{1}{2}r_n^{-1}t_n^2 w_n \in F$$

for each n . Let $v_n := t_n^{-1}(x_n - x) = v + \frac{1}{2}r_n^{-1}t_n w_n$, so that $(v_n) \rightarrow v$, $x + t_n v_n = x_n \in F$. By assumption, there exists some $c > 0$ such that, for n large enough, one has

$$\frac{1}{2}f''(x)vv + f'(x)t_n^{-1}(v_n - v) \geq c,$$

hence

$$\frac{1}{2}r_n f''(x)vv + \frac{1}{2}f'(x)w_n > cr_n > \frac{1}{2}cr > 0$$

as $r > 0$, and the result follows by taking limits. \square

4. Application to mathematical programming. Let us consider in this section the mathematical programming problem

$$(\mathcal{M}) \quad \text{minimize } f(x) : x \in F := g^{-1}(C),$$

where $f : X \rightarrow \mathbb{R}$, $g : X \rightarrow Z$ are twice differentiable mappings, C is a closed convex subset of Z , and X and Z are Banach spaces. Such a formulation encompasses problems in which equality and inequality constraints are present.

We will need a series of preliminary results of some independent interest. The first one gives a characterization of the projective tangent set of order two to the feasible set F . It uses a condition of metric regularity introduced in [36]. Here, for $z \in Z$ we set $d(z, C) = \inf_{c \in C} \|z - c\|$ to denote the distance function to C , and we adopt a similar notation for subsets of X .

PROPOSITION 4.1. *Suppose the following directional metric regularity condition is satisfied for $x \in X$, $v \in X$:*

(DMR) *there exists $\mu > 0$, $\rho > 0$ such that for $t \in (0, \rho)$, $u \in B(v, \rho)$ one has*

$$d(x + tu, g^{-1}(C)) \leq \mu d(g(x + tu), C).$$

Then, for $F = g^{-1}(C)$, one has

$$(w, r) \in \widehat{T}^2(F, x, v) \Leftrightarrow (g'(x)w + rg''(x)vv, r) \in \widehat{T}^2(C, g(x), g'(x)v).$$

Proof. In view of Proposition 2.2 it suffices to prove that $(w, r) \in \widehat{T}^2(F, x, v)$ whenever $(g'(x)w + rg''(x)vv, r) \in \widehat{T}^2(C, g(x), g'(x)v)$. Let $(r_n) \rightarrow r$, $(t_n) \rightarrow 0_+$, $(z_n) \rightarrow z := g'(x)w + rg''(x)vv$ be such that $(r_n^{-1}t_n) \rightarrow 0$, $r_n > 0$ and

$$g(x) + t_n g'(x)v + \frac{1}{2} r_n^{-1} t_n^2 z_n \in C$$

for each n . For n large enough we have $t_n \in (0, \rho)$, $u_n := v + 2^{-1} r_n^{-1} t_n w \in B(v, \rho)$, so that

$$\begin{aligned} d(x + t_n u_n, F) &\leq \mu d(g(x + t_n u_n), C) \\ &\leq \mu \|g(x + t_n u_n) - g(x) - t_n g'(x)v - 2^{-1} r_n^{-1} t_n^2 z_n\| \\ &\leq \frac{1}{2} r_n^{-1} t_n^2 \mu \|g'(x)w + r_n g''(x)vv - z_n\| + o(t_n^2). \end{aligned}$$

Since $(z_n) \rightarrow z$ we can find $x_n \in F$ such that $(r_n t_n^{-2} \|x + t_n u_n - x_n\|) \rightarrow 0$. Defining w_n by $x_n := x + t_n v + 2^{-1} r_n^{-1} t_n^2 w_n$ we get $(w_n) \rightarrow w$, so that $w \in \widehat{T}^2(F, x, v)$. \square

Let us observe that condition (DMR) is a consequence of the following metric regularity condition:

(MR) *there exist $\mu > 0$, $\delta > 0$ such that for each $x' \in B(x, \delta)$ one has*

$$d(x', g^{-1}(C)) \leq \mu d(g(x'), C).$$

This condition is of more common use than the directional metric regularity condition (DMR). In turn, condition (MR) has been shown to be a consequence of the classical Mangasarian–Fromovitz qualification [28], [18] and of its extension to the infinite dimensional case in [40], [32], [6], [16], and [42], which can be written

$$(\mathbb{R}^r) \quad g'(x)(X) - \mathbb{R}_+(C - g(x)) = Z.$$

When the interior $\text{int } C$ of C is nonempty, it has been shown in [32] that the radial tangent cone $T^r(C, x) := \mathbb{R}_+(C - g(x))$ in the preceding condition can be replaced by the usual tangent cone $T(C, g(x)) = \text{cl}(T^r(C, g(x)))$:

$$(R) \quad g'(x)(X) - T(C, g(x)) = Z.$$

However, in general, condition (R) is weaker than condition (R^r) and does not imply (MR). We will use a second-order qualification condition which generalizes the Ben-Tal qualification condition [2]:

$$(TR) \quad g'(x)(X) - T(T(C, g(x)), g'(x)v) = Z,$$

in which v is a given vector of X ; it is still weaker than (R).

We will also need the following duality result.

LEMMA 4.2. *Let P and Q be closed convex cones of the Banach spaces X and Z , resp., and let $A : X \rightarrow Z$, $c : X \rightarrow \mathbb{R}$ be linear and continuous and such that for some $m \in \mathbb{R}$, $b \in Z$*

$$c(x) \geq m \text{ for each } x \in P \cap A^{-1}(b + Q).$$

Then, if $A(P) - Q = Z$, there exists $y \in Q^0$ such that for each $x \in P$

$$c(x) + \langle y, Ax - b \rangle \geq m.$$

Since P is a cone, the conclusion can be written $0 \in c + y \circ A + P^0$ and $\langle y, -b \rangle \geq m$. When $P = X$, we have $c + y \circ A = 0$. Taking $m = 0$, $b = 0$ we get a Farkas lemma:

$$-c \in (A^{-1}(Q) \cap P)^0 \Rightarrow \exists y \in Q^0 : -(c + y \circ A) \in P^0.$$

In what follows we say that $v \in X$ is a *critical vector* at x if $f'(x)v = 0$, $g'(x)v \in T(C, g(x))$, and we write $v \in K(x)$.

THEOREM 4.3. *Let x be a (local) solution to problem (P). Suppose conditions (DMR) and (TR) are satisfied at x . Then, for each critical vector $v \in K(x)$, $v \neq 0$ and each $(z, r) \in \widehat{T}^2(C, g(x), g'(x)v)$ there exists some $y \in N(T(C, g(x)), g'(x)v)$ such that*

$$\begin{aligned} f'(x) + y \circ g'(x) &= 0, \\ r(f''(x)vv + \langle y, g''(x)vv \rangle) &\geq \langle y, z \rangle. \end{aligned}$$

Proof. Given $v \in K(x) \setminus \{0\}$, $(z, r) \in \widehat{T}^2(C, g(x), g'(x)v)$, for each $w \in X$ such that

$$g'(x)w + rg''(x)vv - z \in T(T(C, g(x)), g'(x)v),$$

Proposition 2.3 ensures that

$$(g'(x)w + rg''(x)vv, r) \in \widehat{T}^2(C, g(x), g'(x)v).$$

It follows from Proposition 4.1 that

$$(w, r) \in \widehat{T}^2(F, x, v).$$

Then, by Proposition 3.1, we have

$$f'(x)w \geq -rf''(x)vv.$$

Taking in Lemma 4.2, $A = g'(x)$, $b = z - rg''(x)vv$, $c = f'(x)$, $P = X$, $Q = T(T(C, g(x)), g'(x)v)$, $m = -rf''(x)vv$, and observing that $A(P) - Q = Z$ by condition (TR), we get some $y \in Q^0 = N(T(C, g(x)), g'(x)v)$ such that

$$\begin{aligned} f'(x) + y \circ g'(x) &= 0, \\ \langle y, -z + rg''(x)vv \rangle &\geq -rf''(x)vv. \end{aligned}$$

Thus the result is established. \square

Let us present a variant of the preceding necessary condition.

THEOREM 4.4. *Let x be a (local) solution to problem (P). Suppose conditions (DMR) and (TR) are satisfied at x . Then for each non-null critical vector $v \in K(x)$ and each nonempty closed convex subcone \widehat{Q} of $\widehat{T}^2(C, g(x), g'(x)v)$ not contained in $Z \times \{0\}$, there exists some $y \in N(T(C, g(x)), g'(x)v)$ such that*

$$\begin{aligned} f'(x) + y \circ g'(x) &= 0, \\ \inf_{(z,r) \in \widehat{Q}} [r(f''(x)vv + \langle y, g''(x)vv \rangle) - \langle y, z \rangle] &\geq 0. \end{aligned}$$

Proof. Given $v \in K(x) \setminus \{0\}$, and a cone \widehat{Q} as above, in view of Proposition 2.3, for each $(w, r) \in X \times \mathbb{R}_+$ such that

$$(g'(x)(w) + rg''(x)vv, r) \in \text{cl}(\widehat{Q} + T \times \{0\})$$

with $T := T(T(C, g(x)), g'(x)v)$, we have

$$(g'(x)(w) + rg''(x)vv, r) \in \widehat{T}^2(C, g(x), g'(x)v)$$

since $\widehat{T}^2(C, g(x), g'(x)v)$ is closed. It follows from Proposition 4.1 that

$$(w, r) \in \widehat{T}^2(F, x, v).$$

And then, by Proposition 3.1,

$$f'(x)w + rf''(x)vv \geq 0.$$

Setting $P = X \times \mathbb{R}_+$, $Q = \text{cl}(\widehat{Q} + T \times \{0\})$, and defining A by

$$A(w, r) := (g'(x)w + rg''(x)vv, r)$$

so that $A(P) - Q = Z \times \mathbb{R}$, as is easily seen, it follows from the Farkas lemma recalled above that there exists $(y, -s) \in Q^0 = (\widehat{Q} + T \times \{0\})^0$ such that

$$f'(x)w + rf''(x)vv - rs + \langle y, g'(x)w + rg''(x)vv \rangle \geq 0$$

for each $(w, r) \in X \times \mathbb{R}_+$. It follows that $y \in T^0 := N(T(C, g(x)), g'(x)v)$ and that

$$\begin{aligned} f'(x) + y \circ g'(x) &= 0, \\ r(f''(x)vv + \langle y, g''(x)vv \rangle) &\geq rs. \end{aligned}$$

Since $rs \geq \langle y, z \rangle$ for each $(z, r) \in Q$, the result follows. \square

Since the preceding optimality condition has been derived from Proposition 3.1, and since that criterion is a consequence of the results of [37], one may guess that it is a consequence of the necessary condition of [37] for mathematical programming problems. This is the case. Given $v \in K(x)$ and a nonempty closed convex subcone \widehat{Q}

of $\widehat{T}^2(C, g(x), g'(x)v)$, let us consider two cases. When \widehat{Q} is contained in $Z \times \{0\}$ the condition $-\langle y, z \rangle \geq 0$ for each $z \in \widehat{Q}$ is satisfied by any y in the set $M(x)$ of multipliers, as is easily seen. When $\widehat{Q} \cap Z \times \mathbb{P}$ is nonempty, taking $T = X \times (\widehat{Q} \cap Z \times \{1\})$ in [37, Corollary 3.6] we get some $y \in M(x)$ such that

$$f''(x)vv + \langle y, g''(x)vv \rangle \geq \langle y, z \rangle$$

for each z such that $(z, 1) \in \widehat{Q}$. Taking into account the remarks above and a homogeneity argument, the conclusion follows.

Now, let us turn to sufficient conditions.

THEOREM 4.5. *The following conditions ensure that an element x of F is a strict local minimizer:*

- (a) X is finite dimensional;
- (b) the set $M(x) = \{y \in N(C, g(x)) : f'(x) + y \circ g'(x) = 0\}$ of multipliers at x is nonempty;
- (c) for each $v \in F'(x) \setminus \{0\}$ with $f'(x)v = 0$ and each $(w, r) \in X \times \mathbb{R}_+ \setminus \{(0, 0)\}$ such that $(z, r) := (g'(x)w + rg''(x)vv, r) \in \widehat{T}^2(C, g(x), g'(x)v)$ there exists $y \in M(x)$ such that

$$r(f''(x)vv + \langle y, g''(x)vv \rangle) > \langle y, z \rangle.$$

Proof. The existence of a multiplier y ensures condition (a) of Proposition 3.4 since for any $v \in F'(x)$ we have $g'(x)v \in T(C, g(x))$ and $y \in N(C, g(x))$, hence $\langle y, g'(x)v \rangle \leq 0$ and $f'(x)v \geq 0$.

In order to check condition (b) of Proposition 3.4, let us consider $v \in F'(x) \cap \ker f'(x)$ with $v \neq 0$ and $(w, r) \in \widehat{T}^2(F, x, v)$ with $(w, r) \neq (0, 0)$. Then Proposition 2.2 ensures that $(z, r) \in \widehat{T}^2(C, g(x), g'(x)v)$ for $z = g'(x)w + rg''(x)vv$. Then, taking $y \in M(x)$ as in assumption (c) we get

$$f'(x)w + rf''(x)vv > -\langle y, g'(x)w \rangle + \langle y, z \rangle - r\langle y, g''(x)vv \rangle = 0,$$

and condition (b) is satisfied. \square

5. Comparisons with other works. As mentioned above, the definition we gave for the second-order projective incident cone $\widehat{T}^{ii}(F, x, v)$ to a subset F of X at (x, v) seems to be closely related to Definition 2.2 of [26]: $(w, r) \in TC^{(2)}(F, x, v)$ iff there exist $\varepsilon > 0$ and $\alpha : [0, \varepsilon] \rightarrow X$ such that $\alpha(s) \rightarrow 0$ as $s \rightarrow 0$,

$$x + s\sqrt{r}v + s^2w + s^2\alpha(s) \in F \quad \forall s \in [0, \varepsilon].$$

In fact, supposing X is finite dimensional, so that the weak topology coincides with the strong topology, setting $t = s\sqrt{r}$ we see that for $r > 0$ $(w, r) \in TC^{(2)}(F, x, v)$ iff $(w, r) \in \widehat{T}^{ii}(F, x, v)$ iff $r^{-1}w \in T^{ii}(F, x, v) := \liminf_{t \rightarrow 0_+} 2t^{-2}(F - x - tv)$. However, $(w, 0) \in TC^{(2)}(F, x, v)$ iff $w \in T^i(F, x) := \liminf_{t \rightarrow 0_+} t^{-1}(F - x)$, the first-order incident tangent cone, and there is no relationship with the case $(w, 0) \in \widehat{T}^{ii}(F, x, v)$. Another definition is given in [26], in the style of the Dubovitskii–Milyutin work: $(w, r) \in FC^{(2)}(F, x, v)$ iff there exists $\varepsilon > 0$ such that

$$x + s\sqrt{r}v + s^2B(w, \varepsilon) \subset F \quad \forall s \in [0, \varepsilon].$$

Setting $G := X \setminus F$, we see that for $r > 0$ we have $(w, r) \in FC^{(2)}(G, x, v)$ iff $(w, r) \notin \widehat{T}^2(F, x, v)$. However, for $r = 0$ we have $(w, r) \in FC^{(2)}(G, x, v)$ iff $w \notin T(F, x)$ and there is no connection with $\widehat{T}^2(F, x, v)$.

As mentioned in the introduction, the definition of the projective tangent set we introduced above has been inspired by a notion given in a work of Cambini, Martein, and Komlosi [11] (or, rather, a talk around that paper). With a slight change of notation, their definition is as follows:

$$w \in TC''(F, x, v) \Leftrightarrow (\exists k \in \mathbb{R}_+ \cup \{\infty\} \exists (x_n) \in F^{\mathbb{N}} (x_n) \rightarrow x, \\ \exists (\alpha_n), (\beta_n) \rightarrow \infty : (\alpha_n \beta_n^{-1}) \rightarrow k, (\beta_n [\alpha_n (x_n - x) - v]) \rightarrow w).$$

Clearly, this set is a cone, as is $\widehat{T}^2(F, x, v)$. Given $(w, r) \in \widehat{T}^2(F, x, v)$ and setting $\alpha_n = t_n^{-1}$, $\beta_n = 2r_n t_n^{-1}$ one sees that $w \in TC''(F, x, v)$ so that, denoting by p_X the canonical projection of $X \times \mathbb{R}$ onto X , one has

$$p_X(\widehat{T}^2(F, x, v)) \subset TC''(F, x, v).$$

This inclusion is strict in general as a vector w such that for some sequences $(x_n) \in F^{\mathbb{N}}$, $(x_n) \rightarrow x$, $(\alpha_n), (\beta_n) \rightarrow \infty : (\alpha_n \beta_n^{-1}) \rightarrow 0$, $(\beta_n [\alpha_n (x_n - x) - v]) \rightarrow w$ does not belong to the left-hand side of the preceding relation. The necessary condition of [11] is thus potentially richer than the one of our Proposition 3.1. However, for a vector w as just described, the necessary condition of [11] reads as

$$f'(x)(2kw) + f''(x)(v, v) \geq 0$$

with $k = 0$ or $f''(x)(v, v) \geq 0$. Then, since $x_n = x + \alpha_n^{-1}v + \alpha_n^{-2}(\alpha_n \beta_n^{-1})w_n$, we have $(\alpha_n \beta_n^{-1}w_n) \rightarrow 0$, so that $0 \in T^2(F, x, v)$ (see [12, Observation 7], in this connection) and the conclusion $f''(x)(v, v) \geq 0$ is contained in Proposition 3.1. For a similar reason, the assumptions of their sufficient condition are not more restrictive than the ones of our Proposition 3.4. We refer to [12] for a precise formulation of the optimality conditions of [11] and a number of observations about the second-order tangent sets described above. Among them is the following property [12, Observations 4 and 5]:

$$\widehat{T}^2(F, x, v) + \mathbb{R}T(F, x) \times \{0\} \subset \widehat{T}^2(F, x, v),$$

which is related to the inclusion

$$T^2(F, x, v) + \mathbb{R}T(F, x) \subset T^2(F, x, v)$$

contained in [13, Proposition 3.1].

Moreover, pursuing the line of thought of several papers [8], [9], [10], Cambini, Martein, and Komlosi introduce in [11] a new notion of second-order tangent set and use it for optimality conditions. When applied to mathematical programming problems, another main feature of the approach of [11] is the fact that it takes place in the image of the decision space X by the joint mapping $h := (f, g) : X \rightarrow V := \mathbb{R} \times Z$. In such a setting, X can be an arbitrary topological space, V can be an arbitrary normed vector space, and the following tools address local minimizers rather than global minimizers. Given $x \in X$, let us denote by $T(X, h, x)$ the set of $v \in V$ such that there exist sequences $(x_n) \rightarrow x$, $(t_n) \rightarrow 0_+$, $(v_n) \rightarrow v$ in X , \mathbb{P} , and V , resp., such that $v_n = t_n^{-1}(h(x_n) - h(x_0))$ for each n . Now, given $v \in T(X, h, x)$, let $T^2(X, h, x, v)$ be the set of limits w of sequences $(w_n) = 2t_n^{-2}(h(x_n) - h(x) - t_n v)$, where $(x_n) \rightarrow x$, $(t_n) \rightarrow 0_+$. Clearly,

$$T(X, h, x) \subset T(h(X), h(x)), T^2(X, h, x, v) \subset T^2(h(X), h(x), v),$$

and if X is a normed space,

$$h'(x)(X) \subset T(X, h, x), h'(x)(X) + h''(x)(v, v) \subset T^2(h(X), h(x), v).$$

However, these sets do not seem to be directly related to the projective tangent sets we defined, although they also give rise to optimality conditions in the form

$$T(X, h, x) \cap ((-\mathbb{P}) \times \text{int}C) = \emptyset,$$

$$T^2(X, h, x, v) \cap ((-\mathbb{P}) \times \text{int}C) = \emptyset \quad \forall v \in T(X, h, x) \cap Fr((-\mathbb{P}) \times \text{int}C).$$

On the other hand, conditions in terms of multipliers can be deduced from such relations and from the use of the set A_2 of $v \in T(X, h, x) \setminus \{0\}$ such that there exist $t > 0$ and sequences $(x_n) \rightarrow x$, $(t_n) \rightarrow 0_+$, $(v_n) \rightarrow v$ in X , \mathbb{P} , and V , resp., such that $v_n = t_n^{-1}(h(x_n) - h(x_0))$, $t_n = t\|h(x_n) - h(x)\|$, $\|x_n - x\|^{-2}(h(x_n) - h(x)) \rightarrow 0$. Such a set seems to be more closely related to our projective tangent sets.

Now let us turn to a recent contribution of Bonnans, Cominetti, and Shapiro [5] using a notion of approximation to devise a sufficient optimality condition which we intend to compare with the one in [37]. We recall them briefly. The condition in [37] relies on the notion of *compound tangent set* to $E := (-\mathbb{R}_+) \times C$ (we suppose $f(x) = 0$ for simplicity). Given $u \in X$ one denotes by

$$S_u := \limsup_{(t, u') \rightarrow (0_+, u)} 2t^{-2}(E - h(x) - th'(x)u')$$

the set formed with limits of sequences (w_n) such that there exist sequences $(t_n) \rightarrow 0_+$, $(u_n) \rightarrow u$ in \mathbb{P} and X , resp., with $h(x) + t_n h'(x)u_n + \frac{1}{2}t_n^2 w_n \in E$ for each n . Then one can give a sufficient condition in order that x be an *essential local minimizer of second order* for problem (\mathcal{M}) in the following sense, which differs slightly from the one in [1], [5], [35], [39], and [38]: there exists $\alpha > 0, \beta > 0, \gamma > 0$ such that

$$f(u) \geq f(x) + \alpha\|u - x\|^2 \text{ for any } u \in B(x, \beta) \text{ such that } d(g(u), C) \leq \gamma\|u - x\|^2.$$

We make use of the set $J(x)$ of F. John's multipliers at x for problem (\mathcal{M}) , i.e., the set of $(t, y) \in \mathbb{R}_+ \times N(C, g(x))$ such that

$$tf'(x) + y \circ g'(x) = 0$$

and of the set of *subcritical directions*

$$K^{\leq}(x) := \{u \in X : f'(x)u \leq 0, g'(x)u \in T(C, g(x))\}.$$

This set obviously coincides with the set of critical directions $K(x)$ whenever the set of multipliers $M(x) = \{y : (1, y) \in J(x)\}$ is nonempty.

PROPOSITION 5.1. *The following conditions ensure that an element x of F is an essential local minimizer of second order:*

- (a) X is finite dimensional;
- (b) the set $J(x)$ of John's multipliers at x is nonempty;
- (c) for each $u \in K^{\leq}(x) \setminus \{0\}$ and each $(r, z) \in S_u$ there exists a multiplier $(t, y) \in J(x)$ such that

$$(5) \quad tf''(x)uu + \langle y, g''(x)uu \rangle > rt + \langle y, z \rangle.$$

Proof. Suppose on the contrary that there exist a sequence (x_n) of X and a sequence $(\varepsilon_n) \rightarrow 0_+$ such that $0 < t_n := \|x_n - x\| < \varepsilon_n, d(g(x_n), C) \leq \varepsilon_n t_n^2, f(x_n) < f(x) + t_n^2 \varepsilon_n$ for each n . Without loss of generality we may assume that $(t_n^{-1}(x_n - x))$ converges to some u in X . It is easy to see that $u \in K^{\leq}(x) \setminus \{0\}$ and that $(r, z) := (f''(x)uu, g''(x)uu) \in S_u$. Thus we get a contradiction with (c). \square

Now, in order to present the result of [5] let us introduce the following concepts in which the Pompeiu–Hausdorff *excess* of a subset C over another subset D of a metric space is given by

$$e(C, D) := \sup_{c \in C} d(c, D).$$

DEFINITION 3. A subset A of a metric space is an outer (or upper) hemi-limit of a family $(A_w)_{w \in W}$ of subsets of X parametrized by a subset W of a topological space P as $w \rightarrow w_0 \in \text{cl}W$, $w \in W$ if $e(A_w, A) \rightarrow 0$ as $w \rightarrow w_0$ in W .

Such a limit is not unique: if A' contains A , then A' is again an outer hemi-limit of (A_w) . Moreover, any closed outer hemi-limit of (A_w) contains $\limsup_{w \rightarrow w_0} A_w$, as is easily seen. The concept introduced in [5] can be reformulated as follows (in the case $d = Mu$, which is of interest to us).

DEFINITION 4. Given n.v.s. X and Z , a subset C of Z , a continuous linear mapping $M : X \rightarrow Z$, $u \in X$, $z \in C$, a subset A of Z is said to be an upper (second-order) approximation to C at z with respect to M, z, u if it is an outer hemi-limit of the family $A_{t,u'} := 2t^{-2}(C - z - tMu')$ as $(t, u') \rightarrow (0, u)$ in $\mathbb{P} \times X$.

A simpler notion can be introduced.

DEFINITION 5. Given a subset C of an n.v.s. Z , $z \in C$, $v \in T(C, z)$, a subset A of Z is said to be an outer (second-order) approximation to C at z in the direction v if it is an outer hemi-limit of the family $A_t := 2t^{-2}(C - z - tv)$ as $t \rightarrow 0_+$.

This definition is less demanding than the preceding one: if A is an upper approximation to C at z with respect to M, z, u , and if $v := Au$, then A is an outer approximation to C at z in the direction v .

EXAMPLE. For any convex subset C of Z and any $z \in C$, $v \in T(C, z)$ the cone $T(T(C, z), v)$ is an outer approximation to C at z in the direction v . In fact, for any $t > 0$, $c \in C$, setting $w := 2t^{-2}(c - z - tv)$, $v' := v + (t/2)w = t^{-1}(c - z) \in T(C, z)$, one has $w = 2t^{-1}(v' - v) \in T(T(C, z), v)$. \square

The main result of [5] states that if x is feasible for problem (\mathcal{M}) , if for each $u \in K^\leq(x)$ there exists an upper approximation A to C with respect to $M := g'(x)$, $z := g(x)$, u , and if there exists $(t, y) \in J(x)$ such that

$$(6) \quad tf''(x)uu + \langle y, g''(x)uu \rangle > \sigma(y, A) := \sup_{a \in A} \langle y, a \rangle,$$

then x is a strict locally optimal solution of (\mathcal{M}) . In fact this result can be extended to the case when A is just an outer approximation to C at z in the direction $v := g'(x)u$, and, moreover, it is a simple consequence of the preceding proposition in view of the following lemma.

LEMMA 5.2. If condition (6) holds for some outer approximation to C at $g(x)$ in the direction $v := g'(x)u$, then condition (5) holds.

Proof. It suffices to prove that for any $u \in K^\leq(x)$, any $(r, z) \in S_u$, any $(t, y) \in J(x)$, and any outer approximation A to C at $g(x)$ in the direction $v := g'(x)u$, one has

$$\sigma(y, A) \geq \langle y, z \rangle + rt.$$

Now, since $(r, z) \in S_u$ there exist sequences $(t_n) \rightarrow 0_+$, $(u_n) \rightarrow u$, $(z_n) \rightarrow z$, $(r_n) \rightarrow r$ such that

$$c_n := g(x) + t_n g'(x)u_n + \frac{1}{2}t_n^2 z_n \in C,$$

$$f'(x)u_n + \frac{1}{2}t_n r_n \leq 0.$$

Let $w_n := 2t_n^{-1}(u_n - u)$, and let $q_n := g'(x)w_n + z_n$. Since A is an outer approximation to C at $g(x)$ in the direction $v := g'(x)u$, and since $q_n = 2t_n^{-2}(c_n - g(x) - t_n g'(x)u)$, there exists $a_n \in A$ such that $\varepsilon_n := \|q_n - a_n\| \rightarrow 0$. Then, using the definitions of $J(x)$ and $K^{\leq}(x)$, we get

$$\begin{aligned} \langle y, z_n \rangle + tr_n &= \langle y, q_n \rangle - \langle y, g'(x)w_n \rangle + tr_n \\ &= \langle y, q_n \rangle + t f'(x)w_n + tr_n \\ &= \langle y, q_n \rangle + 2t t_n^{-1} \left(f'(x)u_n + \frac{1}{2} t_n r_n \right) \\ &\leq \langle y, a_n \rangle + \varepsilon_n \|y\|. \end{aligned}$$

Therefore, taking limits, we get

$$\langle y, z \rangle + rt \leq \sigma(y, A).$$

COROLLARY 5.3. *Suppose that for an element x of F the conditions (a) and (b) of the preceding proposition hold while condition (c) is replaced with the following condition (c'). Then x is an essential local minimizer of second order:*

(c') *for each $u \in K^{\leq}(x) \setminus \{0\}$ there exist an outer approximation A of C at $g(x)$ in the direction $g'(x)u$ and a multiplier $(t, y) \in J(x)$ such that*

$$(7) \quad t f''(x)uu + \langle y, g''(x)uu \rangle > \sup_{a \in A} \langle y, a \rangle.$$

REFERENCES

- [1] A. AUSLENDER, *Stability in mathematical programming with nondifferentiable data*, SIAM J. Control Optim., 22 (1984), pp. 239–254.
- [2] A. BEN-TAL, *Second order and related extremality conditions in nonlinear programming*, J. Optim. Theory Appl., 31 (1980), pp. 143–165.
- [3] A. BEN TAL AND J. ZOWE, *A unified theory of first and second-order conditions for extremum problems in topological vector spaces*, Math. Programming Study, 19 (1982), pp. 39–76.
- [4] A. BEN TAL AND J. ZOWE, *Necessary and sufficient optimality conditions for a class of non-smooth minimization problems*, Math. Programming, 24 (1982), pp. 70–91.
- [5] J.F. BONNANS, R. COMINETTI, AND A. SHAPIRO, *Second order necessary conditions and sufficient optimality conditions under abstract constraints*, preprint, 1996.
- [6] J. BORWEIN, *Stability and regular points of inequality systems*, J. Optim. Theory Appl., 48 (1986), pp. 9–52.
- [7] J. BURKE, *Second order necessary and sufficient conditions for convex composite NDO*, Math. Programming, 38 (1987), pp. 287–302.
- [8] A. CAMBINI AND L. MARTEIN, *Second order necessary optimality conditions in the image space: Preliminary results*, in Scalar and Vector Optimization in Economic and Financial Problems, E. Castagnoli and G. Giorgi, eds., 1995, pp. 27–38.
- [9] A. CAMBINI, L. MARTEIN, AND R. CAMBINI, *A new approach to second order optimality conditions in vector optimization*, Technical report 103, Department of Statistics and Applied Math., Univ. of Pisa, 1996.
- [10] R. CAMBINI, *Second order optimality conditions in the image space*, Technical report 99, Department of Statistics and Applied Math., Univ. of Pisa, 1996.
- [11] A. CAMBINI, L. MARTEIN, AND S. KOMLOSI, *Recent developments in second order necessary optimality conditions*, Generalized Convexity, Generalized Monotonicity, J.-P. Crouzeix et al., eds., Kluwer, Amsterdam, 1998, pp. 347–356.
- [12] A. CAMBINI, L. MARTEIN, AND M. VLACH, *Second order tangent sets and optimality conditions*, preprint, Japan Advanced Study of Science and Technology, Hokuriku, Japan, 1997.
- [13] R. COMINETTI, *Metric regularity, tangent sets and second-order optimality conditions*, Applied Math. Optim., 21 (1990), pp. 265–287.
- [14] J.-P. DEDIEU, *Third and fourth-order optimality conditions in optimization*, Optimization, 33 (1995), pp. 97–105.

- [15] J.-P. DEDIEU AND R. JANIN, *A propos des conditions d'optimalité d'ordre trois et quatre pour une fonction de plusieurs variables*, preprint, Univ. Poitiers and Toulouse, 1995.
- [16] A. DMITRUK, A. MILYUTIN, AND N. OSMOLOVSKI, *Lyusternik's theorem and the theory of extrema*, Russian Math. Surveys, 35 (1980), pp. 11–51.
- [17] B. GOLLAN, *Higher order necessary conditions for an abstract optimization problem*, Math. Programming Study, 14 (1981), pp. 69–76.
- [18] S.P. HAN AND O.L. MANGASARIAN, *Exact penalty functions in nonlinear programming*, Math. Programming, 17 (1979), pp. 251–269.
- [19] K.H. HOFFMANN AND H.J. KORNSTAEDT, *Higher order necessary conditions in abstract mathematical programming*, J. Optim. Theory Appl., 26 (1978), pp. 533–569.
- [20] A.D. IOFFE, *Necessary and sufficient conditions for a local minimum 3: Second-order conditions and augmented duality*, SIAM J. Control Optim., 17 (1979), pp. 266–288.
- [21] A.D. IOFFE, *On some recent developments in the theory of second-order optimality conditions*, in Optimization, Proc. Conf. Varetz, 1988, S. Dolecki, ed., Lecture Notes in Math. 1405, Springer-Verlag, Berlin, 1989, pp. 55–68.
- [22] A.D. IOFFE, *Variational analysis of a composite function: A formula for the lower second-order epi-derivative*, J. Math. Anal. Appl., 160 (1991), pp. 379–405.
- [23] R. JANIN, *Conditions d'optimalité d'ordre supérieur en programmation mathématique*, Univ. of Poitiers, 1995, preprint.
- [24] H. KAWASAKI, *An envelop-like effect of infinitely many inequality constraints on second-order necessary conditions for minimization problems*, Math. Programming, 41 (1988), pp. 73–96.
- [25] H. KAWASAKI, *Second-order necessary and sufficient optimality conditions for minimizing a sup type function*, Appl. Math. Optim., 26 (1992), pp. 195–220.
- [26] U. LEDZEWICZ AND H. SCHAETTLER, *Second-order conditions for extremum problems with non-regular equality constraints*, J. Optim. Theory Appl., 86 (1995), pp. 113–144.
- [27] E.S. LEVITIN, A.A. MILYUTIN, AND N.P. OSMOLOVSKII, *Higher order conditions for a local minimum in problems with constraints*, Uspehi Math. Nauk, 33 (1978), pp. 85–148.
- [28] O.L. MANGASARIAN AND S. FROMOVITZ, *The Fritz-John necessary optimality condition in the presence of equality and inequality constraints*, J. Math. Anal. Appl., 7 (1967), pp. 37–47.
- [29] Y. MARUYAMA, *Second-order necessary conditions for nonlinear optimization problems in Banach spaces and their application to an optimal control problem*, Math. Oper. Res., 15 (1990), pp. 467–482.
- [30] H. MAURER, *First and second-order sufficient optimality conditions in mathematical programming and optimal control*, Math. Programming Stud., 14 (1981), pp. 163–177.
- [31] Z. PÁLES AND V.M. ZEIDAN, *Nonsmooth optimum problems with constraints*, SIAM J. Control Optim., 32 (1994), pp. 1476–1502.
- [32] J.-P. PENOT, *On regularity conditions in mathematical programming*, Math. Programming Stud., 19 (1982), pp. 167–199.
- [33] J.-P. PENOT, *A view of second-order extremality conditions*, Third Franco-German Conference in Optimization, C. Lemaréchal, ed., INRIA, Le Chesnay, France, 1984, pp. 34–39.
- [34] J.-P. PENOT, *A Geometric Approach to Higher Order Necessary Conditions*, manuscript, Univ. of Pau, 1984.
- [35] J.-P. PENOT, *Generalized higher order derivatives and higher order optimality conditions*, preprint, Univ. of Santiago, 1984.
- [36] J.-P. PENOT, *Differentiability of relations and differential stability of perturbed optimization problems*, SIAM J. Control Optim., 22 (1984), pp. 529–551.
- [37] J.-P. PENOT, *Optimality conditions in mathematical programming and composite optimization*, Math. Programming, 67 (1994), pp. 225–245.
- [38] J.-P. PENOT, *Sequential derivatives and composite optimization*, Rev. Roumaine Math. Pures Appl., 40 (1995), pp. 501–519.
- [39] J.-P. PENOT, *Central and peripheral results in the study of marginal and performance functions*, in Mathematical Programming with Data Perturbations, A. Fiacco, ed., Marcel Dekker, New York, 1997, pp. 305–337.
- [40] S.M. ROBINSON, *Stability theory for systems of inequalities, Part II: Differentiable nonlinear systems*, SIAM J. Numer. Anal., 13 (1976), pp. 497–513.
- [41] M. STUNIARSKI, *Second-order necessary conditions for optimality in nonsmooth nonlinear programming*, J. Math. Anal. Appl., 154 (1991), pp. 303–317.
- [42] J. ZOWE AND S. KURCYSZ, *Regularity and stability for the mathematical programming problem in Banach space*, Appl. Math. Optim., 5 (1979), pp. 49–62.