Second-Order Consensus for Multiagent Systems With Directed Topologies and Nonlinear Dynamics

Wenwu Yu, *Student Member, IEEE*, Guanrong Chen, *Fellow, IEEE*, Ming Cao, *Member, IEEE*, and Jürgen Kurths

Abstract—This paper considers a second-order consensus problem for multiagent systems with nonlinear dynamics and directed topologies where each agent is governed by both position and velocity consensus terms with a time-varying asymptotic velocity. To describe the system's ability for reaching consensus, a new concept about the generalized algebraic connectivity is defined for strongly connected networks and then extended to the strongly connected components of the directed network containing a spanning tree. Some sufficient conditions are derived for reaching second-order consensus in multiagent systems with nonlinear dynamics based on algebraic graph theory, matrix theory, and Lyapunov control approach. Finally, simulation examples are given to verify the theoretical analysis.

Index Terms—Algebraic connectivity, directed spanning tree, multiagent system, second-order consensus, strongly connected network.

I. Introduction

COLLECTIVE behaviors in networks of autonomous mobile agents have received increasing attention in recent years due to the growing interests in understanding animal group behaviors, such as flocking and swarming [27], [28], and also due to their wide applications in the coordination and control of distributed sensor networks, unmanned-air-vehicle formations, satellite clusters, robotic teams, etc. The study of collective behavior focuses on analyzing how globally coordinated group behavior emerges as a result of local interactions among individuals. In many cooperative multiagent systems, a group of agents only share information with their neighbors locally and simultaneously try to agree on certain global criteria of common interest.

W. Yu and G. Chen are with the Department of Electronic Engineering, City University of Hong Kong, Kowloon, Hong Kong (e-mail: wenwuyu@gmail.com; eegchen@cityu.edu.hk).

M. Cao is with the Faculty of Mathematics and Natural Sciences, Industrial Technology and Management, University of Groningen, 9747 AG Groningen, The Netherlands (e-mail: m.cao@rug.nl).

J. Kurths is with Humboldt University of Berlin, 10099 Berlin, Germany, and also with the Research Domain IV—Transdisciplinary Concepts and Methods, Potsdam Institute for Climate Impact Research (PIK), 14473 Potsdam, Germany (e-mail: Juergen.Kurths@pik-potsdam.de).

Recently, much progress has been made in the study of collective behaviors in multiagent dynamical systems, such as consensus [10], [11], [14], [19], [21], [22], [24]–[26], [40], synchronization [1], [4], [15]–[18], [29]–[38], [41]–[43], and swarming and flocking [20], [27], [28]. The consensus problem usually refers to the problem of how to reach an agreement, such as the position and velocity, among a group of autonomous mobile agents in a dynamical system. In [28], Vicsek *et al.* proposed a simple discrete-time model to simulate a group of autonomous agents moving in the plane with the same speed but different headings. Vicsek's model in essence is a simplified version of the model introduced earlier by Reynolds [27]. Based on the algebraic graph theory [8], it has been shown that the network connectivity is a key factor in reaching consensus [5], [14], [19], [25]. It has also been proved that consensus in a network with a dynamically changing topology can be reached if and only if the time-varying network topology contains a spanning tree frequently enough as the network evolves with time [14], [25].

Up to date, most works on the consensus problem consider the case where the agents are governed by first-order dynamics [4], [14], [17]–[19], [25], [29]–[32], [34], [35], [38]. Nevertheless, the second-order consensus problem has come to be recognized as an important topic [10], [11], [20], [22], [24], where each agent is governed by second-order dynamics. In general, the second-order consensus problem refers to the problem of reaching an agreement among a group of autonomous agents governed by second-order dynamics, such as the position and velocity terms. A detailed analysis of the second-order consensus protocols is a critical step to introduce more complicated dynamics into the model of each individual agent within the general framework of multiagent systems, thus helping researchers and engineers implement distributed cooperative control strategies in networked multiagent systems.

It has been found that, in sharp contrast to the first-order consensus problem, second-order consensus may fail to be achieved in many cases even if the network topology contains a directed spanning tree [25]. More surprisingly, consensus may no longer be reachable within a multiagent system by adding one connection between a chosen pair of agents, which has originally been able to reach consensus. This is inconsistent with the intuition that more connections are helpful for reaching consensus. Some sufficient conditions have been derived for reaching second-order consensus in linear models [22], [24], where the final velocity is constant. In [39], some necessary and sufficient conditions have been obtained for second-order consensus in a network containing a directed spanning tree with delay. It has been found that both the real and imaginary parts

of the eigenvalues of the Laplacian matrix of the network play key roles in reaching second-order consensus in general.

It has also been noticed that there is still little work in the literature on second-order consensus algorithms for directed networks of agents with time-varying velocities. In [23], a second-order consensus algorithm in the multiagent system with directed topology and asymptotic oscillatory velocities was considered where only linear model was studied. This scenario arises naturally when agents have intrinsic nonlinear dynamics [4], [17], [18], [21], [29]–[32], [34], [35], [38], [42], [43]. In this paper, to deal with this challenging scenario, a nonlinear term describing the intrinsic dynamics of each agent is incorporated in the second-order consensus algorithm. As a result, agents move with time-varying velocities, even after a velocity consensus has been reached. A new concept about the generalized algebraic connectivity is defined in the strongly connected network components to describe the ability of reaching consensus in a directed network, and some sufficient conditions are derived for reaching second-order consensus among agents with time-varying velocities.

The rest of this paper is organized as follows. In Section II, some preliminaries on graph theory and the model formulation are given. Second-order consensus algorithms for multiagent dynamical systems in strongly connected networks and networks containing a directed spanning tree are discussed in Sections III and IV, respectively. In Section V, numerical examples are simulated to verify the theoretical analysis. Conclusions are finally drawn in Section VI.

II. PRELIMINARIES

In this section, preliminaries about algebraic graph theory and model formulation are briefly introduced.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, G)$ be a weighted directed network of order N, with the set of nodes $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$, the set of directed edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and a weighted adjacency matrix $G = (G_{ij})_{N \times N}$. A directed edge $\mathcal{E}_{ij}$ in the network $\mathcal{G}$ is denoted by the ordered pair of nodes (v_i, v_j), where v_i and v_j are called the terminal and initial nodes, respectively, which means that node v_i can receive information from node v_j. In view of the definition of adjacency matrices for weighted graphs [12], $G_{ij} > 0$ if and only if there is a directed edge (v_i, v_j) in $\mathcal{G}$. In this paper, only positively weighted networks are considered.

Definition 1 [12]: A network $\mathcal{G}$ is called *undirected* if there is a connection between two nodes v_i and v_j in $\mathcal{G}$, then $G_{ij} = G_{ji} > 0$; otherwise, $G_{ij} = G_{ji} = 0$ ($i \neq j; i, j = 1, 2, \ldots, N$). A network $\mathcal{G}$ is *directed* if there is a connection from node v_j to v_i in $\mathcal{G}$, then $G_{ij} > 0$; otherwise, $G_{ij} = 0$ ($i \neq j; i, j = 1, 2, \ldots, N$).

Note that undirected networks are special cases of directed networks with $G_{ij} = G_{ji}$ for all $i, j = 1, 2, \ldots, N$.

Definition 2 [12]: A directed (undirected) path from node v_j to v_i is a sequence of edges $(v_i, v_{i_1}), (v_{i_1}, v_{i_2}), \ldots, (v_{i_l}, v_j)$ in the directed (undirected) network with distinct nodes v_{i_k}, $k = 1, 2, \ldots, l$. A directed (undirected) network $\mathcal{G}$ is *strongly connected* (connected) if between any pair of distinct nodes v_i and v_j in $\mathcal{G}$, there exists a directed (undirected) path from v_i to v_j, $i, j = 1, 2, \ldots, N$.

Definition 3 [12]: A matrix G in a directed (undirected) network $\mathcal{G}$ is *reducible* if there is a permutation matrix $P \in R^{N \times N}$ and an integer $m, 1 \leq m \leq N - 1$, such that

$$P^{\mathrm{T}} G P = \begin{pmatrix} \widetilde{G}_{11} & 0 \\ \widetilde{G}_{21} & \widetilde{G}_{22} \end{pmatrix}$$

where $\widetilde{G}_{11} \in R^{m \times m}$, $\widetilde{G}_{21} \in R^{(n-m) \times m}$, and $\widetilde{G}_{22} \in R^{(n-m) \times (n-m)}$. Otherwise, G is called *irreducible*.

Definition 4 [3]: A directed network is called a *directed tree* if the underlying network is a tree when the direction of the network is ignored. *A directed rooted tree* is a directed network with at least one root r having the property that, for each node v different from r, there is a unique directed path from r to v. *A directed spanning tree* of a network $\mathcal{G}$ is a directed rooted tree, which contains all the nodes and some edges in $\mathcal{G}$.

The following notations are used throughout this paper for simplicity. Let $\lambda_{\max}(F)$ be the largest eigenvalue of matrix F, I_N (O_N) be the N-dimensional identity (zero) matrix, $1_N \in R^N$ ($0_N \in R^N$) be a vector with each entry being 1 (0), $\mathcal{R}(u)$ and $\mathcal{I}(u)$ be the real and imaginary parts of a complex number u, and $\otimes$ be the Kronecker product [13]. For matrices $\widetilde{A}$ and $\widetilde{B}$ with the same order, $\widetilde{A} > \widetilde{B}$ means that $\widetilde{A} - \widetilde{B}$ is positive definite. A matrix $G \in R^{N \times N}$ is nonnegative if every entry $G_{ij} \geq 0$ ($1 \leq i, j \leq N$) and a vector $x \in R^N$ is positive if every entry $x_i > 0$ ($1 \leq i \leq N$). Finally, let $\rho(A)$ be the spectral radius of matrix A.

The commonly studied second-order consensus protocol is described as follows [10], [11], [22], [24]:

$$
\begin{aligned}
\dot{x}_i(t) &= v_i(t) \\
\dot{v}_i(t) &= \alpha \sum_{j=1, j \neq i}^{N} G_{ij} \left(x_j(t) - x_i(t) \right) \\
&\quad + \beta \sum_{j=1, j \neq i}^{N} G_{ij} \left(v_j(t) - v_i(t) \right), \qquad i = 1, 2, \ldots, N
\end{aligned}
\tag{1}
$$

where $x_i \in R^n$ and $v_i \in R^n$ are the position and velocity states of the ith agent, respectively, $\alpha > 0$ and $\beta > 0$ are the coupling strengths, $G = (G_{ij})_{N \times N}$ is the coupling configuration matrix representing the topological structure of the network, and the Laplacian matrix $L = (L_{ij})_{N \times N}$ is defined by

$$
\begin{aligned}
L_{ii} &= - \sum_{j=1, j \neq i}^{N} L_{ij} \\
L_{ij} &= - G_{ij}, \qquad i \neq j; \; i, j = 1, \ldots, N
\end{aligned}
\tag{2}
$$

which ensures the diffusion property that $\sum_{j=1}^{N} L_{ij} = 0$. For an undirected network, its Laplacian matrix L is positive semidefinite; however, it may not be so for a directed network in general.

When the network reaches second-order consensus in (1), the velocities of all agents converge to $\sum_{j=1}^{N} \xi_j v_j(0)$, which depends only on the initial velocities of the agents, where $\xi = (\xi_1, \ldots, \xi_N)^{\mathrm{T}}$ is the nonnegative left eigenvector of L associated with eigenvalue zero satisfying $\xi^{\mathrm{T}} 1_N = 1$ [22], [24].

However, in most of the applications of multiagent formations, the velocity of each agent is generally not a constant but a time-varying variable. Therefore, in this paper, the following second-order consensus protocol with time-varying velocities is considered:

$$\dot{x}_i(t) = v_i(t)$$

$$\dot{v}_i(t) = f\left(x_i(t), v_i(t), t\right) - \alpha \sum_{j=1}^{N} L_{ij} x_j(t)$$

$$- \beta \sum_{j=1}^{N} L_{ij} v_j(t), \qquad i = 1, 2, \dots, N \qquad (3)$$

where $f : R^n \times R^n \times R^+ \longrightarrow R^n$ is a continuously differentiable vector-valued function. Here, f can be taken as $f = -\nabla U(x, v)$, where $U(x, v)$ is a potential function, then the multiagent system (3) includes many popular swarming and flocking models [9], [20] as special cases.

For convenience, we say a scalar is an irreducible matrix of order one. Next, a lemma is given to show the relation between an irreducible matrix and the corresponding strong connectivity in the network.

Clearly, since $\sum_{j=1}^{N} L_{ij} = 0$, if a consensus can be achieved, the solution $s(t) = (s_1(t), s_2(t)) \in R^{2n}$ of the system (3) must be a possible trajectory of an isolated node satisfying

$$\dot{s}_1(t) = s_2(t)$$

$$\dot{s}_2(t) = f\left(s_1(t), s_2(t), t\right)$$

where $s(t)$ may be an isolated equilibrium point, a periodic orbit, or even a chaotic orbit.

Lemma 1 [3, Th. 3.2.1], [12, Th. 6.2.24]: A matrix G is irreducible if and only if its corresponding network $\mathcal{G}$ is strongly connected.

Lemma 2 [25]: The Laplacian matrix L has a simple eigenvalue zero, and all the other eigenvalues have positive real parts if and only if the directed network has a directed spanning tree.

Lemma 3: Suppose that L is irreducible. Then, $L1_N = 0$, and there is a positive vector $\xi = (\xi_1, \xi_2, \dots, \xi_N)^{\mathrm{T}}$ such that $\xi^{\mathrm{T}} L = 0$. In addition, there exists a positive-definite diagonal matrix $\Xi = \mathrm{diag}(\xi_1, \xi_2, \dots, \xi_N)$ such that $\widehat{L} = (1/2)(\Xi L + L^{\mathrm{T}} \Xi)$ is symmetric, and $\sum_{j=1}^{N} \widehat{L}_{ij} = \sum_{j=1}^{N} \widehat{L}_{ji} = 0$ for all $i = 1, 2, \dots, N$.

Proof: The first statement is proved in [17]. It is easy to see that $\widehat{L}$ is symmetric, namely, $\widehat{L}_{ij} = \widehat{L}_{ji}$ for all $i, j = 1, 2, \dots, N$. Then, one has $\xi = \Xi 1_N$, and $L^{\mathrm{T}} \Xi 1_N = 0$. Therefore, $L^{\mathrm{T}} \Xi$ is a matrix with the sum of the entries in each row being zero. Since $\sum_{j=1}^{N} L_{ij} = 0$, one has $\Xi L 1_N = 0$, and the sum of the entries in each row of ΞL is also zero. Thus, the sum of the entries in each row of matrix $\widehat{L}$ is zero. In addition, since $\widehat{L}$ is symmetric, the sum of the entries in each column of matrix $\widehat{L}$ is also zero. The proof is completed. ∎

Lemma 4 (Schur Complement [2]): The following is a linear matrix inequality:

$$\begin{pmatrix} \mathcal{Q}(x) & \mathcal{S}(x) \\ \mathcal{S}(x)^{\mathrm{T}} & \mathcal{R}(x) \end{pmatrix} > 0$$

where $\mathcal{Q}(x) = \mathcal{Q}(x)^{\mathrm{T}}, \mathcal{R}(x) = \mathcal{R}(x)^{\mathrm{T}}$ is equivalent to one of the following conditions.

1) $\mathcal{Q}(x) > 0, \mathcal{R}(x) - \mathcal{S}(x)^{\mathrm{T}} \mathcal{Q}(x)^{-1} \mathcal{S}(x) > 0$.
2) $\mathcal{R}(x) > 0, \mathcal{Q}(x) - \mathcal{S}(x) \mathcal{R}(x)^{-1} \mathcal{S}(x)^{\mathrm{T}} > 0$.

Lemma 5 [13]: For matrices A, B, C, and D with appropriate dimensions, we have the following conditions.

1) $(\gamma A) \otimes B = A \otimes (\gamma B)$, where γ is a constant.
2) $(A + B) \otimes C = A \otimes C + B \otimes C$.
3) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.
4) $(A \otimes B)^{\mathrm{T}} = A^{\mathrm{T}} \otimes B^{\mathrm{T}}$.

Definition 5: The multiagent system (3) is said to achieve second-order consensus if for any initial conditions

$$\lim_{t \to \infty} \|x_i(t) - x_j(t)\| = 0$$

$$\lim_{t \to \infty} \|v_i(t) - v_j(t)\| = 0 \qquad \forall i, j = 1, 2, \dots, N.$$

III. SECOND-ORDER CONSENSUS IN STRONGLY CONNECTED NETWORKS WITH TIME-VARYING VELOCITIES

In this section, second-order consensus in strongly connected networks with nonlinear dynamics (3) is first investigated.

Assumption 1: There exist nonnegative constants ρ_1 and ρ_2 such that

$$\|f(x, v, t) - f(y, z, t)\| \leq \rho_1 \|x - y\| + \rho_2 \|v - z\|$$

$$\forall x, y, v, z \in R^n; \forall t \geq 0. \quad (4)$$

Note that Assumption 1 is a Lipschitz-type condition, satisfied by many well-known systems.

Let $\widehat{x}_i(t) = x_i(t) - \sum_{k=1}^{N} \xi_k x_k(t)$ and $\widehat{v}_i(t) = v_i(t) - \sum_{k=1}^{N} \xi_k v_k(t)$ represent the position and velocity vectors relative to the average position and velocity of the agents in system (3), where $\xi = (\xi_1, \dots, \xi_N)^{\mathrm{T}}$ is the positive left eigenvector of L associated with eigenvalue zero satisfying $\xi^{\mathrm{T}} 1_N = 1$. Then, one obtains the following error dynamical system:

$$\dot{\widehat{x}}_i(t) = \widehat{v}_i(t),$$

$$\dot{\widehat{v}}_i(t) = f\left(x_i(t), v_i(t), t\right) - \sum_{k=1}^{N} \xi_k f\left(x_k(t), v_k(t), t\right)$$

$$- \alpha \sum_{j=1}^{N} L_{ij} x_j(t) - \beta \sum_{j=1}^{N} L_{ij} v_j(t)$$

$$+ \alpha \sum_{k=1}^{N} \xi_k \sum_{j=1}^{N} L_{kj} x_j(t) + \beta \sum_{k=1}^{N} \xi_k \sum_{j=1}^{N} L_{kj} v_j(t),$$

$$i = 1, 2, \dots, N. \qquad (5)$$

Since $\xi^{\mathrm{T}} L = 0$, one has $\sum_{k=1}^{N} \xi_k \sum_{j=1}^{N} L_{kj} x_j(t) = [(\xi^{\mathrm{T}} L) \otimes I_n] x = 0$ and $\sum_{k=1}^{N} \xi_k \sum_{j=1}^{N} L_{kj} v_j(t) = [(\xi^{\mathrm{T}} L) \otimes I_n] v = 0$.

Note that $\sum_{j=1}^{N} L_{ij} = 0$, so (5) can be written as

$$\dot{\hat{x}}_i(t) = \hat{v}_i(t)$$

$$\dot{\hat{v}}_i(t) = f(x_i(t), v_i(t), t) - \sum_{k=1}^{N} \xi_k f(x_k(t), v_k(t), t)$$

$$- \alpha \sum_{j=1}^{N} L_{ij} \hat{x}_j(t) - \beta \sum_{j=1}^{N} L_{ij} \hat{v}_j(t),$$

$$i = 1, 2, \ldots, N. \qquad (6)$$

Let $\hat{x} = (\hat{x}_1^{\mathrm{T}}, \hat{x}_2^{\mathrm{T}}, \ldots, \hat{x}_N^{\mathrm{T}})^{\mathrm{T}}$, $v = (\hat{v}_1^{\mathrm{T}}, \hat{v}_2^{\mathrm{T}}, \ldots, \hat{v}_N^{\mathrm{T}})^{\mathrm{T}}$, $f(x, v, t) = (f^{\mathrm{T}}(x_1(t), v_1(t), t), \ldots, f^{\mathrm{T}}(x_N(t), v_N(t), t))^{\mathrm{T}}$, and $\hat{y} = (\hat{x}^{\mathrm{T}}, \hat{v}^{\mathrm{T}})^{\mathrm{T}}$. Then, system (6) can be recast in a compact matrix form as follows:

$$\dot{\hat{y}}(t) = F(x, v, t) + (\widetilde{L} \otimes I_n)\hat{y}(t) \qquad (7)$$

where $F(x, v, t) = \begin{pmatrix} 0_{Nn} \\ ((I_N - 1_N \xi^{\mathrm{T}}) \otimes I_n) f(x, v, t) \end{pmatrix}$ and $\widetilde{L} = \begin{pmatrix} O_N & I_N \\ -\alpha L & -\beta L \end{pmatrix}$.

The algebraic graph theory, particularly the notion of algebraic connectivity, has been well developed for undirected networks [8]. For directed graphs, however, it has not been fully developed yet. For example, there are no standard definitions for the algebraic connectivity and consensus convergence rate for directed graphs while their counterparts for undirected graphs have been widely used to study the consensus problem. Some useful concepts have been proposed in [31]. However, in this paper, a new general algebraic connectivity is proposed which can be used to describe consensus ability in multiagent systems. Some additional properties about the derived general algebraic connectivity are also discussed.

Definition 6: For a strongly connected network with Laplacian matrix L, the general algebraic connectivity is defined by

$$a(L) = \min_{x^{\mathrm{T}}\xi = 0, x \neq 0} \frac{x^{\mathrm{T}}\widehat{L}x}{x^{\mathrm{T}}\Xi x} \qquad (8)$$

where $\widehat{L} = (\Xi L + L^{\mathrm{T}}\Xi)/2$, $\Xi = \mathrm{diag}(\xi_1, \ldots, \xi_N)$, $\xi = (\xi_1, \xi_2, \ldots, \xi_N)^{\mathrm{T}} > 0$, and $\xi^{\mathrm{T}}L = 0$, $\sum_{i=1}^{N}\xi_i = 1$.

Note that if $\Xi = \eta I_N$ and the network is undirected, then $a(L) = \lambda_2(L)$.

Lemma 6: Suppose that the matrix $\widehat{L}$ is symmetric, irreducible, and satisfies $\sum_{j=1}^{N} \widehat{L}_{ij} = 0$ with $\widehat{L}_{ij} \geq 0$, $i \neq j$, $i, j = 1, 2, \ldots, N$. Let

$$\widehat{a}(\widehat{L}) = \min_{x^{\mathrm{T}}\xi = 0, x \neq 0} \frac{x^{\mathrm{T}}\widehat{L}x}{x^{\mathrm{T}}x}. \qquad (9)$$

Then, $\lambda_2(\widehat{L}) \geq \widehat{a}(\widehat{L}) \geq 0$. In addition, $\widehat{a}(\widehat{L}) = 0$ if and only if ξ is orthogonal to the left eigenvector of $\widehat{L}$ associated with eigenvalue zero. Moreover, $\widehat{a}(\widehat{L}) = \lambda_2(\widehat{L})$ if ξ is the left eigenvector of $\widehat{L}$ associated with eigenvalue zero.

Proof: From the Courant–Fischer minimum–maximum theorem [12], one has $\lambda_2(\widehat{L}) \geq \widehat{a}(\widehat{L}) \geq 0$. Let Λ be the di-

agonal matrix associated with $\widehat{L}$, i.e., there exists a $P = (p_1, p_2, \ldots, p_N)$ such that $\widehat{L} = P\Lambda P^{\mathrm{T}}$, and let $y = P^{\mathrm{T}}x$. Then

$$\widehat{a}(\widehat{L}) = \min_{x^{\mathrm{T}}\xi = 0, x^{\mathrm{T}}x = 1} x^{\mathrm{T}}\widehat{L}x = \min_{x^{\mathrm{T}}\xi = 0, x^{\mathrm{T}}x = 1} x^{\mathrm{T}}P\Lambda P^{\mathrm{T}}x$$

$$= \min_{x^{\mathrm{T}}\xi = 0, x^{\mathrm{T}}x = 1} \sum_{i=1}^{N} \lambda_i(\widehat{L})y_i^2$$

$$= \min_{y^{\mathrm{T}}P^{\mathrm{T}}\xi = 0, y^{\mathrm{T}}y = 1} \sum_{i=1}^{N} \lambda_i(\widehat{L})y_i^2$$

$$\leq \min_{y^{\mathrm{T}}P^{\mathrm{T}}\xi = 0, y^{\mathrm{T}}y = 1, y_3 = \cdots = y_N = 0} \sum_{i=1}^{2} \lambda_i(\widehat{L})y_i^2$$

$$\leq \lambda_2(\widehat{L}). \qquad (10)$$

The inequalities hold if $\forall y \in R^N$ under the conditions of $y^{\mathrm{T}}P^{\mathrm{T}}\xi = 0$, $y_1 = 0$, $p_2^{\mathrm{T}}\xi = 0$, and $p_1^{\mathrm{T}}\xi \neq 0$. If ξ is the left eigenvector of $\widehat{L}$ associated with eigenvalue zero, then $p_1^{\mathrm{T}}\xi \neq 0$ and $\xi \perp p_i$, $i = 2, \ldots, N$, for any y with $y_1 = 0$. Therefore, $\widehat{a}(\widehat{L}) = \lambda_2(\widehat{L})$ if ξ is the left eigenvector of eigenvalue zero. Similarly

$$\widehat{a}(\widehat{L}) = \min_{y^{\mathrm{T}}P^{\mathrm{T}}\xi = 0, y^{\mathrm{T}}y = 1} \sum_{i=1}^{N} \lambda_i(\widehat{L})y_i^2 \geq \lambda_1(\widehat{L}). \qquad (11)$$

The above inequality holds if and only if $y^{\mathrm{T}}P^{\mathrm{T}}\xi = 0$, i.e., $y^{\mathrm{T}}P^{\mathrm{T}}\xi = \sum_{i=1}^{N} y_i p_i^{\mathrm{T}}\xi$, and $y_2 = \cdots = y_N = 0$. It then follows that $p_1^{\mathrm{T}}\xi = 0$. This completes the proof. ∎

Lemma 7: If the Laplacian matrix L is irreducible, then $a(L) > 0$.

Proof: From Lemma 3, there exist a positive vector $\xi = (\xi_1, \xi_2, \ldots, \xi_N)$ and a positive definite diagonal matrix $\Xi = \mathrm{diag}(\xi_1, \xi_2, \ldots, \xi_N)$, such that $\widehat{L} = (1/2)(\Xi L + L^{\mathrm{T}}\Xi)$ is symmetric and $\sum_{j=1}^{N} \widehat{L}_{ij} = 0$, $i = 1, 2, \ldots, N$. It is easy to see that 1_N is the left eigenvector of eigenvalue zero associated with matrix $\widehat{L}$, and $1_N^{\mathrm{T}}\xi = 1 \neq 0$. From Lemma 6, $a(L) \geq \widehat{a}(\widehat{L})/\max_i \xi_i > 0$. The proof is thus completed. ∎

Lemma 8: The general algebraic connectivity of a strongly connected network can be computed by the following:

$$\begin{aligned} \max \quad & \delta \\ \text{subject to} \quad & Q^{\mathrm{T}}(\widehat{L} - \delta\Xi)Q \geq 0 \end{aligned} \qquad (12)$$

where $Q = \begin{pmatrix} I_{N-1} \\ -\frac{\hat{\xi}^{\mathrm{T}}}{\xi_N} \end{pmatrix} \in R^{N \times (N-1)}$ and $\hat{\xi} = (\xi_1, \ldots, \xi_{N-1})^{\mathrm{T}}$.

Proof: It is easy to see that the columns of Q form a basis of the orthogonal subspace of the vector ξ. Thus, by letting $x = Qz$, one has

$$a(L) = \min_{z \neq 0} \frac{z^{\mathrm{T}}Q^{\mathrm{T}}\widehat{L}Qz}{z^{\mathrm{T}}Q^{\mathrm{T}}\Xi Qz}. \qquad (13)$$

The proof is completed. ∎

Theorem 1: Suppose that the network is strongly connected and Assumption 1 holds. Then, second-order consensus in system (3) is achieved if

$$a(L) > \frac{1}{2} \left(\frac{\rho_1}{\alpha} + \frac{\alpha}{\beta^2} + \frac{\rho_1}{\beta} \right.$$

$$\left. + \sqrt{\left(\frac{\rho_1}{\alpha} - \frac{\alpha}{\beta^2} - \frac{\rho_1}{\beta} \right)^2 + \frac{(\alpha + \beta)^2 \rho_2^2}{\alpha^2 \beta^2}} \right). \quad (14)$$

Proof: Consider the following Lyapunov function candidate:

$$V(t) = \frac{1}{2} \widehat{y}^{\mathrm{T}}(t)(\Omega \otimes I_n)\widehat{y}(t) \quad (15)$$

where $\Omega = \begin{pmatrix} 2\alpha\widehat{L} & \frac{\alpha\Xi}{\beta} \\ \frac{\alpha\Xi}{\beta} & \Xi \end{pmatrix}$. It will be shown that $V(t) \geq 0$ and $V(t) = 0$ if and only if $\widehat{y} = 0$. From the definition of $a(L)$, one has

$$V(t) = \alpha\widehat{x}^{\mathrm{T}}(t)(\widehat{L} \otimes I_n)\widehat{x}(t) + \frac{\alpha}{2\beta}\widehat{x}^{\mathrm{T}}(t)(\Xi \otimes I_n)\widehat{v}(t)$$

$$+ \frac{\alpha}{2\beta}\widehat{v}^{\mathrm{T}}(t)(\Xi \otimes I_n)\widehat{x}(t) + \frac{1}{2}\widehat{v}^{\mathrm{T}}(t)(\Xi \otimes I_n)\widehat{v}(t)$$

$$\geq \frac{1}{2}\widehat{y}^{\mathrm{T}}(t)(\widehat{Q} \otimes I_n)\widehat{y}(t) \quad (16)$$

where $\widehat{Q} = \begin{pmatrix} 2a(L)\alpha\Xi & \frac{\alpha\Xi}{\beta} \\ \frac{\alpha\Xi}{\beta} & \Xi \end{pmatrix}$. By Lemma 4, $\widehat{Q} > 0$ is equivalent to that $\Xi > 0$ and $2a(L)\alpha\Xi - (\alpha^2/\beta^2)\Xi > 0$. From (14), one has $a(L) \geq (1/2)((\rho_1/\alpha) + (\alpha/\beta^2) + (\rho_1/\beta) + |(\rho_1/\alpha) - (\alpha/\beta^2) - (\rho_1/\beta)|) = \max((\rho_1/\alpha), (\alpha/\beta^2) + (\rho_1/\beta))$, and thus $\widehat{Q} > 0$. Consequently, $V(t) \geq 0$ and $V(t) = 0$ if and only if $\widehat{y} = 0$.

Let $\overline{x} = \sum_{j=1}^{N} \xi_j x_j$ and $\overline{v} = \sum_{j=1}^{N} \xi_j v_j$. Taking the derivative of $V(t)$ along the trajectories of (7) yields

$$\dot{V}(t) = \widehat{y}^{\mathrm{T}}(t)(\Omega \otimes I_n) \left[F(x, v, t) + (\widetilde{L} \otimes I_n)\widehat{y} \right]$$

$$= \frac{\alpha}{\beta}\widehat{x}^{\mathrm{T}}(t) \left[(\Xi(I_N - 1_N\xi^{\mathrm{T}})) \otimes I_n \right] f(x, v, t)$$

$$+ \widehat{v}^{\mathrm{T}}(t) \left[(\Xi(I_N - 1_N\xi^{\mathrm{T}})) \otimes I_n \right] f(x, v, t)$$

$$+ \widehat{y}^{\mathrm{T}}(t) \left[(\Omega\widetilde{L}) \otimes I_n \right] \widehat{y}$$

$$= \left[\frac{\alpha}{\beta}\widehat{x}^{\mathrm{T}}(t) + \widehat{v}^{\mathrm{T}}(t) \right] (\Xi \otimes I_n)$$

$$\times \left[f(x, v, t) - 1_N \otimes f(\overline{x}, \overline{v}, t) \right]$$

$$+ \frac{1}{2}\widehat{y}^{\mathrm{T}}(t) \left[(\Omega\widetilde{L} + \widetilde{L}^{\mathrm{T}}\Omega) \otimes I_n \right] \widehat{y}(t)$$

$$- \left[\frac{\alpha}{\beta}\widehat{x}^{\mathrm{T}}(t) + \widehat{v}^{\mathrm{T}}(t) \right] (\Xi \otimes I_n) \left[(1_N\xi^{\mathrm{T}}) \otimes I_n \right]$$

$$\times f(x, v, t) + \left[\frac{\alpha}{\beta}\widehat{x}^{\mathrm{T}}(t) + \widehat{v}^{\mathrm{T}}(t) \right] (\Xi \otimes I_n)$$

$$\times \left[1_N \otimes f(\overline{x}, \overline{v}, t) \right]. \quad (17)$$

Since $\widehat{x}(t) = [(I_N - 1_N\xi^{\mathrm{T}}) \otimes I_n]x(t)$, $\widehat{v}(t) = [(I_N - 1_N\xi^{\mathrm{T}}) \otimes I_n]v(t)$, and $\xi^{\mathrm{T}}1_N = 1$, one has

$$\widehat{x}^{\mathrm{T}}(t)(\Xi \otimes I_n) \left[1_N \otimes f(\overline{x}, \overline{v}, t) \right]$$

$$= \left[1_N^{\mathrm{T}} \otimes f^{\mathrm{T}}(\overline{x}, \overline{v}, t) \right] (\Xi \otimes I_n) \left[(I_N - 1_N\xi^{\mathrm{T}}) \otimes I_n \right] x(t)$$

$$= \left\{ \left[1_N^{\mathrm{T}}\Xi(I_N - 1_N\xi^{\mathrm{T}}) \right] \otimes f^{\mathrm{T}}(\overline{x}, \overline{v}, t) \right\} x(t)$$

$$= \left\{ \left[\xi^{\mathrm{T}}(I_N - 1_N\xi^{\mathrm{T}}) \right] \otimes f^{\mathrm{T}}(\overline{x}) \right\} x(t) = 0 \quad (18)$$

$$\widehat{x}^{\mathrm{T}}(t)(\Xi \otimes I_n) \left[(1_N\xi^{\mathrm{T}}) \otimes I_n \right] f(x, v, t)$$

$$= f^{\mathrm{T}}(x, v, t) \left[(\xi 1_N^{\mathrm{T}}) \otimes I_n \right] (\Xi \otimes I_n)$$

$$\times \left[(I_N - 1_N\xi^{\mathrm{T}}) \otimes I_n \right] x(t)$$

$$= f^{\mathrm{T}}(x, v, t) \left\{ \left[\xi 1_N^{\mathrm{T}}\Xi(I_N - 1_N\xi^{\mathrm{T}}) \right] \otimes I_n \right\} x(t)$$

$$= f^{\mathrm{T}}(x, v, t) \left\{ \left[\xi\xi^{\mathrm{T}}(I_N - 1_N\xi^{\mathrm{T}}) \right] \otimes I_n \right\} x(t) = 0. \quad (19)$$

Similarly, one can obtain

$$\widehat{v}^{\mathrm{T}}(t)(\Xi \otimes I_n) \left[1_N \otimes f(\overline{x}, \overline{v}, t) \right] = 0 \quad (20)$$

$$\widehat{v}^{\mathrm{T}}(t)(\Xi \otimes I_n) \left[(1_N\xi^{\mathrm{T}}) \otimes I_n \right] f(x, v, t) = 0. \quad (21)$$

In addition

$$\Omega\widetilde{L} = \begin{pmatrix} 2\alpha\widehat{L} & \frac{\alpha\Xi}{\beta} \\ \frac{\alpha\Xi}{\beta} & \Xi \end{pmatrix} \begin{pmatrix} O_N & I_N \\ -\alpha L & -\beta L \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-\alpha^2}{\beta}\Xi L & 2\alpha\widehat{L} - \alpha\Xi L \\ -\alpha\Xi L & \frac{\alpha\Xi}{\beta} - \beta\Xi L \end{pmatrix}. \quad (22)$$

Note that $\widehat{L} = (1/2)(\Xi L + L^{\mathrm{T}}\Xi)$, so

$$\frac{1}{2}(\Omega\widetilde{L} + \widetilde{L}^{\mathrm{T}}\Omega) = \begin{pmatrix} -\frac{\alpha^2}{\beta}\widehat{L} & O_N \\ O_N & \frac{\alpha\Xi}{\beta} - \beta\widehat{L} \end{pmatrix}. \quad (23)$$

By Assumption 1, one obtains

$$\widehat{x}^{\mathrm{T}}(t)(\Xi \otimes I_n) \left[f(x, v, t) - 1_N \otimes f(\overline{x}, \overline{v}, t) \right]$$

$$= \sum_{i=1}^{N} (x_i - \overline{x})^{\mathrm{T}} \xi_i \left[f(x_i, v_i, t) - f(\overline{x}, \overline{v}, t) \right]$$

$$\leq \sum_{i=1}^{N} \|\widehat{x}_i\|\xi_i \left(\rho_1\|\widehat{x}_i\| + \rho_2\|\widehat{v}_i\| \right)$$

$$= \rho_1 \sum_{i=1}^{N} \xi_i\|\widehat{x}_i\|^2 + \rho_2 \sum_{i=1}^{N} \xi_i\|\widehat{x}_i\|\|\widehat{v}_i\| \quad (24)$$

$$\widehat{v}^{\mathrm{T}}(t)(\Xi \otimes I_n) \left[f(x, v, t) - 1_N \otimes f(\overline{x}, \overline{v}, t) \right]$$

$$\leq \rho_1 \sum_{i=1}^{N} \xi_i\|\widehat{v}_i\|^2 + \rho_2 \sum_{i=1}^{N} \xi_i\|\widehat{x}_i\|\|\widehat{v}_i\|. \quad (25)$$

Combining (18)–(25), one has

$$
\begin{aligned}
\dot{V}(t) =\; & \left[\frac{\alpha}{\beta}\widehat{x}^{\mathrm{T}}(t) + \widehat{v}^{\mathrm{T}}(t)\right](\Xi \otimes I_n) \\
& \times [f(x,v,t) - 1_N \otimes f(\overline{x},\overline{v},t)] \\
& + \frac{1}{2}\widehat{y}^{\mathrm{T}}(t)\left[(\Omega\widetilde{L} + \widetilde{L}^{\mathrm{T}}\Omega)\otimes I_n\right]\widehat{y}(t) \\
\leq\; & \frac{\alpha}{\beta}\rho_1\sum_{i=1}^{N}\xi_i\|\widehat{x}_i\|^2 + \left(\frac{\alpha}{\beta}+1\right)\rho_2\sum_{i=1}^{N}\xi_i\|\widehat{x}_i\|\|\widehat{v}_i\| \\
& + \rho_1\sum_{i=1}^{N}\xi_i\|\widehat{v}_i\|^2 - \frac{\alpha^2}{\beta}\widehat{x}^{\mathrm{T}}(\widehat{L}\otimes I_n)\widehat{x} \\
& + \widehat{v}^{\mathrm{T}}\left[\left(\frac{\alpha\Xi}{\beta} - \beta\widehat{L}\right)\otimes I_n\right]\widehat{v} \\
\leq\; & \left[\frac{\alpha}{\beta}\rho_1 - \frac{\alpha^2}{\beta}a(L)\right]\sum_{i=1}^{N}\xi_i\|\widehat{x}_i\|^2 \\
& + \left(\frac{\alpha}{\beta}+1\right)\rho_2\sum_{i=1}^{N}\xi_i\|\widehat{x}_i\|\|\widehat{v}_i\| \\
& + \left[\rho_1 + \frac{\alpha}{\beta} - \beta a(L)\right]\sum_{i=1}^{N}\xi_i\|\widehat{v}_i\|^2 \\
=\; & \|\widehat{x}\|^{\mathrm{T}}\overline{Q}\|\widehat{x}\|
\end{aligned}
\tag{26}
$$

where $\|\widehat{x}\| = (\|\widehat{x}_1\|,\ldots,\|\widehat{x}_N\|)^{\mathrm{T}}$ and

$$
\overline{Q} = \begin{pmatrix}
\left[\frac{\alpha}{\beta}\rho_1 - \frac{\alpha^2}{\beta}a(L)\right]\Xi & \frac{1}{2}\left(\frac{\alpha}{\beta}+1\right)\rho_2\Xi \\
\frac{1}{2}\left(\frac{\alpha}{\beta}+1\right)\rho_2\Xi & \left[\rho_1 + \frac{\alpha}{\beta} - \beta a(L)\right]\Xi
\end{pmatrix}.
$$

By Lemma 4, $\overline{Q} < 0$ is equivalent to that

$$
a(L) > \frac{\rho_1}{\alpha}
$$

$$
\left[a(L) - \frac{\rho_1}{\alpha}\right]\left[a(L) - \frac{\alpha}{\beta^2} - \frac{\rho_1}{\beta}\right] > \frac{(\alpha+\beta)^2\rho_2^2}{4\alpha^2\beta^2}.
$$

By simple calculations, one obtains (14). Therefore, the second-order consensus is achieved in system (3) under condition (14). This completes the proof. ■

Corollary 1: Suppose that the network is undirected and Assumption 1 holds. Then, second-order consensus in system (3) is achieved if

$$
\begin{aligned}
\lambda_2(L) > \frac{1}{2}\Bigg(&\frac{\rho_1}{\alpha} + \frac{\alpha}{\beta^2} + \frac{\rho_1}{\beta} \\
& + \sqrt{\left(\frac{\rho_1}{\alpha} - \frac{\alpha}{\beta^2} - \frac{\rho_1}{\beta}\right)^2 + \frac{(\alpha+\beta)^2\rho_2^2}{\alpha^2\beta^2}}\Bigg).
\end{aligned}
$$

Proof: If the network is undirected, then by Definition 6, $a(L) = \lambda_2(L)$. This completes the proof. ■

Remark 1: In Definition 6, the general algebraic connectivity is defined for a strongly connected network, which is shown in

Theorem 1 to be the key factor for reaching network consensus. The right-hand side of condition (14) depends on the coupling strengths α and β, and the nonlinear constants ρ_1 and ρ_2. Thus, $a(L)$ is a key factor concerning the network structure that can be used to describe the ability for reaching consensus.

If $f = 0$, then $\rho_1 = \rho_2 = 0$, and system (3) is reduced to the linear system (1).

Corollary 2: Suppose that the network is strongly connected. Then, second-order consensus in system (1) is achieved if

$$
a(L) > \frac{\alpha}{\beta^2}.
\tag{27}
$$

Up to this point, it is still a challenging problem to compute $a(L)$ and to find the relationship between $a(L)$ and the eigenvalues of L. Fortunately, the following useful result can be obtained as a byproduct of the previous results.

Corollary 3: Suppose that the network is strongly connected. Then, the following statement holds:

$$
\begin{aligned}
a(L) &\leq \min_{2\leq i\leq N}\frac{\mathcal{R}(\mu_i)\left[\mathcal{R}^2(\mu_i) + \mathcal{I}^2(\mu_i)\right]}{\mathcal{I}^2(\mu_i)} \\
&= \min_{2\leq i\leq N}\left[\mathcal{R}(\mu_i) + \frac{\mathcal{R}^3(\mu_i)}{\mathcal{I}^2(\mu_i)}\right].
\end{aligned}
\tag{28}
$$

In addition, if the network is undirected, then $a(L) = \lambda_2(L)$.

Proof: From the results in [39], one knows that second-order consensus in system (1) is achieved if and only if the network contains a directed spanning tree and moreover

$$
\min_{2\leq i\leq N}\frac{\mathcal{R}(\mu_i)\left[\mathcal{R}^2(\mu_i) + \mathcal{I}^2(\mu_i)\right]}{\mathcal{I}^2(\mu_i)} > \frac{\alpha}{\beta^2}.
$$

By Corollary 2, one obtains a sufficient condition $a(L) > (\alpha/\beta^2)$ for the network consensus. Thus, (28) is satisfied. ■

Remark 2: In general, it is very difficult to compute $a(L)$, not to mention finding the relationship between $a(L)$ and the eigenvalues of L. However, if the network is strongly connected, Corollary 3 yields that $a(L) \leq \min_{2\leq i\leq N}[\mathcal{R}(\mu_i) + (\mathcal{R}^3(\mu_i)/\mathcal{I}^2(\mu_i))]$. This result is useful for both theoretical analyses in algebraic graph theory and relevant applications.

IV. SECOND-ORDER CONSENSUS IN NETWORKS CONTAINING A DIRECTED SPANNING TREE WITH TIME-VARYING VELOCITIES

In this section, second-order consensus in networks containing a directed spanning tree with time-varying velocities is further investigated.

Lemma 9 [3]: There exist a permutation matrix P of order N and an integer $m \geq 1$, such that

$$
P^{\mathrm{T}}LP = \begin{pmatrix}
\overline{L}_1 & O & \cdots & O \\
\overline{L}_{21} & \overline{L}_2 & \cdots & O \\
\vdots & \vdots & \ddots & O \\
\overline{L}_{m1} & \overline{L}_{m2} & \cdots & \overline{L}_m
\end{pmatrix}
\tag{29}
$$

where $\overline{L}_1 \in R^{q_1\times q_1}, \overline{L}_2 \in R^{q_2\times q_2},\ldots,\overline{L}_m \in R^{q_m\times q_m}$ are square irreducible matrices, which are uniquely determined

to within simultaneous permutation of their lines, but their ordering is not necessarily unique.

Next, we recall the concept of the condensation network $\mathcal{G}^*$ of a directed network $\mathcal{G}$, which has no closed directed walks [3].

Definition 7 [3]: Let $\mathcal{G}$ be a directed network and let $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_m$ be the strongly connected components of $\mathcal{G}$ with the connection matrices $\overline{L}_1, \overline{L}_2, \ldots, \overline{L}_m$. Then, $\mathcal{G}^*$ is a *condensation network* of $\mathcal{G}$ if there is a connection from a node in $\mathcal{V}(\mathcal{G}_j)$ to a node in $\mathcal{V}(\mathcal{G}_i)$ $(i \neq j)$, then the weight $G^*_{ij} > 0$; otherwise, $G^*_{ij} = 0$ for $i, j = 1, 2, \ldots, m$; $G^*_{ii} = 0$, for $i = 1, 2, \ldots, m$.

Lemma 10: For every $i = 2, 3, \ldots, m$, there is an integer $j < i$ such that $G^*_{ij} > 0$ if and only if the directed network $\mathcal{G}$ contains a directed spanning tree.

Proof: If the directed network $\mathcal{G}$ contains a directed spanning tree, then there is a directed path from the root (a node in $\mathcal{G}_1$) to every other node. If for an integer i, $2 \leq i \leq m$, $G^*_{ij} = 0$ for all $j < i$, then because of the network structure in (29), $G^*_{ij} = 0$ for all $j \neq i$, i.e., there are no paths from the root to the ith strongly connected components. This contradicts the fact that $\mathcal{G}$ contains a directed spanning tree.

If, for every $i = 2, 3, \ldots, m$, there is an integer $j < i$ such that $G^*_{ij} > 0$, then there exists a directed connection from node j to node i in $\mathcal{G}^*$. When $i = 2$, there is a directed path from node 1 to node 2. Suppose there is a directed path from node 1 to all the nodes $2, 3, \ldots, k$. Then, it suffices to prove that there is a directed path from node 1 to node $k + 1$. By assumption, there is an integer $j < k + 1$ and a directed path from j to $k + 1$, such that a directed path from 1 to $k + 1$ exists from 1 to j and, then, to $k + 1$. Therefore, $\mathcal{G}^*$ contains a directed spanning tree with root node 1, which implies that $\mathcal{G}$ has a directed spanning tree with every node in $\mathcal{G}_1$ being a root. ∎

One can change the order of the node indexes to obtain the Frobenius normal form (29). Without loss of generality, assume that the adjacency matrix G of $\mathcal{G}$ is in the Frobenius normal form. Suppose that the condition (14) in Theorem 1 holds in the first strongly connected component, so that the final states of the nodes in this component satisfy [17], [31]

$$
\begin{aligned}
\dot{x}_s(t) &= v_s(t) + \mathcal{O}(e^{\epsilon t}) \\
\dot{v}_s(t) &= f(x_s, v_s, t) + \mathcal{O}(e^{\epsilon t})
\end{aligned}
\tag{30}
$$

where $x_s \in R^n$, $v_s \in R^n$, and $\epsilon < 0$. Let $\overline{L}_i = \overline{L}_{ii} + A_i$, where $\overline{L}_{ii}$ is a zero-row-sum matrix and $A_i \geq 0$ is a diagonal matrix. By Lemma 3, there exists a positive vector $\overline{\xi}_i = (\overline{\xi}_{i1}, \overline{\xi}_{i2}, \ldots, \overline{\xi}_{iq_i})^T$ with appropriate dimensions such that $\overline{\xi}_i^T \overline{L}_{ii} = 0$ and $\sum_{j=1}^{q_i} \overline{\xi}_{ij} = 1$. Let $\widetilde{x}_i = x_i - x_s$, $\widetilde{v}_i = v_i - v_s$, $s_i = \sum_{j=1}^i q_j$, $e_i = (\widetilde{x}_{s_{i-1}}^T, \ldots, \widetilde{x}_{s_i}^T, \widetilde{v}_{s_{i-1}}^T, \ldots, \widetilde{v}_{s_i}^T)^T$,

$e = (e_2^T, \ldots, e_m^T)^T$, $\quad \widetilde{L}^*_{ii} = \begin{pmatrix} O_{q_i} & I_{q_i} \\ -\alpha \overline{L}_i & -\beta \overline{L}_i \end{pmatrix}$, $\quad \widetilde{L}^*_{ij} = \begin{pmatrix} O_{q_i \times q_j} & I_{q_i \times q_j} \\ -\alpha \overline{L}_{ij} & -\beta \overline{L}_{ij} \end{pmatrix}$, $f^*_i(x, v, t) = (f^T(x_{s_{i-1}}, v_{s_{i-1}}, t), \ldots, f^T(x_{s_i}, v_{s_i}, t))^T$, and $F^*_i(x, v, t) = \begin{pmatrix} 0_{q_i n} \\ f^*_i(x, v, t) - 1_{q_i} \otimes f(x_s, v_s, t) \end{pmatrix}$, $i = 1, 2, \ldots, m$. Then, the network (3) can be recast in a compact matrix form as follows:

$$
\dot{e}(t) = F^*(x, v, t) + (\widetilde{L}^* \otimes I_n)e(t) + \mathcal{O}(e^{\epsilon t})
\tag{31}
$$

where $F^*(x, v, t) = (F_2^{*T}(x, v, t), \ldots, F_N^{*T}(x, v, t))^T$, and $\widetilde{L}^* = (\widetilde{L}^*_{ij})_{(m-1) \times (m-1)}$, $2 \leq i, j \leq m$.

Lemma 11: Let

$$
\widetilde{L}^* = \begin{pmatrix} \widetilde{L}^*_{22} & O & \cdots & O \\ \widetilde{L}^*_{32} & \widetilde{L}^*_{33} & \cdots & O \\ \vdots & \vdots & \ddots & O \\ \widetilde{L}^*_{m1} & \widetilde{L}^*_{m2} & \cdots & \widetilde{L}^*_{mm} \end{pmatrix}.
\tag{32}
$$

If there exist positive-definite diagonal matrices

$$
Q^*_j = \begin{pmatrix} 2\alpha \overline{L}_j & \frac{\alpha \overline{\Xi}_j}{\beta} \\ \frac{\alpha \overline{\Xi}_j}{\beta} & \overline{\Xi}_j \end{pmatrix}
$$

where $\overline{\Xi}_j = \text{diag}(\overline{\xi}_{j1}, \overline{\xi}_{j2}, \ldots, \overline{\xi}_{jp_j})$, such that

$$
Q^*_j \widetilde{L}^*_{jj} + \widetilde{L}^{*T}_{jj} Q^*_j < 0, \qquad j = 2, \ldots, m
\tag{33}
$$

then there exists a positive-definite diagonal matrix $\Delta = \text{diag}(\Delta_2 I_{2p_2}, \ldots, \Delta_m I_{2p_m})$ such that

$$
\Delta \widetilde{Q}^* \widetilde{L}^* + \widetilde{L}^{*T} \widetilde{Q}^* \Delta < 0
\tag{34}
$$

where $\widetilde{Q}^* = \text{diag}(\widetilde{Q}^*_2, \ldots, \widetilde{Q}^*_m)$ and Δ_j are positive constants, $j = 2, \ldots, m$.

Proof: Let Φ_i be in (35), shown at the bottom of the page. Then, $\Phi_m = \Delta \widetilde{Q}^* \widetilde{L}^* + \widetilde{L}^{*T} \widetilde{Q}^* \Delta$. By (33), it is easy to see that $\Phi_2 < 0$. Next, it is to prove the lemma by induction. Suppose that $\Phi_i < 0$. It suffices to show that $\Phi_{i+1} < 0$. By Lemma 4, $\Phi_{i+1} < 0$ is equivalent to $Q^*_{i+1} \widetilde{L}^*_{i+1,i+1} + \widetilde{L}^{*T}_{i+1,i+1} Q^*_{i+1} < 0$ according to (33) and

$$
\Phi_i - \Delta_{i+1} \Pi_{i+1} \left(Q^*_{i+1} \widetilde{L}^*_{i+1,i+1} + \widetilde{L}^{*T}_{i+1,i+1} Q^*_{i+1} \right)^{-1} \Pi_{i+1}^T < 0
\tag{36}
$$

where $\Pi_{i+1}^T = Q^*_{i+1}(\widetilde{L}^*_{(i+1)2}, \widetilde{L}^*_{(i+1)3}, \ldots, \widetilde{L}^*_{(i+1)i})$. If Δ_{i+1} is sufficiently smaller than Δ_j for $j < i + 1$, then (36) can be satisfied. Therefore, by choosing Δ_{i+1} sufficiently smaller than

$$
\Phi_i = \begin{pmatrix} \Delta_2 \left(Q^*_2 \widetilde{L}^*_{22} + \widetilde{L}^{*T}_{22} Q^*_2 \right) & \Delta_3 \widetilde{L}^{*T}_{32} Q^*_3 & \cdots & \Delta_i \widetilde{L}^{*T}_{i2} Q^*_i \\ \Delta_3 Q^*_3 \widetilde{L}^*_{32} & \Delta_3 \left(Q^*_3 \widetilde{L}^*_{33} + \widetilde{L}^{*T}_{33} Q^*_3 \right) & \cdots & \Delta_i \widetilde{L}^{*T}_{i3} Q^*_i \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_i Q^*_i \widetilde{L}^*_{i2} & \Delta_i Q^*_i \widetilde{L}^*_{i3} & \cdots & \Delta_i \left(Q^*_i \widetilde{L}^*_{ii} + \widetilde{L}^{*T}_{ii} Q^*_i \right) \end{pmatrix}
\tag{35}
$$

Δ_j for $j < i + 1$, the condition in (36) can be satisfied. The proof is thus completed. ∎

Lemma 12 [38]: If L is irreducible, $L_{ij} = L_{ji} \geq 0$ for $i \neq j$, and $\sum_{j=1}^{N} L_{ij} = 0$ for all $i = 1, 2, \ldots, N$, then all eigenvalues of the matrix

$$\begin{pmatrix} L_{11} + \varepsilon & L_{12} & \ldots & L_{1N} \\ L_{21} & L_{22} & \ldots & L_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ L_{N1} & L_{N2} & \ldots & L_{NN} \end{pmatrix}$$

are positive for any constant $\varepsilon > 0$.

Definition 8: For a network containing a directed spanning tree and the Laplacian matrix in the form of (29), the general algebraic connectivity of the ith strongly connected component ($2 \leq i \leq m$) is defined by

$$\begin{aligned} b(\overline{L}_i) &= \min_{x \neq 0} \frac{x^{\mathrm{T}} \widehat{\overline{L}_i} x}{x^{\mathrm{T}} \overline{\Xi}_i x} \\ &= \min_{x \neq 0} \frac{(\sqrt{\overline{\Xi}_i} x)^{\mathrm{T}} \sqrt{\overline{\Xi}_i}^{-1} \widehat{\overline{L}_i} \sqrt{\overline{\Xi}_i}^{-1} (\sqrt{\overline{\Xi}_i} x)}{(\sqrt{\overline{\Xi}_i} x)^{\mathrm{T}} (\sqrt{\overline{\Xi}_i} x)} \\ &= \min_{y \neq 0} \frac{y^{\mathrm{T}} \sqrt{\overline{\Xi}_i}^{-1} \widehat{\overline{L}_i} \sqrt{\overline{\Xi}_i}^{-1} y}{y^{\mathrm{T}} y} \\ &= \lambda_{\min} \sqrt{\overline{\Xi}_i}^{-1} \widehat{\overline{L}_i} \sqrt{\overline{\Xi}_i}^{-1} \end{aligned} \tag{37}$$

where $\widehat{\overline{L}_i} = (\overline{\Xi}_i \overline{L}_i + \overline{L}_i^{\mathrm{T}} \overline{\Xi}_i)/2$, $\overline{\Xi}_i = \mathrm{diag}(\overline{\xi}_{i1}, \ldots, \overline{\xi}_{iq_i})$, $\sqrt{\overline{\Xi}_i} = \mathrm{diag}(\sqrt{\overline{\xi}_{i1}}, \ldots, \sqrt{\overline{\xi}_{iq_i}})$ $\overline{\xi}_i = (\overline{\xi}_{i1}, \ldots, \overline{\xi}_{iq_i})^{\mathrm{T}} > 0$, and $\overline{\xi}_i^{\mathrm{T}} \overline{L}_i = 0$ with $\sum_{j=1}^{q_i} \overline{\xi}_{ij} = 1$.

Lemma 13: If the Laplacian matrix L has a directed spanning tree, then $\min_{2 \leq j \leq m} \{a(\overline{L}_1), b(\overline{L}_j)\} > 0$.

Proof: From Lemma 7, one has $a(\overline{L}_1) > 0$. It suffices to prove that $\min_{2 \leq j \leq m} b(\overline{L}_j) > 0$. Note that

$$\begin{aligned} b(\overline{L}_i) &= \lambda_{\min} \sqrt{\overline{\Xi}_i}^{-1} \widehat{\overline{L}_i} \sqrt{\overline{\Xi}_i}^{-1} \\ &= \lambda_{\min} \sqrt{\overline{\Xi}_i}^{-1} \left(\frac{\overline{\Xi}_i \overline{L}_{ii} + \overline{L}_{ii}^{\mathrm{T}} \overline{\Xi}_i}{2} + A_i \overline{\Xi}_i \right) \sqrt{\overline{\Xi}_i}^{-1} \end{aligned}$$

where $(\overline{\Xi}_i \overline{L}_{ii} + \overline{L}_{ii}^{\mathrm{T}} \overline{\Xi}_i)/2$ is a zero-row-sum symmetric matrix and $A_i \geq 0$. By Lemma 10, there is at least one positive diagonal entry in A_i. According to Lemma 11, $b(\overline{L}_i) > 0$ for all $i = 2, \ldots, m$. ∎

Theorem 2: Suppose that the network contains a directed spanning tree and Assumption 1 holds. Then, second-order consensus in system (3) is achieved if

$$\min_{2 \leq j \leq m} \{a(\overline{L}_1), b(\overline{L}_j)\}$$

$$> \frac{1}{2} \left(\frac{\rho_1}{\alpha} + \frac{\alpha}{\beta^2} + \frac{\rho_1}{\beta} + \sqrt{\left(\frac{\rho_1}{\alpha} - \frac{\alpha}{\beta^2} - \frac{\rho_1}{\beta} \right)^2 + \frac{(\alpha + \beta)^2 \rho_2^2}{\alpha^2 \beta^2}} \right). \tag{38}$$

Proof: From Theorem 1, one knows that under condition (38), second-order consensus can be achieved in the first strongly connected component. Thus, all the states of the nodes in this component satisfy (30).

Consider the Lyapunov functional candidate

$$V(t) = \sum_{i=2}^{m} \Delta_i e_i^{\mathrm{T}}(t) (Q_i^* \otimes I_n) e_i(t) \tag{39}$$

where $Q_i^* = \begin{pmatrix} 2\alpha \overline{L}_j & \frac{\alpha \overline{\Xi}_i}{\beta} \\ \frac{\alpha \overline{\Xi}_i}{\beta} & \overline{\Xi}_i \end{pmatrix}$, $\overline{\Xi}_i = \mathrm{diag}(\overline{\xi}_{i1}, \overline{\xi}_{i2}, \ldots, \overline{\xi}_{iq_i})$, and Δ_i are positive constants, $i = 2, \ldots, m$.

Taking the derivative of $V(t)$ along (31) and using (23)–(25), one gets (40), shown at the bottom of the page, where $\|e_i\| = (\|\widetilde{x}_{s_{i-1}}\|, \ldots, \|\widetilde{x}_{s_i}\|, \|\widetilde{v}_{s_{i-1}}^{\mathrm{T}}\|, \ldots, \|\widetilde{v}_{s_i}\|)^{\mathrm{T}}$

$$\Gamma_i = \begin{pmatrix} \frac{\alpha}{\beta} \rho_1 \overline{\Xi}_i - \frac{\alpha^2}{\beta} b(\overline{L}_i) \overline{\Xi}_i & \frac{1}{2} \left(\frac{\alpha}{\beta} + 1 \right) \rho_2 \overline{\Xi}_i \\ \frac{1}{2} \left(\frac{\alpha}{\beta} + 1 \right) \rho_2 \overline{\Xi}_i & \left(\rho_1 + \frac{\alpha}{\beta} \right) \overline{\Xi}_i - \beta b(\overline{L}_i) \overline{\Xi}_i \end{pmatrix}$$

and Γ_{ij} are matrices with appropriate dimensions, $2 \leq i, j \leq m$. From Lemma 11, one knows that if $\Gamma_i < 0$ ($2 \leq i \leq m$) and by choosing Δ_{i+1} sufficiently smaller than Δ_j for $j < i + 1$, then second-order consensus can be achieved in system (3). This completes the proof. ∎

$$\begin{aligned} \dot{V} &= 2 \sum_{i=2}^{m} \Delta_i e_i^{\mathrm{T}}(t) (Q_i^* \otimes I_n) \dot{e}_i(t) \leq 2 \sum_{i=2}^{m} \Delta_i e_i^{\mathrm{T}}(t) (Q_i^* \otimes I_n) \left[F_i^*(x, v, t) + \sum_{j=1}^{i} \left(\widetilde{L}_{ij}^* \otimes I_n \right) e_j(t) + \mathcal{O}(e^{\epsilon t}) \right] \\ &\leq 2 \sum_{i=2}^{m} \Delta_i \|e_i\| \begin{pmatrix} \frac{\alpha}{\beta} \rho_1 \overline{\Xi}_i & \frac{1}{2} \left(\frac{\alpha}{\beta} + 1 \right) \rho_2 \overline{\Xi}_i \\ \frac{1}{2} \left(\frac{\alpha}{\beta} + 1 \right) \rho_2 \overline{\Xi}_i & \rho_1 \overline{\Xi}_i \end{pmatrix} \|e_i\| + 2 \sum_{i=2}^{m} \Delta_i e_i^{\mathrm{T}}(t) \left(Q_i^* \widetilde{L}_{ii}^* \otimes I_n \right) e_i(t) \\ &\quad + 2 \sum_{i=2}^{m} \Delta_i \left\| e_i^{\mathrm{T}}(t) (Q_i^* \otimes I_n) \left[\sum_{j=1}^{i-1} \left(\widetilde{L}_{ij}^* \otimes I_n \right) e_j(t) + \mathcal{O}(e^{\epsilon t}) \right] \right\| \\ &\leq 2 \sum_{i=2}^{m} \Delta_i \|e_i\| \Gamma_i \|e_i\| + 2 \sum_{i=2}^{m} \sum_{j=1}^{i-1} \Delta_i \|e_i\| \left[\Gamma_{ij} \|e_j\| + \mathcal{O}(e^{\epsilon t}) \right] \end{aligned} \tag{40}$$
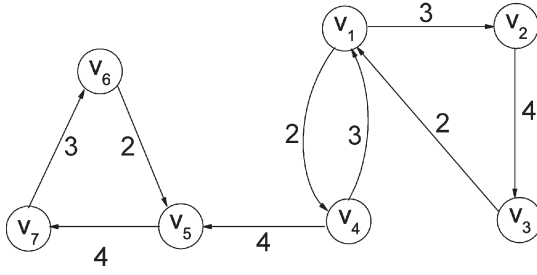
Fig. 1.   Network structure of a network with a directed spanning tree.

Similarly, as in Corollary 3, one can prove the following general result.

Corollary 4: Suppose that the network has a directed spanning tree. Then

$$\min_{2 \leq j \leq m} \left\{ a(\overline{L}_1), b(\overline{L}_j) \right\} \leq \min_{2 \leq i \leq N} \frac{\mathcal{R}(\mu_i) \left[\mathcal{R}^2(\mu_i) + \mathcal{I}^2(\mu_i) \right]}{\mathcal{I}^2(\mu_i)}$$

$$= \min_{2 \leq i \leq N} \left[\mathcal{R}(\mu_i) + \frac{\mathcal{R}^3(\mu_i)}{\mathcal{I}^2(\mu_i)} \right]. \quad (41)$$

Remark 3: In addition to the general algebraic connectivity $a(\overline{L}_1)$ in strongly connected networks, the general algebraic connectivity $b(\overline{L}_i)$ $(2 \leq i \leq m)$ in each strongly connected component of a directed network has also been defined here. It is shown in Theorem 2 that $\min_{2 \leq j \leq m} \{a(\overline{L}_1), b(\overline{L}_j)\}$ plays a key role in reaching consensus and can be used to describe the consensus ability in a network with fixed structure. As a byproduct, (41) is obtained, which is useful in algebraic graph theory in its own right.

Remark 4: Theorem 2 can also be used to study various leader–follower multiagent systems. Suppose that the network has a directed spanning tree and the first strongly connected component has only one node that is a root of the directed network. Then, all the states of the followers converge to that of the leader if

$$\min_{2 \leq j \leq m} \left\{ b(\overline{L}_j) \right\}$$

$$> \frac{1}{2} \left(\frac{\rho_1}{\alpha} + \frac{\alpha}{\beta^2} + \frac{\rho_1}{\beta} + \sqrt{\left(\frac{\rho_1}{\alpha} - \frac{\alpha}{\beta^2} - \frac{\rho_1}{\beta} \right)^2 + \frac{(\alpha+\beta)^2 \rho_2^2}{\alpha^2 \beta^2}} \right).$$

V. SIMULATION EXAMPLES

In this section, a simulation example is given to demonstrate the potentials of our theoretical analysis.

Consider the second-order consensus protocol with time-varying velocities in system (3), where the network structure is shown in Fig. 1 with the weights on the connections. The nonlinear function f is described by Chua's circuit [6]

$$f(x_i(t), v_i(t), t)$$

$$= (\varsigma(-v_{i1} + v_{i2} - l(v_{i1})), v_{i1} - v_{i2} + v_{i3}, -\varrho v_{i2}) \quad (42)$$
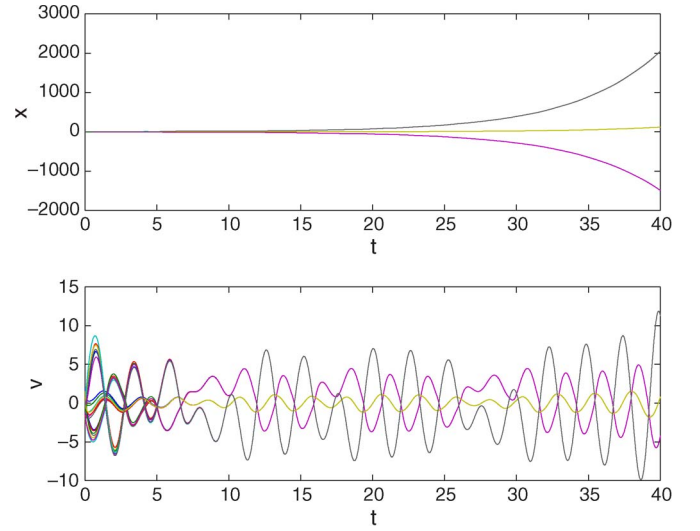

Fig. 2.   Position and velocity states of multiple agents in a network with time-varying velocities.

where $l(v_{i1}) = b v_{i1} + 0.5(a - b)(|v_{i1} + 1| - |v_{i1} - 1|)$. The isolated system (42) is chaotic when $\varsigma = 10$, $\varrho = 18$, $a = -4/3$, and $b = -3/4$, as shown in [35]. In view of Assumption 1, by computation, one obtains $\rho_1 = 4.3871$ and $\rho_2 = 0$. Let $\alpha = 5$ and $\beta = 6$. From Fig. 1, it is easy to see that the network contains a directed spanning tree, where the nodes 1–4 and 5–7 belong to the first and second strongly connected components, respectively. By Lemma 8 and Definition 8, one has $a(\overline{L}_1) = 1.8118$ and $b(\overline{L}_2) = 1.0206$, where $\overline{\xi}_1 = (0.2727, 0.1818, 0.1364, 0.4091)^{\mathrm{T}}$ and $\overline{\xi}_2 = (0.4615, 0.3077, 0.2308)^{\mathrm{T}}$. By Theorem 2, one has that $\min\{a(\overline{L}_1), b(\overline{L}_2)\} = 1.0206 > (\rho_1/\alpha) + (\alpha/\beta^2) + (\rho_1/\beta) + |(\rho_1/\alpha) + (\alpha/\beta^2) + (\rho_1/\beta)| = 0.8701$. Therefore, second-order consensus can be achieved in the multiagent system (3). The position and velocity states of all the agents are shown in Fig. 2.

VI. CONCLUSION

In this paper, some second-order consensus algorithms for multiagent dynamical systems with directed topologies and nonlinear dynamics have been studied. Detailed analysis has been performed on the case in which the second-order dynamics of each agent are determined by both position and velocity terms. Two notions of generalized algebraic connectivity have been introduced to strongly connected network components so as to describe the ability of reaching consensus in a directed network. Some sufficient conditions have also been derived for reaching second-order consensus in multiagent systems with time-varying velocities.

The study of second-order consensus protocols in multiagent systems with directed network topologies is still a challenging problem, and this paper can serve as a stepping stone to study more complicated agent dynamics by combining ideas in algebraic graph theory and control approach. Future works will be on the effects of group behaviors in more complicated networks, such as time-varying networks, stochastic networks, and switching networks, among others.

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Wenwu Yu (S'07) received the B.Sc. degree in information and computing science and the M.Sc. degree in applied mathematics from the Department of Mathematics, Southeast University, Nanjing, China, in 2004 and 2007, respectively. He is currently working toward the Ph.D. degree in the Department of Electronic Engineering, City University of Hong Kong, Kowloon, Hong Kong.

He held several visiting positions in China, Germany, Italy, The Netherlands, and the U.S. He is the author or coauthor of about 30 refereed international journal papers, and a Reviewer of several journals. His research interests include multiagent systems, nonlinear dynamics and control, complex networks and systems, neural networks, cryptography, signal processing, and communications.

Mr. Yu was the recipient of the Best Master Degree Theses Award from Jiangsu Province, China, in 2008 and the Best Student Paper Award in The 5th Chinese Conference on Complex Networks in 2009.

Guanrong Chen (M'89–SM'92–F'97) received the M.Sc. degree in computer science from the Sun Yat-Sen University (Zhongshan), Guangzhou, China, in 1981 and the Ph.D. degree in applied mathematics from Texas A&M University, College Station, in 1987.

He was a tenured Full Professor with the University of Houston, Houston, TX. He is currently a Chair Professor with the Department of Electronic Engineering and the Founding Director of the Centre for Chaos and Complex Networks, City University of Hong Kong, Kowloon, Hong Kong. He is an Honorary Professor at different ranks in more than 20 universities worldwide.

Prof. Chen has served and is serving as Chief Editor, Deputy Chief Editor, Advisory Editor, Features Editor, and Associate Editors for eight international journals, including the IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS, IEEE TRANSACTIONS ON AUTOMATIC CONTROL, and the *International Journal of Bifurcation and Chaos*. He was the recipient of the 1998 Harden–Simons Prize for the Outstanding Journal Paper Award from the American Society of Engineering Education, the 2001 M. Barry Carlton Best Transactions Paper Award from the IEEE Aerospace and Electronic Systems Society, the 2002 Best Paper Award from the Institute of Information Theory and Automation, Academy of Science of the Czech Republic, and the 2005 IEEE Guillemin–Cauer Best Transactions Paper Award from the Circuits and Systems Society. He was also the recipient of the second prize of the 2008 State Natural Science Award of China.

Ming Cao (S'02–M'08) received the B.S. and M.S. degrees in electrical engineering from Tsinghua University, Beijing, China, in 1999 and 2002, respectively, and the Ph.D. degree in electrical engineering from Yale University, New Haven, CT, in 2007.

During the Summer of 2006, he was a Research Intern with the Mathematical Sciences Department, IBM T. J. Watson Research Center, Yorktown Heights, NY. From September 2007 to August 2008, he was a Postdoctoral Research Associate with the Department of Mechanical and Aerospace Engineering, Princeton University, Princeton, NJ. He is currently an Assistant Professor with the Faculty of Mathematics and Natural Sciences, Industrial Technology and Management, University of Groningen, Groningen, The Netherlands. His main research interest is in autonomous agents and multiagent systems, mobile sensor networks, and social robotics.

Dr. Cao is an Associate Editor for *Systems and Control Letters*.

Jürgen Kurths received the Diploma degree in mathematics from the University of Rostock, Rostock, Germany, the Ph.D. degree from the Academy of Sciences of the German Democratic Republic, Potsdam, Germany, in 1983, and the Dr.Habil. degree from the University of Rostock in 1990.

From 1994 to 2008, he was a Full Professor with the University of Potsdam, Potsdam. He is currently a Professor of nonlinear dynamics with Humboldt University of Berlin, Berlin, Germany, and the Chair of the Research Domain IV—Transdisciplinary Concepts and Methods, Potsdam Institute for Climate Impact Research (PIK), Potsdam. His main research interests include complex-synchronization phenomena, complex networks, and time-series analysis and their applications in climatology, sustainability, physiology, and engineering. He has published more than 350 papers and two monographs which are cited more than 12 000 times. He is in the editorial board of more than ten journals, among them *CHAOS*, *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, *European Journal of Physics: Special Topics*, and *Nonlinear Processes in Geophysics*.

Dr. Kurths is a Fellow of the American Physical Society and of the Fraunhofer Society (Germany). He was the recipient of the Humboldt-CSIR Research Price in 2005 and a Dr.h.c. in 2008.