# Second-Order Correct Boundary Conditions for the Numerical Solution of the Mixed Boundary Problem for Parabolic Equations

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**1. Introduction.** Consider the parabolic equation

$$(1)\quad \frac{\partial^2 u}{\partial x^2} - a(x,t)\frac{\partial u}{\partial t} + b(x,t)\frac{\partial u}{\partial x} + c(x,t)u = d(x,t), \quad 0 < x < 1, 0 < t \leqq T,$$

and the initial condition

$$(2)\quad u(x,0) = f(x), \qquad 0 \leqq x \leqq 1.$$

Assume that $a(x,t) > 0$. It is well known (Douglas [1], Rose [5]) that the Dirichlet problem (1), (2), with boundary conditions

$$(3)\quad \begin{cases} u(0,t) = g(t), \\ u(1,t) = h(t), \end{cases} \qquad 0 < t \leqq T,$$

can be approximated by the solution of the difference equation

$$(4)\quad \begin{aligned} &\Delta_x^2 w_{in} - a_{in}\Delta_t w_{in} + b_{in}\Delta_x w_{in} \\ &\qquad + c_{in} w_{in} = d_{in}, \quad i = 1, \cdots, I-1, n = 1, \cdots, N, \end{aligned}$$

subject to the initial condition

$$(5)\quad w_{i0} = f_i, \qquad i = 0, \cdots, I,$$

and the boundary conditions

$$(6)\quad \begin{cases} w_{0n} = g_n, \\ w_{In} = h_n, \end{cases} \qquad n = 1, \cdots, N.$$

The subscripts $i$ and $n$ indicate that the function is evaluated at the point $(ih, nk)$ where $h = I^{-1}$, $k = TN^{-1}$. The difference operators in (4) are defined by

$$(7)\quad \begin{cases} \Delta_x^2 w_{in} = \dfrac{1}{h^2}(w_{i-1,n} - 2w_{in} + w_{i+1,n}), \\ \Delta_t w_{in} = \dfrac{1}{k}(w_{in} - w_{i,n-1}), \\ \Delta_x w_{in} = \dfrac{1}{2h}(w_{i+1,n} - w_{i-1,n}). \end{cases}$$

If* $u \varepsilon C^{4,2}([0,1] \times [0,T])$, then the error

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* $\varphi(x,t) \varepsilon C^{\alpha,\beta}(R)$ if and only if $\varphi$ is continuously differentiable $\alpha$ times with respect to $x$ and $\beta$ times with respect to $t$ in the region $R$.

$$z_{in} = u_{in} - w_{in} \,, \tag{8}$$

satisfies

$$\max_{i,n} | z_{in} | = O(h^2 + k). \tag{9}$$

If the conditions (3) are replaced by the Neumann conditions

$$\begin{cases} \dfrac{\partial u}{\partial x}(0, t) = g(t), \\ \dfrac{\partial u}{\partial x}(1, t) = h(t), \end{cases} \qquad 0 < t \leqq T, \tag{10}$$

then the solution $w_{in}$ of equations (4) and (5) with boundary conditions

$$\begin{cases} \dfrac{w_{1n} - w_{0n}}{h} = g_n \,, & n = 1, \cdots, N, \\ \dfrac{w_{In} - w_{I-1,n}}{h} = h_n \,, & n = 1, \cdots, N, \end{cases} \tag{11}$$

converges to $u_{in}$, but the error is $O(h + k)$ (Douglas [3]). From the analysis, it is clear that the $h$ (instead of $h^2$) arises in the first order correctness of the boundary conditions.

Recently, Isaacson [4] has shown that an approximation that is second order correct in $h$ can be obtained by replacing conditions (11) with

$$\begin{cases} \dfrac{w_{1n} - w_{-1,n}}{2h} = g_n \,, & n = 1, \cdots, N, \\ \dfrac{w_{I+1,n} - w_{I-1,n}}{2h} = h_n \,, & n = 1, \cdots, N. \end{cases} \tag{12}$$

This result is not entirely pleasing, however, for it requires the assumption that $u$ can be extended to satisfy sufficient continuity conditions in $[-h, 1 + h] \times [0, T]$.

**2. Interior Approximations.** In the present paper, it is shown that if the centered differences in (12) are replaced by one-sided, second order correct differences, the error is $O(h^2 + k)$. This result applies (as do those mentioned above) if the Neumann conditions (10) are replaced by the mixed boundary conditions

$$\begin{aligned} -p(t)u(0, t) + q(t) \frac{\partial u}{\partial x}(0, t) &= g(t), & 0 < t \leqq T, \\ -r(t)u(1, t) - s(t) \frac{\partial u}{\partial x}(1, t) &= h(t), & 0 < t \leqq T. \end{aligned} \tag{13}$$

It is necessary to assume that $p$, $q$, $r$, and $s$ are non-negative, and that $p + q$ and $r + s$ are bounded away from zero. It is not necessary to assume, as do both Isaacson [4] and Rose [6], that one or more of the coefficients $p$, $q$, $r$, $s$ is bounded away from zero.

Assume that the quantities $a$, $b$, $c$, $d$, $p$, $q$, $r$, and $s$ are bounded, and that

$u \,\varepsilon\, C^{4,2}([0, 1] \times [0, T])$. By Taylor's theorem

$$(14)\quad \begin{aligned} &\Delta_x^2 u_{in} - a_{in}\Delta_t u_{in} + b_{in}\Delta_x u_{in} \\ &\qquad + c_{in}u_{in} = d_{in} + A_{in}, \quad i = 1, \cdots, I-1, n = 1, \cdots, N, \end{aligned}$$

where $|A_{in}| < A(h^2 + k)$ and $A$ is a constant. Similarly,

$$(15)\quad \begin{cases} \dfrac{1}{2h}(-3u_{0n} + 4u_{In} - u_{2n}) = \left.\dfrac{\partial u}{\partial x}\right|_{0n} + B_n^+, \\ \dfrac{1}{2h}(u_{I-2,n} - 4u_{I-1,n} + 3u_{In}) = \left.\dfrac{\partial u}{\partial x}\right|_{In} + B_n^-, \end{cases} \qquad n = 1, \cdots, N,$$

where $B_n^+$ and $B_n^-$ are bounded by a constant multiple of $h^2$. For simplicity let

$$(16)\quad \begin{cases} \Delta_x^+ u_{0n} = \dfrac{1}{2h}(-3u_{0n} + 4u_{1n} - u_{2n}), \\ \Delta_x^- u_{In} = \dfrac{1}{2h}(u_{I-2,n} - 4u_{I-1,n} + 3u_{In}). \end{cases}$$

Then

$$(17)\quad \begin{cases} -p_n u_{0n} + q_n\Delta_x^+ u_{0n} = g_n + B_{0n}, \\ -r_n u_{In} - s_n\Delta_x^- u_{In} = h_n + B_{In}, \end{cases} \qquad n = 1, \cdots, N,$$

where $|B_{in}| \leqq Bh^2$ and $B$ is a constant.

Approximate $u_{in}$ by the solution $w_{in}$ of (4) and (5) with boundary conditions

$$(18)\quad \begin{cases} -p_n w_{0n} + q_n\Delta_x^+ w_{0n} = g_n, \\ -r_n w_{in} - s_n\Delta_x^- w_{in} = h_n, \end{cases} \qquad n = 1, \cdots, N.$$

Then the error (8) satisfies

$$(19)\quad \begin{cases} \Delta_x^2 z_{in} - a_{in}\Delta_t z_{in} + b_{in}\Delta_x z_{in} + c_{in}z_{in} = A_{in}, & i = 1, \cdots, I-1, \\ & n = 1, \cdots, N, \\ -p_n z_{0n} + q_n\Delta_x^+ z_{0n} = B_{0n}, & n = 1, \cdots, N, \\ -r_n z_{In} - s_n\Delta_x^- z_{In} = B_{In}, & n = 1, \cdots, N, \\ z_{i0} = 0, & i = 1, \cdots, I-1. \end{cases}$$

In order to bound $z_{in}$ we prove the following lemmas.

LEMMA 1. *Let* $v_{in}$ *satisfy*

$$(20)\quad \begin{cases} \Delta_x^2 v_{in} - a_{in}\Delta_t v_{in} + b_{in}\Delta_x v_{in} + c_{in}v_{in} \leqq 0, & i = 1, \cdots, I-1, \\ & n = 1, \cdots, N, \\ -p_n v_{0n} + q_n\Delta_x^+ v_{0n} \leqq 0, & n = 1, \cdots, N, \\ -r_n v_{In} - s_n\Delta_x^- v_{In} \leqq 0, & n = 1, \cdots, N, \\ v_{i0} \geqq 0, & i = 1, \cdots, I-1. \end{cases}$$

*If, for all* $i = 1, \cdots, I - 1, n = 1, \cdots, N$,

$$
(21) \qquad \begin{cases} 0 < \alpha \leqq a_{in}, \\ |b_{in}| < \beta \leqq \dfrac{1}{h}, \\ 0 \leqq -c_{in} < \gamma \leqq \dfrac{\alpha}{k} \\ p_n, q_n, r_n, s_n \geqq 0, \\ p_n + q_n > 0, \\ r_n + s_n > 0, \end{cases}
$$

*where* $\alpha, \beta, \gamma$ *are constants, then*

$$
(22) \qquad v_{in} \geqq 0, \qquad i = 1, \cdots, I - 1, n = 1, \cdots, N.
$$

Note that $v_{in}$ is non-negative only in the *interior* of the region. With little difficulty, one can construct examples for which (22) holds, but for which $v_{0n} < 0$ and $v_{1n} < 0$, for some $n$.

*Proof*. Suppose the lemma is false. Let

$$
(23) \qquad n_0 = \min\{n \mid v_{in} < 0 \quad \text{for some} \quad i, \qquad 1 \leqq i \leqq I - 1\}.
$$

Then $n_0 \geqq 1$. Let $i_0$ denote a value such that $v_{i_0,n_0}$ is a local negative minimum with respect to $i$. There are three cases.

Case 1: $i_0 = 0$. Since $v_{0,n_0} < 0$,

$$
(24) \qquad 0 \geqq -p_{n_0}v_{0,n_0} + q_{n_0}\Delta_x^+ v_{0,n_0} \geqq q_{n_0}\Delta_x^+ v_{0,n_0}.
$$

If $q_{n_0} = 0$, then $p_{n_0} > 0$ and it follows that $v_{0,n_0} \geqq 0$, contradicting the hypothesis. Thus $q_{n_0} > 0$, and by (24),

$$
(25) \qquad \Delta_x^+ v_{0,n_0} \leqq 0.
$$

Therefore,

$$
(26) \qquad \begin{aligned} \Delta_x^2 v_{1,n_0} &= \frac{1}{h^2}(v_{0,n_0} - 2v_{1,n_0} + v_{2,n_0}) \\ &= \frac{1}{h}\left[\frac{1}{2h}(v_{2,n_0} - v_{0,n_0}) - \Delta_x^+ v_{0,n_0}\right] \\ &\geqq \frac{1}{h}\left[\frac{1}{2h}(v_{2,n_0} - v_{0,n_0})\right] \\ &= \frac{1}{h}\Delta_x v_{1,n_0}. \end{aligned}
$$

From the second of conditions (21),

$$
(27) \qquad \Delta_x^2 v_{1,n_0} \geqq -b_{in}\Delta_x v_{1,n_0}.
$$

Thus, the first of inequalities (20) gives

$$
(28) \qquad -a_{1,n_0}\Delta_t v_{1,n_0} + c_{1,n_0}v_{1,n_0} \leqq 0.
$$

Since $c_{1,n_0} \leqq 0$ and $v_{1,n_0-1} \geqq 0$, it follows that $v_{1,n_0} \geqq 0$.

Case 2: $i_0 = I$. By an argument analogous to that of case 1, it follows that $v_{I-1,n_0} \geqq 0$.

Case 3: $1 \leqq i_0 \leqq I - 1$. Here a maximum principle argument is used. From the first of inequalities (20),

$$\begin{aligned}\left(\frac{1}{k}a_{i_0,n_0} - c_{i_0,n0}\right)v_{i_0,n_0} &\geqq \left(\frac{1}{h^2} - \frac{1}{2h}b_{i_0,n_0}\right)\left(v_{i_0-1,n_0} - v_{i_0,n_0}\right) \\ &+ \left(\frac{1}{h^2} + \frac{1}{2h}b_{i_0,n_0}\right)\left(v_{i_0+1,n_0} - v_{i_0,n_0}\right) + \frac{1}{k}a_{i_0,n_0}v_{i_0,n_0-1}.\end{aligned} \tag{29}$$

Since every term on the right is non-negative, it follows that $v_{i_0,n_0} \geqq 0$. This is a contradiction. Q. E. D.

LEMMA 2. *Under conditions* (21) *and the conditions*

i) *for some* $\delta$,

$$\tfrac{1}{4}p_n + q_n \geqq \delta > 0 \quad and \quad \tfrac{1}{4}r_n + s_n \geqq \delta > 0, \tag{30}$$

ii) $k < \dfrac{\alpha}{4\gamma}$,

*there exists a funtcion* $\zeta(x, t)$ *such that*

$$\begin{cases}\Delta_x^2\zeta_{in} - a_{in}\Delta_t\zeta_{in} + b_{in}\Delta_x\zeta_{in} + c_{in}\zeta_{in} \leqq -1, & i = 1, \cdots, I-1, \\ & n = 1, \cdots, N, \\ -p_n\zeta_{0n} + q_n\Delta_x^+\zeta_{0n} \leqq -1, & n = 1, \cdots, N, \\ -r_n\zeta_{In} - s_n\Delta_x^-\zeta_{In} \leqq -1, & n = 1, \cdots, N, \\ \zeta_{i0} \geqq 0, & i = 1, \cdots, I-1,\end{cases} \tag{31}$$

*and*

$$0 \leqq \zeta(x, t) \leqq M_0, \qquad 0 \leqq x \leqq 1,\ 0 \leqq t \leqq T, \tag{32}$$

*where* $M_0$ *is a constant depending on* $\alpha, \beta, \gamma, \delta$ *and* $T$.

*Proof.* Let

$$\zeta^*(x, t) = (\tfrac{1}{2} - x)^2. \tag{33}$$

Then

$$\begin{aligned}\Delta_x^2\zeta^*_{in} - a_{in}\Delta_t\zeta^*_{in} + b_{in}\Delta_x\zeta^*_{in} + c_{in}\zeta^*_{in} &= 2 - 2b_{in}(\tfrac{1}{2} - x_i) \\ + c_{in}(\tfrac{1}{2} - x_i)^2 &\leqq 2 + \beta + \tfrac{1}{4}\gamma, \quad i = 1, \cdots, I-1, \\ & \qquad n = 1, \cdots, N,\end{aligned} \tag{34}$$

and

$$\begin{cases}-p_n\zeta^*_{0n} + q_n\Delta_x^+\zeta^*_{0n} = -\tfrac{1}{4}p_n - q_n \leqq -\delta, \\ -r_n\zeta^*_{In} - s_n\Delta_x^-\zeta^*_{In} = -\tfrac{1}{4}r_n - s_n \leqq -\delta,\end{cases} \qquad n = 1, \cdots, N. \tag{35}$$

Let $\zeta^{**}(x, t) = e^{\sigma t}$, $\sigma > 0$. Then

$$\begin{aligned}&\Delta_x^2\zeta^{**}_{in} - a_{in}\Delta_t\zeta^{**}_{in} + b_{in}\Delta_x\zeta^{**}_{in} + c_{in}\zeta^{**}_{in} \\ &= e^{\sigma t_n}\left[-\frac{1}{k}a_{in}(1 - e^{-\sigma k}) + c_{in}\right] \leqq e^{\sigma t_n}\left[-\frac{\alpha}{k}(1 - e^{-\sigma k}) + \gamma\right].\end{aligned} \tag{36}$$

By Taylor's theorem,

$$(37)\qquad e^{-\sigma k} = 1 - \sigma k + \frac{\sigma^2 k^2}{2} e^{-\sigma k'} < 1 - \sigma k + \frac{\sigma^2 k^2}{2},$$

where $0 < k' < k$. Since $k < \frac{1}{4}\alpha\gamma^{-1}$, for $\sigma = 2\alpha^{-1}\gamma$ it follows that

$$(38)\quad \Delta_x^2 \zeta_{in}^{**} - a_{in}\Delta_t \zeta_{in}^{**} + b_{in}\Delta_x \zeta_{in}^{**} + c_{in}\zeta_{in}^{**} < -e^{\sigma t_n}\frac{\gamma}{2} < -\frac{\gamma}{2} < 0.$$

Also,

$$(39)\qquad \begin{cases} -p_n\zeta_{0n}^{**} + q_n\Delta_x^{+}\zeta_{0n}^{**} = -p_n e^{\sigma t_n} \leqq 0, \\ -r_n\zeta_{0n}^{**} - s_n\Delta_x^{-}\zeta_{0n}^{**} = -r_n e^{\sigma t_n} \leqq 0. \end{cases}$$

Let $M_1$ and $M_2$ be constants satisfying

$$(40)\qquad \begin{cases} M_1 \geqq \dfrac{1}{\delta}, \\ M_2 \geqq \dfrac{2}{\gamma}\left[1 + M_1\left(2 + \beta + \dfrac{1}{4}\gamma\right)\right]. \end{cases}$$

Then

$$(41)\qquad \zeta = M_1\zeta^{*} + M_2\zeta^{**}$$

satisfies the conditions of the lemma. Q. E. D.

THEOREM 1. *If $u \varepsilon C^{4,2}([0, 1] \times [0, T])$ is a solution of* (1) *with initial condition* (2) *and boundary conditions* (13), *if there exist constants $\alpha, \beta, \gamma, \delta$ such that*

$$(42)\qquad \begin{cases} 0 < \alpha \leqq a(x, t), & 0 < x < 1, 0 < t \leqq T, \\ |b(x, t)| < \beta, & 0 < x < 1, 0 < t \leqq T, \\ 0 \leqq -c(x, t) < \gamma, & 0 < x < 1, 0 < t \leqq T, \\ p(t), q(t), r(t), s(t) \geqq 0, & 0 < t \leqq T, \\ p(t) + q(t) \geqq \delta > 0, & 0 < t \leqq T, \\ r(t) + s(t) \geqq \delta > 0, & 0 < t \leqq T, \end{cases}$$

*and if $h$ and $k$ are sufficiently small, then*

$$(43)\qquad \max_{0 \leqq i \leqq I} |z_{in}| \leqq M(h^2 + k), \qquad n = 1, \cdots, N$$

*where $M$ is a constant that depends on $\alpha, \beta, \gamma, \delta, T$.*

*Proof.* Let $\zeta(x, t)$ be a function given by Lemma 2.

Let $M_3 = \max(A, B)$ and let

$$(44)\qquad \begin{cases} v_{in}^{+} = M_3(h^2 + k)\zeta_{in} + z_{in}, & i = 0, \cdots, I \\ v_{in}^{-} = M_3(h^2 + k)\zeta_{in} - z_{in}, & n = 0, \cdots, N. \end{cases}$$

By (19) if $h$ and $k$ are so small that $\beta \leqq h^{-1}$, $\gamma \leqq \alpha k^{-1}$, and $k < \frac{1}{4}\alpha\gamma^{-1}$, then $v_{in}^{+}$ and

$v_{in}^-$ satisfy the conditions of Lemma 1. Hence

$$\begin{cases} v_{in}^+ \geqq 0, & i = 1, \cdots, I-1, \\ v_{in}^- \geqq 0, & n = 1, \cdots, N, \end{cases} \tag{45}$$

whence

$$|z_{in}| \leqq M_0 M_3(h^2 + k), \qquad i = 1, \cdots, I-1 \quad n = 1, \cdots, N. \tag{46}$$

From (19),

$$\begin{aligned} z_{0n} &= \left(-p_n - \frac{3}{2h} q_n\right)^{-1} \left[\frac{q_n}{2h}(z_{2n} - 4 z_{1n}) + B_{0n}\right] \\ &= -(2h\, p_n + 3\, q_n)^{-1} [q_n(z_{2n} - 4 z_{1n}) + 2h\, B_{0n}] \\ &\leqq M_4 [\max(|z_{1,n}|, |z_{2,n}|) + h^3], \qquad n = 1, \cdots, N, \end{aligned} \tag{47}$$

where $M_4$ is a constant. A similar inequality holds for $z_{In}$. The bound (43) follows from these inequalities and (46). Q. E. D.

THEOREM 2. *If the coefficients satisfy the conditions of Lemma* 1, *the difference system* (4), (5), (18) *has a unique solution.*

*Proof.* Uniqueness is an immediate consequence of Lemma 1. Existence follows by the Fredholm alternative.

**3. Generalizations.** The restriction $c(x, y) \leqq 0$ can be removed as follows. Let $z_{in}$ satisfy (19). Then

$$\zeta_{in} = e^{\lambda t_n} z_{in} \tag{48}$$

satisfies (19) with $c_{in}$ replaced by

$$c_{in}^* = c_{in} - a_{in} \frac{e^{\lambda t_n} - e^{\lambda t_{n-1}}}{k}, \tag{49}$$

with $a_{in}$ replaced by $e^{-\lambda h} a_{in}$, and with each of $A_{in}$, $B_{0n}$, $B_{In}$ multiplied by $e^{\lambda t_n}$.

If $c(x, y)$ is bounded, $\lambda$ can be chosen large enough so that

$$\frac{e^{\lambda t_n} - e^{\lambda t_{n-1}}}{k} > \frac{1}{\alpha} \sup c(x, y) \tag{50}$$

for all $k$ sufficiently small; in particular for $k < \frac{1}{4}\alpha\gamma^{-1}$. Thus $c_{in}^* < 0$. Therefore, Theorem 1 applies to $\zeta_{in}$, and, *a fortiori*, to $z_{in}$.

The arguments above can be extended to the problem, considered by Lotkin [5] and Isaacson [4], of the parabolic equation (1) in two regions $0 < x < x_0$ and $x_0 < x < 1$, with conditions (2), (13) and

$$\begin{cases} u(x_0 -, t) = u(x_0 +, t), \\ \dfrac{\partial u}{\partial x}(x_0 -, t) = \kappa(t) \dfrac{\partial u}{\partial x}(x_0 +, t), \end{cases} \tag{51}$$

the derivatives in the second equation being replaced by either the centered differences (7) or the uncentered difference (16). An appropriate auxiliary function $\zeta$

can be constructed as in the proof of Lemma 2 if $\zeta^*$ in equation (33) is replaced by

$$\zeta^*(x,t) = \begin{cases} x_0^2(1-x)^2, & 0 \leqq x \leqq x_0, \\ x^2(1-x_0)^2, & x_0 < x \leqq 1. \end{cases}$$

**4. The Non-Linear Problem.** The results above can be extended to include the non-linear system

$$\text{(52)} \qquad \begin{cases} F(x, t, u, u_x, u_{xx}, u_t) = 0, & 0 < x < 1, 0 < t \leqq T, \\ G(t, u, u_x) = 0, & x = 0, 0 < t \leqq T, \\ H(t, u, u_x) = 0, & x = 1, 0 < t \leqq T, \\ u(x, 0) = f(x), & 0 \leqq x \leqq 1, \end{cases}$$

provided $F$, $G$, $H$, and $u$ satisfy certain continuity conditions. Indeed, if $u \,\varepsilon\, C^{4,2}([0,1] \times [0,T])$, then

$$\text{(53)} \qquad \begin{cases} F(x, t, u, u_x, u_{xx}, u_t) = F(x, t, u, \Delta_x u + \delta_1, \Delta_x^2 u + \delta_2, \Delta_t u + \delta_3), \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 < x < 1, 0 < t \leqq T, \\ G(t, u, u_x) = G(t, u, \Delta_x^+ u + \delta_4), \qquad x = 0, 0 < t \leqq T, \\ H(t, u, u_x) = H(t, u, \Delta_x^- u + \delta_5), \qquad x = 0, 0 < t \leqq T, \end{cases}$$

where, for some constant $A$,

$$\text{(54)} \qquad \begin{cases} |\delta_1|, |\delta_2|, |\delta_4|, |\delta_5| \leqq Ah^2, \\ \qquad\qquad\qquad |\delta_3| \leqq Ak. \end{cases}$$

Let $w$, an approximation to $u$, satisfy

$$\text{(55)} \qquad \begin{cases} F(x_i, t_n, w_{in}, \Delta_x w_{in}, \Delta_x^2 w_{in}, \Delta_t w_{in}) = 0, \\ \qquad\qquad i = 1, \cdots, I-1, n = 1, \cdots, N, \\ G(t_n, w_{0n}, \Delta_x^+ w_{0n}) = 0, \qquad n = 1, \cdots, N, \\ H(t_n, w_{In}, \Delta_x^- w_{In}) = 0, \qquad n = 1, \cdots, N, \\ \qquad w_{i0} = f_i, \qquad i = 0, \cdots, I. \end{cases}$$

Suppose that $F$, $G$, and $H$ are continuous in $[0, 1] \times [0, T]$, and that the derivatives $F_3$, $F_4$, $F_5$, $F_6$, $G_2$, $G_3$, $H_2$, and $H_3$ exist in $(0, 1) \times (0, T)$. Then the mean value theorem applied to the difference of the respective equations in (53) and (55) yields

$$\text{(56)} \qquad \begin{cases} F_3 \cdot (u_{in} - w_{in}) + F_4 \cdot [\Delta_x(u_{in} - w_{in}) + \delta_1] \\ \qquad + F_5 \cdot [\Delta_x^2(u_{in} - w_{in}) + \delta_2] + F_6 \cdot [\Delta_t(u_{in} - w_{in}) + \delta_3] = 0, \\ \qquad\qquad i = 1, \cdots, I-1, \\ \qquad\qquad n = 1, \cdots, N, \\ G_2 \cdot (u_{0n} - w_{0n}) + G_3 \cdot [\Delta_x^+(u_{0n} - w_{0n}) + \delta_4] = 0, \quad n = 1, \cdots, N, \\ H_2 \cdot (u_{In} - w_{In}) + H_3 \cdot [\Delta_x^-(u_{In} - w_{In}) + \delta_5] = 0, \quad n = 1, \cdots, N, \\ (u_{i0} - w_{i0}) = 0, \quad i = 0, \cdots, I, \end{cases}$$

where the values of the arguments of $F$, $G$, and $H$ lie between the values of the corresponding arguments in (53) and (55). Assume that all derivatives $F_3$, $F_4$, $F_5$, $F_6$, $G_2$, $G_3$, $H_2$, and $H_3$ are bounded, and that the relations

$$(57) \qquad \begin{cases} F_5 > 0, \\ \dfrac{F_6}{F_5} \leqq -\alpha < 0, \\ -G_2, G_3, -H_2, -H_3 \geqq 0, \\ -G_2 + G_3 \geqq \delta > 0, \\ -H_2 - H_3 \geqq \delta > 0, \end{cases}$$

hold throughout $[0, 1] \times [0, T]$. Then it is seen that equations (56) are identical with equations (19) (except that the coefficients now depend on $u$ and $w$ as well as $x$ and $t$) and that Theorem 1 holds. Thus the error is $O(h^2 + k)$.

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