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SECOND-ORDER DIFFERENTIAL EQUATIONS: CONDITIONS OF COMPLETE INTEGRABILITY [1]

Vasilij Petrovich Ermakov (1845–1922)

from Lectures on Integration of Differential Equations

1

Linear second-order equations with variable coefficients can be completely integrated only in very rare cases. We consider the most important of them.

To begin we prove that, if a particular integral of the equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+A\frac{\mathrm{d}y}{\mathrm{d}x}+By=0 \tag{1}$$

is known, then the determination of a complete integral is reduced to a quadrature.

Let u be a particular integral of this equation, *ie*

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2}+A\frac{\mathrm{d}u}{\mathrm{d}x}+Bu=0. \tag{2}$$

We eliminate B from (1) and (2) to obtain

$$u\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-y\,\frac{\mathrm{d}^2u}{\mathrm{d}x^2}+A\left(u\,\frac{\mathrm{d}y}{\mathrm{d}x}-y\,\frac{\mathrm{d}u}{\mathrm{d}x}\right)=0. \tag{3}$$

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[1] Translated from Russian by A. O. Harin, under redaction by P. G. L. Leach

A first integral of (3) is

$$u\,\frac{\mathrm{d}y}{\mathrm{d}x}-y\,\frac{\mathrm{d}u}{\mathrm{d}x}=C_1\exp\left(-\int A\,\mathrm{d}x\right).$$

When we integrate again, we obtain the complete integral[2]

$$y=C_1u\int\exp\left(-\int A\,\mathrm{d}x\right)\frac{\mathrm{d}x}{u^2}+C_2u.$$

2

Any linear differential equation of the second order, *videlicet*

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+A\,\frac{\mathrm{d}y}{\mathrm{d}x}+By=0,$$

can always be reduced by a transformation of the dependent variable to a form in which the first derivative is absent. Explicitly we set

$$y=z\exp\left(-\frac{1}{2}\int A\,\mathrm{d}x\right)$$

to obtain

$$\frac{\mathrm{d}^2z}{\mathrm{d}x^2}=\left(\frac{1}{4}\,A^2+\frac{1}{2}\,\frac{\mathrm{d}A}{\mathrm{d}x}-B\right)z.$$

We see below that this form makes it easy to discover conditions of integrability for differential equations.

3

The majority of differential equations for which it is possible to find conditions of integrability reduce to the form

$$(4)\qquad \big(Ax^2+Bx+C\big)\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(Dx+E)\,\frac{\mathrm{d}y}{\mathrm{d}x}+Fy=0,$$

where the uppercase coefficients are parameters independent of the variables, x and y. When we take the nth derivative of (4) and set

$$z=\frac{\mathrm{d}^ny}{\mathrm{d}x^n},$$

we obtain

$$(5)\quad \big(Ax^2+Bx+C\big)\,\frac{\mathrm{d}^2z}{\mathrm{d}x^2}\big((D+2An)x+E+Bx\big)\frac{\mathrm{d}z}{\mathrm{d}x}+\big(F+An+Dn+An^2\big)\,z=0.$$

[2]Editor's Note: The integral for the second solution is known as Abel's formula.

Thus, *if an integral of* (4) *is known, we can find an integral of* (5) *when n is a natural number.*

An integral of (4) can always be found in the case that $F = 0$, *ie* the equation has the form

$$(6) \qquad \left(Ax^2 + Bx + C\right) \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (Dx + E) \frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

We write

$$\int \frac{Dx + E}{Ax^2 + Bx + C} \,\mathrm{d}x = -\varphi(x).$$

Then we determine an integral of (5) of the form

$$y = \alpha \int \exp\left(\varphi(x)\right) \mathrm{d}x + \beta,$$

where α and β are the arbitrary constants of integration. Consequently we have proven that, *if n is a natural number, then a particular integral of the equation*

$$\left(Ax^2 + Bx + C\right) \frac{\mathrm{d}^2 z}{\mathrm{d}x^2} \left((D + 2An)x + E + Bx\right) \frac{\mathrm{d}z}{\mathrm{d}x} + n\,(D + A + An)\, z = 0$$

is given by the formula

$$z = \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}} \Big(\exp\left(\varphi(x)\right) \Big).$$

4

We pass to a more thorough investigation of particular cases. From what we proved above it is evident that the differential equation,

$$(7) \quad (x+a)(x+b) \frac{\mathrm{d}^2 z}{\mathrm{d}x^2} + \left((n-\lambda)(x+b) + (n-\mu)(x+a)\right) \frac{\mathrm{d}z}{\mathrm{d}x} + n(n-1-\lambda-\mu)z = 0,$$

is completely integrable if n is a natural number. In the present case

$$\varphi(x) = \int \frac{\lambda(x+b) + \mu(x+a)}{(x+a)(x+b)} \,\mathrm{d}x = \lambda \log(x+a) + \mu \log(x+b).$$

Hence a particular integral of the equation is given by the formula

$$z = \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}} \left((x+a)^{\lambda} (x+b)^{\mu}\right).$$

We apply the transformation of §2 to (7). When we set

$$z = (x+a)^{(\lambda-n)/2} (x+b)^{(\mu-n)/2} y,$$

we reduce (7) to the form

$$(x+a)(x+b)\,\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = \Bigg(\frac{(n-\lambda-1)^2-1}{4}\,\frac{b-a}{x+a} + \frac{(n-\mu-1)^2-1}{4}\,\frac{a-b}{x+b} + \frac{(\lambda+\mu+1)^2-1}{4}\Bigg) z.$$

When we compare this equation with

$$(x+a)(x+b)\,\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \left(\frac{\alpha}{x+a} + \frac{\beta}{x+b} + \gamma\right) y,$$

we obtain three algebraic equations the solutions of which are

$$\begin{aligned} n &= \frac{1}{2}\left(1 \pm \sqrt{1+\frac{4\alpha}{b-a}} \pm \sqrt{1+\frac{4\beta}{a-b}} \pm \sqrt{1+4\gamma}\right), \\ \lambda &= -\frac{1}{2}\left(1 \pm \sqrt{1+\frac{4\alpha}{b-a}} \mp \sqrt{1+\frac{4\beta}{a-b}} \mp \sqrt{1+4\gamma}\right), \\ \mu &= -\frac{1}{2}\left(1 \mp \sqrt{1+\frac{4\alpha}{b-a}} \pm \sqrt{1+\frac{4\beta}{a-b}} \mp \sqrt{1+4\gamma}\right). \end{aligned} \tag{8}$$

Before each of the roots in these equations either upper or lower sign can be taken so that altogether there are eight solutions which leads to the following result.

To find the complete integrability conditions of the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{Ax^2+Bx+C}{(x+a)^2(x+b)^2}\, y \tag{9}$$

decompose the fraction

$$\frac{Ax^2+Bx+C}{(x+a)(x+b)}$$

into partial fractions according to

$$\frac{Ax^2+Bx+C}{(x+a)(x+b)} = \frac{\alpha}{x+a} + \frac{\beta}{x+b} + \gamma.$$

The equation can be completely integrated if one of the eight expressions

$$\sqrt{1+\frac{4\alpha}{b-a}} \pm \sqrt{1+\frac{4\beta}{a-b}} \pm \sqrt{1+4\gamma}$$

is an odd integer.

If this condition be satisfied, a particular integral of (9) is

$$y = (x+a)^{(n-\lambda)/2}(c+b)^{(n-\mu)/2}\,\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}\big((a+a)^{\lambda}(x+b)^{\mu}\big), \tag{10}$$

where n, λ and μ are given by the formulæ (8) in which, naturally, signs should be chosen so that n be a positive integer. If two of the numbers in (10) are odd integers, we can find two independent particular integrals and consequently the complete integral without the use of quadratures.

5

The conditions of integrability found above are not unique. We demonstrate the existence of other conditions.

It is easy to verify that the complete integral of the equation

$$(t^2+b-a)\,\frac{\mathrm{d}^2z}{\mathrm{d}t^2}+t\,\frac{\mathrm{d}z}{\mathrm{d}t}-\delta^2z=0 \tag{11}$$

is given by the formula

$$C_1\left(t+\sqrt{t^2+b-a}\right)^{\delta}+C_2\left(t-\sqrt{t^2+b-a}\right)^{\delta}. \tag{12}$$

As was proven in §3, the nth derivative of (12) is the complete integral of the equation

$$(t^2+b-a)\,\frac{\mathrm{d}^2z}{\mathrm{d}t^2}+(2n+1)\,t\,\frac{\mathrm{d}z}{\mathrm{d}t}+\left(n^2-\delta^2\right)z=0. \tag{13}$$

The nth derivative of (12) is nothing but the coefficient of u^n in the expansion of the expression,

$$C_1\left(t+u+\sqrt{t^2+b-a}\right)^{\delta}+C_2\left(t+u-\sqrt{t^2+b-a}\right)^{\delta}, \tag{14}$$

in increasing powers of u.

Under the change of independent variable $t \longrightarrow \sqrt{x+a}$ equation (13) becomes

$$(x+a)(x+b)\,\frac{\mathrm{d}^2z}{\mathrm{d}x^2}+\frac{1}{2}\left((x+b)+(2n+1)(x+a)\right)\frac{\mathrm{d}z}{\mathrm{d}x}+\frac{1}{4}\left(n^2-\delta^2\right)z=0. \tag{15}$$

When we substitute for t into (14), we find that the complete integral of (15) is the coefficient of u^n in the expansion of the expression

$$\begin{aligned}C_1&\left(\sqrt{x+a}+u+\sqrt{x+b+2u\sqrt{x+a}+u^2}\,\right)^{\delta}\\&+C_2\left(\sqrt{x+a}+u-\sqrt{x+b+2u\sqrt{x+a}+u^2}\,\right)^{\delta}\end{aligned} \tag{16}$$

in increasing powers of u.

When we take the mth derivative of (15) and set $s = \mathrm{d}^m z/\mathrm{d}x^m$, we obtain

$$(17)\qquad (x+a)(x+b)\,\frac{\mathrm{d}^2 s}{\mathrm{d}x^2} + \frac{1}{2}\big((2m+1)(x+b) + (2m+2n+1)(x+a)\big)\,\frac{\mathrm{d}s}{\mathrm{d}x} + \frac{1}{4}\big((2m+n)^2 - \delta^2\big)z = 0.$$

The complete integral of (18) is the mth derivative of the coefficient of u^n in the expansion of (16) in increasing powers of u.

We apply the transformation of §2 to (18). When we set

$$s = (x+a)^{-(2m+1)/4}(x+b)^{-(2m+2n+1)/4}y,$$

we obtain

$$(18)\qquad (x+a)(x+b)\,\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \bigg(\frac{(2m-1)^2-4}{16}\,\frac{b-a}{x+a} + \frac{(2m+2n-1)^2-4}{16}\,\frac{a-b}{x+b} + \frac{\delta^2-1}{4}\bigg)y.$$

If we compare (18) with

$$(19)\qquad (x+a)(x+b)\,\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \left(\frac{\alpha}{x+a} + \frac{\beta}{x+b} + \gamma\right)y,$$

we make the identifications

$$m = \frac{1}{2} + \sqrt{1 + \frac{4\alpha}{b-a}}, \quad m+n = \frac{1}{2} + \sqrt{1 + \frac{4\beta}{a-b}} \quad \text{and} \quad \delta = \sqrt{1+4\gamma}.$$

Thus (19) is completely integrable if

$$\frac{1}{2} + \sqrt{1 + \frac{4\alpha}{b-a}} \quad \text{and} \quad \frac{1}{2} + \sqrt{1 + \frac{4\beta}{a-b}}$$

are integers. In this case we obtain the complete integral through multiplying

$$(x+a)^{(2m+1)/4}(x+b)^{(2m-2n+1)/4}$$

by the mth derivative of the coefficient of u^n in the expansion of (16) in increasing powers of u.

6

We have found that the equation

$$(20)\qquad (x+a)(x+b)\,\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \left(\frac{\alpha}{x+a} + \frac{\beta}{x+b} + \gamma\right)y,$$

can be completely integrated if

$$\frac{1}{2}+\sqrt{1+\frac{4\alpha}{b-a}} \quad \text{and} \quad \frac{1}{2}+\sqrt{1+\frac{4\beta}{a-b}}$$

are whole numbers. To find further conditions for integrability we transform the variables of (20) according to

$$x+a \longrightarrow \frac{(b-a)^2}{t+a} \quad \text{and} \quad y \longrightarrow \frac{z}{t+a}$$

to obtain

$$(t+a)(t+b)\,\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = \left(\frac{\gamma(b-a)}{t+a}+\frac{\beta}{t+b}+\frac{\alpha}{b-a}\right) z. \tag{21}$$

This equation is of the same form as (20) and hence is completely integrable if

$$\frac{1}{2}+\sqrt{1+4\gamma} \quad \text{and} \quad \frac{1}{2}+\sqrt{1+\frac{4\beta}{a-b}}$$

are whole numbers. With this condition (20) is also integrable. In the same way we can prove that (20) is also integrable in the case that

$$\frac{1}{2}+\sqrt{1+4\gamma} \quad \text{and} \quad \frac{1}{2}+\sqrt{1+\frac{4\alpha}{a-b}}$$

are integers. Thus we obtain the result that

The equation

$$(x+a)(x+b)\,\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \left(\frac{\alpha}{x+a}+\frac{\beta}{x+b}+\gamma\right) y,$$

in addition to the cases indicated in §4, *is completely integrable if two among the three numbers*,

$$\frac{1}{2}+\sqrt{1+\frac{4\alpha}{b-a}}, \quad \frac{1}{2}+\sqrt{1+\frac{4\beta}{a-b}} \quad \text{and} \quad \sqrt{1+4\gamma},$$

are whole numbers.

7

The equation

$$(x+a)(x+b)(x+c)\,\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \left(\frac{\alpha}{x+a}+\frac{\beta}{x+b}+\frac{\gamma}{x+c}\right) y \tag{22}$$

can be transformed to the form of the equation, (20), examined above by a transformation of variables. When we make the change

$$x + c = -\frac{(c-a)(c-b)}{t+a+b-c}, \quad y = \frac{z}{t+a+b-c}, \tag{23}$$

we obtain

$$(t+a)(t+b)\,\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = \left(\frac{\alpha}{(c-a)(t+a)} + \frac{\beta}{(c-b)(t+b)} + \frac{\gamma}{(c-a)(c-b)}\right) z. \tag{24}$$

The integrability conditions for this equation can be found following the rules given in §4 and §6. Thus we obtain the following result.

To find the integrability conditions for the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{Ax^2 + Bx + C}{(x+a)^2(x+b)^2(x+c)^2}\, y$$

decompose the fraction

$$\frac{Ax^2 + Bx + C}{(x+a)^2(x+b)^2(x+c)^2}$$

into partial fractions according to

$$\frac{Ax^2 + Bx + C}{(x+a)^2(x+b)^2(x+c)^2} = \frac{\alpha}{x+a} + \frac{\beta}{x+b} + \frac{\gamma}{x+c}.$$

The equation is completely integrable if

$$\sqrt{1 + \frac{4\alpha}{(a-b)(a-c)}} \pm \sqrt{1 + \frac{4\beta}{(b-a)(b-c)}} \pm \sqrt{1 + \frac{4\gamma}{(c-a)(c-b)}}$$

is an odd integer. The equation is completely integrable also in the case when two of the three numbers,

$$\frac{1}{2} + \sqrt{1 + \frac{4\alpha}{(a-b)(a-c)}}, \quad \frac{1}{2} + \sqrt{1 + \frac{4\beta}{(b-a)(b-c)}} \quad \text{and} \quad \frac{1}{2} + \sqrt{1 + \frac{4\gamma}{(c-a)(c-b)}},$$

are whole numbers.

8

We now pass to a new form of the equation

$$(x+a)\,\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} + \big(n - \mu - \lambda(x-a)\big)\frac{\mathrm{d}z}{\mathrm{d}x} - \lambda n z = 0. \tag{25}$$

This equation, as was shown in §3, is completely integrable if n is a positive integer. In the present case

$$\varphi(x) = \int \left(\lambda + \frac{\mu}{x+a}\right) \mathrm{d}x = \lambda x + \mu \log(x+a). \tag{26}$$

Therefore a particular integral of (25) is given by the formula

$$z = \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}} \left((x+a)^{\mu} \mathrm{e}^{\lambda x}\right). \tag{27}$$

We apply the transformation of §2 to (25). When we set

$$z = (x+a)^{(\mu-n)} \mathrm{e}^{\lambda x/2}, \tag{28}$$

(25) becomes

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \left(\frac{\lambda^2}{4} + \frac{\lambda(n+\mu)}{2(x+a)} + \frac{(n-\mu-1)^2-1}{4(x+a)^2}\right) y. \tag{29}$$

Comparing this equation with the equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \left(\alpha + \frac{\beta}{x+a} + \frac{\gamma}{(x+a)^2}\right) y \tag{30}$$

we make the identifications

$$\begin{aligned} n &= \frac{1}{2} \pm \frac{1}{2}\sqrt{1+4\gamma} \pm \frac{\beta}{2\sqrt{\alpha}}, \\ \mu &= -\frac{1}{2} \mp \frac{1}{2}\sqrt{1+4\gamma} \pm \frac{\beta}{2\sqrt{\alpha}} \quad \text{and} \\ \lambda &= \pm 2\sqrt{\alpha}. \end{aligned} \tag{31}$$

Thus we obtain the result that

The differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \left(\alpha + \frac{\beta}{x+a} + \frac{\gamma}{(x+a)^2}\right) y$$

is completely integrable if

$$\sqrt{1+4\gamma} \pm \frac{\beta}{\sqrt{\alpha}}$$

is an integer.

Given that this condition holds, a particular integral of the equation is

$$y = (x+a)^{(n-\mu)/2} \mathrm{e}^{-\lambda x/2} \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}} (x+a)^{\mu} \mathrm{e}^{\lambda x}, \tag{32}$$

where n, λ and μ are given by (31).

9

The integrability condition found in §8 is not unique. Moreover it does not hold in the case that $\alpha = 0$, *ie* when the equation has the form

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \left(\frac{\beta}{x+a} + \frac{\gamma}{(x+a)^2} \right) y. \tag{33}$$

To find the integrability condition for this equation we apply the transformation

$$x + a = \frac{1}{2}(t+a)^2, \quad y = z\sqrt{t+a}. \tag{34}$$

Then (33) becomes

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = \left(2\beta + \frac{3+16\gamma}{4(t+a)^2} \right) z. \tag{35}$$

Equation (35), hence (33), is completely integrable if $2\sqrt{1+4\gamma}$ is an odd integer. Thus we obtain

The differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \left(\alpha + \frac{\beta}{x+a} + \frac{\gamma}{(x+a)^2} \right) y$$

is completely integrable if $\alpha = 0$ *and* $2\sqrt{1+4\gamma}$ *is an odd integer.*

In addition to the two cases indicated the equation is also completely integrable when $\alpha = \beta = 0$, *ie*, the equation is

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\gamma}{(x+a)^2}\, y. \tag{36}$$

The solution of (36) is

$$y = C_1(x+a)^{\mu} + C_2(x+a)^{1-\mu}, \tag{37}$$

where

$$\mu = \frac{1}{2}\left(1 + \sqrt{1+4\gamma} \right).$$

10

We demonstrate the determination of the integrability condition and solution given in §9.

It is easy to show that

$$s(x) = C_1 \exp\left(2\delta\sqrt{x+a}\right) + C_2 \exp\left(-2\delta\sqrt{x+a}\right) \tag{38}$$

is the solution of the equation

$$(x+a)\,\frac{\mathrm{d}^2 s}{\mathrm{d}x^2}+\frac{1}{2}\,\frac{\mathrm{d}s}{\mathrm{d}x}-\delta^2 s=0. \tag{39}$$

We differentiate (39) n times and set

$$z=\frac{\mathrm{d}^n s}{\mathrm{d}x^n}\,.$$

Then the equation becomes

$$(x+a)\,\frac{\mathrm{d}^2 z}{\mathrm{d}x^2}+\left(n+\frac{1}{2}\right)\frac{\mathrm{d}z}{\mathrm{d}x}-\delta^2 z=0. \tag{40}$$

The solution of (40) is given by the formula

$$z(x)=\frac{\mathrm{d}^n}{\mathrm{d}x^n}\Big(C_1\exp\big(2\delta\sqrt{x+a}\,\big)+C_2\exp\big(-2\delta\sqrt{x+a}\,\big)\Big). \tag{41}$$

We apply the transformation of §2 to (41). When we set

$$z=(x+a)^{-(1+2n)/4}y, \tag{42}$$

we obtain

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}=\left(\frac{\delta^2}{x+a}+\frac{(2n-1)^2-4}{16(x+a)^2}\right)y. \tag{43}$$

On comparison of (43) with

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}=\left(\frac{\beta}{x+a}+\frac{\gamma}{(x+a)^2}\right)y \tag{44}$$

we obtain

$$\delta=\sqrt{\beta}\quad\text{and}\quad n=\frac{1}{2}+\sqrt{1+4\gamma}\,.$$

For n a positive integer the solution of (43) is

$$y(x)=(x+a)^{(1+2n)/4}\,\frac{\mathrm{d}^n}{\mathrm{d}x^n}\Big(C_1\exp\big(2\delta\sqrt{x+a}\,\big)+C_2\exp\big(-2\delta\sqrt{x+a}\,\big)\Big). \tag{45}$$

11

The differential equation

$$(x+b)^2\,\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}=\left(\frac{\alpha}{(x+b)^2}+\frac{\beta}{(x+a)(x+b)}+\frac{\gamma}{(x+a)^2}\right) \tag{46}$$

can be transformed by a change of variables to the form of the equation examined in the three previous sections. When we apply the transformation

$$x+b=\frac{a-b}{t+a-1} \quad \text{and} \quad y=\frac{z}{t+a-1} \tag{47}$$

to (46), we obtain

$$(a-b)^2\frac{\mathrm{d}^2z}{\mathrm{d}t^2}=\left(\alpha+\frac{\beta}{t+a}+\frac{\gamma}{(t+a)^2}\right). \tag{48}$$

The integrability conditions for (48) can be found according to the rules developed in the three previous sections. Thus we obtain

The differential equation

$$(x+b)^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\left(\frac{\alpha}{(x+b)^2}+\frac{\beta}{(x+a)(x+b)}+\frac{\gamma}{(x+a)^2}\right)$$

is completely integrable in the following three cases:

1. *if*
$$\sqrt{1+\frac{4\gamma}{(a-b)^2}}\pm\frac{\beta}{(a-b)\sqrt{\alpha}}$$
is an odd integer,
2. *if* $\alpha=0$ *and*
$$\frac{1}{2}+\sqrt{1+\frac{4\gamma}{(a-b)^2}}$$
is an integer and
3. *if* $\alpha=\beta=0$.

12

The differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\left(\frac{\alpha}{(x+a)^4}+\frac{\beta}{(x+a)^3}+\frac{\gamma}{(x+a)^2}\right)y \tag{49}$$

can be transformed by a change of variables to the form of the equation examined in §§8, 9 and 10. If we set

$$x+a=\frac{1}{t+a} \quad \text{and} \quad y=\frac{z}{t+a},$$

(49) becomes

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = \left(\alpha + \frac{\beta}{t+a} + \frac{\gamma}{(t+a)^2}\right) z. \tag{50}$$

Thus we obtain

The differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \left(\frac{\alpha}{(x+a)^4} + \frac{\beta}{(x+a)^3} + \frac{\gamma}{(x+a)^2}\right) y$$

is completely integrable in the following three cases:

1. *if*
$$\sqrt{1+4\gamma} \pm \frac{\beta}{\sqrt{\alpha}}$$
is an odd integer,
2. *if* $\alpha = 0$ *and* $2\sqrt{1+4\gamma}$ *is an odd integer and*
3. *if* $\alpha = \beta = 0$.

13

The first of the integrability conditions given in §12 and the solution of (50) can also be obtained as below.

As was shown in §3, the differential equation

$$(x+a)^2 \frac{\mathrm{d}^2 z}{\mathrm{d}x^2} + \big((2n-\lambda)(x+a) + \mu\big) \frac{\mathrm{d}z}{\mathrm{d}x} + n(n-1-\lambda)z = 0 \tag{51}$$

is completely integrable if n is an integer. In the present case

$$\varphi(x) = \int \left(\frac{\lambda}{x+a} - \frac{\mu}{(x+a)^2}\right) \mathrm{d}x = \lambda \log(x+a) + \frac{\mu}{x+a}. \tag{52}$$

A particular integral of the equation is expressed by the formula

$$z(x) = \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}} \left((x+a)^{\lambda} \exp\left(\frac{\mu}{x+a}\right)\right). \tag{53}$$

When we apply the transformation of §2, namely

$$z = (x+a)^{(\lambda-2n)/2} \exp\left(\frac{\mu}{2(x+a)}\right) y,$$

we obtain

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \left(\frac{\mu^2}{4(x+a)^4} + \frac{\mu(2n-\lambda-2)}{2(x+a)^3} + \frac{(\lambda+1)^2-1}{4(x+a)^2}\right) y. \tag{54}$$

On comparison with (49), namely

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \left(\frac{\alpha}{(x+a)^4} + \frac{\beta}{(x+a)^3} + \frac{\gamma}{(x+a)^2}\right) y,$$

we obtain

$$n = \frac{1}{2}\left(1 \pm \sqrt{1+4\gamma} \pm \frac{\beta}{2\sqrt{\alpha}}\right), \quad \lambda = -1 \pm \sqrt{1+4\gamma} \quad \text{and} \quad \mu = \pm 2\sqrt{\alpha}.$$

If n is found to be a positive integer, a particular integral of the equation may be written as

$$y = (x+a)^{(2n-\lambda)/2} \exp\left(-\frac{\mu}{2(x+a)}\right) \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}} \left((x+a)^{\lambda} \exp\left(\frac{\mu}{x+a}\right)\right). \tag{55}$$

14

The differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \left(\alpha x^2 + \beta x + \gamma\right) y \tag{56}$$

can be transformed by a change of variables to a particular form of the equation examined in §8. If we set

$$x + \frac{\beta}{2\alpha} = \sqrt{t+a} \quad \text{and} \quad y = z(t+a)^{-1/4},$$

we obtain

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = \left(\frac{\alpha}{4} + \frac{4\alpha\gamma - \beta^2}{16\alpha(t+a)} - \frac{3}{16(t+a)^2}\right) z. \tag{57}$$

As has been proved in §8, we obtain

The differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \left(\alpha x^2 + \beta x + \gamma\right) y$$

is completely integrable if

$$\frac{1}{2}\left(1 \pm \frac{4\alpha\gamma - \beta}{4\alpha\sqrt{\alpha}}\right)$$

is an odd integer.

15

The integrability condition given in the previous section and the solution can be found as follows.

As has been shown in §3, the differential equation

$$\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} - (2\lambda + \mu)\frac{\mathrm{d}z}{\mathrm{d}x} - 2\lambda n z = 0 \tag{58}$$

is completely integrable if n is a positive integer. In the case of (58)

$$\varphi(x) = \int (2\lambda x + \mu)\,\mathrm{d}x = \lambda x^2 + \mu x.$$

Therefore a particular integral of the equation is given by

$$z = \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}} \exp\left(\lambda x^2 + \mu x\right).$$

When we apply the transformation of §2, *videlicet*

$$z = \exp\left(\frac{1}{2}(2\lambda x + \mu)\right) y,$$

we obtain

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \left(\lambda^2 x^2 + \lambda\mu x + \frac{\mu^2}{4} - \lambda + 2\lambda n\right) y. \tag{59}$$

When we compare (59) with (56), *videlicet*

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \left(\alpha x^2 + \beta x + \gamma\right) y,$$

we find that

$$\lambda = \pm\sqrt{\alpha}, \quad \mu = \pm\frac{\beta}{\sqrt{\alpha}} \quad \text{and} \quad n = \frac{1}{2}\left(1 \pm \frac{4\alpha\gamma - \beta}{4\alpha\sqrt{\alpha}}\right).$$

If n is a positive integer, the solution of (56) is

$$y(x) = \exp\left(-\frac{1}{2}(2\lambda x + \mu)\right) \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}\left(\exp(\lambda x^2 + \mu x)\right). \tag{60}$$

16

The differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \left(\frac{\alpha}{(x+a)^6} + \frac{\beta}{(x+a)^5} + \frac{\gamma}{(x+a)^4}\right) \tag{61}$$

can be transformed to (56) by the change of variables

$$x + a = \frac{1}{t} \quad \text{and} \quad y = \frac{z}{t}.$$

Specifically we obtain

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = \left(\alpha t^2 + \beta t + \gamma\right) z.$$

As a consequence of the result of §15 we have

The differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \left(\frac{\alpha}{(x+a)^6} + \frac{\beta}{(x+a)^5} + \frac{\gamma}{(x+a)^4}\right)$$

can be completely integrated if

$$\frac{1}{2}\left(1 \pm \frac{4\alpha\gamma - \beta}{4\alpha\sqrt{\alpha}}\right)$$

is an integer.

17

All of the differential equations examined thus far can be expressed in terms of a single general formula, *videlicet*

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{Ax^2 + Bx + C}{(Dx^3 + Ex^2 + Fx + G)^2}\, y. \tag{62}$$

The methods of integration of (62) and its integrability conditions essentially depend upon the roots of the equation

$$Dx^3 + Ex^2 + Fx + G = 0. \tag{63}$$

The sections in which we examined the different particular cases are

1. §7 when all roots of (63) are different,
2. §11 when (63) has two roots equal,
3. §16 when (63) has three roots equal,
4. §§4 and 6 when $D = 0$ and the roots of the equation

   $$Ex^2 + Fx + G = 0 \tag{64}$$

   are unequal,

5. §§12 and 13 when $D = 0$ and the roots of (64) are equal,
6. §§8, 9 and 10 when $D = E = 0$ and
7. §§14 and 15 when $D = E = F = 0$.

We can transform the differential equation,

$$(65) \qquad \left(Ex^2 + Fx + G\right) \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (Hx + K) \frac{\mathrm{d}y}{\mathrm{d}x} + Ly = 0,$$

to the standard form by the application of the transformation of §2, *videlicet*

$$y = z \exp\left(\int -\frac{1}{2} \frac{Hx + K}{Ex^2 + Fx + G} \, \mathrm{d}x \right).$$

Equation (65) becomes

$$(66) \qquad \frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = \frac{Ax^2 + Bx + C}{(Ex^2 + Fx + G)^2} \, y$$

which is a particular case of (62) with $D = 0$.

18

There are many differential equations which can be reduced to the equations examined above by a change of variables. We consider some of them.

The differential equation

$$(67) \qquad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{a\mathrm{e}^{2x} + b\mathrm{e}^{x} + c}{(\alpha \mathrm{e}^{x} + \beta)^2} \, y$$

is transformed to

$$(68) \qquad \frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = \left(-\frac{1}{4t^2} + \frac{at^2 + bt + c}{t^2(\alpha t + \beta)} \right) z$$

by the transformation

$$x = \log t \quad \text{and} \quad y = \frac{z}{\sqrt{t}} .$$

The integrability conditions for this equation can be found according to the rules developed in §§4, 6, 8, 9, 10, 12 and 13.

The differential equation

$$(69) \qquad x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{a(\log x)^2 + b \log x + c}{(\alpha \log x + \beta)^2} \, y$$

is transformed to

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = \left(\frac{1}{4} + \frac{at^2 + bt + c}{(\alpha t + \beta)}\right) z \tag{70}$$

by the transformation

$$x = \mathrm{e}^t \quad \text{and} \quad y = z \exp(t/2).$$

The integrability conditions for (69) can be found according to the rules developed in §§8, 9, 10, 14 and 15.

The differential equation

$$\cos^2 x \, \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \left(a \sin^2 x + b \sin x + c\right) y \tag{71}$$

is transformed to

$$\left(t^2 - 1\right)^2 \frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = \left(\left(a - \frac{1}{4}\right) t^2 + bt + c - \frac{1}{2}\right) z \tag{72}$$

by the transformation

$$\sin x = t \quad \text{and} \quad y = z \left(1 - t^2\right)^{-1/4}.$$

According to the rule developed in §4 (72), hence (71), is completely integrable if

$$\sqrt{\frac{1}{4} + a - b + c} \pm \sqrt{\frac{1}{4} + a + b + c} \pm \sqrt{a}$$

is an odd integer. According to the rule developed in §6 this equation is also completely integrable in the case when two of the three numbers

$$\frac{1}{2} + \sqrt{\frac{1}{4} + a - b + c}, \quad \frac{1}{2} + \sqrt{\frac{1}{4} + a + b + c} \quad \text{and} \quad \frac{1}{2} + \sqrt{a}$$

are integers.

The differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \left(a \tan^2 x + b \tan x + c\right) y \tag{73}$$

is transformed to

$$\left(t^2 + 1\right)^2 \frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = \left(at^2 + bt + c\right) z \tag{74}$$

by the transformation

$$\tan x = t \quad \text{and} \quad y = z \left(1 + t^2\right)^{-1/2}.$$

According to the rule developed in §4 (74), hence (73), is completely integrable if

$$\sqrt{1+4a} \pm \sqrt{a-c+ib} \pm \sqrt{a-c-ib}$$

is an odd integer.

PFAFF's equation,

$$(75) \qquad \left(ax^{\delta}+b\right) x^{2} \frac{\mathrm{d}^{2} y}{\mathrm{d} x^{2}}+\left(c x^{\delta}+e\right) x \frac{\mathrm{d} y}{\mathrm{d} x}+\left(f x^{\delta}+g\right) y=0,$$

can be transformed to the form

$$(76) \qquad \left(ax^{\delta}+b\right) x^{2} \frac{\mathrm{d}^{2} z}{\mathrm{d} x^{2}}=\left(\alpha x^{2 \delta}+\beta x^{\delta}+\gamma\right) z$$

by using the transformation of §2. Under the change of variables

$$(77) \qquad x=t^{1 / \delta} \quad \text{and} \quad z=s t^{(1-\delta) /(2 \delta)}$$

equation (76) becomes

$$(78) \qquad \delta^{2} \frac{\mathrm{d}^{2} s}{\mathrm{d} t^{2}}=\left(\frac{1-\delta^{2}}{4 t^{2}}+\frac{\alpha t^{2}+\beta t+\gamma}{t^{2}(a t+b)^{2}}\right) s.$$

The integrability conditions for (78) can be found according to the rules developed in §§4, 6, 8, 9, 10, 12 and 13. When we set $a=\beta=\gamma=0$ and $b=1$ in (76), we obtain

$$(79) \qquad \frac{\mathrm{d}^{2} z}{\mathrm{d} x^{2}}=\alpha x^{2(\delta-1)} z,$$

which is known as RICCATI's equation. When we apply the transformation (77), equation (79) becomes

$$\delta^{2} \frac{\mathrm{d}^{2} s}{\mathrm{d} t^{2}}=\left(\alpha+\frac{1-\delta^{2}}{4 t^{2}}\right) s.$$

According to the rule developed in §8 we find that RICCATI's equation is completely integrable if $1/\delta$ is an odd integer.

19

Some nonlinear differential equations of the first and second orders are reduced to linear form by a transformation of the dependent variable. The equation

$$(80) \qquad \frac{\mathrm{d} y}{\mathrm{d} x}=y^{2}+A y+B$$

is reduced to the linear second-order equation

$$(81) \qquad \frac{\mathrm{d}^{2} z}{\mathrm{d} x^{2}}-A \frac{\mathrm{d} z}{\mathrm{d} x}+B z=0$$

under the transformation
$$y = -\frac{1}{z}\,\frac{\mathrm{d}z}{\mathrm{d}x}\,.$$

The more general equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = Ay^2 + By + C, \tag{82}$$

where A, B and C may be functions of x, can be reduced to (80) by a change of the independent variable. We set
$$y = \frac{z}{A}$$
to obtain
$$\frac{\mathrm{d}z}{\mathrm{d}x} = z^2 + \frac{1}{A}\left(B + \frac{\mathrm{d}A}{\mathrm{d}x}\right) z + AC.$$
As has been shown above, this equation can be reduced to a linear second-order differential equation.

The differential equation

$$y\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + A\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + By\,\frac{\mathrm{d}y}{\mathrm{d}x} + Cy^2 = 0 \tag{83}$$

can be transformed to the form of (82) by a change of the dependent variable. If we set

$$y = \exp\left(-\int z\,\mathrm{d}x\right), \tag{84}$$

we obtain
$$\frac{\mathrm{d}z}{\mathrm{d}x} = (A+1)z^2 - Bz + C.$$
As we have proven above, this equation can be reduced to the linear second-order differential equation.

The particular case of (83) with $A = -1$, *videlicet*

$$y\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + By\,\frac{\mathrm{d}y}{\mathrm{d}x} + Cy^2 = 0, \tag{85}$$

deserves special attention. Equation (85) can always be completely integrated since it reduces to the linear first-order equation

$$\frac{\mathrm{d}z}{\mathrm{d}x} + Bz = C$$

under the transformation (84).

20

Some nonlinear differential equations are related to linear second-order differential equations.

If an integral of the equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = My \tag{86}$$

is known, one can find an integral of the equation

$$\frac{\mathrm{d}^2z}{\mathrm{d}x^2} = Mz + \frac{\alpha}{z^3}\,, \tag{87}$$

where α *is some constant.*

We eliminate M from (86) and (87) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y\,\frac{\mathrm{d}z}{\mathrm{d}x} - z\,\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\alpha y}{z^3}\,. \tag{88}$$

When we multiply both sides of (88) by

$$2\left(y\,\frac{\mathrm{d}z}{\mathrm{d}x} - z\,\frac{\mathrm{d}y}{\mathrm{d}x}\right),$$

(88) becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y\,\frac{\mathrm{d}z}{\mathrm{d}x} - z\,\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = -\frac{2\alpha y}{z}\,\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{z}\right).$$

We integrate this to obtain

$$\left(y\frac{\mathrm{d}z}{\mathrm{d}x} - z\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = C - \frac{\alpha y^2}{z^2}. \tag{89}$$

If y_1 and y_2 are two particular integrals of (86), we obtain two first integrals of (87) when we substitute them for y in (89). The integrals are

$$\left(y_1\,\frac{\mathrm{d}z}{\mathrm{d}x} - z\,\frac{\mathrm{d}y_1}{\mathrm{d}x}\right)^2 = C_1 - \frac{\alpha y_1^2}{z^2} \quad\text{and}\quad \left(y_2\,\frac{\mathrm{d}z}{\mathrm{d}x} - z\,\frac{\mathrm{d}y_2}{\mathrm{d}x}\right)^2 = C_2 - \frac{\alpha y_2^2}{z^2}.$$

On the elimination of $\mathrm{d}z/\mathrm{d}x$ from these two first integrals we obtain the solution of (87).

The solution of (87) can also be obtained as follows. From (89) we obtain

$$\mathrm{d}x = \frac{y\,\mathrm{d}z - z\,\mathrm{d}y}{\sqrt{C - \dfrac{\alpha y^2}{z^2}}}.$$

When we divide both sides by y^2, this becomes

$$\frac{\mathrm{d}x}{y^2} = \frac{\frac{z}{y}\,\mathrm{d}\left(\frac{z}{y}\right)}{\sqrt{C\,\frac{z^2}{y^2} - \alpha}}\,.$$

We multiply by C and integrate both sides to obtain

$$C\int\frac{\mathrm{d}x}{y^2} + C_0 = \sqrt{C\,\frac{z^2}{y^2} - \alpha}\,, \tag{90}$$

where C_0 is the second constant of integration. In (90) we have the solution of (87). Instead of y it is sufficient to take any particular solution[3] of (86).

Conversely, if a particular solution of (87) *is known, we can find the complete solution of* (86).

Since it is sufficient to find particular integrals of (86), we can set $C = 0$ in (89). Thus we obtain

$$y\,\frac{\mathrm{d}z}{\mathrm{d}x} - z\,\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{y}{z}\,\sqrt{-\alpha}$$

which is variables separable and can be written as

$$\frac{\mathrm{d}y}{y} = \frac{\mathrm{d}z}{z} \pm \frac{\mathrm{d}x\,\sqrt{-\alpha}}{z^2}\,.$$

On integration this becomes

$$\log y = \log z \pm \sqrt{-\alpha}\int\frac{\mathrm{d}x}{z^2}$$

from which it follows that

$$y = z\exp\left(\pm\sqrt{-\alpha}\int\frac{\mathrm{d}x}{z^2}\right).$$

We take the upper and then the lower sign to obtain two particular integrals of (86).

21

The theorem proven in §20 can be generalised as below.

If p is some known function of x and f is some other given function, then the solution of the equation

$$p\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - y\,\frac{\mathrm{d}^2p}{\mathrm{d}x^2} = \frac{1}{p^2}\,f\left(\frac{y}{p}\right) \tag{91}$$

[3] Editor's Note: In (90) one has a generalisation of Abel's formula for a second solution of a linear second-order differential equation given a particular solution.

can be determined by means of quadratures.

The multiplication of (91) by

$$2\left(p\,\frac{\mathrm{d}y}{\mathrm{d}x}-y\,\frac{\mathrm{d}p}{\mathrm{d}x}\right)\mathrm{d}x$$

leads to the differential form

$$\mathrm{d}\left(p\,\frac{\mathrm{d}y}{\mathrm{d}x}-y\,\frac{\mathrm{d}p}{\mathrm{d}x}\right)^2=2f\left(\frac{y}{p}\right)\,\mathrm{d}\left(\frac{y}{p}\right).$$

We integrate this to obtain

$$\left(p\frac{\mathrm{d}y}{\mathrm{d}x}-y\frac{\mathrm{d}p}{\mathrm{d}x}\right)^2=\varphi\left(\frac{y}{p}\right)+C,$$

where $\varphi(z)=2\int f(z)\,\mathrm{d}z$. This is the expression for a first integral of (91). We solve for $\mathrm{d}x$, namely

$$\mathrm{d}x=\frac{p\,\mathrm{d}y-y\,\mathrm{d}p}{\sqrt{\varphi\left(\frac{y}{p}\right)+C}},$$

divide by p^2 and integrate to obtain

$$\int\frac{\mathrm{d}x}{p^2}+C_0=\int\frac{\mathrm{d}\left(\frac{y}{p}\right)}{\sqrt{\varphi\left(\frac{y}{p}\right)+C}}, \tag{92}$$

where C_0 is the second constant of integration. Equation (92) represents the complete integral of the equation.

In the particular case for which $p=x$ equation (91) takes the form

$$x^3\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=f\left(\frac{y}{x}\right).$$

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Translator: A. O. Harin
Redactor: P. G. L. Leach
School of Mathematical Sciences,
University of KwaZulu-Natal,
Private Bag X54001 Durban 4000,
Republic of South Africa
E–mail: leachp@ukzn.ac.za and leach@math.aegean.gr

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