SECOND ORDER DISCRETIZATION SCHEMES OF STOCHASTIC DIFFERENTIAL SYSTEMS FOR THE COMPUTATION OF THE INVARIANT LAW*

Denis TALAY

INRIA
Avenue Emile Hugues
Sophia Antipolis
F-06565 Valbonne (France)

Abstract

We discretize in time with step-size h a Stochastic Differential Equation whose solution has a unique invariant probability measure μ; if $(\overline{X}_p^h, p \in \mathbb{N})$ is the solution of the discretized system, we give an estimate of

$$|\int f(x)d\mu(x) - \lim_{N \longrightarrow +\infty} \frac{1}{N} \sum_{p=1}^{N} f(\overline{X}_p^h)| \tag{1}$$

in terms of h for several discretization methods.

In particular, methods which are of second order for the approximation of $Ef(X_t)$ in finite time are shown to be generically of second order for the ergodic criterion (1).

Résumé

Nous discrétisons en temps avec un pas de temps noté h une Equation Différentielle Stochastique dont la solution possède une unique mesure invariante μ ; si $(\overline{X}_p^h, p \in \mathbb{N})$ est la solution du système discrétisé, nous estimons en fonction de h l'erreur

$$|\int f(x)d\mu(x) - \lim_{N \longrightarrow +\infty} \frac{1}{N} \sum_{p=1}^{N} f(\overline{X}_p^h)|$$

*Stochastics and Stochastic Reports, 29(1):13–36, 1990.

correspondant à plusieurs méthodes de discrétisation.

En particulier, nous montrons que les méthodes du second ordre pour l'approximation de $Ef(X_t)$ en temps fini sont génériquement du second ordre pour le critère ergodique (1).

1 Introduction

We consider the Stochastic Differential System of dimension d, driven by a Wiener process of dimension r :

$$X_t = X_0 + \int_0^t b(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s \tag{2}$$

The solution starting at $x \in \mathbb{R}^d$ will be denoted by $(X_t(x))$.

At our knowledge, up to now the numerical analysis of the Stochastic Differential Systems has essentially followed five directions : mean-square approximation (Clark and Cameron [2], Milshtein [10], Platen [17], Rumelin [18]), pathwise approximation (Talay [19]), approximation of expectations of the solution (Milshtein [11], Milshtein [12], Talay [20], Talay [21]), construction of schemes asymptotically efficient for the minimization of the normalized quadratic mean error (Clark [3], Newton [13]), numerical computation of Lyapunov exponents of bilinear systems (Pardoux & Talay [16]).

A review of the main results concerning the first three points can be found in Pardoux & Talay [15].

Here, we will suppose that the solution of the system (2) has a unique invariant measure μ.

For some applications, it is interesting to compute the integral of a given function f with respect to μ, for example in order to get the asymptotic value of $Ef(X_t)$.

Under the hypotheses of this paper, μ will have a density, p. One way to compute $\int f(x)d\mu(x)$ could be to solve the stationary Focker-Planck equation $L^*p = 0$, where L^* is the adjoint of the infinitesimal generator of the process (X_t).

But the stationary Focker-Planck equation is a P.D.E., and its numerical resolution could be extremely difficult or impossible, especially when the dimension of the state-space, d, is large.

In [5], Gerardi, Marchetti & Rosa propose to approximate (X_t) by a sequence of pure jump processes which converge in law.

We propose an alternative strategy.

We discretize in time the system (2), so that the solution of the discretized system can easily be simulated on a computer; h denoting the step-size of the discretization, $(\overline{X}_p^h, p \in \mathbb{N})$ will denote the approximating process. We will see that, the discretization method being conveniently chosen, $(\overline{X}_p^h, p \in \mathbb{N})$ has a unique invariant probability measure too, denoted by $\overline{\mu}^h$.

Then we choose a large enough N, and we compute :

$$\frac{1}{N}\sum_{p=1}^{N} f(\overline{X}_p^h) \tag{3}$$

(when N goes to infinity, the above average converges to $\int f(x)d\overline{\mu}^h(x)$).

The error due to this method cumulates, first the "discretization error"

$$|\int f(x)d\mu(x) - \int f(x)d\overline{\mu}^h(x)|$$

related to the choosen discretization scheme, and, second, the error

$$|\int f(x)d\overline{\mu}^h(x) - \frac{1}{N}\sum_{p=1}^{N} f(\overline{X}_p^h)|$$

which only depends on the choice of the integration time Nh.

We will see in Section (4.2) that the estimation of the second error corresponding to each choice of N is extremely difficult, and up to now we do not know how to optimize the choice of N corresponding to a given wished accuracy.

Here our objective is to present efficient numerical schemes which lead to a weak discretization error. More precisely, we consider a family of discretization schemes, and we give, in terms of h, a bound for the discretization error :

$$|\int f(x)d\mu(x) - \lim_{N\longrightarrow+\infty}\frac{1}{N}\sum_{p=1}^{N} f(\overline{X}_p^h)|$$

In particular, we show schemes such that this error is of order h^2. In addition, these schemes seem to be more numerically stable that first-order schemes.

The paper is organized as follows :

Section 2 : we present discretization methods of Stochastic Differential Equations, and for a particular S.D.E., we compute the order of the errors (1) ;

Section 3 : we formulate our hypotheses, and we state our main results ;

Section 4 : we give results of illustrative numerical experiments ;

Sections 5,6,7 : these Sections are devoted to the proofs.

2 Discretization Methods of S.D.E.

2.1 Notations and definitions

- for any real process (Y_t) :

$$\Delta^h_{p+1} Y := Y_{(p+1)\,h} - Y_{p\,h}$$

- for any matrix σ, σ_j will denote the j^{th} column of σ ; if $\sigma(x_1, \ldots, x_d)$ is a matrix-valued application, $\partial\sigma_j$ denotes the matrix-valued application whose element of the i^{th} row and k^{th} column is $\partial_k \sigma^i_j$.

- the sequence

$$(U^j_{p+1}\, ,\, Z^{kj}_{p+1}\, ,\, j,k = 1, \ldots, r\, ,\, p \in \mathbb{N})$$

will be a family of independent random variables ; the (U^j_{p+1}) are i.i.d. and must satisfy the following conditions :

$$E[U^j_{p+1}] = E[U^j_{p+1}]^3 = E[U^j_{p+1}]^5 = 0 \quad (4)$$

$$E[U^j_{p+1}]^2 = 1 \quad (5)$$

$$E[U^j_{p+1}]^4 = 3 \quad (6)$$

$$E[U^j_{p+1}]^6 < +\infty \quad (7)$$

the (Z^{kj}_{p+1}) are i.i.d., their common law being defined by :

$$P(Z^{kj}_p = \frac{1}{2}) = P(Z^{kj}_p = -\frac{1}{2}) = \frac{1}{2}$$

for example, one could choose :

$$U_{p+1}^j = \frac{1}{\sqrt{h}} \Delta_{p+1}^h W^j$$

but also one could choose for U_{p+1}^j the discrete law of mass $\frac{2}{3}$ at 0 and of mass $\frac{1}{6}$ at the points $+\sqrt{3}$ and $-\sqrt{3}$;

- the family (S_p^{kj}) is defined by :

$$\begin{aligned} S_{p+1}^{kj} &= \frac{1}{2} U_{p+1}^k U_{p+1}^j + Z_{p+1}^{kj} \quad , \quad k < j \\ S_{p+1}^{kj} &= \frac{1}{2} U_{p+1}^k U_{p+1}^j - Z_{p+1}^{jk} \quad , \quad k > j \\ S_{p+1}^{jj} &= \frac{1}{2} \left[(U_{p+1}^j)^2 - 1 \right] \end{aligned}$$

Finally, in all the sequel of this Section, we will suppose that the functions b, σ are of class $\mathcal{C}^\infty$ with bounded derivatives.

2.2 Milshtein scheme

The "Milshtein scheme" is defined by :

$$\overline{X}_{p+1}^h = \overline{X}_p^h + \sum_{j=1}^r \sigma_j(\overline{X}_p^h) U_{p+1}^j \sqrt{h} + b(\overline{X}_p^h) h + \sum_{j,k=1}^r \partial\sigma_j(\overline{X}_p^h) \sigma_k(\overline{X}_p^h) S_{p+1}^{kj} h \quad (8)$$

Under the above assumptions on b and σ, for all function f of class $\mathcal{C}^\infty$ such that f and all its derivatives have an at most polynomial growth at infinity, one can show (see Talay [20] or [21], Milshtein [12]) :

$$\forall p > 0 \ , \ \exists C_p \ , \ \forall h < 1 \quad : \quad |Ef(X_{ph}) - Ef(\overline{X}_p^h)| \leq C_p h$$

2.3 Two examples of second-order discretization schemes

We define the matrix a and the vectors A_j by (with the usual convention for the summation indices) :

$$\begin{aligned} a &= \sigma\sigma^* \\ A_j &= \frac{1}{2} a_l^k \partial_{kl} \sigma_j = \frac{1}{2} \sum_{k,l=1}^d a_l^k \partial_{kl} \sigma_j \end{aligned}$$

Besides, we will denote by L the infinitesimal generator of the process (X_t) :

$$L = b^i \partial_i + \frac{1}{2} a^i_j \partial_{ij}$$

We consider the scheme defined by :

$$\begin{aligned} \overline{X}^h_{p+1} &= \overline{X}^h_p + \sum_{j=1}^{r} \sigma_j(\overline{X}^h_p) U^j_{p+1} \sqrt{h} + b(\overline{X}^h_p)\, h + \sum_{j,k=1}^{r} \partial\sigma_j(\overline{X}^h_p) \sigma_k(\overline{X}^h_p) S^{kj}_{p+1} h \\ &\quad + \frac{1}{2} \sum_{j=1}^{r} \left\{ \partial b(\overline{X}^h_p)\sigma_j(\overline{X}^h_p) + \partial\sigma_j(\overline{X}^h_p) b(\overline{X}^h_p) + A_j(\overline{X}^h_p) \right\} U^j_{p+1} h^{\frac{3}{2}} + \frac{1}{2} Lb(\overline{X}^h_p) h^2 \end{aligned} \tag{9}$$

Then, under the same hypotheses on b, σ and f as above, one can show (see Talay [20] or [21],Milshtein [12]) :

$$\forall p > 0 \,,\, \exists C_p \,,\, \forall h < 1 \;:\; |Ef(X_{ph}) - Ef(\overline{X}^h_p)| \leq C_p h^2 \tag{10}$$

Another example of a scheme satisfying the previous property is the “MCRK” scheme of Talay [21].

2.4 Second-order discretization schemes

In this Section, $\mathcal{F}_p$ will be the σ-algebra generated by $(\overline{X}^h_0, ..., \overline{X}^h_p)$.

In Talay [21], it is shown that a sufficient condition for a scheme to satisfy (10) is the set of hypotheses (C1), (C2), (C3) below (which is satisfied by the Monte-Carlo and the MCRK schemes) :

(C1) $\overline{X}^h_0 = X_0$

(C2) $\forall n \in \mathbb{N} \,,\, \forall N \in \mathbb{N} \,,\, \exists C > 0 \,,\, \forall p \leq N \quad,\quad E|\overline{X}^h_p|^n \leq C$

(C3) the following properties are satisfied for all $p \in \mathbb{N}$, where all the right-side terms of the equalities must be understood evaluated at $\overline{X}^h_p$:

$$\begin{aligned} E\left(\Delta^h_{p+1}\overline{X} | \mathcal{F}_p\right) &= bh + \frac{1}{2}(Lb)h^2 + \xi_{p+1} \quad,\quad E|\xi_{p+1}| \leq Ch^3 \\ E\left((\Delta^h_{p+1}\overline{X})^{i_1}(\Delta^h_{p+1}\overline{X})^{i_2}) | \mathcal{F}_p\right) &= \sigma^{i_1}_j \sigma^{i_2}_j h + (b^{i_1} b^{i_2} + \frac{1}{2}\partial_{k_1}\sigma^{i_1}_j \partial_{k_2}\sigma^{i_2}_j \sigma^{k_1}_l \sigma^{k_2}_l \end{aligned}$$

$$
\begin{aligned}
& + \frac{1}{2}\partial_k b^{i_2}\sigma_j^{i_1}\sigma_j^k + \frac{1}{2}\partial_k b^{i_1}\sigma_j^{i_2}\sigma_j^k \\
& + \frac{1}{2}\sigma_j^{i_1}\partial_k\sigma_j^{i_2}b^k + \frac{1}{2}\sigma_j^{i_2}\partial_k\sigma_j^{i_1}b^k \\
& + \frac{1}{4}\sigma_j^{i_1}\partial_{kl}\sigma_j^{i_2}\sigma_n^k\sigma_n^l + \frac{1}{4}\sigma_j^{i_2}\partial_{kl}\sigma_j^{i_1}\sigma_n^k\sigma_n^l)h^2 \\
& + \xi_{p+1}^{i_1 i_2} \quad , \quad E|\xi_{p+1}^{i_1 i_2}| \leq Ch^3
\end{aligned}
$$

$$
\begin{aligned}
E\left((\Delta_{p+1}^h\overline{X})^{i_1}\dots(\Delta_{p+1}^h\overline{X})^{i_3})|\mathcal{F}_p\right) &= (b^{i_1}\sigma_j^{i_2}\sigma_j^{i_3} + b^{i_2}\sigma_j^{i_3}\sigma_j^{i_1} + b^{i_3}\sigma_j^{i_1}\sigma_j^{i_2} \\
& + \frac{1}{2}\sigma_l^{i_2}\partial_k\sigma_l^{i_3}\sigma_j^{i_1}\sigma_j^k + \frac{1}{2}\sigma_l^{i_3}\partial_k\sigma_l^{i_2}\sigma_j^{i_1}\sigma_j^k \\
& + \frac{1}{2}\sigma_l^{i_3}\partial_k\sigma_l^{i_1}\sigma_j^{i_2}\sigma_j^k + \frac{1}{2}\sigma_l^{i_1}\partial_k\sigma_l^{i_3}\sigma_j^{i_2}\sigma_j^k \\
& + \frac{1}{2}\sigma_l^{i_1}\partial_k\sigma_l^{i_2}\sigma_j^{i_3}\sigma_j^k + \frac{1}{2}\sigma_l^{i_2}\partial_k\sigma_j^{i_1}\sigma_j^{i_3}\sigma_j^k)h^2 \\
& + \xi_{p+1}^{i_1 i_2 i_3} \quad , \quad E|\xi_{p+1}^{i_1 i_2 i_3}| \leq Ch^3 \\
E\left((\Delta_{p+1}^h\overline{X})^{i_1}\dots(\Delta_{p+1}^h\overline{X})^{i_4})|\mathcal{F}_p\right) &= (\sigma_j^{i_1}\sigma_j^{i_2}\sigma_l^{i_3}\sigma_l^{i_4} + \sigma_j^{i_1}\sigma_j^{i_3}\sigma_l^{i_2}\sigma_l^{i_4} + \sigma_j^{i_1}\sigma_j^{i_4}\sigma_l^{i_2}\sigma_l^{i_3})h^2 \\
& + \xi_{p+1}^{i_1\dots i_4} \quad , \quad E|\xi_{p+1}^{i_1\dots i_4}| \leq Ch^3 \\
E\left((\Delta_{p+1}^h\overline{X})^{i_1}\dots(\Delta_{p+1}^h\overline{X})^{i_5})|\mathcal{F}_p\right) &= \xi_{p+1}^{i_1\dots i_5} \quad , \quad E|\xi_{p+1}^{i_1\dots i_5}| \leq Ch^3 \\
E\left((\Delta_{p+1}^h\overline{X})^{i_1}\dots(\Delta_{p+1}^h\overline{X})^{i_6})|\mathcal{F}_p\right) &= \xi_{p+1}^{i_1\dots i_6} \quad , \quad E|\xi_{p+1}^{i_1\dots i_6}| \leq Ch^3
\end{aligned}
$$

Definition

A discretization scheme will be called a "*second-order scheme*" if it satisfies the Conditions $(C2)$ and $(C3)$.

2.5 Ergodic situation : one example

Let us consider the Ornstein-Uhlenbeck process solution of :

$$
X_t = X_0 - \int_0^t X_s ds + \sqrt{2}W_t
$$

Its invariant measure μ is the Gaussian law $\mathcal{N}(0,1)$.

Proposition 2.1 *For any continuous function f which has an at most polynomial growth at infinity, and for any starting point x :*

1. *if* $(\overline{X}_p^h(x))$ *is defined by the Milshtein scheme (8) with initial value*

$$\overline{X}_0^h(x) = x$$

with the particular choice

$$\sqrt{h}U_{p+1}^j = \Delta_{p+1}^h W$$

then :

$$\lim_{N\to\infty} \frac{1}{N} \sum_{p=1}^{N} f(\overline{X}_p^h(x)) = \int f(x)d\mu(x) + O(h) \quad , \quad a.s. \tag{11}$$

2. *if* $(\overline{X}_p^h(x))$ *is defined by the scheme (9), with* $\sqrt{h}U_{p+1}^j = \Delta_{p+1}^h W$*,then :*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{p=1}^{N} f(\overline{X}_p^h(x)) = \int f(x)d\mu(x) + O(h^2) \quad , \quad a.s. \tag{12}$$

Proof

Let us show (11).

The gaussian measure $\overline{\mu}^h$ with mean zero and variance equal to

$$\frac{1}{1-\frac{h}{2}}$$

is invariant for the Markov chain $(\overline{X}_p^h)$.

All the transition probabilities of that chain are equivalent to Lebesgue measure. Therefore, any invariant measure has the same property ; hence, as a consequence of the ergodic theorem, $\overline{\mu}^h$ is the unique invariant probability measure of $(\overline{X}_p^h)$, and (11) follows directly.

The proof of (12) is similar , except that the invariant probability measure is the gaussian measure with mean zero and variance equal to

$$\frac{1-h+\frac{h^2}{4}}{1-h+\frac{h^2}{2}-\frac{h^3}{8}}$$

3 Main Results

3.1 Hypotheses

We recall that we will denote by L the infinitesimal generator of the process (X_t), given by :

$$L = b^i \partial_i + \frac{1}{2} a^i_j \partial_{ij}$$

We suppose :

(H1) the functions b, σ are of class $\mathcal{C}^\infty$ with bounded derivatives of any order ; the function σ is bounded

(H2) the operator L is uniformly elliptic : there exists a positive constant α such that :

$$\forall x, \xi \in \mathbb{R}^d \quad , \quad \sum_{i,j} a^i_j(\xi) x_i x_j \geq \alpha |x|^2$$

(H3) there exists a strictly positive constant β and a compact set K such that :

$$\forall x \in \mathbb{R}^d - K \quad , \quad x \cdot b(x) \leq -\beta |x|^2$$

It is well known that (H1) and (H3) is a (even too strong) sufficient condition for (X_t) to be ergodic (see Hasminskii [6] e.g.) : (X_t) has a unique invariant probability measure, μ, and (H2) implies the existence of a smooth density $p(x)$ for μ.

Moreover :

Proposition 3.1 *(i) Under (H1) and (H3), the following holds :*

$$\forall n \in \mathbb{N} \ , \ \exists C_n > 0 \ , \ \exists \gamma_n > 0 \ : \ E|X_t(x)|^n \leq C_n (1 + |x|^n \exp(-\gamma_n t)) \quad , \quad \forall t \ , \ \forall x \tag{13}$$

(ii) The unique invariant probability measure of (X_t), μ, has a smooth density $p(x)$ and finite moments of any order.

Proof

(i) The inequality can easily be proved by recurrence and by applying the Ito formula.

(ii) We just remark that for any compact set K :

$$\int_K |x|^n p(x)dx = \lim_{t \to +\infty} E(|X_t(x)|^n 1_K(X_t(x))) \leq C_n$$

where C_n is the constant in (13).

It remains to let K increase to $\mathbb{R}^d$. □

3.2 Statement of the Theorems

We will say that a discretization scheme is *ergodic* if the Markov chain defined by the scheme is ergodic.

The common law of the family (U^j_{p+1}) may be so singular that the associated scheme is not ergodic.

Let us give an example of such a situation : let us consider the one-dimensional system defined by $b(x) = sign(x)$ and $\sigma(x) \equiv 1$, $h = \sqrt{3}$, Milshtein scheme, and let us choose the discrete law defined in Section 2.1 ; then the law of $(\overline{X}^h_p(x))$ charges the set $x + \sqrt{3}\mathcal{Z}$ (where $\mathcal{Z}$ is the set of relative integers); therefore, the process has an infinite number of invariant probability measures.

But that degenerate situation cannot arise with the natural choice for defining (U^j_{p+1}) (other choices are also possible, but there is no reason to simulate more complicated laws than gaussian laws on a computer, so we have not searched to state a more general result).

Theorem 3.2 *Suppose*

$$\sqrt{h} U^j_{p+1} = \Delta^h_{p+1} W$$

Then, for all step-size h small enough, the Milshtein scheme, as well as the second-order schemes Monte-Carlo and MCRK of Section (2.4), are ergodic.

Now we will state our main result.

The space $\mathcal{C}_p^\infty$ will denote the space of numerical functions f of $\mathbb{R}^d$ of class $\mathcal{C}^\infty$, which have the property that f, as well as all its derivatives, have an at most polynomial growth at infinity.

Theorem 3.3 *Suppose that the hypotheses (H1), (H2), (H3) hold.*

Then, for any function f of $\mathcal{C}_p^\infty$:

1. *if the Milshtein scheme (8) is ergodic, it satisfies :*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{p=1}^{N} f(\overline{X}_p^h(x)) = \int f(x)d\mu(x) + O(h) \quad , \quad a.s.$$

2. *any ergodic second-order scheme satisfies :*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{p=1}^{N} f(\overline{X}_p^h(x)) = \int f(x)d\mu(x) + O(h^2) \quad , \quad a.s. \tag{14}$$

We will need to establish a technical result about $Ef(X_t(x))$, which is the key of the proof of the above Theorem. This result is interesting by itself : it precisely describes the asymptotic behaviour of $Ef(X_t(x)) - \int f(x)d\mu(x)$.

Notation

In the statement below and in all the sequel, the operators ∂, ∇ and D applied to a function $u(t,x)$ always refer to derivations with respect to spatial coordinates.

We will often write $u(t)$ instead of $u(t,x)$.

Theorem 3.4 *Suppose that the hypotheses (H1), (H2), (H3) hold, and let f be a function of the space $\mathcal{C}_p^\infty$.*

Let $u(t,x) = Ef(X_t(x))$.

Then, for any multi-index I, there exists an integer s_I and strictly positive constants Γ_I and γ_I such that the spatial derivative $\partial_I u(t,x)$ satisfies :

$$|\partial_I u(t,x)| \leq \Gamma_I(1+|x|^{s_I})\exp(-\gamma_I t) \tag{15}$$

Remark

As the proof of the previous Theorem will show it, the following result also holds (but we will not use it in the sequel) :

$$\exists s \in \mathbb{N}\ ,\ \exists \Gamma > 0\ ,\ \exists \gamma > 0\ ,\ |u(t,x) - \int f(x) d\mu(x)| \leq \Gamma(1+|x|^s) \exp(-\gamma t)\ \ ,\ \ \forall t\ ,\ \forall x$$

4 Numerical experiments

4.1 The discretized system

We have choosen a 2-dimensional system defined by :

$$b^i(x^1, x^2) = -\frac{1}{2}x^i - \frac{1}{4}x^j \quad , \quad j \neq i$$

and

$$\sigma(x^1, x^2) = \begin{bmatrix} \sin(x^1 + x^2) & cos(x^1 + x^2) \\ \sin(x^1 + x^2 + \frac{\pi}{3}) & \cos(x^1 + x^2 + \frac{\pi}{3}) \end{bmatrix}$$

The invariant law of (X_t), μ, is gaussian $\mathcal{N}(0, Id)$.

The function f is $\|x\|^2 - 1$.

4.2 Choice of N

As said before, the error of the method presented here is the sum of the error due to the discretization, and the error due to the approximation of $\int f(x) d\overline{\mu}^h(x)$ by :

$$\frac{1}{N} \sum_{p=1}^{N} f(\overline{X}_p^h)$$

A convenient choice for N depends on the wished accuracy on the value of $\int f(x) d\mu(x)$. Unfortunately, it is a critical point. Actually, we are going to see that, to get an estimation of a N, one should solve a P.D.E. whose complexity is equivalent to that of the Fokker-Planck equation, and moreover this P.D.E. involves the unknown $\int f(x) d\mu(x)$.

More precisely, the method used in Florens-Zmirou [4] to prove a central-limit theorem for

$$\frac{1}{\sqrt{t}}\int_0^t [f(X_s) - \int f(x)d\mu(x)]ds$$

can easily be extended in the multidimensional case, so that, if

$$\tilde{f} = f - \int f(x)d\mu(x)$$

we have :

$$\frac{1}{\sqrt{t}}\int_0^t \tilde{f}(X_s)ds \stackrel{t\to+\infty}{\longrightarrow} \mathcal{N}(0, V(f))$$

where, if $v(x)$ is solution of the Poisson P.D.E. $Lv = -\tilde{f}$:

$$V(f) = 2\int \tilde{f}(x)v(x)d\mu(x)$$

We do not know how to cleverly proceed to approximate $V(f)$ (even roughly, it would be enough to have an idea about N).

Let us just present a naive approach.

The solution $v(x)$ satisfies : $v(x) = \int_0^{+\infty} E\tilde{f}(X_t(x))dt$.

It may happen that one a priori knows where the measure μ is concentrated (for example, for reasons related to the underlying physical problem). Then one may construct a piecewise constant approximation of the function v on a bounded domain of $\mathbb{R}^d$ in the following manner : in each subdomain of the discretized domain, one chooses one point x, then one chooses integers J and N as small as possible, and a time discretization step h as large as possible, in order to simulate several independent paths of the process $(\overline{X}_p^h(x))$, $(\overline{X}_p^{h,j}(x))$, and to approximate $v(x)$ by

$$v(x) \simeq \frac{h}{J}\sum_{j=1}^{J}\sum_{p=1}^{N}[f(\overline{X}_p^{h,j}(x)) - \frac{1}{N}\sum_{p=1}^{N} f(\overline{X}_p^{h,1}(x))]$$

Finally one uses one of the previous simulated paths to roughly approximate $V(f)$ by

$$V(f) \simeq \frac{2}{N}\sum_{p=1}^{N} v(\overline{X}_p^h(x))[f(\overline{X}_p^h(x)) - \frac{1}{N}\sum_{p=1}^{N} f(\overline{X}_p^{h,1}(x))]$$

The above central-limit theorem then permits to fix the definitive value of N, and a new single path of $(\overline{X}_p^h(x))$ is simulated, corresponding to a small value of h.

If the estimation of the constant $V(f)$ appears to be impossible, a procedure to stop the algorithm may be to wait for oscillations of $\frac{1}{N}\sum_{p=1}^{N} f(\overline{X}_p^h(x))$ with weak amplitudes around a value which is decided to be its limit.

4.3 Numerical results

We have tested the Milshtein scheme and the MCRK scheme.

First we have estimated the constant $V(f)$, by the procedure described above. The scheme was the MCRK scheme, the time discretization step was $h = 0.01$, and the final time $Nh = 100.0$. The computed value has been $V(f) = 1.2$.

Our objective being to get a precision of order 10^{-2}, we have choosen the final time of the algorithm $Nh = 500.0$.

At the time $Nh = 500.0$ with $h = 0.01$, the error due to the MCRK scheme is less than 0.005 , whereas the error due to the Milshtein scheme is more than 0.18. To get an error equal to 0.005 with the Milshtein scheme, we might choose $h = 0.001$.

Remark : if the discretization step h is choosen too large, the theoretical precision of the method is alterated, but also it may appear that the discretization scheme becomes numerically unstable. From that point of view also, the MCRK has a better behaviour than the Milshtein scheme (at least in our example).

The Fortran programs have been generated by a system of automatic generation of programs of simulation of solutions of Stochastic Differential Systems (see Leblond & Talay [8] for a presentation).

5 Proof of Theorem (3.2)

The basic fact (which is false for the chain given in the example of Section 3.2) is the existence of an irreducibility measure, i.e a measure such that any compact set of strictly positive measure can be reached in finite time from any starting point x with a strictly positive probability.

Namely this measure will be the measure of density $1_K(\cdot)$ w.r.t. the Lebesgue measure, where K is the compact set of (H3).

Actually, we are going to show that K_0, any open set included in K, can be reached in finite time from any starting point x with a strictly positive probability.

Indeed, first it is easy to deduce from (H3) and the boundedness of σ that, for any starting point x, the chain reaches the compact set K in finite time with a strictly positive probability. Let $\tilde{x}$ be the reached point in K.

Then, as the choosen law of the U_{p+1}^j's is equivalent to the Lebesgue measure, under $(H2)$, one can show that there exists h_0 independent of $\tilde{x}$ and K_0 such that, for any $h < h_0$, $(\overline{X}_t^h)$ reaches K_0 from $\tilde{x}$ in finite time with a strictly positive probability.

Moreover, let us denote by P^h the transition probability of the considered Markov chain.

One can check that for all small enough h, there exists ε positive , such that for all x outside K :

$$\int P^h(x, dy)|y|^2 \leq |x|^2 - \varepsilon$$

Then a result of Tweedie [22] implies the ergodicity of the chain. $\square$

6 Proof of Theorem (3.4)

Preliminaries

1. First, it is well known that $u(t, x)$ is a classical solution of the P.D.E. :

$$\begin{aligned} \frac{d}{dt} u(t, x) &= L\, u(t, x) \\ u(0, x) &= f(x) \end{aligned} \tag{16}$$

Differentiating the solution with respect to the initial condition, one can show (cf Kunita [7], Section I-3 e.g.) that the function $u(t, x)$ satisfies :

$$\forall n \in \mathbb{N}\,,\ \exists s_n \in \mathbb{N}\,,\ \forall t > 0\,,\ \exists C_n(t) > 0\ :\ |D^n u(\theta, x)| \leq C_n(t)(1+|x|^{s_n})\ ,\ \ \forall \theta \leq t \tag{17}$$

Therefore, Proposition (3.1) implies that the functions f and $D^n u(t,x)$ (for any n) belong to $\mathcal{L}^2(\mathbb{R}^d, \mu)$.

2. The functions $u(t,x)$ and $u(t,x) - \int f(x) d\mu(x)$ have the same derivatives, therefore **in all that Section we will suppose that**

$$\int f(x) d\mu(x) = 0 \tag{18}$$

3. For an integer s (depending on I) to be defined below, we define

$$\pi_s(x) = \frac{1}{(1+|x|^2)^s}$$

Plan of the proof

The proof will be divided in 3 parts :

1. in Lemma (6.1), we will show that for any ball B, there exists strictly positive constants C and λ such that

$$\forall t > 0 \quad , \quad \forall x \in B \quad , \quad |u(t,x)| \leq C \exp(-\lambda t)$$

2. then, in Lemma (6.2), we will show that there exists strictly positive constants C and λ such that

$$\forall t > 0 \quad , \quad \int |u(t,x)|^2 \pi_s(x) dx \leq C \exp(-\lambda t)$$

3. we prove that the previous inequality also holds for the spatial derivatives of $u(t,x)$ (with other constants) and then we deduce (15).

6.1 First Lemma

In this Section, we will prove the

Lemma 6.1 *Under the hypotheses of Theorem (3.4), for any ball B, there exists strictly positive constants C and λ such that*

$$\forall t > 0 \quad , \quad \forall x \in B \quad , \quad |u(t,x)| \leq C \exp(-\lambda t) \tag{19}$$

The proof is in 2 parts :

1. $p(x)$ being the density of the measure μ, we show that there exists strictly positive constants C and λ such that

$$\forall t > 0 \quad , \quad \int |u(t,x)|^2 p(x) dx \leq C \exp(-\lambda t) \tag{20}$$

2. then we deduce that for any multi-index J, there exists strictly positive constants C_J and λ_J such that :

$$\int |\partial_J u(t,x)|^2 p(x) dx \leq C_J \exp(-\lambda_J t) \tag{21}$$

These results imply (19) : since $p(x)$ is a strictly positive continuous function on any ball $B = B(O,R)$:

$$\|\partial_J u(t)\|^2_{L^2(B)} \leq C \int |\partial_J u(t,x)|^2 p(x) dx$$

and we conclude by applying the Sobolev imbedding Theorem.

6.1.1 Proof of (20)

Remark : a similar result is obtained in Bouc & Pardoux [1] (Corollary 1.10) with another set of hypotheses, including the following, unsatisfying in our context, since the density p is unknown :

$$\exists C > 0 \quad , \quad \exists M > 0 \quad , \quad |x| \geq M \Longrightarrow x_i \partial_j (a^i_j p)(x) \leq -C p(x)|x|$$

In the present context, let us choose a positive real number θ and let us consider the sequence (X_{t_n}) with $t_n = n\theta$; it is an ergodic Markov chain; moreover, (H3) implies :

$$\exists \alpha > 0 \, , \, \exists B = B(0,R) \supset K \quad , \quad \sup_{x \in \mathbb{R}^d - B} E[(1+\alpha\theta)|X_{t_{n+1}}|^2 - |X_{t_n}|^2 | X_{t_n} = x] < 0$$

Then (cf e.g. Nummelin [14], Chapters 5,6) the chain is geometrically recurrent and (cf Tweedie [23]) for any function ϕ of the space $\mathcal{C}^\infty_p$ and satisfying (18), there exists $C > 0$, $\lambda > 0$ such that :

$$\forall n \quad , \quad \int |E\phi(X_{t_n}(x))| p(x) dx \leq C \exp(-\lambda t_n)$$

In particular, this inequality is true for $\phi = f$ (under (18)).

Using Proposition (3.1), as f is of growth at most polynomial at infinity, we remark :

$$\exists C_0 > 0 \ , \ \exists N \in \mathbb{N} \ , \ \exists \gamma > 0 \quad , \quad |u(t,x)| \leq C_0(1 + |x|^N \exp(-\gamma t))$$

Therefore :

$$\int |u(t_n,x)|^2 p(x)dx \leq C_0 C \exp(-\lambda t_n) + C_0^2 exp(-\gamma t) \int (1+|x|^N \exp(-\gamma t))|x|^N p(x)dx$$

so that we get :

$$\forall n \quad , \quad \int |u(t_n,x)|^2 p(x)dx \leq C_1 \exp(-\lambda_1 t_n)$$

To conclude, we use the fact that the function $\int |u(t,x)|^2 p(x)dx$ is decreasing ; actually, the first preliminary remark of that Section justifies the following inequality :

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int |u(t,x)|^2 p(x)dx &= \int u(t,x)Lu(t,x)p(x)dx \\ &\leq -\frac{1}{2}\int a_j^i(x)\partial_i u(t,x)\partial_j u(t,x)p(x)dx \\ &\leq 0 \end{aligned}$$

6.1.2 Proof of (21)

We will prove this inequality by recurrence over the length $l(J)$ of the multi-index J.

First step : $l(J) = 1$ **(Lions [9])**

Let us begin by proving :

$$\exists \delta > 0 \quad , \quad \int_0^{+\infty} e^{\delta t} (\int |\nabla u(t)|^2 d\mu) dt < +\infty \tag{22}$$

Using the usual convention on the summation of indices, one may write :

$$\frac{d}{dt}|u(t)|^2 - L(|u(t)|^2) = -a_j^i(\partial_i u(t))(\partial_j u(t))$$

Multiplying the previous equality by $e^{\delta t}$, integrating with respect to μ, one gets :

$$e^{\delta t}\frac{d}{dt}\int |u(t)|^2 d\mu + Ce^{\delta t}\int |\nabla u(t)|^2 d\mu \leq 0 \tag{23}$$

Now, let us choose an arbitrarily large time T and integrate from 0 to T the previous inequality :

$$e^{\delta T}\int |u(T)|^2 d\mu + C\int_0^T e^{\delta t}(\int |\nabla u(t)|^2 d\mu)dt \leq \int |f|^2 d\mu + \delta\int_0^T e^{\delta\, t}(\int |u(t)|^2 d\mu)dt$$

Therefore, it just remains to use (20) to get (22) (for $\delta < \lambda$).

Now, let us remark one can choose positive constants C_1 and C_2 such that :

$$\begin{aligned}\frac{d}{dt}|\nabla u(t)|^2 - L(|\nabla u(t)|^2) &= -a^i_j(\partial_{ik}u(t))(\partial_{jk}u(t)) + (\partial_k a^i_j)(\partial_{ij}u(t))(\partial_k u(t)) \\ &\quad + 2(\partial_k b^i)(\partial_i u(t))(\partial_k u(t)) \\ &\leq -C_1|D^2u(t)|^2 + C_2|\nabla u(t)|^2\end{aligned}$$

Now let us choose $\gamma < \delta$ and proceed as above ; we get :

$$e^{\gamma T}\int |\nabla u(T)|^2 d\mu + C_1\int_0^T e^{\gamma t}(\int |D^2u(t)|^2 d\mu)dt \leq \int |\nabla f|^2 d\mu + (C_2+\gamma)\int_0^T e^{\gamma t}(\int |\nabla u(t)|^2 d\mu)dt$$

Thus we have shown :

$$\int |\nabla u(t,x)|^2 p(x)dx \leq C\exp(-\gamma t) \tag{24}$$

Second step : Recurrence

Now, let us suppose that for all $k \leq m$, there exists strictly positive constants C_k and γ_k such that :

$$\int |D^k u(t,x)|^2 p(x)dx \leq C_k \exp(-\gamma_k t)$$

Let us show that a similar inequality holds for $m+1$.

First :

$$|D^m u(t,x)|^2 = \sum_{\substack{J \\ l(J)=m}} (\partial_J u(t,x))^2$$

Now, it appears that, for every multi-index J of length $m+1$, there exists a family F_J of multi-indices of length at most equal to $m+1$, and a family of bounded functions (ϕ^J_{KL} , $K, L \in F_J$) such that

$$\begin{aligned} \frac{d}{dt} D_J u(t,x) &= D_J L u(t,x) \\ &= b^i \partial_{J\cup\{i\}} u(t,x) + \frac{1}{2} a^i_j \partial_{J\cup\{ij\}} u(t,x) + \\ &\quad + \sum_{\substack{K,L \\ l(K)+l(L)\leq 2m+1}} \phi^J_{KL} \partial_K u(t,x) \partial_L u(t,x) \end{aligned}$$

Therefore :

$$\begin{aligned} \frac{d}{dt}|D^m u(t,x)|^2 - L|D^m u(t,x)|^2 &= -a^i_j (\partial_{J\cup\{i\}} u(t,x))(\partial_{J\cup\{j\}} u(t,x)) \\ &\quad + \sum_{\substack{K,L \\ l(K)+l(L)\leq 2m+1}} \phi^J_{KL} \partial_K u(t,x) \partial_L u(t,x) \\ &\leq -C^m_1 |D^{m+1} u(t,x)|^2 + C^m_2 \sum_{k\leq m} |D^k u(t,x)|^2 \end{aligned}$$

Now, we proceed as above : we choose a strictly positive constant δ_{m+1} small enough, we multiply the previous inequality by $e^{\delta_{m+1}t}$, and then we integrate with respect to μ, so that we obtain :

$$\int_0^{+\infty} e^{\delta_{m+1}t} (\int |D^{m+1} u(t)|^2 d\mu) dt < +\infty$$

Then we write

$$\frac{d}{dt}|D^{m+1} u(t,x)|^2 - L|D^{m+1} u(t,x)|^2 \leq -C^{m+1}_1 |D^{m+2} u(t,x)|^2 + C^{m+1}_2 \sum_{k\leq m+1} |D^k u(t,x)|^2$$

we choose $\gamma_{m+1} < \delta_{m+1}$ and we proceed as at the end of the first step. $\square$

6.2 Second Lemma

In this Section, we will prove the

Lemma 6.2 *Under the hypotheses of Theorem (3.4), there exists strictly positive constants C and λ such that*

$$\forall t > 0 \quad , \quad \int |u(t,x)|^2 \pi_s(x) dx \leq C \exp(-\lambda t) \tag{25}$$

First step

Let us recall the property (17) :

$$\forall n \in \mathbb{N} \, , \, \exists s_n \in \mathbb{N} \, , \, \forall t > 0 \, , \, \exists C_n(t) > 0 \; : \; |D^n u(\theta, x)| \leq C_n(t)(1+|x|^{s_n}) \; , \; \forall \theta \leq t$$

Thus, for any integer $n \geq 0$, there exists an integer s_n such that, for any $0 \leq m \leq n$ and any $t \geq 0$:

$$|D^m u(t,x)| \pi_{s_n}(x) \in L^2(\mathbb{R}^d) \tag{26}$$

Second, we remark that for any multi-index J and any integer s, there exists a smooth function $\psi_{J,s}(x)$ such that :

1. the derivative $\partial_J \pi_s(x)$ can be written
$$\partial_J \pi_s(x) = \psi_{J,s}(x) \pi_s(x)$$
2. $\psi_{J,s}(x) \longrightarrow 0$ when $|x| \longrightarrow +\infty$

Let M_I be the integer defined by :

$$l(I) = [M_I - d/2]$$

Then (26) implies that it is possible to choose an integer s_0 such that

$$\forall t > 0 \quad , \quad \forall s \geq s_0 \quad , \quad \forall m \leq M_I, \quad D^m(u(t)\pi_s) \in L^2(\mathbb{R}^d) \tag{27}$$

Second step

Let s be an arbitrary integer larger than s_0.

Our definition of the integer M_I implies that $|D(u(t)\pi_s)| \in L^2(\mathbb{R}^d)$; therefore the following holds (with a summation w.r.t. all the indices) :

$$\begin{aligned}
\int u(t,x)Lu(t,x)\pi_s(x)dx &= -\frac{1}{2}\int(\partial_i b^i)|u(t)|^2\pi_s dx - \frac{1}{2}\int b^i|u(t)|^2(\partial_i\pi_s)dx \\
&\quad - \frac{1}{2}\int(\partial_i a^i_j)(\partial_j u(t))u(t)\pi_s dx - \frac{1}{2}\int a^i_j(\partial_i u(t))(\partial_j u(t))\pi_s dx \\
&\quad - \frac{1}{2}\int a^i_j(\partial_j u(t))u(t)(\partial_i\pi_s)dx
\end{aligned}$$

so that, by (H2) :

$$\begin{aligned}
\int u(t,x)Lu(t,x)\pi_s(x)dx &\leq -\frac{1}{2}\int(\partial_i b^i)|u(t)|^2\pi_s dx + \int s\frac{b\cdot x}{1+|x|^2}|u(t)|^2\pi_s dx \\
&\quad + \frac{1}{4}\int(\partial_{ij}a^i_j)|u(t)|^2\pi_s dx + \frac{1}{4}\int(\partial_i a^i_j)|u(t)|^2\psi_j\pi_s dx \\
&\quad - \frac{1}{2}\alpha\int|Du(t)|^2\pi_s dx \\
&\quad + \frac{1}{4}\int(\partial_j a^i_j)|u(t)|^2\psi_i\pi_s dx + \frac{1}{4}\int a^i_j|u(t)|^2\psi_{ij}\pi_s dx \\
&= \int(\phi_1(x)+\phi_2(x)+s\frac{b\cdot x}{1+|x|^2})|u(t)|^2\pi_s dx - \frac{1}{2}\alpha\int|Du(t)|^2\pi_s dx
\end{aligned}$$

where

- $\phi_1(x)$ is a bounded function independent of s
- $\phi_2(x)$ is a function depending on s, but tending to 0 when $|x| \longrightarrow +\infty$

Now we fix $s \geq s_0$ in order to get the following inequality, possible under (H3) :

$$\limsup_{|x|\to+\infty}(\phi_1(x)+\phi_2(x)+s\frac{b\cdot x}{1+|x|^2}) < 0$$

For any ball $B = B(0,R)$:

$$\begin{aligned}
\int(\phi_1(x)+\phi_2(x)+s\frac{b\cdot x}{1+|x|^2})|u(t)|^2\pi_s dx &= \int_B(\phi_1(x)+\phi_2(x)+s\frac{b\cdot x}{1+|x|^2})|u(t)|^2\pi_s dx \\
&\quad + \int_{\mathbb{R}^d-B}(\phi_1(x)+\phi_2(x)+s\frac{b\cdot x}{1+|x|^2})|u(t)|^2\pi_s dx
\end{aligned}$$

Choosing R large enough (in terms of s), using (19), one may deduce that there exists strictly positive constants such that :

$$\int (\phi_1(x) + \phi_2(x) + s\frac{b \cdot x}{1+|x|^2})|u(t)|^2 \pi_s dx \leq -C_1 \int |u(t)|^2 \pi_s dx + C_2 \exp(-\lambda t)$$

so that :

$$\frac{1}{2}\frac{d}{dt}\int |u(t)|^2 \pi_s dx \leq -C_1 \int |u(t)|^2 \pi_s dx + C_2 \exp(-\lambda t)$$

Thus we may deduce (25).

6.3 End of the proof of Theorem (3.4)

First step

Let us suppose that we have shown

$$\exists C_I > 0\,,\ \exists \lambda_I > 0\ :\ \forall m \leq M_I\,,\ \forall t > 0\ ,\ \int |D^m u(t,x)|^2 \pi_s(x) dx \leq C_I \exp(-\lambda_I t) \tag{28}$$

We already have remarked that for any multi-index J :

$$\partial_J \pi_s(x) = \psi_J(x)\pi_s(x) \quad , \quad \psi_J(x) \ \ \textit{bounded} \tag{29}$$

Therefore we would get

$$\exists C_I\ ,\ \lambda_I\ :\ \forall m \leq M_I\ ,\ \forall t > 0\ ,\ \int |D^m(u(t,x)\pi_s(x))|^2 dx \leq C_I \exp(-\lambda_I t)$$

Then we could deduce (15) as a consequence of the previous inequality and of the Sobolev imbedding Theorem.

Second step : proof of (28)

Again we use (29) to remark that a sufficient result would be : there exists strictly positive constants C_I and λ_I such that, for any multi-index J of length $l(J) \leq M_I$

$$\int |\partial_J u(t,x)|^2 \pi_s(x) dx \leq C_J \exp(-\lambda_J t)$$

Then let us fix such a J, and begin with the case $: l(J) = 1$.

The proof is very similar to that of Section (6.1).

In Section (6.1.2), we used the fact that

$$L^* p(x) = 0$$

in order to get the inequality (22) and the next ones..

Here, we remark that there exists functions $\phi_1(x)$ and $\phi_2(x)$ such that

- $\phi_1(x)$ is a bounded function independent of s
- $\phi_2(x)$ is a function depending on s, but tending to 0 when $|x| \longrightarrow +\infty$
- the following equality holds (since s satisfies (27)) :

$$\begin{aligned} \int L|u(t)|^2 \pi_s dx &= \int |u(t)|^2 L^* \pi_s dx \\ &= \int (\phi_1(x) + \phi_2(x) + 2s \frac{b \cdot x}{1+|x|^2}) |u(t,x)|^2 \pi_s(x) dx \end{aligned}$$

As in the previous Section, after having possibly increased the value of s, we can choose a ball $B = B(0, R)$ such that :

$$\forall x \in \mathbb{R}^d - B \quad , \quad \phi_1(x) + \phi_2(x) + s \frac{b \cdot x}{1+|x|^2} < 0$$

Using (19), we deduce that there exists positive constants C_0 and λ_0 satisfying :

$$\int L|u(t)|^2 \pi_s dx \leq C_0 \exp(-\lambda_0 t)$$

Proceeding as in Section (6.1.2), we can show that the inequality (22) remains true with $\pi_s(x)dx$ instead of $d\mu(x)$ and δ small enough, and then show that (24) remains true with $\pi_s(x)dx$ instead of $d\mu(x)$ and γ small enough.

Again, a recurrence permits to generalize to the derivatives of higher order.

7 Proof of Theorem (3.3)

7.1 Moments of the approximating process

The approximating process satisfies an analogous property to (13).

Proposition 7.1 *For Milshtein scheme or any second-order scheme with initial condition x, for all integer n :*

$$\exists C_n > 0 \ , \ \exists \gamma_n > 0 \ , \ \exists H > 0 \ , \ \ \forall h \leq H \ , \ E|\overline{X}_p^h(x)|^n \leq C_n(1+|x|^n \exp(-\gamma_n ph)) \ , \ \forall p \ , \ \forall x \tag{30}$$

Proof

Using (H3) and (C3), it is easy to show the existence of strictly positive constants C_1 and C_2 satisfying, for any h small enough :

$$E|\overline{X}_{p+1}^h|^2 \leq (1 - C_1 h)\, E|\overline{X}_p^h|^2 + C_2 h$$

so, iterating the previous inequality, one proves the Lemma for $n = 2$.

By recurrence, one shows the result for any integer n. □

Therefore, the unique invariant probability measure μ^h of the approximating process has finite moments of any order.

Moreover, as f has an at most polynomial growth at infinity, the previous Proposition implies that the sequence

$$\frac{1}{N} \sum_{p=1}^{N} f(\overline{X}_p^h(x))$$

is equiintegrable.

Therefore one may deduce that for any deterministic initial condition x :

$$\begin{aligned} \int f(x) d\mu^h(x) &= \lim_{n \longrightarrow +\infty} \frac{1}{N} \sum_{p=1}^{N} f(\overline{X}_p^h(x)) \quad a.s. \\ &= \lim_{n \longrightarrow +\infty} \frac{1}{N} \sum_{p=0}^{N} Ef(\overline{X}_p^h(x)) \end{aligned} \tag{31}$$

7.2 End of the proof of Theorem (3.3)

First, we define the symbol $\stackrel{E}{=}$: we will write $X \stackrel{E}{=} Y$ instead of $E(X) = E(Y)$.

We will just treat the case of an ergodic second-order scheme, the case of Milshtein scheme leading to simpler computations.

Let us perform a Taylor expansion up to order 6 of the function u solution of (16). As shown in Talay [21]:

$$u(jh, \overline{X}^h_{p+1}(x)) \stackrel{E}{=} u(jh, \overline{X}^h_p(x)) + Lu(jh, \overline{X}^h_p(x))h + \frac{1}{2}L^2u(jh, \overline{X}^h_p(x))h^2 + r^h_{j,p+1}h^3$$

with the remainder term $r^h_{j,p+1}$ expressed as a sum of terms, each one being of the form :

$$Constant \times E\left[\psi(\overline{X}^h_p(x))\partial_J u(jh, \overline{X}^h_p(x) + \theta(\overline{X}^h_{p+1}(x) - \overline{X}^h_p(x)))\right]$$

where :

- $\psi(x)$ is a function equal to a product of functions among the set constituted by the coordinates of b, σ and their derivatives
- $0 < \theta < 1$

Thus, using (H1), (15) and (30), one can check that the above remainder term satisfies, as soon as $h \leq H$ (where H has been defined in the previous Proposition) :

$$\exists \lambda > 0 \text{ , } \exists s \in \mathbb{N} \text{ , } \sum_{j=0}^{+\infty} |r^h_{j,p+1}| \leq \frac{C_0}{1 - e^{-\lambda h}} E(1 + |\overline{X}^h_p(x)|^s + |\overline{X}^h_{p+1}(x)|^s) \leq \frac{C}{h}(1 + |x|^s)$$

Now we use the equation (16) in order to write :

$$u((j+1)h, \overline{X}^h_p(x)) \stackrel{E}{=} u(jh, \overline{X}^h_p(x)) + Lu(jh, \overline{X}^h_p(x))h + \frac{1}{2}L^2u(jh, \overline{X}^h_p(x))h^2 + \tilde{r}^h_{j,p+1}h^3$$

with a remainder term $\tilde{r}^h_{j,p+1}$ which can be expressed in the same manner as $r^h_{j,p+1}$.

Therefore, if we define $R^h_{j,p+1}$ by

$$R^h_{j,p+1} = r^h_{j,p+1} - \tilde{r}^h_{j,p+1}$$

$R^h_{j,p+1}$ satisfies (by (30)) :

$$\sum_{j=0}^{+\infty} |R^h_{j,p+1}| \leq \frac{C_1}{h}(1 + |x|^s) \tag{32}$$

and moreover :

$$u(jh, \overline{X}^h_{p+1}(x)) \overset{E}{=} u((j+1)h, \overline{X}^h_p(x)) + R^h_{j,p+1} h^3 \tag{33}$$

Remarking :

$$\frac{1}{N}\sum_{p=1}^{N} f(\overline{X}^h_p(x)) = \frac{1}{N}\sum_{p=1}^{N} u(0, \overline{X}^h_p(x))$$

with successive uses of (33) one obtains :

$$\frac{1}{N}\sum_{p=1}^{N} f(\overline{X}^h_p(x)) \overset{E}{=} \frac{1}{N}\sum_{p=1}^{N} u(ph, x) + \frac{1}{N}\sum_{p=1}^{N}\sum_{j=0}^{p-1} R^h_{j,p} h^3$$

But :

- (X_t) being ergodic and $u(t,x)$ satisfying

  $$u(t,x) = Ef(X_t(x))$$

  we know :

  $$\lim_{N\to\infty} \frac{1}{N}\sum_{p=1}^{N} u(ph, x) = \int f(x) d\mu(x)$$

- the estimation (32) implies

  $$\frac{1}{N}\sum_{p=0}^{N}\sum_{j=0}^{p-1} R^h_{j,p} h^3 \leq C_1(1 + |x|^s) h^2$$

To have proved (14), now it just remains to use (31). $\square$

8 Conclusion

We have built schemes which are of second-order for the ergodic criterion (1), and tested them numerically.

There is no theoretical reason for which one cannot build schemes of higher order, but such schemes would be very costful in computation time. Besides, the error due to the necessarily reasonable number of integration steps N would likely mask the gain in precision due to the scheme.

From a practical point of view, an important problem remains : the choice of N, and a further research in that direction is necessary.

References

[1] R.BOUC & E.PARDOUX : Asymptotic Analysis of P.D.E.s with wide-band noise disturbances, and expansion of the moments , Stochastic Analysis and Applications 2,4,1984.

[2] J.M.C. CLARK & R.J. CAMERON : The maximaum rate of convergence of discrete approximations for stochastic differential equations , *Stochastic Differential Systems – Filtering and Control*, B. Grigelionis (Ed.), Proceedings of the IFIP Working Conference, Vilnius, Lithuania, USSR, 1978, Lecture Notes in Control and Information Sciences 25, Springer-Verlag, 1980.

[3] J.M.C. CLARK : An efficient approximation for a class of stochastic differential equations , *Advances in Filtering and Optimal Stochastic Control*, W. Fleming & L. Gorostiza (Eds.), Proceedings of the IFIP Working Conference, Cocoyoc, Mexico, 1982, Lecture Notes in Control and Information Sciences 42, Springer-Verlag, 1982.

[4] D. FLORENS-ZMIROU : Théorème de limite centrale pour une diffusion et sa discrétisée , Compte Rendu de l'Académie des Sciences de Paris, t.299, Série I, 19, 1984.

[5] A.GERARDI & F.MARCHETTI & A.M.ROSA : Simulation of diffusions with boundary conditions , Systems & Control Letters 4, 253,1984.

[6] R.Z.HASMINSKI : *Stochastic stability of differential equations* , Sijthoff & Noordhoff, 1980.

[7] H.KUNITA : Stochastic differential equations and stochastic flows of diffeomorphisms , *Cours de l'Ecole d'été de Probabilités de Saint-Flour 1982* , Lecture Notes in Mathematics, Springer, 1984.

[8] J.LEBLOND & D.TALAY : Simulation of diffusion processes with Presto, building systems like Presto with Adagio , Proceedings of the 7th French-Belgian Statisticians Meeting, Rouen, France, 1986.

[9] P.L.LIONS : private communication.

[10] G.N. MILSHTEIN : Approximate integration of stochastic differential equations , Theory of Probability and Applications, 19, 557-562, 1974.

[11] G.N. MILSHTEIN : A method of second-order accuracy integration of stochastic differential equations , Theory of Probability and Applications, 23, 1976.

[12] G.N. MILSHTEIN : Weak approximation of solutions of systems of stochastic differential equations , Theory of Probability and Applications, 30, 750-766, 1985.

[13] N.J. NEWTON : An asymptotically efficient difference formula for solving stochastic differential equations , Stochastics 19, 175-206, 1986.

[14] E. NUMMELIN : *General irreducible Markov chains and non-negative operators* , Cambridge University Press, 1984.

[15] E.PARDOUX & D.TALAY : Discretization and simulation of S.D.E. , Acta Applicandae Mathematicae 3,23, 1985.

[16] E.PARDOUX & D.TALAY : Stability of linear differential systems with parametric excitation , *Nonlinear Stochastic Dynamic Engineering Systems* , G.I. Schueller & F. Ziegler (Eds.), Proceedings of the IUTAM Symposium, Innsbruck, 1987, Springer-Verlag (to appear).

[17] E. PLATEN : An approximation method for a class of Ito processes, Lietuvos Matemematikos Rinkiniys, 1981.

[18] W. RUMELIN : Numerical treatment of stochastic differential equations , SIAM Journal on Numerical Analysis 19-3, 604-613, 1982.

[19] D.TALAY : Résolution trajectorielle et analyse numérique des équations différentielles stochastiques , Stochastics, 9, 275-306, 1983.

[20] D.TALAY : Efficient numerical schemes for the approximation of expectations of functionals of S.D.E. , *Filtering and Control of Random Processes* , H.Korezlioglu & G.Mazziotto & J.Szpirglas (Eds.), Proceedings of the ENST-CNET Colloquium, Paris, 1983, Lecture Notes in Control and Information Sciences 61, Springer-Verlag, 1984.

[21] D.TALAY : Discretisation d'une E.D.S. et calcul approché d' espérances de fonctionnelles de la solution , M2AN 20-1, 141, 1986.

[22] R.L.TWEEDIE : Sufficient conditions for ergodicity and recurrence of Markov chains on a general state space , Stochastic Processes and Applications 3, 385, 1975.

[23] R.L.TWEEDIE : The existence of moments for stationary Markov chains , J.Appl.Prob. 20,191-196, 1983.

Contents