Progress of Theoretical Physics, Vol. 113, No. 3, March 2005

Second-Order Gauge Invariant Perturbation Theory

—— *Perturbative Curvatures in the Two-Parameter Case* ——

Kouji Nakamura

Department of Astronomical Science, The Graduate University for Advanced Studies, Mitaka 181-8588, Japan

(Received October 6, 2004)

Based on the gauge invariant variables proposed in our previous paper [K. Nakamura, Prog. Theor. Phys. **110** (2003), 723], some formulae for the perturbative curvatures of each order are derived. We follow the general framework of the second-order gauge invariant perturbation theory on an arbitrary background spacetime to derive these formulae. It is found that these perturbative curvatures have the same form as those given in the definitions of gauge invariant variables for arbitrary perturbative fields, which were proposed in the above paper. As a result, we explicitly see that any perturbative Einstein equation can be given in terms of a gauge invariant form. We briefly discuss physical situations to which this framework should be applied.

§1. Introduction

In many theories of physics, realistic situations are often difficult to describe by an exact solution of a theory, because theories in physics and their exact solutions are often too idealized to properly represent natural phenomena. Given this situation, we have to consider perturbative approaches to investigate realistic situations. General relativity is one theory in which the construction of exact solutions is not so easy. Though there are many exact solutions to the Einstein equation,[1] these are often too idealized. For this reason, general relativistic perturbation theory is a useful technique to investigate natural phenomena.[2),3)]

In addition to this technical problem, general relativity is based on the concept of general covariance. Intuitively, the principle of general covariance states that there is no preferred coordinate system in nature, though the notion of general covariance is mathematically included in the definition of a spacetime manifold in a trivial way. This is based on the philosophy that coordinate systems are originally chosen by us, and natural phenomena have nothing to do with our coordinate system. Due to this general covariance, the *gauge degree of freedom*, which is an unphysical degree of freedom of perturbations, arises in general relativistic perturbations. To obtain physically meaningful results, we have to fix this gauge degrees of freedom or to extract the *gauge invariant part of perturbations.* A similar situation has been found in recent investigations of the oscillatory behavior of a gravitating Nambu-Goto membrane,[4),5)] which concern the dynamical degrees of freedom of extended gravitating objects.

On the other hand, higher-order multi-parameter perturbations, in which there are two or more small parameters, can be applied to many physical situations. One well-known application of two-parameter perturbation theory is perturbations of a spherical star,[6] in which we choose the gravitational field of the spherical star as the background spacetime for the perturbations, one of the parameters for the perturbations corresponds to the rotation of the star, and the other is its pulsation amplitude. The effects due to the rotation-pulsation coupling are described at higher orders. A similar perturbation theory on the Minkowski background spacetime was developed by the present author to study the comparison of the oscillatory behavior of a gravitating string with that of a test string.[5] Even in the one-parameter case, it is interesting to consider higher-order perturbations. In particular, Gleiser et al.[7] reported that second-order perturbations yield accurate wave forms of gravitational waves. Hence, it is worthwhile to investigate higher-order multi-parameter perturbation theory from a general point of view.

Motivated by these physical applications, the general relativistic gauge invariant multi-parameter perturbation theory has been developed in a number of papers.[5),8),9] In particular, the procedure to find gauge invariant variables for higher-order perturbations on a generic background spacetime was proposed by the present author[8] assuming that we have already known the procedure to find gauge invariant variables for a linear-order metric perturbations. The contents of this paper are based on this proposal. The main purpose of this paper is to present some formulae of the second-order perturbative curvatures within the two-parameter perturbation theory that are useful in some physical applications. When we derive these formulae, we follow the general framework of the second-order gauge invariant perturbation theory on an arbitrary background spacetime. This framework is originally proposed by Stewart et al.[10] and developed by Bruni et al.,[9),11] and the present author.[8] These perturbative curvatures have the same form as those given in the definitions of gauge invariant variables for arbitrary perturbative fields which are proposed in a previous paper.[8] As in that paper, we do not make any specific assumption regarding the background spacetime and the physical meaning of the two-parameter family. Because we make no assumption concerning the background spacetime, this framework has a wide area of applications.

The organization of this paper is as follows. In §2, we review the general framework of the second order gauge invariant perturbation theory. We mainly review the one-parameter perturbation theory. We emphasize that the review in §2 of this paper is based on the idea of Stewart et al.[10] and the development carried out by Bruni et al.[9),11] In §3, we present some formulae for the second-order perturbative curvatures within the two-parameter perturbation theory. We also give a derivation of these formulae. The final section, §4, is devoted to a summary and brief discussion of physical situations to which this framework of higher-order perturbation theory should be applied. We employ the notation of our previous paper[8] and use the abstract index notation.[12]

§2. Gauge degree of freedom in perturbation theory

In this section, we briefly review the gauge degree of freedom in general relativistic perturbations. This was originally discussed by Stewart et al.[10] To explain the *gauge degree of freedom* in perturbation theories, we have to keep in mind what we are doing when we consider perturbations. We first comment on the intuitive explanation of the gauge degree of freedom in §2.1. Next, in §2.2, we review the more precise mathematical formulation of the perturbations in the theories with general covariance. The explanation given here is based on the works of Bruni et al.,[11)] which represent extensions of the idea of Stewart et al. When we consider the perturbations in the theory with general covariance, we have to exclude these gauge degrees of freedom in perturbations. To accomplish this, gauge invariant quantities of the perturbations are useful, and these are regarded as physically meaningful quantities. In §2.3, based on the mathematical preparation given in §2.2, we review the procedure to find gauge invariant quantities of perturbations, which was developed by the present author.[8)]

2.1. *What is "gauge"?*

In perturbation theory, we always treat two spacetime manifolds. One is the physical spacetime, which we attempt to describe by perturbations, and the other is the background spacetime, which we prepare for perturbative analyses. Let us denote the physical spacetime by $(\mathcal{M}, \bar{g}_{ab})$ and the background spacetime by $(\mathcal{M}_0, g_{ab})$. Keeping in mind these two spacetime manifolds, let us formally denote the spacetime metric and the other physical tensor fields on the physical spacetime $\mathcal{M}$ by Q. As the perturbation of the physical variable Q, we always write equations of the form

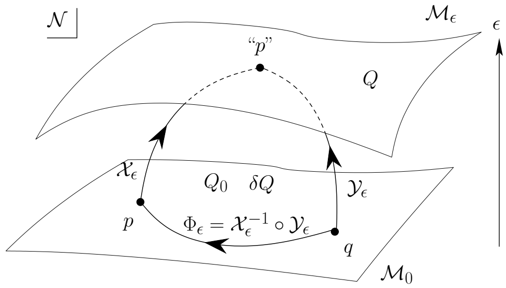

Fig. 1. "Gauge choice" in one-parameter perturbation.

$$Q(\text{“}p\text{”}) = Q_0(p) + \delta Q(p). \tag{2·1}$$

Usually, this equation is regarded as the relation between the physical variable Q and its background value Q_0 of the same field, or simply as the definition of the deviation δQ of Q from its background value, Q_0. In fact, through Eq. (2·1), we have implicitly assigned a correspondence between points of the physical and the background spacetime, since this equation gives a relation between field variables Q, Q_0 and δQ. More specifically, $Q(\text{“}p\text{”})$ in the left-hand side of Eq. (2·1) is a field on the physical spacetime $\mathcal{M}$, and "p" $\in \mathcal{M}$. On the other hand, we should regard the background value $Q_0(p)$ of $Q(\text{“}p\text{”})$ and its deviation $\delta Q(p)$ from $Q_0(p)$, which are on the right-hand side of Eq. (2·1), as fields on the background spacetime $\mathcal{M}_0$ and $p \in \mathcal{M}_0$. Because Eq. (2·1) is for field variables, it implicitly states that the points "p" $\in \mathcal{M}$ and $p \in \mathcal{M}_0$ are same. This is an implicit assumption of the existence of a map $\mathcal{M}_0 \to \mathcal{M} : p \in \mathcal{M}_0 \mapsto$ "p" $\in \mathcal{M}$, which is usually called a "*gauge choice*" in perturbation theory.[10)] Clearly, this is more than the usual assignment of coordinate labels to points within the single spacetime.

It is important to note that the correspondence between points on each $\mathcal{M}_\epsilon$, which is established by such a relation as Eq. (2·1), is not unique to the theory in which general covariance is imposed. Rather, Eq. (2·1) involves the degree of freedom corresponding to the choice of the map $\mathcal{X} : \mathcal{M}_0 \to \mathcal{M}$ (the choice of the point identification map $\mathcal{M}_0 \to \mathcal{M}$). This is called the *gauge degree of freedom*. Further, such a degree of freedom always exists in the perturbations of a theory in which we impose general covariance, unless, there is a preferred coordinate system in the theory and we naturally introduce this coordinate system onto both $\mathcal{M}_0$ and $\mathcal{M}$. Then, we can choose the identification map $\mathcal{X}$ using this coordinate system. However, there is no such coordinate system, due to general covariance, and we have no guiding principle to choose the identification map $\mathcal{X}$. Therefore, we may identify "p" $\in \mathcal{M}$ with $q \in \mathcal{M}_0$ ($q \neq p$) instead of $p \in \mathcal{M}_0$. (See Fig. 1.) A gauge transformation is simply a change of the map $\mathcal{X}$.

2.2. *More precise formulation of perturbations*

In this section, we review the more precise formulation concerning "*gauge degree of freedom*" based on the above understanding of "*gauges*".[10),11)] We mainly review the one-parameter perturbation theory in §2.2.1 and comment on the results in two-parameter perturbation theory in §2.2.2. The essential part of the multi-parameter perturbations is completely similar to the one-parameter case.[8),11)] Details can be seen in the papers by Bruni et al.[11)] and by the present author.[8)]

2.2.1. One-parameter perturbation theory

We denote the perturbation parameter by ϵ, and we consider the $m+1$-dimensional manifold $\mathcal{N} = \mathcal{M} \times \mathbb{R}$, where $m = \dim \mathcal{M}$ and $\epsilon \in \mathbb{R}$, as depicted in Fig. 1. By this construction, the manifold $\mathcal{N}$ is foliated by m-dimensional submanifolds $\mathcal{M}_\epsilon$ that are diffeomorphic to the physical spacetime $\mathcal{M}$. The background $\mathcal{M}_0 = \mathcal{N}|_{\epsilon=0}$ and the physical spacetime $\mathcal{M} = \mathcal{M}_\epsilon = \mathcal{N}|_{\mathbb{R}=\epsilon}$ are also submanifolds embedded in $\mathcal{N}$. Each point on $\mathcal{N}$ is identified by a pair (p, ϵ), where $p \in \mathcal{M}_\epsilon$, and each point on the background spacetime $\mathcal{M}_0$ in $\mathcal{N}$ is identified by $\epsilon = 0$. The manifold $\mathcal{N}$ has a natural differentiable structure consisting of the direct product of $\mathcal{M}$ and $\mathbb{R}$. By this construction, the perturbed spacetimes $\mathcal{M}_\epsilon$ for each ϵ must have the same differential structure. In other words, we require that perturbations be continuous in the sense that $(\mathcal{M}, \bar{g}_{ab})$ and $(\mathcal{M}_0, g_{ab})$ are connected by a continuous curve within the extended spacetime $\mathcal{N}$. Hence, the changes of the differential structure resulting from the perturbation, for example the formation of singularities and singular perturbations in the sense of fluid mechanics, are excluded from the study carried out in this paper.

Let us consider the set of field equations

$$\mathcal{E}[Q_\epsilon] = 0 \tag{2·2}$$

on the physical spacetime $\mathcal{M}_\epsilon$ for the physical variables Q_ϵ on $\mathcal{M}_\epsilon$. The field equation (2·2) formally represents the Einstein equation for the metric on $\mathcal{M}_\epsilon$ and the equations for matter fields on $\mathcal{M}_\epsilon$. If a tensor field Q_ϵ is given on each $\mathcal{M}_\epsilon$, Q_ϵ is automatically extended to a tensor field on $\mathcal{N}$ by $Q(p, \epsilon) := Q_\epsilon(p)$, where $p \in \mathcal{M}_\epsilon$. In this extension, the field equation (2·2) is regarded as the equation on the extended manifold $\mathcal{N}$. Thus, we have extended an arbitrary tensor field and field equations (2·2) on each $\mathcal{M}_\epsilon$ to those on the extended manifold $\mathcal{N}$.

Tensor fields on $\mathcal{N}$ obtained by the above construction are necessarily "tangent" to each $\mathcal{M}_\epsilon$, i.e., their normal component to each $\mathcal{M}_\epsilon$ identically vanishes. To consider the basis of the tangent space of $\mathcal{N}$, we introduce the normal form of each $\mathcal{M}_\epsilon$ in $\mathcal{N}$ and its dual. These are denoted by $(d\epsilon)_a$ and $(\partial/\partial\epsilon)^a$, respectively, and they satisfy

$$(d\epsilon)_a \left(\frac{\partial}{\partial\epsilon}\right)^a = 1. \tag{2·3}$$

The form $(d\epsilon)_a$ and its dual $(\partial/\partial\epsilon)^a$ are normal to any tensor field that is extended from the tangent space on each $\mathcal{M}_\epsilon$ by the above construction. The set consisting of $(d\epsilon)_a$, $(\partial/\partial\epsilon)^a$ and the basis of the tangent space on each $\mathcal{M}_\epsilon$ is regarded as the basis of the tangent space of $\mathcal{N}$.

To define the perturbation of an arbitrary tensor field Q, we compare Q on the physical spacetime $\mathcal{M}_\epsilon$ with Q_0 on the background spacetime, and it is necessary to identify the points of $\mathcal{M}_\epsilon$ with those of $\mathcal{M}_0$. This point identification map is the so-called *gauge choice* in the context of perturbation theories, as mentioned in §2.1. The gauge choice is accomplished by assigning a diffeomorphism $\mathcal{X}_\epsilon : \mathcal{N} \to \mathcal{N}$ such that $\mathcal{X}_\epsilon : \mathcal{M}_0 \to \mathcal{M}_\epsilon$. Following the paper of Bruni et al.,[11)] we introduce a gauge choice $\mathcal{X}_\epsilon$ as one of the one-parameter groups of diffeomorphisms that satisfy the property

$$\mathcal{X}_{\epsilon_1+\epsilon_2} = \mathcal{X}_{\epsilon_1} \circ \mathcal{X}_{\epsilon_2} = \mathcal{X}_{\epsilon_2} \circ \mathcal{X}_{\epsilon_1}. \tag{2·4}$$

This one-parameter group of diffeomorphisms is generated by the vector field ${}^{\mathcal{X}}\!\eta^a_{(\epsilon)}$. This vector field ${}^{\mathcal{X}}\!\eta^a_{(\epsilon)}$, which we call a *generator*, is defined by the action of the corresponding pull-back $\mathcal{X}^*_\epsilon$ for a generic tensor field Q on $\mathcal{M} \times \mathbb{R}$:

$$\pounds_{{}^{\mathcal{X}}\!\eta_{(\epsilon)}} Q := \lim_{\epsilon\to 0} \frac{\mathcal{X}^*_\epsilon Q - Q}{\epsilon}, \tag{2·5}$$

and it is decomposed as

$${}^{\mathcal{X}}\!\eta^a_{(\epsilon)} =: \left(\frac{\partial}{\partial\epsilon}\right)^a + \theta^a, \quad \theta^a (d\epsilon)_a = 0, \quad \pounds_{\frac{\partial}{\partial\epsilon}} \theta^a = 0. \tag{2·6}$$

The third condition in (2·6) is imposed merely for simplicity. Except for the conditions in (2·6), we may regard θ^a as an arbitrary vector field on $\mathcal{M}_\epsilon$ (not on $\mathcal{N}$); i.e., the arbitrariness of the gauge choice is given by that of the vector field θ^a.

The Taylor expansion of the pull-back $\mathcal{X}^*_\epsilon Q$ is given by

$$\mathcal{X}^*_\epsilon Q = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \left[\frac{\partial^k}{\partial\epsilon^k} \mathcal{X}^*_\epsilon Q\right]_{\epsilon=0} = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \pounds^k_{{}^{\mathcal{X}}\!\eta_{(\epsilon)}} Q. \tag{2·7}$$

Once the definition of the pull-back of the gauge choice $\mathcal{X}_\epsilon$ is given, the perturbation $\varDelta^{\mathcal{X}} Q_\epsilon$ of a tensor field Q under the gauge choice $\mathcal{X}_\epsilon$ is simply defined as

$$\varDelta^{\mathcal{X}} Q_\epsilon := \mathcal{X}^*_\epsilon Q|_{\mathcal{M}_0} - Q_0. \tag{2·8}$$

We note that all the variables in this definition are defined on $\mathcal{M}_0$. The first term on the right-hand side of (2·8) can be Taylor-expanded as

$$\mathcal{X}_\epsilon^* Q_\epsilon|_{\mathcal{M}_0} = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} {}^{(k)}_{\mathcal{X}}Q. \tag{2·9}$$

Equations (2·8) and (2·9) define the perturbation of $O(k)$ of a physical variable Q_ϵ under the gauge choice $\mathcal{X}$ and its background value ${}^{(0)}_{\mathcal{X}}Q = Q_0$. Through Eqs. (2·7) and (2·9), each order perturbation ${}^{(k)}_{\mathcal{X}}Q$ under the gauge choice $\mathcal{X}_\epsilon$ is given by

$${}^{(k)}_{\mathcal{X}}Q = \pounds^k_{{}^{\mathcal{X}}\eta} Q\Big|_{\mathcal{M}_0}. \tag{2·10}$$

The above understanding of the gauge choice and perturbations naturally leads to the *gauge transformation rules* between *different gauge choices* and the concept of *gauge invariance* as follows.

Suppose that $\mathcal{X}_\epsilon$ and $\mathcal{Y}_\epsilon$ are two one-parameter groups of diffeomorphisms with the generators ${}^{\mathcal{X}}\eta^a$ and ${}^{\mathcal{Y}}\eta^a$ on $\mathcal{N}$, respectively, i.e., $\mathcal{X}_\epsilon$ and $\mathcal{Y}_\epsilon$ are two gauge choices. These generators are decomposed in the same manner as Eq. (2·6):

$${}^{\mathcal{X}}\eta^a = \left(\frac{\partial}{\partial\epsilon}\right)^a + \theta^a, \quad {}^{\mathcal{Y}}\eta^a = \left(\frac{\partial}{\partial\epsilon}\right)^a + \iota^a. \tag{2·11}$$

The integral curves of each ${}^{\mathcal{X}}\eta^a$ and ${}^{\mathcal{Y}}\eta^a$ in $\mathcal{N}$ are the orbit of the action of the gauge choices $\mathcal{X}_\epsilon$ and $\mathcal{Y}_\epsilon$, respectively. Since the generators ${}^{\mathcal{X}}\eta^a$ and ${}^{\mathcal{Y}}\eta^a$ are transverse to each $\mathcal{M}_\epsilon$ everywhere on $\mathcal{N}$, the integral curves of these vector field intersect with each $\mathcal{M}_\epsilon$. Therefore, points lying on the same integral curve of either of the two are to be regarded as *the same point* within the respective gauges (see Fig. 1). Hence, $\mathcal{X}_\epsilon$ and $\mathcal{Y}_\epsilon$ are both point identification maps. When $\theta^a \neq \iota^a$, these point identification maps are regarded as *two different gauge choices*.

Suppose that $\mathcal{X}_\epsilon$ and $\mathcal{Y}_\epsilon$ are two different gauge choices which are generated by the vector fields ${}^{\mathcal{X}}\eta^a$ and ${}^{\mathcal{Y}}\eta^a$, respectively. These gauge choices also pull back a generic tensor field Q on $\mathcal{N}$ to two other tensor fields, $\mathcal{X}_\epsilon^* Q$ and $\mathcal{Y}_\epsilon^* Q$, for any given value of ϵ. In particular, on $\mathcal{M}_0$, we now have three tensor fields associated with a tensor field Q; i.e., one is the background value Q_0 of Q, and the other two are the pulled back variables of Q from $\mathcal{M}_\epsilon$ to $\mathcal{M}_0$ by the two different gauge choices,

$${}^{\mathcal{X}}Q_\epsilon := \mathcal{X}_\epsilon^* Q|_{\mathcal{M}_0} = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} {}^{(k)}_{\mathcal{X}}Q = Q_0 + \Delta^{\mathcal{X}} Q_\epsilon, \tag{2·12}$$

$${}^{\mathcal{Y}}Q_\epsilon := \mathcal{Y}_\epsilon^* Q|_{\mathcal{M}_0} = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} {}^{(k)}_{\mathcal{Y}}Q = Q_0 + \Delta^{\mathcal{Y}} Q_\epsilon. \tag{2·13}$$

Here, we have used Eqs. (2·8) and (2·9). Because $\mathcal{X}_\epsilon$ and $\mathcal{Y}_\epsilon$ are gauge choices that map the background spacetime $\mathcal{M}_0$ into the physical spacetime $\mathcal{M}_\epsilon$, ${}^{\mathcal{X}}Q_\epsilon$ and ${}^{\mathcal{Y}}Q_\epsilon$ are the representations on $\mathcal{M}_0$ of the perturbed tensor field Q in the two different gauges. The quantities ${}^{(k)}_{\mathcal{X}}Q$ and ${}^{(k)}_{\mathcal{Y}}Q$ in Eqs. (2·12) and (2·13) are the perturbations of $O(k)$ in the gauges $\mathcal{X}$ and $\mathcal{Y}$, respectively.

Now, we consider the concept of *gauge invariance*. Following the paper of Bruni et al.,[9] we consider the concept of *gauge invariance up to order n*. We say that Q is *gauge invariant up to order n* iff for any two gauges $\mathcal{X}$ and $\mathcal{Y}$

$${}^{(k)}_{\mathcal{X}}Q = {}^{(k)}_{\mathcal{Y}}Q \quad \forall k, \quad \text{with} \quad k < n. \tag{2·14}$$

From this definition, we can prove that the nth-order perturbation of a tensor field Q is gauge invariant up to order n iff in a given gauge $\mathcal{X}$ we have $\pounds_{\xi}{}^{(k)}_{\mathcal{X}}Q = 0$ for any vector field ξ^a defined on $\mathcal{M}_0$ and for any $k < n$. As a consequence, the nth-order perturbation of a tensor field Q is gauge invariant up to order n iff Q_0 and all its perturbations of lower than nth order are, in any gauge, either vanishing or constant scalars, or a combination of Kronecker deltas with constant coefficients.[9)–11)]

In general, the representation ${}^{\mathcal{X}}Q_\epsilon$ on $\mathcal{M}_0$ of the perturbed variable Q on $\mathcal{M}_\epsilon$ depends on the gauge choice $\mathcal{X}_\epsilon$. If we apply a different gauge choice, the representation of Q_ϵ on $\mathcal{M}_0$ may change. Recalling that the gauge choice $\mathcal{X}$ is a point identification map from $\mathcal{M}_0$ to $\mathcal{M}_\epsilon$ (see Fig. 1), the change of the gauge choice from $\mathcal{X}_\epsilon$ to $\mathcal{Y}_\epsilon$ is represented by the diffeomorphism

$$\Phi_\epsilon := (\mathcal{X}_\epsilon)^{-1} \circ \mathcal{Y}_\epsilon. \tag{2·15}$$

This diffeomorphism Φ_ϵ is the map $\Phi_\epsilon : \mathcal{M}_0 \to \mathcal{M}_0$ for each value of $\epsilon \in \mathbb{R}$. As shown in Fig. 1, the diffeomorphism Φ_ϵ changes the point identification, as expected from the understanding of the gauge choice discussed in §2.1. Therefore, the diffeomorphism Φ_ϵ is regarded as the gauge transformation $\Phi_\epsilon : \mathcal{X}_\epsilon \to \mathcal{Y}_\epsilon$.

The gauge transformation Φ_ϵ induces a pull-back from the representation ${}^{\mathcal{X}}Q_\epsilon$ of the perturbed tensor field in the gauge choice $\mathcal{X}_\epsilon$ to the representation ${}^{\mathcal{Y}}Q_\epsilon$ in the gauge choice $\mathcal{Y}_\epsilon$. Actually, the tensor fields ${}^{\mathcal{X}}Q_\epsilon$ and ${}^{\mathcal{Y}}Q_\epsilon$, which are defined on $\mathcal{M}_0$, are connected by the linear map Φ^*_ϵ as

$$\begin{aligned} {}^{\mathcal{Y}}Q_\epsilon = \mathcal{Y}^*_\epsilon Q|_{\mathcal{M}_0} &= \left(\mathcal{Y}^*_\epsilon \left(\mathcal{X}_\epsilon \mathcal{X}^{-1}_\epsilon\right)^* Q\right)\Big|_{\mathcal{M}_0} \\ &= \left(\mathcal{X}^{-1}_\epsilon \mathcal{Y}_\epsilon\right)^* \left(\mathcal{X}^*_\epsilon Q\right)\Big|_{\mathcal{M}_0} = \Phi^*_\epsilon {}^{\mathcal{X}}Q_\epsilon. \end{aligned} \tag{2·16}$$

According to generic arguments concerning the Taylor expansion of the pull-back of a tensor field on the same manifold,[8),11)] the gauge transformation $\Phi^*_\epsilon {}^{\mathcal{X}}Q_\epsilon$ should be given by the form

$$\Phi^*_\epsilon {}^{\mathcal{X}}Q = {}^{\mathcal{X}}Q + \epsilon \pounds_{\xi_1} {}^{\mathcal{X}}Q + \frac{\epsilon^2}{2}\left\{\pounds_{\xi_2} + \pounds^2_{\xi_1}\right\} {}^{\mathcal{X}}Q + O(\epsilon^3), \tag{2·17}$$

where the vector fields ξ^a_1 and ξ^a_2 are the generators of the gauge transformation Φ_ϵ.

Comparing the representation (2·17) of the expansion in terms of the generators ξ^a_p of the pull-back $\Phi^*_\epsilon {}^{\mathcal{X}}Q$ and that in terms of the generators ${}^{\mathcal{X}}\eta^a_{(\epsilon)}$ and ${}^{\mathcal{Y}}\eta^a_{(\epsilon)}$ of the pull-back $\mathcal{Y}^*_\epsilon \circ \left(\mathcal{X}^{-1}_\epsilon\right)^* Q$ $(= \Phi^*_\epsilon {}^{\mathcal{X}}Q)$, we easily find explicit expressions for the generators ξ^a_p of the gauge transformation $\Phi = \mathcal{X}^{-1} \circ \mathcal{Y}$ in terms of the generators

${}^{\mathcal{X}}\eta^a_{(\epsilon)}$ and ${}^{\mathcal{Y}}\eta^a_{(\epsilon)}$ of the gauge choices. Further, because the gauge transformation Φ_ϵ is a map within the background spacetime $\mathcal{M}_0$, the generator should be given as vector fields on $\mathcal{M}_0$. The explicit expression of the generators ξ^a_p in terms of the components of the generators of the gauge choices is given in some papers.[8),11)]

We can now derive the relation between the perturbations in the two different gauges. Up to second order, these relations are derived by substituting (2·12) and (2·13) into (2·17):

$$\overset{(1)}{\underset{\mathcal{Y}}{}}Q - \overset{(1)}{\underset{\mathcal{X}}{}}Q = \pounds_{\xi_1} Q_0, \tag{2·18}$$

$$\overset{(2)}{\underset{\mathcal{Y}}{}}Q - \overset{(2)}{\underset{\mathcal{X}}{}}Q = 2\pounds_{\xi_{(1)}} \overset{(1)}{\underset{\mathcal{X}}{}}Q + \left\{ \pounds_{\xi_{(2)}} + \pounds^2_{\xi_{(1)}} \right\} Q_0. \tag{2·19}$$

These results are, of course, consistent with the concept of gauge invariance up to order n, as introduced above. Inspecting these gauge transformation rules, we can define the gauge invariant variables.

2.2.2. Two-parameter perturbation theory

Here, we briefly review the two-parameter case. We denote the two parameters for the perturbation by ϵ and λ. In this case, we have to consider the extended manifold $\mathcal{N} = \mathcal{M} \times \mathbb{R}^2$, instead of $\mathcal{N} = \mathcal{M} \times \mathbb{R}$ used in the one-parameter case, where $(\epsilon, \lambda) \in \mathbb{R}^2$. As in the one-parameter case, the gauge choice $\mathcal{X}_{\epsilon,\lambda}$ is a point identification map $\mathcal{X}_{\epsilon,\lambda} : \mathcal{M}_0 \to \mathcal{M}$ on $\mathcal{N}$. This gauge choice $\mathcal{X}_{\epsilon,\lambda}$ has the property

$$\mathcal{X}_{\epsilon_1,\lambda_1} \circ \mathcal{X}_{\epsilon_2,\lambda_2} = \mathcal{X}_{\epsilon_1+\epsilon_2,\lambda_1+\lambda_2} \quad \lambda_1, \lambda_2, \epsilon_1, \epsilon_2 \in \mathbb{R}. \tag{2·20}$$

This property implies that

$$\mathcal{X}_{\epsilon,\lambda} = \mathcal{X}_{\epsilon,0} \circ \mathcal{X}_{0,\lambda} = \mathcal{X}_{0,\lambda} \circ \mathcal{X}_{\epsilon,0}, \tag{2·21}$$

where $\mathcal{X}_{\epsilon,0}$ and $\mathcal{X}_{0,\lambda}$ are two one-parameter groups of diffeomorphisms defined by the property Eq. (2·4).

We denote the generators of $\mathcal{X}_{\epsilon,0}$ and $\mathcal{X}_{0,\lambda}$ by ${}^{\mathcal{X}}\eta^a_{(\epsilon)}$ and ${}^{\mathcal{X}}\eta^a_{(\lambda)}$, respectively. We also introduce the basis $(\partial/\partial\epsilon)^a$, $(d\epsilon)_a$, $(\partial/\partial\lambda)^a$, $(d\lambda)_a$ as vector fields on $\mathcal{N}$, which satisfy conditions similar to those in (2·3).[8)] Using these bases, the generators ${}^{\mathcal{X}}\eta^a_{(\epsilon)}$ (${}^{\mathcal{X}}\eta^a_{(\lambda)}$) of the one-parameter group of diffeomorphisms $\mathcal{X}_{\epsilon,0}$ ($\mathcal{X}_{0,\lambda}$) are decomposed in the same manner as in Eq. (2·6). The property (2·21) is expressed by

$$[{}^{\mathcal{X}}\eta_{(\epsilon)}, {}^{\mathcal{X}}\eta_{(\lambda)}]^a = 0 \tag{2·22}$$

in terms of these generators. The Taylor expansion of the pull-back $\mathcal{X}^*_{\epsilon,\lambda}Q$ is given by

$$\mathcal{X}^*_{\epsilon,\lambda}Q = \sum_{k,k'=0}^{\infty} \frac{\lambda^k \epsilon^{k'}}{k!k'!} \left[\frac{\partial^{k+k'}}{\partial\lambda^k \partial\epsilon^{k'}} \mathcal{X}^*_{\epsilon,\lambda}Q \right]_{\lambda=\epsilon=0} \tag{2·23}$$

$$= \sum_{k,k'=0}^{\infty} \frac{\lambda^k \epsilon^{k'}}{k!k'!} \pounds^k_{{}^{\mathcal{X}}\eta_{(\lambda)}} \pounds^{k'}_{{}^{\mathcal{X}}\eta_{(\epsilon)}} Q. \tag{2·24}$$

The perturbation $\Delta_0^{\mathcal{X}} Q_{\epsilon,\lambda}$ of an arbitrary tensor field Q in terms of the gauge choice $\mathcal{X}_{\epsilon,\lambda}$ is given by

$$\Delta_0^{\mathcal{X}} Q_{\epsilon,\lambda} := \mathcal{X}^*_{\epsilon,\lambda} Q\big|_{\mathcal{M}_0} - Q_0, \quad \mathcal{X}^*_{\epsilon,\lambda} Q\big|_{\mathcal{M}_0} = \sum_{k,k'=0}^{\infty} \frac{\lambda^k \epsilon^{k'}}{k!k'!} \, {}^{(k,k')}_{\quad\mathcal{X}}Q, \tag{2·25}$$

where ${}^{(k,k')}_{\quad\mathcal{X}}Q$ is the perturbation of order (k,k') and ${}^{(0,0)}_{\quad\mathcal{X}}Q = Q_0$. Together with the expansion given in (2·24) and (2·25), each order perturbation ${}^{(k,k')}_{\quad\mathcal{X}}Q$ with the gauge choice $\mathcal{X}_{\epsilon,\lambda}$ is given by

$${}^{(k,k')}_{\quad\mathcal{X}}Q = \pounds^k_{{}^{\mathcal{X}}\eta_{(\lambda)}} \pounds^{k'}_{{}^{\mathcal{X}}\eta_{(\epsilon)}} Q\Big|_{\mathcal{M}_0}. \tag{2·26}$$

The concept of different gauges, a gauge transformation, gauge invariance, and the definition of gauge transformation rules in the two-parameter case are similar to those in the one-parameter case. For this reason, we can derive the following gauge transformation rules:[8)]

$${}^{(p,q)}_{\quad\mathcal{Y}}Q - {}^{(p,q)}_{\quad\mathcal{X}}Q = \pounds_{\xi_{(p,q)}} Q_0 \qquad \text{for} \quad (p,q) = (1,0),(0,1), \tag{2·27}$$

$${}^{(p,q)}_{\quad\mathcal{Y}}Q - {}^{(p,q)}_{\quad\mathcal{X}}Q = 2\pounds_{\xi_{(\frac{p}{2},\frac{q}{2})}} {}^{(\frac{p}{2},\frac{q}{2})}_{\quad\mathcal{X}}Q + \left\{ \pounds_{\xi_{(p,q)}} + \pounds^2_{\xi_{(\frac{p}{2},\frac{q}{2})}} \right\} Q_0, \qquad \text{for} \quad (p,q) = (2,0),(0,2), \tag{2·28}$$

$${}^{(1,1)}_{\quad\mathcal{Y}}Q - {}^{(1,1)}_{\quad\mathcal{X}}Q = \pounds_{\xi_{(1,0)}} {}^{(0,1)}_{\quad\mathcal{X}}Q + \pounds_{\xi_{(0,1)}} {}^{(1,0)}_{\quad\mathcal{X}}Q + \left\{ \pounds_{\xi_{(1,1)}} + \frac{1}{2}\pounds_{\xi_{(1,0)}}\pounds_{\xi_{(0,1)}} + \frac{1}{2}\pounds_{\xi_{(0,1)}}\pounds_{\xi_{(1,0)}} \right\} Q_0, \tag{2·29}$$

where the $\xi^a_{(p,q)}$ are the generators for gauge transformation $\Phi_{\epsilon,\lambda} := (\mathcal{X}_{\epsilon,\lambda})^{-1} \circ \mathcal{Y}_{\epsilon,\lambda}$.

In this paper, we treat these gauge transformation rules of two-parameter perturbation theory as mentioned in the introduction (§1), because the one-parameter case considered above can be treated as a special case of this two-parameter case.

2.3. *Gauge invariant variables*

Inspecting the gauge transformation rules (2·27)–(2·29), we can define the gauge invariant variables for a metric perturbation and for arbitrary matter fields. Employing the idea of gauge invariance up to order n for nth-order perturbations,[11)] we proposed the procedure to construct gauge invariant variables of higher-order perturbations.[8)] This proposal is as follows. First, we construct gauge invariant variables for the metric perturbation. Then, we define the gauge invariant variables for perturbations of an arbitrary field, excluding perturbations of the metric. The procedure to find the gauge invariant part of a higher-order perturbation is a simple extension of that for linear-order perturbations.

To consider the metric perturbation, we expand the metric on the physical spacetime $\mathcal{M}$, which is pulled back to the background spacetime $\mathcal{M}_0$ using a gauge choice in the form given in (2·25):

$$\mathcal{X}^{*}_{\epsilon,\lambda}\bar{g}_{ab} = \sum_{k',k=0}^{\infty} \frac{\epsilon^{k}\lambda^{k'}}{k!k'!} {}^{(k,k')}_{\quad\mathcal{X}}h_{ab} \tag{2·30}$$

$$= g_{ab} + \epsilon {}^{(1,0)}_{\quad\mathcal{X}}h_{ab} + \lambda {}^{(0,1)}_{\quad\mathcal{X}}h_{ab} + \frac{\epsilon^2}{2} {}^{(2,0)}_{\quad\mathcal{X}}h_{ab} + \epsilon\lambda {}^{(1,1)}_{\quad\mathcal{X}}h_{ab} + \frac{\lambda^2}{2} {}^{(0,2)}_{\quad\mathcal{X}}h_{ab} + O^3(\epsilon,\lambda), \tag{2·31}$$

where ${}^{(0,0)}h_{ab} = g_{ab}$ is the metric on the background spacetime $\mathcal{M}_0$. Of course, the expansion (2·31) of the metric depends entierly on the gauge choice $\mathcal{X}$. Nevertheless, we do not explicitly express the index of the gauge choice $\mathcal{X}$ in an expression if there is no possibility of confusion.

Our starting point to construct gauge invariant variables is the assumption that *we already know the procedure to find gauge invariant variables for the linear metric perturbations.* Then, linear metric perturbations ${}^{(1,0)}h_{ab}$ (${}^{(0,1)}h_{ab}$) are decomposed as

$${}^{(p,q)}h_{ab} =: {}^{(p,q)}\mathcal{H}_{ab} + 2\nabla_{(a}\, {}^{(p,q)}X_{b)}, \quad (p,q) = (1,0),(0,1), \tag{2·32}$$

where ${}^{(p,q)}\mathcal{H}_{ab}$ and ${}^{(p,q)}X_a$ are the gauge invariant and variant parts of the linear-order metric perturbations.[8] Hence, under the gauge transformation (2·27), these are transformed as ${}^{(p,q)}_{\quad\mathcal{Y}}\mathcal{H}_{ab} - {}^{(p,q)}_{\quad\mathcal{X}}\mathcal{H}_{ab} = 0$ and ${}^{(p,q)}_{\quad\mathcal{Y}}X^a - {}^{(p,q)}_{\quad\mathcal{X}}X^a = \xi^a_{(p,q)}$.

As emphasized in a previous paper,[8] the above assumption is quite strong and it is not trivial to carry out the systematic decomposition (2·32) on an arbitrary background spacetime, as this procedure depends completely on the background spacetime $(\mathcal{M}_0, g_{ab})$. However, this procedure is known in the perturbation theory on some simple background spacetimes, for example the cosmological perturbations of homogeneous and isotropic universes[2] or perturbations of spherically symmetric spacetimes.[3] Further, from a general point of view, knowledge of linear perturbation theory is always necessary to carry out the second-order perturbations. For these reasons, we start from this assumption in spite of the fact that it is quite strong.

Once we accept this assumption, we can always find gauge invariant variables for higher-order perturbations.[8] As shown in a previous paper,[8] at second-order, the metric perturbations are decomposed as

$${}^{(p,q)}h_{ab} =: {}^{(p,q)}\mathcal{H}_{ab} + 2\pounds_{{}^{(\frac{p}{2},\frac{q}{2})}X}\, {}^{(\frac{p}{2},\frac{q}{2})}h_{ab} + \left(\pounds_{{}^{(p,q)}X} - \pounds^2_{{}^{(\frac{p}{2},\frac{q}{2})}X}\right) g_{ab}, \qquad (p,q) = (2,0),(0,2); \tag{2·33}$$

$${}^{(1,1)}h_{ab} =: {}^{(1,1)}\mathcal{H}_{ab} + \pounds_{{}^{(0,1)}X}\, {}^{(1,0)}h_{ab} + \pounds_{{}^{(1,0)}X}\, {}^{(0,1)}h_{ab} + \left\{\pounds_{{}^{(1,1)}X} - \frac{1}{2}\pounds_{{}^{(1,0)}X}\pounds_{{}^{(0,1)}X} - \frac{1}{2}\pounds_{{}^{(0,1)}X}\pounds_{{}^{(1,0)}X}\right\} g_{ab}, \tag{2·34}$$

where ${}^{(p,q)}\mathcal{H}_{ab}$ and ${}^{(p,q)}X_a$ are the gauge invariant and variant parts of the metric perturbations under the gauge transformation rules (2·27)–(2·29).

Furthermore, using the gauge variant parts ${}^{(p,q)}X_a$ of metric perturbations,[8] gauge invariant variables for an arbitrary field Q excluding the metric are given by[8]

$$
\begin{aligned}
{}^{(p,q)}\mathcal{Q} &:= {}^{(p,q)}Q - \pounds_{{}_{(p,q)}X} Q_0, \qquad (p,q) = (1,0),(0,1), && (2\cdot35)\\
{}^{(p,q)}\mathcal{Q} &:= {}^{(p,q)}Q - 2\pounds_{{}_{(\frac{p}{2},\frac{q}{2})}X}{}^{(\frac{p}{2},\frac{q}{2})}Q \\
&\quad - \left\{\pounds_{{}_{(p,q)}X} - \pounds^2_{{}_{(\frac{p}{2},\frac{q}{2})}X}\right\} Q_0, \qquad (p,q) = (2,0),(0,2), && (2\cdot36)\\
{}^{(1,1)}\mathcal{Q} &:= {}^{(1,1)}Q - \pounds_{{}_{(1,0)}X}{}^{(0,1)}Q - \pounds_{{}_{(0,1)}X}{}^{(1,0)}Q \\
&\quad - \left\{\pounds_{{}_{(1,1)}X} - \frac{1}{2}\pounds_{{}_{(1,0)}X}\pounds_{{}_{(0,1)}X} - \frac{1}{2}\pounds_{{}_{(0,1)}X}\pounds_{{}_{(1,0)}X}\right\} Q_0. && (2\cdot37)
\end{aligned}
$$

It is straightforward to confirm that the variables ${}^{(p,q)}\mathcal{Q}$ defined by (2·35)–(2·37) are gauge invariant under the gauge transformation rules (2·27)–(2·29). In this paper, we derive some formulae for second-order perturbations of curvatures from the expansion of the metric perturbation on a generic spacetime. The starting point of this derivation is the decomposition (2·32)–(2·34) of the metric perturbations in terms of the gauge invariant and variant variables. As a result, we find that all formulae have forms which are similar to those given in the definitions (2·35)–(2·37) of the gauge invariant variables for arbitrary matter fields.

§3. Formulae of perturbative curvatures

Now, we derive the formulae for the perturbative curvatures at each order in two parameter perturbation theory, following the standard derivation of the perturbative curvature.[12]

The starting point of the derivation is simply the definition of the curvature $\bar{R}_{abc}{}^{d}$ on the physical spacetime $(\mathcal{M}, \bar{g}_{ab})$

$$\left(\bar{\nabla}_a\bar{\nabla}_b - \bar{\nabla}_b\bar{\nabla}_a\right)\bar{\omega}_c = \bar{\omega}_d \bar{R}_{abc}{}^{d}, \qquad (3\cdot1)$$

where $\bar{\nabla}_a$ is the covariant derivative associated with the metric $\bar{g}_{ab}$ on the physical spacetime $\mathcal{M}$ and $\bar{\omega}_c$ is an arbitrary one-form on the physical spacetime $\mathcal{M}$. We similarly define the curvature $R_{abc}{}^{d}$ on the background spacetime $(\mathcal{M}_0, g_{ab})$,

$$\left(\nabla_a\nabla_b - \nabla_b\nabla_a\right)\omega_c = \omega_d R_{abc}{}^{d}, \qquad (3\cdot2)$$

where ∇_a is the covariant derivative associated with the metric g_{ab} on the background spacetime $\mathcal{M}_0$, and ω_c is an arbitrary one-form on the background spacetime $\mathcal{M}_0$. Our task is to compare $\bar{R}_{abc}{}^{d}$ and $R_{abc}{}^{d}$. To accomplish this, we have to consider the difference between the gauge choices for the physical spacetime $\mathcal{M}$ and the background spacetime $\mathcal{M}_0$ as discussed above.

To compare the Riemann curvature (3·1) of the physical spacetime $\mathcal{M}$ and that (3·2) of the background spacetime $\mathcal{M}_0$, we introduce the derivative operator $\mathcal{X}^*\bar{\nabla}_a\left(\mathcal{X}^{-1}\right)^*$ on the background spacetime $\mathcal{M}_0$. This derivative operator $\mathcal{X}^*\bar{\nabla}_a\left(\mathcal{X}^{-1}\right)^*$ is the pull-back of the covariant derivative $\bar{\nabla}_a$ associated with the metric $\bar{g}_{ab}$ on the physical spacetime $\mathcal{M}$. The property of the derivative operator $\mathcal{X}^*\bar{\nabla}_a\left(\mathcal{X}^{-1}\right)^*$ as the covariant derivative is given by

$$\mathcal{X}^*\bar{\nabla}_a\left(\left(\mathcal{X}^{-1}\right)^*\mathcal{X}^*\bar{g}_{ab}\right)=0, \tag{3·3}$$

where $\mathcal{X}^*\bar{g}_{ab}$ is the pull-back of the metric on the physical spacetime $\mathcal{M}$, which is expanded as Eq. (2·31). Through the introduction of this operator $\mathcal{X}^*\bar{\nabla}_a\left(\mathcal{X}^{-1}\right)^*$, we can regard the definition of the Riemann curvature (3·1) on the physical spacetime $\mathcal{M}$ as an equation on the background spacetime. Since the pull-back $\mathcal{X}^*\bar{\nabla}_a\left(\mathcal{X}^{-1}\right)^*$ on the background spacetime $\mathcal{M}_0$ of the covariant derivative $\bar{\nabla}_a$ on the physical spacetime $\mathcal{M}$ is linear, satisfies the Leibnitz rule, commutes with contraction, is consistent with the concept of tangent vectors, and is torsion free,[*)] we can regard it as a derivative operator on the background spacetime $\mathcal{M}_0$.[12)] Of course, the representation of this derivative operator $\mathcal{X}^*\bar{\nabla}_a\left(\mathcal{X}^{-1}\right)^*$ on the background spacetime $\mathcal{M}_0$ depends entirely on the gauge choice $\mathcal{X}$. Though we should keep in mind that we have already chosen a gauge when we regard Eq. (3·1) as an equation on the background spacetime $\mathcal{M}_0$, we do not explicitly express the index of the gauge choice $\mathcal{X}$ in any expression, again.

Since $\bar{\nabla}_a$ $(=\mathcal{X}^*\bar{\nabla}_a\left(\mathcal{X}^{-1}\right)^*)$ may be regarded as a derivative operator on the background spacetime that satisfies $\bar{\nabla}_a\bar{g}_{bc}=0$, there exists a tensor field $C^c{}_{ab}$ on the background spacetime $\mathcal{M}_0$ such that

$$\bar{\nabla}_a\omega_b=\nabla_a\omega_b-C^c{}_{ab}\omega_c, \tag{3·4}$$

where ω_c is an arbitrary one-form on the background spacetime $\mathcal{M}_0$. From the property (3·3) of the covariant derivative operator $\bar{\nabla}_a$ on $\mathcal{M}$, the tensor field $C^c{}_{ab}$ on $\mathcal{M}_0$ is given by

$$C^c{}_{ab}=\frac{1}{2}\bar{g}^{cd}\left(\nabla_a\bar{g}_{db}+\nabla_b\bar{g}_{da}-\nabla_d\bar{g}_{ab}\right). \tag{3·5}$$

We note that the gauge dependence of the derivative $\bar{\nabla}_a$ as an operator on $\mathcal{M}_0$ is included only in this tensor field $C^c{}_{ab}$. From Eq. (3·1), the Riemann curvature $\bar{R}_{abc}{}^d$ associated with the metric $\bar{g}_{ab}$ is given by the Riemann curvature $R_{abc}{}^d$ on the background spacetime and the tensor field $C^c{}_{ab}$ as follows:

$$\bar{R}_{abc}{}^d=R_{abc}{}^d-2\nabla_{[a}C^d{}_{b]c}+2C^e{}_{c[a}C^d{}_{b]e}. \tag{3·6}$$

To obtain a perturbative expression of the curvatures, we first calculate the expansion of the inverse metric $\bar{g}^{ab}$, and then the perturbative expression of the tensor $C^c{}_{ab}$ by using Eq. (3·5). Next, we derive an expression of the perturbative curvature.

In this paper, we present some formulae for the second-order perturbative curvature in the two-parameter perturbation theory. To derive the second-order formulae, we first calculate the $O(\epsilon\lambda)$ formulae, since the other second-order formulae $[O(\epsilon^2)$ and $O(\lambda^2)]$ are easily derived from these for $O(\epsilon\lambda)$ through a simple replacement of

*) In this paper, we do not treat the torsion tensor. If we wish to consider a spacetime with torsion, we have to extend the formulation to that including the torsion tensor.

the perturbative variables. We also note that all variables on the physical spacetime $\mathcal{M}$ are pulled-back to the background spacetime $\mathcal{M}_0$ using a gauge choice $\mathcal{X}$. In this sense, all variables treated below are tensor fields defined on the background spacetime $\mathcal{M}_0$. We also denote the perturbative expansion of the pull-back of the variable $\bar{Q}$ on the physical spacetime $\mathcal{M}$ by

$$\bar{Q} = \sum_{k',k=0}^{\infty} \frac{\epsilon^k \lambda^{k'}}{k!k'!} {}^{(k,k')}\bar{Q}, \tag{3·7}$$

as in Eq. (2·25).

Once we have derived the formulae of the perturbative Riemann curvature (see §3.1) of each order, it is straightforward to derive corresponding formulae of the Ricci curvature (§3.2), scalar curvature (§3.3), Einstein tensor (§3.4), and Weyl curvature (§3.5). We also derive the perturbative form of the divergence of an arbitrary tensor field of the second rank to check the perturbative Bianchi identities.

3.1. *Expansion of the inverse metric and the Riemann curvature*

Following the outline of the calculations explained above, we first calculate the perturbative expansion of the inverse metric. The expression for the inverse metric can be readily derived from the expansion (2·31) of the metric $\bar{g}_{ab}$ and the definition of the inverse metric

$$\bar{g}^{ab}\bar{g}_{bc} = \delta^a{}_c. \tag{3·8}$$

We also expand the inverse metric $\bar{g}^{ab}$ in the form (3·7). Then, each term of the expansion of the inverse metric is given by

$$ {}^{(p,q)}\bar{g}^{ab} = -{}^{(p,q)}h^{ab}, \qquad (p,q) = (1,0),(0,1) \tag{3·9}$$

$$ {}^{(p,q)}\bar{g}^{ab} = 2{}^{(\frac{p}{2},\frac{q}{2})}h^{ac}\,{}^{(\frac{p}{2},\frac{q}{2})}h_c{}^b - {}^{(p,q)}h^{ab}, \qquad (p,q) = (2,0),(0,2) \tag{3·10}$$

$$ {}^{(1,1)}\bar{g}^{ab} = {}^{(0,1)}h^{ca}\,{}^{(1,0)}h_c{}^b + {}^{(0,1)}h^{cb}\,{}^{(1,0)}h_c{}^a - {}^{(1,1)}h^{ab}. \tag{3·11}$$

To derive the formulae for the perturbative expansion of the Riemann curvature, we have to derive the formulae for the perturbative expansion of the tensor $C^c{}_{ab}$ defined in Eq. (3·5). The tensor $C^c{}_{ab}$ is also expanded in the same form as Eq. (3·7). The first-order perturbations of $C^c{}_{ab}$ have the well-known form[12)]

$$ {}^{(p,q)}C^c{}_{ab} = \nabla_{(a}\,{}^{(p,q)}h_{b)}{}^c - \frac{1}{2}\nabla^c\,{}^{(p,q)}h_{ab} =: H_{ab}{}^c\left[{}^{(p,q)}h\right], \tag{3·12}$$

where $(p,q) = (1,0),(0,1)$, and ${}^{(p,q)}h$ in the brackets of the variable $H_{ab}{}^c\left[{}^{(p,q)}h\right]$ indicates that $H_{ab}{}^c\left[{}^{(p,q)}h\right]$ is constituted of three covariant derivatives of the perturbative metric ${}^{(p,q)}h_{ab}$. In terms of the tensor field defined by Eq. (3·12), the second-order perturbations of $C^c{}_{ab}$ are given by

$$ {}^{(p,q)}C^c{}_{ab} = H_{ab}{}^c\left[{}^{(p,q)}h\right] - 2{}^{(\frac{p}{2},\frac{q}{2})}h^{cd}\,H_{abd}\left[{}^{(\frac{p}{2},\frac{q}{2})}h\right], \tag{3·13}$$

$$ {}^{(1,1)}C^c{}_{ab} = H_{ab}{}^c\left[{}^{(1,1)}h\right] - {}^{(1,0)}h^{cd}\,H_{abd}\left[{}^{(0,1)}h\right] - {}^{(0,1)}h^{cd}\,H_{abd}\left[{}^{(1,0)}h\right], \tag{3·14}$$

where $(p,q)=(2,0),(0,2)$ in Eq. (3·13).

The Riemann curvature (3·6) on the physical spacetime $\mathcal{M}$ can be expanded in the form (3·7). The forms of the perturbative Riemann curvature up to second order are given by

$$ {}^{(p,q)}\bar{R}_{abc}{}^{d} = -2\nabla_{[a}\,{}^{(p,q)}C^{d}{}_{b]c}, \qquad (p,q)=(1,0),(0,1), \quad (3·15)$$

$$ {}^{(p,q)}\bar{R}_{abc}{}^{d} = -2\nabla_{[a}\,{}^{(p,q)}C^{d}{}_{b]c} + 4\,{}^{(\frac{p}{2},\frac{q}{2})}C^{e}{}_{c[a}\,{}^{(\frac{p}{2},\frac{q}{2})}C^{d}{}_{b]e}, \quad (p,q)=(2,0),(0,2), \quad (3·16)$$

$$ {}^{(1,1)}\bar{R}_{abc}{}^{d} = -2\nabla_{[a}\,{}^{(1,1)}C^{d}{}_{b]c} + 2\,{}^{(1,0)}C^{e}{}_{c[a}\,{}^{(0,1)}C^{d}{}_{b]e} + 2\,{}^{(0,1)}C^{e}{}_{c[a}\,{}^{(0,1)}C^{d}{}_{b]e}. \quad (3·17)$$

Substituting Eqs. (3·12)–(3·14) into Eqs. (3·15)–(3·17), we obtain the perturbative form of the Riemann curvature in terms of the variables defined by Eq. (3·12). This perturbative form of linear-order is simply given by the replacement

$$ {}^{(p,q)}C^{c}{}_{ab} \rightarrow H_{ab}{}^{c}\left[{}^{(p,q)}h\right] \quad (3·18)$$

in Eq. (3·15). On the other hand, the perturbative form of the $O(\epsilon^2)$ and $O(\lambda^2)$ Riemann curvatures are derived from the perturbative form of $O(\epsilon\lambda)$. For these reasons, we only present the derivation of the perturbative form of $O(\epsilon\lambda)$ in terms of the variables defined by Eq. (3·12),

$$\begin{aligned} {}^{(1,1)}\bar{R}_{abc}{}^{d} = & -2\nabla_{[a}H_{b]c}{}^{d}\left[{}^{(1,1)}h\right] \\ & +2H_{[a}{}^{de}\left[{}^{(1,0)}h\right]H_{b]ce}\left[{}^{(0,1)}h\right] + 2H_{[a}{}^{de}\left[{}^{(0,1)}h\right]H_{b]ce}\left[{}^{(1,0)}h\right] \\ & +2{}^{(1,0)}h^{de}\nabla_{[a}H_{b]ce}\left[{}^{(0,1)}h\right] + 2{}^{(0,1)}h^{de}\nabla_{[a}H_{b]ce}\left[{}^{(1,0)}h\right], \end{aligned} \quad (3·19)$$

as the second-order perturbative curvature.

To write down the perturbative curvatures (3·15) and (3·19) in terms of the gauge invariant and variant variables defined by Eqs. (2·32)–(2·34), we first derive an expression for the tensor field $H_{ab}{}^{c}\left[{}^{(p,q)}h\right]$ in terms of the gauge invariant variables, and then, we derive a perturbative expression for the Riemann curvature.

First, we consider the linear-order perturbation (3·15) of the Riemann curvature. Using the decomposition (2·32) and the identity $R_{[abc]}{}^{d}=0$, we can easily derive the relation

$$ H_{abc}\left[{}^{(p,q)}h\right] = H_{abc}\left[{}^{(p,q)}\mathcal{H}\right] + \nabla_a\nabla_b\,{}^{(p,q)}X_c + R_{bca}{}^{d}\,{}^{(p,q)}X_d, \quad (3·20)$$

where the variable $H_{abc}\left[{}^{(p,q)}\mathcal{H}\right]$ is defined by

$$ H_{abc}\left[{}^{(p,q)}\mathcal{H}\right] := g_{cd}H_{ab}{}^{d}\left[{}^{(p,q)}\mathcal{H}\right], \quad (3·21)$$

$$ H_{ab}{}^{c}\left[{}^{(p,q)}\mathcal{H}\right] := \nabla_{(a}\,{}^{(p,q)}\mathcal{H}_{b)}{}^{c} - \frac{1}{2}\nabla^{c}\,{}^{(p,q)}\mathcal{H}_{ab}. \quad (3·22)$$

Clearly, the variable $H_{ab}{}^{c}\left[{}^{(p,q)}\mathcal{H}\right]$ is gauge invariant. Taking the derivative of H_{abc} and using the Bianchi identity $\nabla_{[a}R_{bc]de}=0$, we obtain

$$ {}^{(p,q)}\bar{R}_{abc}{}^{d} = -2\nabla_{[a}H_{b]c}{}^{d}\left[{}^{(p,q)}\mathcal{H}\right] + \pounds_{{}^{(p,q)}X}R_{abc}{}^{d}, \quad (3·23)$$

where $(p,q) = (1,0),(0,1)$.

Next, we consider the second-order curvature perturbation. We first consider the $O(\epsilon\lambda)$ metric perturbation as mentioned above. Inspecting the definition (2·34) of the gauge invariant variable of $O(\epsilon\lambda)$ metric perturbation, we first define the variable

$$
\begin{aligned}
{}^{(1,1)}\widehat{\mathcal{H}}_{ab} := {}^{(1,1)}h_{ab} - \pounds_{{}^{(0,1)}X}{}^{(1,0)}h_{ab} - \pounds_{{}^{(1,0)}X}{}^{(0,1)}h_{ab} \\
+\frac{1}{2}\left(\pounds_{{}^{(1,0)}X}\pounds_{{}^{(0,1)}X} + \pounds_{{}^{(0,1)}X}\pounds_{{}^{(1,0)}X}\right) g_{ab}.
\end{aligned}
\tag{3·24}
$$

As in the case of linear order, we evaluate the tensor $H_{ab}{}^{c}\left[{}^{(1,1)}\widehat{\mathcal{H}}\right]$ and obtain

$$
\begin{aligned}
2H_{ab}{}^{c}\left[{}^{(1,1)}\widehat{\mathcal{H}}\right] = {} & 2H_{ab}{}^{c}\left[{}^{(1,1)}h\right] \\
& -\pounds_{{}^{(0,1)}X}\left(H_{ab}{}^{c}\left[{}^{(1,0)}h\right] + H_{ab}{}^{c}\left[{}^{(1,0)}\mathcal{H}\right]\right) \\
& -\pounds_{{}^{(1,0)}X}\left(H_{ab}{}^{c}\left[{}^{(0,1)}h\right] + H_{ab}{}^{c}\left[{}^{(0,1)}\mathcal{H}\right]\right) \\
& +\left(H_{abd}\left[{}^{(1,0)}h\right] + H_{abd}\left[{}^{(1,0)}\mathcal{H}\right]\right)\pounds_{{}^{(0,1)}X}g^{cd} \\
& +\left(H_{ab}{}^{c}\left[{}^{(0,1)}h\right] + H_{ab}{}^{c}\left[{}^{(0,1)}\mathcal{H}\right]\right)\pounds_{{}^{(1,0)}X}g^{cd} \\
& -\left({}^{(1,0)}h_{d}{}^{e} + {}^{(1,0)}\mathcal{H}_{d}{}^{e}\right)\left(\nabla_a\nabla_b{}^{(0,1)}X^{d} - R_{eab}{}^{d}{}^{(0,1)}X^{e}\right) \\
& -\left({}^{(0,1)}h_{d}{}^{e} + {}^{(0,1)}\mathcal{H}_{d}{}^{e}\right)\left(\nabla_a\nabla_b{}^{(1,0)}X^{d} - R_{eab}{}^{d}{}^{(1,0)}X^{e}\right).
\end{aligned}
\tag{3·25}
$$

In the derivation of this expression, some formulae, which are summarized in Appendix A, are useful. After straightforward calculations, we obtain

$$
\begin{aligned}
{}^{(1,1)}\bar{R}_{abc}{}^{d} = {} & -2\nabla_{[a}H_{b]c}{}^{d}\left[{}^{(1,1)}\widehat{\mathcal{H}}\right] \\
& +2H_{[a}{}^{de}\left[{}^{(1,0)}\mathcal{H}\right]H_{b]ce}\left[{}^{(0,1)}\mathcal{H}\right] + 2H_{[a}{}^{de}\left[{}^{(0,1)}\mathcal{H}\right]H_{b]ce}\left[{}^{(1,0)}\mathcal{H}\right] \\
& +{}^{(1,0)}\mathcal{H}_{e}{}^{d}\left(\pounds_{{}^{(0,1)}X}R_{abc}{}^{e} - {}^{(0,1)}\bar{R}_{abc}{}^{e}\right) \\
& +{}^{(0,1)}\mathcal{H}_{e}{}^{d}\left(\pounds_{{}^{(1,0)}X}R_{abc}{}^{e} - {}^{(1,0)}\bar{R}_{abc}{}^{e}\right) \\
& +\pounds_{{}^{(0,1)}X}\left({}^{(1,0)}\bar{R}_{abc}{}^{d} - \frac{1}{2}\pounds_{{}^{(1,0)}X}R_{abc}{}^{d}\right) \\
& +\pounds_{{}^{(1,0)}X}\left({}^{(0,1)}\bar{R}_{abc}{}^{d} - \frac{1}{2}\pounds_{{}^{(0,1)}X}R_{abc}{}^{d}\right).
\end{aligned}
\tag{3·26}
$$

The first term on the right-hand side, $-2\nabla_{[a}H_{b]c}{}^{d}\left[{}^{(1,1)}\widehat{\mathcal{H}}\right]$, includes the gauge degree of freedom, because the variable ${}^{(1,1)}\widehat{\mathcal{H}}_{ab}$ is transformed as a linear-order metric perturbation. Since we have already assume that the linear metric perturbation is decomposed as Eq. (2·32), we can also decomposed the variable ${}^{(1,1)}\widehat{\mathcal{H}}_{ab}$ as

$$
{}^{(1,1)}\widehat{\mathcal{H}}_{ab} =: {}^{(1,1)}\mathcal{H}_{ab} + 2\nabla_{(a}{}^{(1,1)}X_{b)},
\tag{3·27}
$$

as pointed out in a previous paper.[8] The variables ${}^{(1,1)}\mathcal{H}_{ab}$ and ${}^{(1,1)}X_b$ are gauge invariant and variant parts of $O(\epsilon\lambda)$ metric perturbation. Then, as in the case of linear order, we obtain

$$-2\nabla_{[a}H_{b]c}{}^{d}\left[{}^{(1,1)}\hat{\mathcal{H}}\right] = -2\nabla_{[a}H_{b]c}{}^{d}\left[{}^{(1,1)}\mathcal{H}\right] + \pounds_{{}^{(1,1)}X}R_{abc}{}^{d}. \tag{3·28}$$

Further, using Eq. (3·23), we reach the final form of the perturbative Riemann curvature of $O(\epsilon\lambda)$:

$$\begin{aligned}
{}^{(1,1)}\bar{R}_{abc}{}^{d} = & -2\nabla_{[a}H_{b]c}{}^{d}\left[{}^{(1,1)}\mathcal{H}\right] \\
& +2H_{[a}{}^{de}\left[{}^{(1,0)}\mathcal{H}\right]H_{b]ce}\left[{}^{(0,1)}\mathcal{H}\right] + 2H_{[a}{}^{de}\left[{}^{(0,1)}\mathcal{H}\right]H_{b]ce}\left[{}^{(1,0)}\mathcal{H}\right] \\
& +2{}^{(1,0)}\mathcal{H}_{e}{}^{d}\nabla_{[a}H_{b]c}{}^{e}\left[{}^{(0,1)}\mathcal{H}\right] + 2{}^{(0,1)}\mathcal{H}_{e}{}^{d}\nabla_{[a}H_{b]c}{}^{e}\left[{}^{(1,0)}\mathcal{H}\right] \\
& +\pounds_{{}^{(0,1)}X}{}^{(1,0)}\bar{R}_{abc}{}^{d} + \pounds_{{}^{(1,0)}X}{}^{(0,1)}\bar{R}_{abc}{}^{d} \\
& +\left(\pounds_{{}^{(1,1)}X} - \frac{1}{2}\pounds_{{}^{(0,1)}X}\pounds_{{}^{(1,0)}X} - \frac{1}{2}\pounds_{{}^{(1,0)}X}\pounds_{{}^{(0,1)}X}\right)R_{abc}{}^{d}.
\end{aligned} \tag{3·29}$$

The first three lines on the right-hand side of Eq. (3·29) are the gauge invariant part and the remaining two lines are the gauge variant part of the perturbative Riemann curvature of $O(\epsilon\lambda)$.

The perturbative Riemann curvatures of $O(\epsilon^2)$ and $O(\lambda^2)$ are simply obtained through the replacement of the variables in Eq. (3·29) of $O(\epsilon\lambda)$. To obtain the Riemann curvature of $O(\epsilon^2)$, we consider the replacements of the variables

$${}^{(0,1)}X^a \rightarrow {}^{(1,0)}X^a, \quad {}^{(0,1)}\mathcal{H}_{ab} \rightarrow {}^{(1,0)}\mathcal{H}_{ab} \tag{3·30}$$

in Eq. (3·29). Similarly, to obtain the $O(\lambda^2)$ Riemann curvature, we consider the replacements of the variables

$${}^{(1,0)}X^a \rightarrow {}^{(0,1)}X^a, \quad {}^{(1,0)}\mathcal{H}_{ab} \rightarrow {}^{(0,1)}\mathcal{H}_{ab} \tag{3·31}$$

in Eq. (3·29). These replacements are consistent with the definitions (2·33) and (2·34) of the gauge invariant variables of $O(\epsilon^2)$ and $O(\lambda^2)$. Hence, we obtain the perturbative forms of the Riemann curvatures of $O(\epsilon^2)$ and $O(\lambda^2)$ as

$$\begin{aligned}
{}^{(p,q)}\bar{R}_{abc}{}^{d} = & -2\nabla_{[a}H_{b]c}{}^{d}\left[{}^{(p,q)}\mathcal{H}\right] \\
& +4H_{[a}{}^{de}\left[{}^{(\frac{p}{2},\frac{p}{2})}\mathcal{H}\right]H_{b]ce}\left[{}^{(\frac{p}{2},\frac{p}{2})}\mathcal{H}\right] + 4{}^{(\frac{p}{2},\frac{p}{2})}\mathcal{H}_{e}{}^{d}\nabla_{[a}H_{b]c}{}^{e}\left[{}^{(\frac{p}{2},\frac{p}{2})}\mathcal{H}\right] \\
& +2\pounds_{{}^{(\frac{p}{2},\frac{p}{2})}X}{}^{(\frac{p}{2},\frac{p}{2})}\bar{R}_{abc}{}^{d} + \left(\pounds_{{}^{(p,q)}X} - \pounds^2_{{}^{(\frac{p}{2},\frac{p}{2})}X}\right)R_{abc}{}^{d},
\end{aligned} \tag{3·32}$$

where $(p,q) = (2,0), (0,2)$.

Equations (3·23), (3·29) and (3·32) show that all variables defined by

$${}^{(p,q)}\mathcal{R}_{abc}{}^{d} := {}^{(p,q)}\bar{R}_{abc}{}^{d} - \pounds_{{}^{(p,q)}X}R_{abc}{}^{d} \tag{3·33}$$

for $(p,q)=(1,0),(0,1)$,

$$ {}^{(p,q)}\mathcal{R}_{abc}{}^{d} := {}^{(p,q)}\bar{R}_{abc}{}^{d} - 2\pounds_{(\frac{p}{2},\frac{p}{2})X}{}^{(\frac{p}{2},\frac{p}{2})}\bar{R}_{abc}{}^{d} - \left(\pounds_{(p,q)X} - \pounds^{2}_{(\frac{p}{2},\frac{p}{2})X}\right) R_{abc}{}^{d} \quad (3\cdot34) $$

for $(p,q)=(2,0),(0,2)$, and

$$ \begin{aligned} {}^{(1,1)}\mathcal{R}_{abc}{}^{d} := {}^{(1,1)}\bar{R}_{abc}{}^{d} - \pounds_{(0,1)X}{}^{(1,0)}\bar{R}_{abc}{}^{d} - \pounds_{(1,0)X}{}^{(0,1)}\bar{R}_{abc}{}^{d} \\ - \left(\pounds_{(1,1)X} - \frac{1}{2}\pounds_{(0,1)X}\pounds_{(1,0)X} - \frac{1}{2}\pounds_{(1,0)X}\pounds_{(0,1)X}\right) R_{abc}{}^{d} \end{aligned} \quad (3\cdot35) $$

are gauge invariant. These indeed do have the same forms as the definitions (2·35)–(2·37) for each order gauge invariant variable for an arbitrary field.

Here, we derive the perturbative formulae of the Riemann curvature $\bar{R}_{abcd}$, which are used in the derivation of the Weyl curvature $\bar{C}_{abcd}$. For this purpose, we first expand the definition

$$ \bar{R}_{abcd} = \bar{g}_{ed}\bar{R}_{abc}{}^{e}. \quad (3\cdot36) $$

The form of the perturbation of $\bar{R}_{abcd}$ at each order is derived from the formulae

$$ {}^{(p,q)}\bar{R}_{abcd} = {}^{(p,q)}\bar{g}_{ed}R_{abc}{}^{e} + g_{ed}{}^{(p,q)}\bar{R}_{abc}{}^{e}, \quad (p,q)=(1,0),(0,1), \quad (3\cdot37) $$

$$ \begin{aligned} {}^{(1,1)}\bar{R}_{abcd} = {}^{(1,1)}\bar{g}_{ed}R_{abc}{}^{e} + {}^{(1,0)}\bar{g}_{ed}{}^{(0,1)}\bar{R}_{abc}{}^{e} \\ + {}^{(0,1)}\bar{g}_{ed}{}^{(1,0)}\bar{R}_{abc}{}^{e} + g_{ed}{}^{(1,1)}\bar{R}_{abc}{}^{e}. \end{aligned} \quad (3\cdot38) $$

The formulae for ${}^{(p,q)}\bar{R}_{abcd}$ with $(p,q)=(2,0),(0,2)$ are derived using the replacements (3·30) and (3·31) of the perturbative variables as in the case of the Riemann curvature $\bar{R}_{abc}{}^{d}$. The explicit form of each order ${}^{(p,q)}\bar{R}_{abcd}$ is as follows:

$$ {}^{(p,q)}\bar{R}_{abcd} = -2\nabla_{[a}H_{b]cd}\left[{}^{(p,q)}\mathcal{H}\right] + {}^{(p,q)}\mathcal{H}_{d}{}^{e}R_{abce} + \pounds_{(p,q)X}R_{abcd} \quad (3\cdot39) $$

for $(p,q)=(1,0),(0,1)$,

$$ \begin{aligned} {}^{(p,q)}\bar{R}_{abcd} = -2\nabla_{[a}H_{b]cd}\left[{}^{(p,q)}\mathcal{H}\right] + R_{abce}{}^{(p,q)}\mathcal{H}_{d}{}^{e} + 4H_{d[a}{}^{e}\left[{}^{(p,q)}\mathcal{H}\right]H_{b]ce}\left[{}^{(p,q)}\mathcal{H}\right] \\ +2\pounds_{(\frac{p}{2},\frac{q}{2})X}{}^{(\frac{p}{2},\frac{q}{2})}R_{abcd} + \left(\pounds_{(p,q)X} - \pounds^{2}_{(\frac{p}{2},\frac{q}{2})X}\right) R_{abcd} \end{aligned} \quad (3\cdot40) $$

for $(p,q)=(2,0),(0,2)$, and

$$ \begin{aligned} {}^{(1,1)}\bar{R}_{abcd} = -2\nabla_{[a}H_{b]cd}\left[{}^{(1,1)}\mathcal{H}\right] + R_{abce}{}^{(1,1)}\mathcal{H}_{d}{}^{e} \\ +2H_{d[a}{}^{e}\left[{}^{(1,0)}\mathcal{H}\right]H_{b]ce}\left[{}^{(0,1)}\mathcal{H}\right] + 2H_{d[a}{}^{e}\left[{}^{(0,1)}\mathcal{H}\right]H_{b]ce}\left[{}^{(1,0)}\mathcal{H}\right] \\ +\pounds_{(1,0)X}{}^{(0,1)}R_{abcd} + \pounds_{(0,1)X}{}^{(1,0)}R_{abcd} \\ +\left(\pounds_{(1,1)X} - \frac{1}{2}\pounds_{(1,0)X}\pounds_{(0,1)X} - \frac{1}{2}\pounds_{(1,0)X}\pounds_{(0,1)X}\right) R_{abcd}. \end{aligned} \quad (3\cdot41) $$

The perturbative forms (3·39)–(3·41) also show that the variables defined by

$$ {}^{(p,q)}\mathcal{R}_{abcd} := {}^{(p,q)}\bar{R}_{abcd} - \pounds_{(p,q)X}R_{abcd}, \quad (p,q)=(1,0),(0,1), \quad (3\cdot42) $$

$$
\begin{aligned}
{}^{(p,q)}\mathcal{R}_{abcd} :=\; & {}^{(p,q)}\bar{R}_{abcd} - 2\pounds_{(\frac{p}{2},\frac{p}{2})X}{}^{(\frac{p}{2},\frac{p}{2})}\bar{R}_{abcd} \\
& - \left(\pounds_{(p,q)X} - \pounds^2_{(\frac{p}{2},\frac{p}{2})X}\right) R_{abcd}, \qquad (p,q) = (2,0),(0,2), \qquad (3{\cdot}43)\\
{}^{(1,1)}\mathcal{R}_{abcd} :=\; & {}^{(1,1)}\bar{R}_{abcd} - \pounds_{(0,1)X}{}^{(1,0)}\bar{R}_{abcd} - \pounds_{(1,0)X}{}^{(0,1)}\bar{R}_{abcd} \\
& - \left(\pounds_{(1,1)X} - \frac{1}{2}\pounds_{(0,1)X}\pounds_{(1,0)X} - \frac{1}{2}\pounds_{(1,0)X}\pounds_{(0,1)X}\right) R_{abcd} \qquad (3{\cdot}44)
\end{aligned}
$$

are gauge invariant.

3.2. *Ricci curvature*

Contracting the indices b and d in Eqs. (3·23), (3·29) and (3·32) of the perturbative Riemann curvature, we can derive the formulae for the expansion of the Ricci curvature,

$$
{}^{(p,q)}\bar{R}_{ab} = -2\nabla_{[a}H_{c]b}{}^{c}\left[{}^{(p,q)}\mathcal{H}\right] + \pounds_{(p,q)X}R_{ab} \qquad (3{\cdot}45)
$$

for first order and

$$
\begin{aligned}
{}^{(p,q)}\bar{R}_{ab} =\; & -2\nabla_{[a}H_{c]b}{}^{c}\left[{}^{(p,q)}\mathcal{H}\right] + 4H_{[a}{}^{cd}\left[{}^{(\frac{p}{2},\frac{p}{2})}\mathcal{H}\right] H_{c]bd}\left[{}^{(\frac{p}{2},\frac{p}{2})}\mathcal{H}\right] \\
& +4{}^{(\frac{p}{2},\frac{p}{2})}\mathcal{H}_{d}{}^{c}\nabla_{[a}H_{b]c}{}^{d}\left[{}^{(\frac{p}{2},\frac{p}{2})}\mathcal{H}\right] \\
& +2\pounds_{(\frac{p}{2},\frac{p}{2})X}{}^{(\frac{p}{2},\frac{p}{2})}\bar{R}_{ab} + \left(\pounds_{(p,q)X} - \pounds^2_{(\frac{p}{2},\frac{p}{2})X}\right) R_{ab}, \qquad (3{\cdot}46)\\
{}^{(1,1)}\bar{R}_{ab} =\; & -2\nabla_{[a}H_{c]b}{}^{c}\left[{}^{(1,1)}\mathcal{H}\right] \\
& +2H_{[a}{}^{cd}\left[{}^{(1,0)}\mathcal{H}\right] H_{c]bd}\left[{}^{(0,1)}\mathcal{H}\right] + 2H_{[a}{}^{cd}\left[{}^{(0,1)}\mathcal{H}\right] H_{c]bd}\left[{}^{(1,0)}\mathcal{H}\right] \\
& +2{}^{(1,0)}\mathcal{H}_{d}{}^{c}\nabla_{[a}H_{c]b}{}^{d}\left[{}^{(0,1)}\mathcal{H}\right] + 2{}^{(0,1)}\mathcal{H}_{d}{}^{c}\nabla_{[a}H_{c]b}{}^{d}\left[{}^{(1,0)}\mathcal{H}\right] \\
& +\pounds_{(0,1)X}{}^{(1,0)}\bar{R}_{ab} + \pounds_{(1,0)X}{}^{(0,1)}\bar{R}_{ab} \\
& + \left(\pounds_{(1,1)X} - \frac{1}{2}\pounds_{(0,1)X}\pounds_{(1,0)X} - \frac{1}{2}\pounds_{(1,0)X}\pounds_{(0,1)X}\right) R_{ab}, \qquad (3{\cdot}47)
\end{aligned}
$$

for second order, where $(p,q) = (1,0),(0,1)$ in Eq. (3·45) and $(p,q) = (2,0),(0,2)$ in Eq. (3·46).

It is trivial from the derivation that Eqs. (3·45), (3·46) and (3·47) show that the variables defined by

$$
{}^{(p,q)}\mathcal{R}_{ab} = {}^{(p,q)}\bar{R}_{ab} - \pounds_{(p,q)X}R_{ab} \qquad (3{\cdot}48)
$$

for $(p,q) = (1,0),(0,1)$,

$$
{}^{(p,q)}\mathcal{R}_{ab} = {}^{(p,q)}\bar{R}_{ab} - 2\pounds_{(\frac{p}{2},\frac{p}{2})X}{}^{(\frac{p}{2},\frac{p}{2})}\bar{R}_{ab} - \left(\pounds_{(p,q)X} - \pounds^2_{(\frac{p}{2},\frac{p}{2})X}\right) R_{ab} \qquad (3{\cdot}49)
$$

for $(p,q) = (2,0),(0,2)$, and

$$
\begin{aligned}
{}^{(1,1)}\mathcal{R}_{ab} =\; & {}^{(1,1)}\bar{R}_{ab} - \pounds_{(0,1)X}{}^{(1,0)}\bar{R}_{ab} - \pounds_{(1,0)X}{}^{(0,1)}\bar{R}_{ab} \\
& - \left(\pounds_{(1,1)X} - \frac{1}{2}\pounds_{(0,1)X}\pounds_{(1,0)X} - \frac{1}{2}\pounds_{(1,0)X}\pounds_{(0,1)X}\right) R_{ab} \qquad (3{\cdot}50)
\end{aligned}
$$

are gauge invariant. These also have the same forms as those in the definitions (2·35)–(2·37) of each order gauge invariant variable for the perturbation of an arbitrary field.

3.3. *Scalar curvature*

The scalar curvature on the physical spacetime $\mathcal{M}$ is given by

$$\bar{R} := \bar{g}^{ab}\bar{R}_{ab}. \tag{3·51}$$

To obtain the perturbative form of the scalar curvature, the left-hand side of Eq. (3·51) is expanded in the form (3·7) and the right-hand side is expanded by using the Leibniz rule. Then, the perturbative formula for the scalar curvature at each order is derived from perturbative form of the inverse metric (3·9)–(3·11) and the Ricci curvature (3·45)–(3·47). Straightforward calculations lead to the expansion of the scalar curvature.

Using (3·9) and (3·45), we obtain the first-order perturbative form of the scalar curvature as

$${}^{(p,q)}\bar{R} = -2\nabla_{[a}H_{b]}{}^{ab}\left[{}^{(p,q)}\mathcal{H}\right] - R_{ab}{}^{(p,q)}\mathcal{H}^{ab} + \pounds_{(p,q)X}R, \tag{3·52}$$

where $(p,q) = (0,1),(1,0)$. Then, using (3·9), (3·11), (3·45) and (3·47), the perturbative scalar curvature of $O(\epsilon\lambda)$ is found to be given by

$$\begin{aligned}
{}^{(1,1)}\bar{R} = & -2\nabla_{[a}H_{b]}{}^{ab}\left[{}^{(1,1)}\mathcal{H}\right] + R^{ab}\left(2{}^{(1,0)}\mathcal{H}_{c(a}{}^{(0,1)}\mathcal{H}_{b)}{}^{c} - {}^{(1,1)}\mathcal{H}_{ab}\right) \\
& +2H_{[a}{}^{cd}\left[{}^{(0,1)}\mathcal{H}\right]H_{c]}{}^{a}{}_{d}\left[{}^{(1,0)}\mathcal{H}\right] + 2H_{[a}{}^{cd}\left[{}^{(1,0)}\mathcal{H}\right]H_{c]}{}^{a}{}_{d}\left[{}^{(0,1)}\mathcal{H}\right] \\
& +2{}^{(1,0)}\mathcal{H}_{c}{}^{b}\nabla_{[a}H_{b]}{}^{ac}\left[{}^{(0,1)}\mathcal{H}\right] + 2{}^{(0,1)}\mathcal{H}_{c}{}^{b}\nabla_{[a}H_{b]}{}^{ac}\left[{}^{(1,0)}\mathcal{H}\right] \\
& +2{}^{(1,0)}\mathcal{H}^{ab}\nabla_{[a}H_{d]b}{}^{d}\left[{}^{(0,1)}\mathcal{H}\right] + 2{}^{(0,1)}\mathcal{H}^{ab}\nabla_{[a}H_{d]b}{}^{d}\left[{}^{(1,0)}\mathcal{H}\right] \\
& +\pounds_{(0,1)X}{}^{(1,0)}\bar{R} + \pounds_{(1,0)X}{}^{(0,1)}\bar{R} \\
& +\left(\pounds_{(1,1)X} - \frac{1}{2}\pounds_{(0,1)X}\pounds_{(1,0)X} - \frac{1}{2}\pounds_{(1,0)X}\pounds_{(0,1)X}\right)R.
\end{aligned} \tag{3·53}$$

To derive the perturbative scalar curvature of $O(\epsilon^2)$, the replacement (3·30) of the variables is applied to Eq. (3·53). Similarly, the replacement (3·31) is applied to Eq. (3·53) when we derive the perturbative scalar curvature of $O(\lambda^2)$. Then we obtain the perturbative form of the scalar curvatures of $O(\epsilon^2)$ and $O(\lambda^2)$:

$$\begin{aligned}
{}^{(p,q)}\bar{R} = & -2\nabla_{[a}H_{b]}{}^{ab}\left[{}^{(p,q)}\mathcal{H}\right] + R^{ab}\left(2{}^{(\frac{p}{2},\frac{p}{2})}\mathcal{H}_{ca}{}^{(\frac{p}{2},\frac{p}{2})}\mathcal{H}_{b}{}^{c} - {}^{(p,q)}\mathcal{H}_{ab}\right) \\
& +4H_{[a}{}^{cd}\left[{}^{(\frac{p}{2},\frac{p}{2})}\mathcal{H}\right]H_{c]}{}^{a}{}_{d}\left[{}^{(\frac{p}{2},\frac{p}{2})}\mathcal{H}\right] + 4{}^{(\frac{p}{2},\frac{p}{2})}\mathcal{H}_{c}{}^{b}\nabla_{[a}H_{b]}{}^{ac}\left[{}^{(\frac{p}{2},\frac{p}{2})}\mathcal{H}\right] \\
& +4{}^{(\frac{p}{2},\frac{p}{2})}\mathcal{H}^{ab}\nabla_{[a}H_{d]b}{}^{d}\left[{}^{(\frac{p}{2},\frac{p}{2})}\mathcal{H}\right] \\
& +2\pounds_{(\frac{p}{2},\frac{p}{2})X}{}^{(\frac{p}{2},\frac{p}{2})}\bar{R} + \left(\pounds_{(p,q)X} - \pounds^{2}_{(\frac{p}{2},\frac{p}{2})X}\right)R,
\end{aligned} \tag{3·54}$$

where $(p,q) = (2,0),(0,2)$.

It is also trivial from the derivation that Eqs. (3·52)–(3·54) show that the variables defined by

$$ {}^{(p,q)}\mathcal{R} = {}^{(p,q)}\bar{R} - \pounds_{{}_{(p,q)}X} R \tag{3·55}$$

for $(p,q) = (1,0), (0,1)$,

$$ {}^{(p,q)}\mathcal{R} = {}^{(p,q)}\bar{R} - 2\pounds_{{}_{(\frac{p}{2},\frac{p}{2})}X} {}^{(\frac{p}{2},\frac{p}{2})}\bar{R} - \left(\pounds_{{}_{(p,q)}X} - \pounds^2_{{}_{(\frac{p}{2},\frac{p}{2})}X}\right) R \tag{3·56}$$

for $(p,q) = (2,0), (0,2)$, and

$$ {}^{(1,1)}\mathcal{R} = {}^{(1,1)}\bar{R} - \pounds_{{}_{(0,1)}X} {}^{(1,0)}\bar{R} - \pounds_{{}_{(1,0)}X} {}^{(0,1)}\bar{R} \\ - \left(\pounds_{{}_{(1,1)}X} - \frac{1}{2}\pounds_{{}_{(0,1)}X}\pounds_{{}_{(1,0)}X} - \frac{1}{2}\pounds_{{}_{(1,0)}X}\pounds_{{}_{(0,1)}X}\right) R \tag{3·57}$$

are gauge invariant. These too have the same forms as those given in the definitions (2·35)–(2·37) of each order gauge invariant variable for the perturbations of an arbitrary field.

3.4. *Einstein tensor*

Next, we consider the perturbative form of the Einstein tensor. The Einstein tensor on the physical spacetime $\mathcal{M}$ is defined by

$$ \bar{G}_{ab} := \bar{R}_{ab} - \frac{1}{2}\bar{g}_{ab}\bar{R}. \tag{3·58}$$

As in the case of the scalar curvature, the left-hand side of Eq. (3·58) is expanded in the form (3·7), and the second term on right-hand side of Eq. (3·58) is expanded by using the Leibniz rule. Then, the perturbative formula for the Einstein tensor at each order is derived from the perturbative form of the metric (2·32)–(2·34), those of the Ricci curvature (3·45)–(3·47), and those of the Ricci scalar (3·52)–(3·54).

The linear order Einstein tensor is given by

$$ {}^{(p,q)}\bar{G}_{ab} = -2\nabla_{[a}\mathcal{H}_{d]b}{}^{d}\left[{}^{(p,q)}\mathcal{H}\right] + g_{ab}\nabla_{[c}\mathcal{H}_{d]}{}^{cd}\left[{}^{(p,q)}\mathcal{H}\right] - \frac{1}{2}R\,{}^{(p,q)}\mathcal{H}_{ab} + \frac{1}{2}g_{ab}R_{cd}\,{}^{(p,q)}\mathcal{H}^{cd} \\ +\pounds_{{}_{(p,q)}X} G_{ab}, \tag{3·59}$$

where $(p,q) = (1,0), (0,1)$. Next, using Eqs. (2·32)–(2·34), (3·47), (3·52) and (3·53), the Einstein tensor of $O(\epsilon\lambda)$ is found to be given by

$$ \begin{aligned} {}^{(1,1)}\bar{G}_{ab} = & -2\nabla_{[a}H_{c]b}{}^{c}\left[{}^{(1,1)}\mathcal{H}\right] \\ & +2H_{[a}{}^{de}\left[{}^{(1,0)}\mathcal{H}\right]H_{d]be}\left[{}^{(0,1)}\mathcal{H}\right] + 2H_{[a}{}^{de}\left[{}^{(0,1)}\mathcal{H}\right]H_{d]be}\left[{}^{(1,0)}\mathcal{H}\right] \\ & +2{}^{(1,0)}\mathcal{H}_{e}{}^{d}\nabla_{[a}H_{d]b}{}^{e}\left[{}^{(0,1)}\mathcal{H}\right] + 2{}^{(0,1)}\mathcal{H}_{e}{}^{d}\nabla_{[a}H_{d]b}{}^{e}\left[{}^{(1,0)}\mathcal{H}\right] \\ & -\frac{1}{2}g_{ab}\left(-2\nabla_{[c}H_{d]}{}^{cd}\left[{}^{(1,1)}\mathcal{H}\right] + R_{de}\left(2{}^{(0,1)}\mathcal{H}_{c}{}^{d}\,{}^{(1,0)}\mathcal{H}^{ec} - {}^{(1,1)}\mathcal{H}^{de}\right)\right. \\ & \qquad +2H_{[c}{}^{de}\left[{}^{(1,0)}\mathcal{H}\right]H_{d]}{}^{c}{}_{e}\left[{}^{(0,1)}\mathcal{H}\right] + 2H_{[c}{}^{de}\left[{}^{(0,1)}\mathcal{H}\right]H_{d]}{}^{c}{}_{e}\left[{}^{(1,0)}\mathcal{H}\right] \end{aligned}$$

$$
\begin{aligned}
&\qquad +2^{(1,0)}\mathcal{H}_e{}^d\nabla_{[c}H_{d]}{}^{ce}\left[{}^{(0,1)}\mathcal{H}\right]+2^{(0,1)}\mathcal{H}_e{}^d\nabla_{[c}H_{d]}{}^{ce}\left[{}^{(1,0)}\mathcal{H}\right] \\
&\qquad +2^{(1,0)}\mathcal{H}^{ce}\nabla_{[c}H_{d]e}{}^d\left[{}^{(0,1)}\mathcal{H}\right]+2^{(0,1)}\mathcal{H}^{ce}\nabla_{[c}H_{d]e}{}^d\left[{}^{(1,0)}\mathcal{H}\right]\Big) \\
&+{}^{(0,1)}\mathcal{H}_{ab}\left(\nabla_{[c}H_{d]}{}^{cd}\left[{}^{(1,0)}\mathcal{H}\right]+\frac{1}{2}R_{cd}{}^{(0,1)}\mathcal{H}^{cd}\right) \\
&+{}^{(1,0)}\mathcal{H}_{ab}\left(\nabla_{[c}H_{d]}{}^{cd}\left[{}^{(0,1)}\mathcal{H}\right]+\frac{1}{2}R_{cd}{}^{(1,0)}\mathcal{H}^{cd}\right) \\
&-\frac{1}{2}R\,{}^{(1,1)}\mathcal{H}_{ab} \\
&+\pounds_{{}^{(0,1)}X}{}^{(1,0)}\bar{G}_{ab}+\pounds_{{}^{(1,0)}X}{}^{(0,1)}\bar{G}_{ab} \\
&+\left(\pounds_{{}^{(1,1)}X}-\frac{1}{2}\pounds_{{}^{(0,1)}X}\pounds_{{}^{(1,0)}X}-\frac{1}{2}\pounds_{{}^{(1,0)}X}\pounds_{{}^{(0,1)}X}\right)G_{ab}.
\end{aligned}
\tag{3·60}
$$

Then, through the replacements (3·30) and (3·31), the perturbative forms of the Einstein tensor of $O(\epsilon^2)$ and $O(\lambda^2)$ are given by

$$
\begin{aligned}
{}^{(p,q)}\bar{G}_{ab}=&-2\nabla_{[a}H_{c]b}{}^c\left[{}^{(p,q)}\mathcal{H}\right] \\
&+4H_{[a}{}^{cd}\left[{}^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}\right]H_{c]bd}\left[{}^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}\right]+4^{(p,q)}\mathcal{H}_c{}^d\nabla_{[a}H_{d]b}{}^c\left[{}^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}\right] \\
&-\frac{1}{2}g_{ab}\Big(-2\nabla_{[c}H_{d]}{}^{cd}\left[{}^{(p,q)}\mathcal{H}\right]+R_{de}\left(2^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}_c{}^d\,{}^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}^{ec}-{}^{(p,q)}\mathcal{H}^{de}\right) \\
&\qquad +4H_{[c}{}^{de}\left[{}^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}\right]H_{d]}{}^c{}_e\left[{}^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}\right] \\
&\qquad +4^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}_e{}^d\nabla_{[c}H_{d]}{}^{ce}\left[{}^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}\right]+4^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}^{ce}\nabla_{[c}H_{d]e}{}^d\left[{}^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}\right]\Big) \\
&+2^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}_{ab}\nabla_{[c}H_{d]}{}^{cd}\left[{}^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}\right]+{}^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}_{ab}R_{cd}{}^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}^{cd}-\frac{1}{2}R\,{}^{(p,q)}\mathcal{H}_{ab} \\
&+2\pounds_{{}^{(\frac{p}{2},\frac{q}{2})}X}{}^{(\frac{p}{2},\frac{q}{2})}\bar{G}_{ab}+\left(\pounds_{{}^{(p,q)}X}-\pounds^2_{{}^{(\frac{p}{2},\frac{q}{2})}X}\right)G_{ab},
\end{aligned}
\tag{3·61}
$$

where $(p,q)=(2,0),(0,2)$.

Further, we also derive the formulae for the perturbation of the Einstein tensor

$$
\bar{G}_a{}^b:=\bar{g}^{bc}\bar{G}_{ac} \tag{3·62}
$$

on $\mathcal{M}$. Because the derivation of these formulae is similar to the above perturbative curvatures, we only present the final results:

$$
{}^{(p,q)}\bar{G}_a{}^b={}^{(1)}\mathcal{G}_a{}^b\left[{}^{(p,q)}\mathcal{H}\right]+\pounds_{{}^{(p,q)}X}\;G_a{}^b \tag{3·63}
$$

for $(p,q)=(0,1),(1,0)$,

$$
\begin{aligned}
{}^{(p,q)}\bar{G}_a{}^b=&{}^{(1)}\mathcal{G}_a{}^b\left[{}^{(p,q)}\mathcal{H}\right]+{}^{(2)}\mathcal{G}_a{}^b\left[{}^{(p,q)}\mathcal{H},{}^{(p,q)}\mathcal{H}\right] \\
&+2\pounds_{{}^{(\frac{p}{2},\frac{q}{2})}X}\;{}^{(\frac{p}{2},\frac{q}{2})}\bar{G}_a{}^b+\left\{\pounds_{{}^{(p,q)}X}-\pounds^2_{{}^{(\frac{p}{2},\frac{q}{2})}X}\right\}G_a{}^b
\end{aligned}
\tag{3·64}
$$

for $(p,q) = (0,2), (2,0)$, and

$$
\begin{aligned}
{}^{(1,1)}\bar{G}_a{}^b &= {}^{(1)}\mathcal{G}_a{}^b\left[{}^{(1,1)}\mathcal{H}\right] + {}^{(2)}\mathcal{G}_a{}^b\left[{}^{(1,0)}\mathcal{H}, {}^{(0,1)}\mathcal{H}\right] \\
&\quad + \pounds_{{}^{(1,0)}X}\ {}^{(0,1)}\bar{G}_a{}^b + \pounds_{{}^{(0,1)}X}\ {}^{(1,0)}\bar{G}_a{}^b \\
&\quad + \left\{\pounds_{{}^{(1,1)}X} - \frac{1}{2}\pounds_{{}^{(1,0)}X}\pounds_{{}^{(0,1)}X} - \frac{1}{2}\pounds_{{}^{(0,1)}X}\pounds_{{}^{(1,0)}X}\right\} G_a{}^b, \qquad (3\cdot65)
\end{aligned}
$$

where

$$
\begin{aligned}
{}^{(1)}\mathcal{G}_a{}^b[A] &:= -2\nabla_{[a}H_{d]}{}^{bd}[A] - A^{cb}R_{ac} \\
&\quad + \frac{1}{2}\delta_a{}^b\left(2\nabla_{[e}H_{d]}{}^{ed}[A] + R_{ed}A^{ed}\right), \qquad (3\cdot66) \\
{}^{(2)}\mathcal{G}_a{}^b[A,B] &:= 2R_{ad}B_c{}^{(b}A^{d)c} + 2H_{[a}{}^{de}[A]H_{d]}{}^b{}_e[B] + 2H_{[a}{}^{de}[B]H_{d]}{}^b{}_e[A] \\
&\quad + 2A_e{}^d\nabla_{[a}H_{d]}{}^{be}[B] + 2B_e{}^d\nabla_{[a}H_{d]}{}^{be}[A] \\
&\quad + 2A_c{}^b\nabla_{[a}H_{d]}{}^{cd}[B] + 2B_c{}^b\nabla_{[a}H_{d]}{}^{cd}[A] \\
&\quad - \delta_a{}^b\left(R_{de}B_f{}^{(d}A^{e)f} + H_{[f}{}^{de}[A]H_{d]}{}^f{}_e[B] + H_{[f}{}^{de}[B]H_{d]}{}^f{}_e[A]\right. \\
&\quad \left. + 2A_e{}^d\nabla_{[f}H_{d]}{}^{[fe]}[B] + 2B_e{}^d\nabla_{[f}H_{d]}{}^{[fe]}[A]\right). \qquad (3\cdot67)
\end{aligned}
$$

We note that ${}^{(1)}\mathcal{G}_a{}^b[*]$ and ${}^{(2)}\mathcal{G}_a{}^b[*,*]$ in Eqs. (3·63)–(3·65) are the gauge invariant parts of the perturbative Einstein tensors, and each expression (3·63)–(3·65) has a form similar to Eqs. (2·35)–(2·37), respectively.

3.5. *Weyl curvature*

Here, we consider a perturbation of the Weyl curvature, which is useful to study some physical situations. In m-dimensional spacetime, the Weyl curvature is defined by

$$
\bar{C}_{abcd} := \bar{R}_{abcd} - \frac{2}{m-2}\left(\bar{g}_{a[c}\bar{R}_{d]b} - \bar{g}_{b[c}\bar{R}_{d]a}\right) + \frac{2}{(m-1)(m-2)}\bar{R}\bar{g}_{a[c}\bar{g}_{d]b}. \qquad (3\cdot68)
$$

Using the perturbative formulae for each order perturbation of the Riemann curvature (3·39)–(3·41), of the Ricci curvature (3·48)–(3·50), of scalar curvature (3·55)–(3·57), and of the metric perturbation (2·32)–(2·34), we can explicitly confirm that the perturbative forms of the Weyl curvature at each order are given by

$$
\begin{aligned}
{}^{(p,q)}\bar{C}_{abcd} &= {}^{(p,q)}\mathcal{C}_{abcd} + \pounds_{{}^{(p,q)}X}C_{abcd} \qquad \text{for} \quad (p,q) = (0,1), (1,0), \qquad (3\cdot69) \\
{}^{(p,q)}\bar{C}_{abcd} &= {}^{(p,q)}\mathcal{C}_{abcd} + 2\pounds_{{}^{(\frac{p}{2},\frac{q}{2})}X}\,{}^{(\frac{p}{2},\frac{q}{2})}\bar{C}_{abcd} + \left(\pounds_{{}^{(p,q)}X} - \pounds^2_{{}^{(\frac{p}{2},\frac{q}{2})}X}\right)C_{abcd} \\
&\qquad \text{for} \quad (p,q) = (0,2), (2,0), \qquad (3\cdot70) \\
{}^{(1,1)}\bar{C}_{abcd} &= {}^{(1,1)}\mathcal{C}_{abcd} + \pounds_{{}^{(0,1)}X}\,{}^{(1,0)}\bar{C}_{abcd} + \pounds_{{}^{(1,0)}X}\,{}^{(0,1)}\bar{C}_{abcd} \\
&\quad + \left(\pounds_{{}^{(1,1)}X} - \frac{1}{2}\pounds_{{}^{(1,0)}X}\pounds_{{}^{(0,1)}X} - \frac{1}{2}\pounds_{{}^{(0,1)}X}\pounds_{{}^{(1,0)}X}\right)C_{abcd}, \qquad (3\cdot71)
\end{aligned}
$$

where

$$
\begin{aligned}
{}^{(p,q)}\mathcal{C}_{abcd} = {}^{(p,q)}\mathcal{R}_{abcd} - \frac{2}{m-2}\Big\{ & {}^{(p,q)}\mathcal{H}_{a[c}R_{d]b} - {}^{(p,q)}\mathcal{H}_{b[c}R_{d]a} \\
& + g_{a[c}{}^{(p,q)}\mathcal{R}_{d]b} - g_{b[c}{}^{(p,q)}\mathcal{R}_{d]a} \Big\} \\
+ \frac{2}{(m-1)(m-2)}\Big\{ & {}^{(p,q)}\mathcal{R} g_{a[c}g_{d]b} + R\ {}^{(p,q)}\mathcal{H}_{a[c}g_{d]b} \\
& + R\ g_{a[c}{}^{(p,q)}\mathcal{H}_{d]b} \Big\}
\end{aligned}
\tag{3·72}
$$

for $(p,q) = (0,1), (1,0)$,

$$
\begin{aligned}
{}^{(p,q)}\mathcal{C}_{abcd} = {}^{(p,q)}\mathcal{R}_{abcd} - \frac{2}{m-2}\Big[& {}^{(p,q)}\mathcal{H}_{a[c}R_{d]b} - {}^{(p,q)}\mathcal{H}_{b[c}R_{d]a} \\
& + g_{a[c}\ {}^{(p,q)}\mathcal{R}_{d]b} - g_{b[c}\ {}^{(p,q)}\mathcal{R}_{d]a} \\
& + 2{}^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}_{a[c}\ {}^{(\frac{p}{2},\frac{q}{2})}\mathcal{R}_{d]b} - 2{}^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}_{b[c}\ {}^{(\frac{p}{2},\frac{q}{2})}\mathcal{R}_{d]b} \Big] \\
+ \frac{2}{(m-1)(m-2)}\Big[& {}^{(p,q)}\mathcal{R} g_{a[c}g_{d]b} + R\ {}^{(p,q)}\mathcal{H}_{a[c}g_{d]b} + R g_{a[c}\ {}^{(p,q)}\mathcal{H}_{d]b} \\
& + 2{}^{(\frac{p}{2},\frac{q}{2})}\mathcal{R} g_{a[c}\ {}^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}_{d]b} + 2{}^{(\frac{p}{2},\frac{q}{2})}\mathcal{R}\ {}^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}_{a[c}g_{d]b} \\
& + 2R\ {}^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}_{a[c}\ {}^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}_{d]b} \Big]
\end{aligned}
\tag{3·73}
$$

for $(p,q) = (0,2), (2,0)$, and

$$
\begin{aligned}
{}^{(1,1)}\mathcal{C}_{abcd} = {}^{(1,1)}\mathcal{R}_{abcd} - \frac{2}{m-2}\Big[& {}^{(1,1)}\mathcal{H}_{a[c}R_{d]b} - {}^{(1,1)}\mathcal{H}_{b[c}R_{d]a} \\
& + g_{a[c}\ {}^{(1,1)}\mathcal{R}_{d]b} - g_{b[c}\ {}^{(1,1)}\mathcal{R}_{d]a} \\
& + {}^{(0,1)}\mathcal{H}_{a[c}\ {}^{(1,0)}\mathcal{R}_{d]b} + {}^{(1,0)}\mathcal{H}_{a[c}\ {}^{(0,1)}\mathcal{R}_{d]b} \\
& - {}^{(0,1)}\mathcal{H}_{b[c}\ {}^{(1,0)}\mathcal{R}_{d]a} - {}^{(1,0)}\mathcal{H}_{b[c}\ {}^{(0,1)}\mathcal{R}_{d]a} \Big] \\
+ \frac{2}{(m-1)(m-2)}\Big[& {}^{(1,1)}\mathcal{R} g_{a[c}g_{d]b} + R\ {}^{(1,1)}\mathcal{H}_{a[c}g_{d]b} + R g_{a[c}\ {}^{(1,1)}\mathcal{H}_{d]b} \\
& + {}^{(0,1)}\mathcal{R}\ g_{a[c}\ {}^{(1,0)}\mathcal{H}_{d]b} + {}^{(0,1)}\mathcal{R}\ {}^{(1,0)}\mathcal{H}_{a[c}\ g_{d]b} \\
& + {}^{(1,0)}\mathcal{R}\ {}^{(0,1)}\mathcal{H}_{a[c}\ g_{d]b} + {}^{(1,0)}\mathcal{R}\ g_{a[c}\ {}^{(0,1)}\mathcal{H}_{d]b} \\
& + R\ {}^{(1,0)}\mathcal{H}_{a[c}{}^{(0,1)}\mathcal{H}_{d]b} + R\ {}^{(0,1)}\mathcal{H}_{a[c}{}^{(1,0)}\mathcal{H}_{d]b} \Big].
\end{aligned}
\tag{3·74}
$$

We also derive the formulae for the perturbative expansion of $C_{abc}{}^{d}$ from

$$\bar{C}_{abc}{}^{d} = \bar{g}^{ed}\bar{C}_{abce}. \tag{3·75}$$

Since the Weyl curvature is traceless, i.e., $\bar{C}_{adc}{}^{d} = 0$, we can also verify the formulae derived here by confirming this traceless property.

Actually, we explicitly confirm this traceless property with the following formulae for the perturbative Weyl tensor at each order:

$${}^{(p,q)}\bar{C}_{abc}{}^{d} = {}^{(p,q)}\mathcal{C}_{abc}{}^{d} + \pounds_{{}^{(p,q)}X} C_{abc}{}^{d} \qquad \text{for} \quad (p,q) = (0,1), (1,0), \tag{3·76}$$

$$ {}^{(p,q)}\bar{C}_{abc}{}^{d} = {}^{(p,q)}\mathcal{C}_{abc}{}^{d} + 2\pounds_{(\frac{p}{2},\frac{q}{2})X}{}^{(\frac{p}{2},\frac{q}{2})}\bar{C}_{abc}{}^{d} + \left(\pounds_{(p,q)X} - \pounds^{2}_{(\frac{p}{2},\frac{q}{2})X}\right) C_{abc}{}^{d} \quad \text{for} \quad (p,q) = (0,2),(2,0), \qquad (3\cdot77) $$

$$ {}^{(1,1)}\bar{C}_{abc}{}^{d} = {}^{(1,1)}\mathcal{C}_{abc}{}^{d} + \pounds_{(0,1)X}{}^{(1,0)}\bar{C}_{abc}{}^{d} + \pounds_{(1,0)X}{}^{(0,1)}\bar{C}_{abc}{}^{d} + \left(\pounds_{(1,1)X} - \frac{1}{2}\pounds_{(1,0)X}\pounds_{(0,1)X} - \frac{1}{2}\pounds_{(0,1)X}\pounds_{(1,0)X}\right) C_{abc}{}^{d}, \qquad (3\cdot78) $$

where

$$ {}^{(p,q)}\mathcal{C}_{abc}{}^{d} := -{}^{(p,q)}\mathcal{H}^{de}C_{abce} + g^{de(p,q)}\bar{C}_{abce}, \qquad (3\cdot79) $$

$$ {}^{(p,q)}\mathcal{C}_{abc}{}^{d} = -\left({}^{(p,q)}\mathcal{H}^{de} - 2{}^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}_{f}{}^{e}\,{}^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}^{df}\right) C_{abce} + g^{de}\,{}^{(p,q)}\mathcal{C}_{abce} - 2{}^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}^{de}\,{}^{(\frac{p}{2},\frac{q}{2})}\mathcal{C}_{abce}, \qquad (3\cdot80) $$

$$ {}^{(1,1)}\mathcal{C}_{abc}{}^{d} = -\left({}^{(1,1)}\mathcal{H}^{de} - {}^{(1,0)}\mathcal{H}_{f}{}^{e}\,{}^{(0,1)}\mathcal{H}^{df} - {}^{(1,0)}\mathcal{H}_{f}{}^{d}\,{}^{(0,1)}\mathcal{H}^{ef}\right) C_{abce} + g^{de}\,{}^{(1,1)}\mathcal{C}_{abce} - {}^{(0,1)}\mathcal{H}^{de}\,{}^{(1,0)}\mathcal{C}_{abce} - {}^{(1,0)}\mathcal{H}^{de}\,{}^{(0,1)}\mathcal{C}_{abce}. \qquad (3\cdot81) $$

By using the fact that $C_{abc}{}^{b} = 0$ for the Weyl curvature on the background $\mathcal{M}_0$, Eqs. (3·72), (3·45), (3·52) and (3·39), a straightforward calculation yields

$$ {}^{(p,q)}\mathcal{C}_{abc}{}^{b} = 0 \qquad (3\cdot82) $$

for $(p,q) = (0,1),(1,0)$. Further, with straightforward calculations using Eqs. (3·74), (3·72), (3·57), (3·52), (3·50), (3·45), (3·41) and (3·39), we can explicitly confirm the identity

$$ {}^{(1,1)}\mathcal{C}_{abc}{}^{b} = 0, \qquad (3\cdot83) $$

and hence, we obtain

$$ {}^{(p,q)}\mathcal{C}_{abc}{}^{b} = 0 \qquad (3\cdot84) $$

for $(p,q) = (0,2),(2,0)$ through the replacements (3·30) and (3·31) of the perturbative variables. The property ${}^{(p,q)}\mathcal{C}_{abc}{}^{b} = 0$ is trivial from the definition of the gauge invariant part of the Weyl curvature. However, this trivial result gives us great confidence in the formulae derived here.

3.6. *Divergence of an arbitrary tensor of second rank and the Bianchi identity*

Here, we consider the perturbation of the Bianchi identity and the divergence of the energy momentum tensor, which are derived from the divergence $\bar{\nabla}_a\bar{T}_b{}^a$ of an arbitrary tensor field $\bar{T}_b{}^a$ of second rank. The operator $\bar{\nabla}_a$ are the covariant derivative associated with the metric $\bar{g}_{ab}$ on the physical spacetime $\mathcal{M}$. As discussed above, $\bar{\nabla}_a$ is pulled back to the background spacetime $\mathcal{M}_0$ as the derivative operator $\mathcal{X}^*\bar{\nabla}_a\left(\mathcal{X}^{-1}\right)^*$ by choosing a gauge $\mathcal{X}$. Further, the operation of $\bar{\nabla}_a$ as an operator on $\mathcal{M}_0$ is represented by the covariant derivative ∇_a, which is associated with the background metric g_{ab} on $\mathcal{M}_0$, and the tensor field $C^c{}_{ab}$ defined by Eq. (3·5). Hence, we may concentrate on the Taylor expansion of the equation

$$ \bar{\nabla}_a\bar{T}_b{}^a = \nabla_a\bar{T}_b{}^a + C^a{}_{ca}\bar{T}_b{}^c - C^c{}_{ba}\bar{T}_c{}^a \qquad (3\cdot85) $$

to derive the perturbative form of the divergence of an arbitrary tensor field of second rank. The tensor field $\bar{T}_a{}^b$, which is pulled back from the physical spacetime $\mathcal{M}_\epsilon$ to the background spacetime $\mathcal{M}_0$, is expanded as Eq. (3·7). Following the definitions (2·35)–(2·37) of gauge invariant variables, the gauge invariant variables ${}^{(p,q)}\mathcal{T}_b{}^a$ for each order perturbation ${}^{(p,q)}\bar{T}_a{}^b$ are defined by

$$
\begin{aligned}
{}^{(p,q)}\mathcal{T}_b{}^a &:= {}^{(p,q)}\bar{T}_b{}^a - \pounds_{{}_{(p,q)}X} T_b{}^a, \qquad (p,q) = (1,0),(0,1), \qquad (3·86)\\
{}^{(p,q)}\mathcal{T}_b{}^a &:= {}^{(p,q)}\bar{T}_b{}^a - 2\pounds_{{}_{(\frac{p}{2},\frac{q}{2})}X} {}^{(\frac{p}{2},\frac{q}{2})}\bar{T}_b{}^a \\
&\quad - \left\{ \pounds_{{}_{(p,q)}X} - \pounds^2_{{}_{(\frac{p}{2},\frac{q}{2})}X} \right\} T_b{}^a, \qquad (p,q) = (2,0),(0,2), \qquad (3·87)\\
{}^{(1,1)}\mathcal{T}_b{}^a &:= {}^{(1,1)}\bar{T}_b{}^a - \pounds_{{}_{(1,0)}X} {}^{(0,1)}\bar{T}_b{}^a - \pounds_{{}_{(0,1)}X} {}^{(1,0)}\bar{T}_b{}^a \\
&\quad - \left\{ \pounds_{{}_{(1,1)}X} - \frac{1}{2}\pounds_{{}_{(1,0)}X}\pounds_{{}_{(0,1)}X} - \frac{1}{2}\pounds_{{}_{(0,1)}X}\pounds_{{}_{(1,0)}X} \right\} T_b{}^a. \qquad (3·88)
\end{aligned}
$$

We also expand

$$\bar{\nabla}_a \bar{T}_b{}^a = \sum_{k',k=0}^{\infty} \frac{\epsilon^k \lambda^{k'}}{k!k'!} {}^{(k,k')}\!\left(\bar{\nabla}_a \bar{T}_b{}^a\right). \qquad (3·89)$$

3.6.1. Linear order

A simple expansion of Eq. (3·85) yields

$$
\begin{aligned}
{}^{(p,q)}\!\left(\bar{\nabla}_a \bar{T}_b{}^a\right) &= \nabla_a {}^{(p,q)}\bar{T}_b{}^a + \left(H_{ca}{}^a \left[{}^{(p,q)}\mathcal{H}\right] + \nabla_c \nabla_a {}^{(p,q)}X^a \right) T_b{}^c \\
&\quad - \left(H_{bac} \left[{}^{(p,q)}\mathcal{H}\right] + \nabla_b \nabla_a {}^{(p,q)}X_c + R_{acb}{}^e {}^{(p,q)}X_e \right) T^{ca} \qquad (3·90)
\end{aligned}
$$

for $(p,q) = (0,1),(1,0)$. Then, using the gauge invariant variable defined by Eq. (3·86), we obtain

$$
\begin{aligned}
{}^{(p,q)}\!\left(\bar{\nabla}_a \bar{T}_b{}^a\right) &= \nabla_a {}^{(p,q)}\mathcal{T}_b{}^a + H_{ca}{}^a \left[{}^{(p,q)}\mathcal{H}\right] T_b{}^c - H_{ba}{}^c \left[{}^{(p,q)}\mathcal{H}\right] T_c{}^a \\
&\quad + \pounds_{{}_{(p,q)}X} \nabla_a T_b{}^a \qquad (3·91)
\end{aligned}
$$

for linear order $[(p,q) = (1,0),(0,1)]$, where we have used the formula (A·4).

Now, we check the linear-order perturbation of the Bianchi identity by using Eq. (3·91). To do this, we regard $T_a{}^b$ as the Einstein tensor $G_a{}^b$. Further, the gauge invariant part of the tensor $T_a{}^b$ is regarded as the gauge invariant part of the linear order Einstein tensor:

$$
\begin{aligned}
T_a{}^b &\to G_a{}^b, \qquad (3·92)\\
{}^{(p,q)}\mathcal{T}_a{}^b &\to {}^{(1)}\mathcal{G}_a{}^b \left[{}^{(p,q)}\mathcal{H}\right]. \qquad (3·93)
\end{aligned}
$$

Through these replacements, we obtain

$$
\begin{aligned}
{}^{(p,q)}\!\left(\bar{\nabla}_a \bar{G}_b{}^a\right) &= \nabla_a {}^{(1)}\mathcal{G}_a{}^b \left[{}^{(p,q)}\mathcal{H}\right] + H_{ca}{}^a \left[{}^{(p,q)}\mathcal{H}\right] G_b{}^c - H_{ba}{}^c \left[{}^{(p,q)}\mathcal{H}\right] G_c{}^a \\
&\quad + \pounds_{{}_{(p,q)}X} \nabla_a G_b{}^a. \qquad (3·94)
\end{aligned}
$$

On the other hand, a direct calculation using Eq. (3·66) yields

$$\nabla_a {}^{(1)}\mathcal{G}_a{}^b\left[{}^{(p,q)}\mathcal{H}\right] = -H_{ca}{}^a\left[{}^{(p,q)}\mathcal{H}\right]G_b{}^c + H_{ba}{}^c\left[{}^{(p,q)}\mathcal{H}\right]G_c{}^a \tag{3·95}$$

as an identity. Therefore, the linear-order expansion of the divergence of the Einstein tensor is given by

$${}^{(p,q)}\!\left(\bar{\nabla}_a\bar{G}_b{}^a\right) = \pounds_{(p,q)X}\nabla_a G_b{}^a, \quad (p,q) = (1,0),(0,1). \tag{3·96}$$

Because $\nabla_a G_b{}^a = 0$ for an arbitrary spacetime, we can easily see that ${}^{(p,q)}\!\left(\bar{\nabla}_a\bar{G}_b{}^a\right) = 0$, identically, at linear order.

3.6.2. Second order

Next, we consider the $O(\epsilon\lambda)$ perturbation of $\nabla_a T_b{}^a$. A straightforward calculations yields

$$\begin{aligned}
{}^{(1,1)}\!\left(\bar{\nabla}_a\bar{T}_b{}^a\right) = & \nabla_a {}^{(1,1)}\mathcal{T}_b{}^a \\
& -\left(H_{cad}\left[{}^{(0,1)}\mathcal{H}\right]{}^{(1,0)}\mathcal{H}^{da} + H_{cad}\left[{}^{(1,0)}\mathcal{H}\right]{}^{(0,1)}\mathcal{H}^{da} - H_{ca}{}^a\left[{}^{(1,1)}\mathcal{H}\right]\right)T_b{}^c \\
& +\left(H_{bad}\left[{}^{(0,1)}\mathcal{H}\right]{}^{(1,0)}\mathcal{H}^{dc} + H_{bad}\left[{}^{(1,0)}\mathcal{H}\right]{}^{(0,1)}\mathcal{H}^{dc} - H_{ba}{}^c\left[{}^{(1,1)}\mathcal{H}\right]\right)T_c{}^a \\
& -H_{ba}{}^c\left[{}^{(1,0)}\mathcal{H}\right]{}^{(0,1)}\mathcal{T}_c{}^a + H_{ca}{}^a\left[{}^{(1,0)}\mathcal{H}\right]{}^{(0,1)}\mathcal{T}_b{}^c \\
& -H_{ba}{}^c\left[{}^{(0,1)}\mathcal{H}\right]{}^{(1,0)}\mathcal{T}_c{}^a + H_{ca}{}^a\left[{}^{(0,1)}\mathcal{H}\right]{}^{(1,0)}\mathcal{T}_b{}^c \\
& +\pounds_{(1,0)X}{}^{(0,1)}\!\left(\bar{\nabla}_a\bar{T}_b{}^a\right) + \pounds_{(0,1)X}{}^{(1,0)}\!\left(\bar{\nabla}_a\bar{T}_b{}^a\right) \\
& +\left\{\pounds_{(1,1)X} - \frac{1}{2}\pounds_{(0,1)X}\pounds_{(1,0)X} - \frac{1}{2}\pounds_{(1,0)X}\pounds_{(0,1)X}\right\}\left(\nabla_a T_b{}^a\right).
\end{aligned} \tag{3·97}$$

Applying the replacements (3·30) and (3·31) of the perturbative variables to Eq. (3·97), the $O(\epsilon^2)$ and $O(\lambda^2)$ perturbations of the divergence of a tensor $T_a{}^b$ are obtained as

$$\begin{aligned}
{}^{(p,q)}\!\left(\bar{\nabla}_a\bar{T}_b{}^a\right) = & \nabla_a {}^{(p,q)}\mathcal{T}_b{}^a \\
& -\left(2H_{cad}\left[{}^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}\right]{}^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}^{da} - H_{ca}{}^a\left[{}^{(p,q)}\mathcal{H}\right]\right)T_b{}^c \\
& +\left(2H_{bad}\left[{}^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}\right]{}^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}^{dc} - H_{ba}{}^c\left[{}^{(p,q)}\mathcal{H}\right]\right)T_c{}^a \\
& -2H_{ba}{}^c\left[{}^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}\right]{}^{(\frac{p}{2},\frac{q}{2})}\mathcal{T}_c{}^a + 2H_{ca}{}^a\left[{}^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}\right]{}^{(\frac{p}{2},\frac{q}{2})}\mathcal{T}_b{}^c \\
& +2\pounds_{(\frac{p}{2},\frac{q}{2})X}{}^{(\frac{p}{2},\frac{q}{2})}\!\left(\bar{\nabla}_a\bar{T}_b{}^a\right) + \left\{\pounds_{(p,q)X} - \pounds^2_{(\frac{p}{2},\frac{q}{2})X}\right\}\left(\nabla_a T_b{}^a\right),
\end{aligned} \tag{3·98}$$

where $(p,q) = (2,0),(0,2)$.

As in the linear-order case, we can also check the Bianchi identities of the second order perturbations using Eqs. (3·98) and (3·97). To do this, we regard $T_a{}^b$ as the Einstein tensor $G_a{}^b$ and apply the following replacement:

$$T_a{}^b \rightarrow G_a{}^b, \tag{3·99}$$

$${}^{(p,q)}\mathcal{T}_a{}^b \rightarrow {}^{(1)}\mathcal{G}_a{}^b\left[{}^{(p,q)}\mathcal{H}\right] \qquad \text{for} \quad (p,q) = (1,0),(0,1), \tag{3·100}$$

$$ {}^{(p,q)}\mathcal{T}_a{}^b \rightarrow {}^{(1)}\mathcal{G}_a{}^b\left[{}^{(p,q)}\mathcal{H}\right] + {}^{(2)}\mathcal{G}_a{}^b\left[{}^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}, {}^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}\right] \quad \text{for} \quad (p,q) = (2,0),(0,2), \tag{3·101} $$

$$ {}^{(1,1)}\mathcal{T}_a{}^b \rightarrow {}^{(1)}\mathcal{G}_a{}^b\left[{}^{(1,1)}\mathcal{H}\right] + {}^{(2)}\mathcal{G}_a{}^b\left[{}^{(1,0)}\mathcal{H}, {}^{(0,1)}\mathcal{H}\right]. \tag{3·102} $$

First, we consider the perturbative Bianchi identity of $O(\epsilon\lambda)$:

$$ \begin{aligned} {}^{(1,1)}\!\left(\bar{\nabla}_a \bar{G}_b{}^a\right) = \nabla_a & \left({}^{(1)}\mathcal{G}_b{}^a\left[{}^{(1,1)}\mathcal{H}\right] + {}^{(2)}\mathcal{G}_b{}^a\left[{}^{(1,0)}\mathcal{H}, {}^{(0,1)}\mathcal{H}\right]\right) \\ & - \left(H_{cad}\left[{}^{(0,1)}\mathcal{H}\right]\ {}^{(1,0)}\mathcal{H}^{da} + H_{cad}\left[{}^{(1,0)}\mathcal{H}\right]\ {}^{(0,1)}\mathcal{H}^{da} - H_{ca}{}^a\left[{}^{(1,1)}\mathcal{H}\right]\right) G_b{}^c \\ & + \left(H_{bad}\left[{}^{(0,1)}\mathcal{H}\right]\ {}^{(1,0)}\mathcal{H}^{dc} + H_{bad}\left[{}^{(1,0)}\mathcal{H}\right]\ {}^{(0,1)}\mathcal{H}^{dc} - H_{ba}{}^c\left[{}^{(1,1)}\mathcal{H}\right]\right) G_c{}^a \\ & - H_{ba}{}^c\left[{}^{(1,0)}\mathcal{H}\right]\ {}^{(1)}\mathcal{G}_c{}^a\left[{}^{(0,1)}\mathcal{H}\right] + H_{ca}{}^a\left[{}^{(1,0)}\mathcal{H}\right]\ {}^{(1)}\mathcal{G}_b{}^c\left[{}^{(0,1)}\mathcal{H}\right] \\ & - H_{ba}{}^c\left[{}^{(0,1)}\mathcal{H}\right]\ {}^{(1)}\mathcal{G}_c{}^a\left[{}^{(1,0)}\mathcal{H}\right] + H_{ca}{}^a\left[{}^{(0,1)}\mathcal{H}\right]\ {}^{(1)}\mathcal{G}_b{}^c\left[{}^{(1,0)}\mathcal{H}\right] \\ & + \pounds_{{}^{(1,0)}X}\ {}^{(0,1)}\!\left(\bar{\nabla}_a \bar{G}_b{}^a\right) + \pounds_{{}^{(0,1)}X}\ {}^{(1,0)}\!\left(\bar{\nabla}_a \bar{G}_b{}^a\right) \\ & + \left\{\pounds_{{}^{(1,1)}X} - \frac{1}{2}\pounds_{{}^{(0,1)}X}\pounds_{{}^{(1,0)}X} - \frac{1}{2}\pounds_{{}^{(1,0)}X}\pounds_{{}^{(0,1)}X}\right\}\left(\nabla_a G_b{}^a\right). \end{aligned} \tag{3·103} $$

Here, we note that the identity (3·95) implies

$$ \nabla_a {}^{(1)}\mathcal{G}_b{}^a\left[{}^{(1,1)}\mathcal{H}\right] = -H_{ca}{}^a\left[{}^{(1,1)}\mathcal{H}\right] G_b{}^c + H_{ba}{}^c\left[{}^{(1,1)}\mathcal{H}\right] G_c{}^a. \tag{3·104} $$

Further, straightforward calculations lead to the following identity:

$$ \begin{aligned} & \nabla_a {}^{(2)}\mathcal{G}_b{}^a\left[{}^{(1,0)}\mathcal{H}, {}^{(0,1)}\mathcal{H}\right] \\ = & -H_{ca}{}^a\left[{}^{(1,0)}\mathcal{H}\right]\ {}^{(1)}\mathcal{G}_b{}^c\left[{}^{(0,1)}\mathcal{H}\right] - H_{ca}{}^a\left[{}^{(0,1)}\mathcal{H}\right]\ {}^{(1)}\mathcal{G}_b{}^c\left[{}^{(1,0)}\mathcal{H}\right] \\ & + H_{ba}{}^e\left[{}^{(1,0)}\mathcal{H}\right]\ {}^{(1)}\mathcal{G}_e{}^a\left[{}^{(0,1)}\mathcal{H}\right] + H_{ba}{}^e\left[{}^{(0,1)}\mathcal{H}\right]\ {}^{(1)}\mathcal{G}_e{}^a\left[{}^{(1,0)}\mathcal{H}\right] \\ & - \left(H_{bad}\left[{}^{(0,1)}\mathcal{H}\right]\ {}^{(1,0)}\mathcal{H}^{dc} + H_{bad}\left[{}^{(1,0)}\mathcal{H}\right]\ {}^{(0,1)}\mathcal{H}^{dc}\right) G_c{}^a \\ & + \left(H_{cad}\left[{}^{(0,1)}\mathcal{H}\right]\ {}^{(1,0)}\mathcal{H}^{ad} + H_{cad}\left[{}^{(1,0)}\mathcal{H}\right]\ {}^{(0,1)}\mathcal{H}^{ad}\right) G_b{}^c. \end{aligned} \tag{3·105} $$

Using the identities (3·104) and (3·105) , we easily find

$$ \begin{aligned} {}^{(1,1)}\!\left(\bar{\nabla}_a \bar{G}_b{}^a\right) = & \pounds_{{}^{(1,0)}X}\ {}^{(0,1)}\!\left(\bar{\nabla}_a \bar{G}_b{}^a\right) + \pounds_{{}^{(0,1)}X}\ {}^{(1,0)}\!\left(\bar{\nabla}_a \bar{G}_b{}^a\right) \\ & + \left\{\pounds_{{}^{(1,1)}X} - \frac{1}{2}\pounds_{{}^{(0,1)}X}\pounds_{{}^{(1,0)}X} - \frac{1}{2}\pounds_{{}^{(1,0)}X}\pounds_{{}^{(0,1)}X}\right\}\left(\nabla_a G_b{}^a\right). \end{aligned} \tag{3·106} $$

Since the Bianchi identities $\nabla_a G_b{}^a = 0$ on the background spacetime $\mathcal{M}_0$ and those of the linear-order perturbations ${}^{(p,q)}(\nabla_a G_b{}^a) = 0$ [for $(p,q) = (1,0),(0,1)$] have already been confirmed, we have also confirmed the identity

$$ {}^{(1,1)}\!\left(\bar{\nabla}_a \bar{G}_b{}^a\right) = 0. \tag{3·107} $$

Applying the replacements (3·30) and (3·31) of the perturbative variables to Eq. (3·103), we obtain the $O(\epsilon^2)$ and $O(\lambda^2)$ Bianchi identities:

$$\begin{aligned}
{}^{(p,q)}\!\left(\bar{\nabla}_a\bar{G}_b{}^a\right) = {} & \nabla_a\left({}^{(1)}\mathcal{G}_b{}^a\left[{}^{(p,q)}\mathcal{H}\right] + {}^{(2)}\mathcal{G}_b{}^a\left[{}^{(\frac{p}{2},\frac{q}{2})}\mathcal{H},{}^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}\right]\right) \\
& -\left(2H_{cad}\left[{}^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}\right]\ {}^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}^{da} - H_{ca}{}^a\left[{}^{(p,q)}\mathcal{H}\right]\right)G_b{}^c \\
& +\left(2H_{bad}\left[{}^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}\right]\ {}^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}^{dc} - H_{ba}{}^c\left[{}^{(p,q)}\mathcal{H}\right]\right)G_c{}^a \\
& -2H_{ba}{}^c\left[{}^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}\right]\ {}^{(1)}\mathcal{G}_c{}^a\left[{}^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}\right] + 2H_{ca}{}^a\left[{}^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}\right]\ {}^{(1)}\mathcal{G}_b{}^c\left[{}^{(\frac{p}{2},\frac{q}{2})}\mathcal{H}\right] \\
& +2\pounds_{{}^{(\frac{p}{2},\frac{q}{2})}X}\ {}^{(\frac{p}{2},\frac{q}{2})}\!\left(\bar{\nabla}_a\bar{G}_b{}^a\right) + \left\{\pounds_{{}^{(p,q)}X} - \pounds^2_{{}^{(\frac{p}{2},\frac{q}{2})}X}\right\}\left(\nabla_a G_b{}^a\right) \\
= {} & 0,
\end{aligned} \tag{3·108}$$

where $(p,q) = (0,2),(2,0)$.

3.7. *Einstein equations*

Finally, we consider perturbations of the Einstein equation at each order. First, we expand the energy momentum tensor as Eq. (3·7) and impose the perturbed Einstein equation of each order,

$$ {}^{(p,q)}G_a{}^b = 8\pi G\ {}^{(p,q)}T_a{}^b. \tag{3·109}$$

Then, we define the gauge invariant variable ${}^{(p,q)}\mathcal{T}_a{}^b$ for the perturbative energy momentum tensor at each order by Eqs. (3·86)–(3·88). In terms of the gauge invariant variables, the perturbative Einstein equation at each order is given by

$$ {}^{(1)}\mathcal{G}_a{}^b\left[{}^{(p,q)}\mathcal{H}\right] = 8\pi G\ {}^{(p,q)}\mathcal{T}_a{}^b \tag{3·110}$$

for linear order $[(p,q) = (0,1),(1,0)]$ and

$$ {}^{(1)}\mathcal{G}_a{}^b\left[{}^{(1,1)}\mathcal{H}\right] + {}^{(2)}\mathcal{G}_a{}^b\left[{}^{(1,0)}\mathcal{H},{}^{(0,1)}\mathcal{H}\right] = 8\pi G\ {}^{(1,1)}\mathcal{T}_a{}^b, \tag{3·111}$$

$$ {}^{(1)}\mathcal{G}_a{}^b\left[{}^{(p,q)}\mathcal{H}\right] + {}^{(2)}\mathcal{G}_a{}^b\left[{}^{(p,q)}\mathcal{H},{}^{(p,q)}\mathcal{H}\right] = 8\pi G\ {}^{(p,q)}\mathcal{T}_a{}^b \tag{3·112}$$

for second order $[(p,q) = (0,2),(2,0)]$. These explicitly show that, order by order, the Einstein equations are necessarily obtained in terms of gauge invariant variables only.

§4. Summary and discussion

In summary, we have derived some formulae for second-order gauge invariant perturbations, namely those of the Riemann, Ricci, scalar, Einstein and Weyl curvature tensors. We also derived the formulae for the divergence of an arbitrary tensor field of second rank for perturbations at each order. These perturbative curvatures have the same forms as those in the definitions (2·35)–(2·37) of the gauge invariant variables for arbitrary perturbative fields which are proposed in a previous paper.[8] These are useful for investigating physical problems using perturbation theory.

Through the derivation of these formulae, we have confirmed some facts. First, if linear-order gauge invariant perturbation theory is well established, its extension to higher orders and multi-parameter perturbations is straightforward. Second, the perturbative Weyl curvature at each order preserves the property of the tracelessness of the Weyl curvature. Third, the perturbative Einstein curvature of each order satisfies the perturbative Bianchi identities at each order. Fourth, the perturbative Einstein equations at each order necessarily take in gauge invariant forms. These properties we have confirmed are trivial from their derivations. In particular, the fourth result is trivial because of the following reason: Any equation can be written in a form in which the right-hand side is equal to "zero" in any gauge. This "zero" is gauge invariant. Hence, the left-hand side should be gauge invariant. However, we emphasize that these trivial results imply that the formulae derived here and the framework of higher-order perturbation theory applied here are mathematically consistent at this level.

Further, we note that in our framework, we specify nothing about the background spacetime nor about the physical meaning of the parameters for the perturbations. Our framework is based only on general covariance. For this reason, this framework is applicable to any theory in which general covariance is imposed, and thus it has very many applications. Actually, we are planning to apply it to some physical problems. The following are candidates of the physical situations to which the second order perturbation theory should be applied: the radiation reaction problem in gravitational wave emission;[13] stationary axisymmetric ideal MHD flow around a black hole or a star;[14] the correspondence between observables in experiment and gauge invariant variables; dynamics of gravitating membranes (for example, topological defects,[4] brane world,[15] and so on); perturbations of a compact star with rotation and pulsation;[6] Post-Minkowski expansion alternative to post-Newtonian expansion;[16] higher-order cosmological perturbations and primordial non-Gaussianity.[17]

In particular, the gauge invariant form of the second-order perturbation of the divergence of the energy momentum tensor should be useful in considering the gauge problem in the radiation reaction of gravitational wave emission. In astrophysical contexts, it is natural to consider the situation in which a solar mass object falls into a supermassive black hole of mass $\sim 10^6 M_\odot$. This is one of the target phenomena of the observation of gravitational wave by LISA (laser interferometer space antenna for gravitational wave measurements).[18] In such a situation, the perturbation parameter is the ratio of the mass of the compact object to that of the central supermassive black hole. We may regard the second order perturbations as describing the radiation reaction effect of gravitational wave emission. By applying the gauge invariant formulation discussed here, we can exclude gauge freedom completely and thus there is no gauge ambiguities in the results. It would be quite interesting to apply the formulation discussed here to this radiation reaction problem. We leave this application as a future work.

In addition to the radiation reaction problem of gravitational wave emission, there are many applications, some of which are listed above. Indeed, because there are few assumptions in our treatment, it is natural to expect that there are many applications that are not listed above. We also expect that the formulae derived here will be found to be very powerful tools in many applications.

Acknowledgements

The author acknowledges Prof. L. Kofman and Prof. R.M. Wald and their colleagues for the hospitality during my visit to CITA and the Relativity Group at the University of Chicago. The author also thanks Prof. M. Sasaki for the valuable comments on the title of this paper. Finally, the author deeply thanks all members of Division of Theoretical Astronomy at NAOJ and other colleagues for their continuous encouragement.

Appendix A
—— *Useful Formulae* ——

In the calculations in the main text, the knowledge of the commutation relations of the covariant derivative and Lie derivative are useful. Here, we summarize the commutation relations which are used in the derivation in the main text:

$$\nabla_a \pounds_X t_b = \pounds_X \nabla_a t_b + X^c R_{acb}{}^d t_d + t_c \nabla_a \nabla_b X^c, \tag{A·1}$$

$$\begin{aligned}\nabla_a \pounds_X t_{bc} = & \pounds_X \nabla_a t_{bc} + X^d R_{adb}{}^e t_{ec} + X^d R_{adc}{}^e t_{be} \\ & + t_{dc} \nabla_a \nabla_b X^d + t_{bd} \nabla_a \nabla_c X^d,\end{aligned} \tag{A·2}$$

$$\begin{aligned}\nabla_a \pounds_X t_b{}^c = & \pounds_X \nabla_a t_b{}^c + X^d R_{adb}{}^e t_e{}^c - X^d R_{ade}{}^c t_b{}^e \\ & + t_d{}^c \nabla_a \nabla_b X^d - t_b{}^d \nabla_a \nabla_d X^c,\end{aligned} \tag{A·3}$$

$$\begin{aligned}\nabla_a \pounds_X t_{bcd} = & \pounds_X \nabla_a t_{bcd} + X^e R_{aeb}{}^f t_{fcd} + X^e R_{aec}{}^f t_{bfd} + X^e R_{aed}{}^f t_{bcf} \\ & + t_{ecd} \nabla_a \nabla_b X^e + t_{bed} \nabla_a \nabla_c X^e + t_{bce} \nabla_a \nabla_d X^e,\end{aligned} \tag{A·4}$$

$$\begin{aligned}\nabla_a \pounds_X t_{bc}{}^d = & \pounds_X \nabla_a t_{bc}{}^d + X^e R_{aeb}{}^f t_{fc}{}^d + X^e R_{aec}{}^f t_{bf}{}^d - X^e R_{aef}{}^d t_{bc}{}^f \\ & + t_{ec}{}^d \nabla_a \nabla_b X^e + t_{be}{}^d \nabla_a \nabla_c X^e - t_{bc}{}^e \nabla_a \nabla_e X^d.\end{aligned} \tag{A·5}$$

References

1) H. Stephani, D. Kramer, M. A. H. MacCallum, C. Hoenselaers and E. Herlt, *Exact Solutions of Einstein's Field Equations Second Edition*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, 2003).
2) J. M. Bardeen, Phys. Rev. D **22** (1980), 1882.
H. Kodama and M. Sasaki, Prog. Theor. Phys. Suppl. No. 78 (1984), 1.
V. F. Mukhanov, H. A. Feildman and R. H. Brandenberger, Phys. Rep. **215** (1992), 203.
3) U. H. Gerlach and U. K. Sengupta, Phys. Rev. D **19** (1979), 2268; Phys. Rev. D **20** (1979), 3009; Phys. Rev. D **22** (1980), 1300; J. Math. Phys. **20** (1979), 2540.
C. Gundlach and J. M. Martíne-García, Phys. Rev. D **61** (2000), 084024.
J. M. Martíne-García and C. Gundlach, Phys. Rev. D **64** (2001), 024012.
S. Chandrasekhar, *The mathematical theory of black holes* (Oxford: Clarendon Press, 1983).
4) K. Nakamura, A. Ishibashi and H. Ishihara, Phys. Rev. D **62** (2000), 101502(R).
K. Nakamura and H. Ishihara, Phys. Rev. D **63** (2001), 127501.
K. Nakamura, Class. Quantum Grav. **19** (2002), 783; Phys. Rev. D **66** (2002), 084005.
5) K. Nakamura, Prog. Theor. Phys. **110** (2003), 201.

6) Y. Kojima, Prog. Theor. Phys. Suppl. No. 128 (1997), 251.
A. Passamonti, M. Bruni, L. Gualtieri and C. F. Sopuerta, gr-qc/0407108.
7) R. J. Gleiser, C. O. Nicasio, R. H. Price and J. Pullin, Phys. Rep. **325** (2000), 41.
8) K. Nakamura, Prog. Theor. Phys. **110** (2003), 723.
9) M. Bruni, L. Gualtieri and C. F. Sopuerta, Class. Quantum Grav. **20** (2003), 535.
C. F. Sopuerta, M. Bruni and L. Gualtieri, Phys. Rev. D **70** (2004), 064002.
10) J. M. Stewart and M. Walker, Proc. R. Soc. London A **341** (1974), 49.
J. M. Stewart, Class. Quantum Grav. **7** (1990), 1169.
J. M. Stewart, *Advanced General Relativity* (Cambridge University Press, Cambridge, 1991).
11) M. Bruni, S. Matarrese, S. Mollerach and S. Sonego, Class. Quantum Grav. **14** (1997), 2585.
M. Bruni and S. Sonego, Class. Quantum Grav. **16** (1999), L29.
12) R. M. Wald, *General Relativity* (Chicago, IL: University of Chicago Press, 1984).
13) Y. Mino, M. Sasaki and T. Tanaka, Phys. Rev. D **55** (1997), 3457.
T. C. Quinn and R. M. Wald, Phys. Rev. D **56** (1997), 3381.
T. C. Quinn and R. M. Wald, Phys. Rev. D **60** (1999), 064009.
E. Poisson, Living Rev. Relativity **7** (2004), 6 and references therein.
14) S. Nitta, M. Takahashi and A. Tomimatsu, Phys. Rev. D **44** (1991), 2295.
K. Ioka and M. Sasaki, Phys. Rev. D **67** (2003), 124026.
K. Ioka and M. Sasaki, Astrophys. J. **600** (2004), 296.
15) L. Randall and R. Sundrum, Phys. Rev. Lett. **83** (1999), 4690.
L. Randall and R. Sundrum, Phys. Rev. Lett. **83** (1999), 3370.
J. Garriga and T. Tanaka, Phys. Rev. Lett. **84** (2000), 2778.
16) S. Detweiler and L. H. Brown, Jr, Phys. Rev. D **56** (1997), 826.
17) K. A. Malik and D. Wands, Class. Quantum Grav. **21** (2004), L65.
N. Bartolo, S. Matarrese and A. Riotto, astro-ph/0407505.
N. Bartolo, E. Komatsu, S. Matarrese and A. Riotto, astro-ph/0406398.
N. Bartolo, S. Matarrese and A. Riotto, Phys. Rev. D **69** (2004), 043503.
N. Bartolo, S. Matarrese and A. Riotto, J. High Energy Phys. **04** (2004), 006.
18) `http://lisa.jpl.nasa.gov/index.html`