Second order partial differential equations (SOPDEs) and nonlinear connections on the tangent bundle of k^1-velocities of a manifold

Florian Munteanu

Abstract. In this paper I will describe two distinct ways by which it obtain a sequence of SOPDEs and a sequences of nonlinear connections on the tangent bundle of k^1-velocties T_k^1M, starting from a given SOPDE, follows the ideas of papers [25], [26], [27], [28]. Some properties about this sequences is also presented. Interesting cases appear in the presence of a regular Lagrangian on T_k^1M.

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1 Introduction

In this section we present the main notions used in the paper, with a strong accent on the necessary tools for obtain the results from the second section.

1.1 The tangent bundle of k^1-velocities of a manifold M

An almost tangent structure J on a $2n$-dimensional manifold M is tensor field of type $(1,1)$ of constant rank n such that $J^2 = 0$. The manifold M is then called an almost tangent manifold. Almost tangent structures were introduced by Clark and Bruckheimer [5] and Eliopoulos [6] around 1960 and have been studied by numerous authors (see [10, 11, 12, 13, 14, 15, 16, 17, 18]).

The canonical model of these structures is the tangent bundle $\tau_M : TM \to M$ of an arbitrary manifold M. Recall that for a vector X_x at a point $x \in M$ its vertical lift is the vector on TM given by

$$X_x^V(v_x) = \frac{d}{dt}(v_x + tX_x)_{|t=0} \in T_{v_x}(TM)$$

for all points $v_x \in TM$.

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The canonical tangent structure J on TM is defined by

$$J_{v_x}(Z_{v_x}) = ((\tau_M)_*(v_x)Z_{v_x})^V_{v_x}$$

for all vectors $Z_{v_x} \in T_{v_x}(TM)$, and it is locally given by

$$J = \frac{\partial}{\partial v^i} \otimes dx^i \tag{1.1}$$

with respect the bundle coordinates on TM. This tensor J can be regarded as the vertical lift of the identity tensor on M to TM [19].

The almost k-tangent structures were introduced as generalization of the almost tangent structures [7, 8].

Definition 1.1 *An almost k-tangent structure J on a manifold M of dimension $n + kn$ is a family $(J^1, \dots, J^k)$ of tensor fields of type $(1, 1)$ such that*

$$J^A \circ J^B = J^B \circ J^A = 0, \qquad rank\, J^A = n, \qquad Im\, J^A \cap (\oplus_{B \neq A} Im\, J^B) = 0, \tag{1.2}$$

for $1 \leq A, B \leq k$. In this case the manifold M is then called an almost k-tangent manifold.

The canonical model of these structures is the k-tangent vector bundle $T^1_k M = J^1_0(R^k, M)$ of an arbitrary manifold M, that is the vector bundle with total space the manifold of 1-jets of maps with source at $0 \in R^k$ and with projection map $\tau : T^1_k M \to M$, $\tau(j^1_0\sigma) = \sigma(0)$. This bundle is also known as the *tangent bundle of k^1-velocities* of M [19].

The manifold $T^1_k M$ can be canonically identified with the Whitney sum of k copies of TM, that is

$$\begin{array}{rcc} T^1_k M & \equiv & TM \oplus \cdots \oplus TM, \\ j^1_0\sigma & \equiv & (j^1_0\sigma_1 = v_1, \dots, j^1_0\sigma_k = v_k) \end{array}$$

where $\sigma_A = \sigma(0, \dots, t, \dots, 0)$ with $t \in R$ at position A and $v_A = (\sigma_A)_*(0)(\frac{d}{dt}_0)$.

If (x^i) are local coordinates on $U \subseteq M$ then the induced local coordinates (x^i, v^i_A), $1 \leq i \leq n$, $1 \leq A \leq k$, on $\tau^{-1}(U) \equiv T^1_k U$ are given by

$$x^i(j^1_0\sigma) = x^i(\sigma(0)), \qquad v^i_A(j^1_0\sigma) = \frac{d}{dt}(x^i \circ \sigma_A)|_{t=0} = v_A(x^i)\,.$$

A local change of coordinates on $T^1_k M$, $(x^i, v^i_A) \to (\tilde{x}^i, \tilde{v}^i_A)$, is given by the rule:

$$\left\{ \begin{array}{ll} \tilde{x}^i = \tilde{x}^i(x^1, \dots, x^n) & , rank \left(\dfrac{\partial \tilde{x}^i}{\partial x^j} \right) = n\,, \\ \tilde{v}^i_A = \dfrac{\partial \tilde{x}^i}{\partial x^j} v^j_A & , 1 \leq A \leq k,\, 1 \leq i \leq n. \end{array} \right. \tag{1.3}$$

If $T_u T^1_k M$ is the tangent space to $T^1_k M$ in u, then $\left\{ \dfrac{\partial}{\partial x^i}(u), \dfrac{\partial}{\partial v^i_A}(u) \right\}$ is its local basis and $\left\{ dx^i(u), dv^i_A(u) \right\}$ is its dual local basis (local cobasis). With respect to (1.3) the basis and the cobasis have the following rules of changing:

$$\left\{\begin{array}{l}\dfrac{\partial}{\partial x^i}=\dfrac{\partial \tilde{x}^j}{\partial x^i}\dfrac{\partial}{\partial \tilde{x}^j}+\dfrac{\partial \tilde{v}^j_A}{\partial x^i}\dfrac{\partial}{\partial \tilde{v}^j_A}\,,\\ \dfrac{\partial}{\partial v^i_A}=\dfrac{\partial \tilde{x}^j}{\partial x^i}\dfrac{\partial}{\partial \tilde{v}^j_A}\,,\end{array}\right. \tag{1.4}$$

and

$$\left\{\begin{array}{l} dx^i=\dfrac{\partial \tilde{x}^i}{\partial x^j}d\tilde{x}^j\,,\\ dv^i_A=\dfrac{\partial v^i_A}{\partial \tilde{x}^j}d\tilde{x}^j+\dfrac{\partial x^i}{\partial \tilde{x}^j}d\tilde{v}^j_A\,.\end{array}\right. \tag{1.5}$$

It is observe that $\dfrac{\partial \tilde{x}^j}{\partial x^i}=\dfrac{\partial \tilde{v}^j_A}{\partial v^i_A}$.

Definition 1.2 *For a vector X_x at M we define its vertical A-lift $(X_x)^A$ as the vector on T^1_kM given by*

$$(X_x)^A(j^1_0\sigma)=\frac{d}{dt}((v_1)_x,\dots,(v_{A-1})_x,(v_A)_x+tX_x,(v_{A+1})_x\dots,(v_k)_x)|_{t=0}\in T_{j^1_0\sigma}(T^1_kM)$$

for all points $j^1_0\sigma\equiv((v_1)_x,\dots,(v_k)_x))\in T^1_kM$.

In local coordinates we have

$$(X_x)^A=\sum_{i=1}^{n}a^i\frac{\partial}{\partial v^i_A} \tag{1.6}$$

for a vector $X_x=a^i\,\partial/\partial x^i$.

The *canonical vertical vector fields* C^A_B on T^1_kM are defined by

$$C^A_B(x,X_1,X_2,\dots,X_k)=(X_B)^A \tag{1.7}$$

and are locally given by $C^A_B=v^i_B\frac{\partial}{\partial v^i_A}$.

The *canonical k-tangent structure* $(J^1,\dots,J^k)$ on T^1_kM is defined by

$$J^A(Z_{j^1_0\sigma})=(\tau_*(Z_{j^1_0\sigma}))^A$$

for all vectors $Z_{j^1_0\sigma}\in T_{j^1_0\sigma}(T^1_kM)$. In local coordinates we have

$$J^A=\frac{\partial}{\partial v^i_A}\otimes dx^i \tag{1.8}$$

The tensors J^A can be regarded as the $(0,\dots,1_A,\dots,0)$-lift of the identity tensor on M to T^1_kM defined in [19].

1.2 The cotangent bundle of k^1-covelocities of M and $(T^1_k)^*M$

Almost cotangent structures were introduced by Bruckheimer [4]. An *almost cotangent structure* on a $2m$-dimensional manifold M consists of a pair (ω, V) where ω is a symplectic form and V is a distribution such that

$$(i) \quad \omega\rfloor_{V\times V} = 0, \qquad (ii) \quad \ker\omega = \{0\}$$

The canonical model of this structure is the cotangent bundle $\tau^*_M : T^*M \to M$ of an arbitrary manifold M, where ω is the canonical symplectic form $\omega_0 = -d\theta_0$ on T^*M and V is the vertical distribution. Let us recall the definition of the Liouville form θ_0 in T^*M:

$$\theta_0(\alpha)(\tilde{X}_\alpha) = \alpha((\tau^*_M)_*(\alpha)(\tilde{X}_\alpha)),$$

for all vectors $\tilde{X}_\alpha \in T_\alpha(T^*M)$. In local coordinates (x^i, p_i) on T^*M

$$(1.9) \qquad \theta_0 = p_i dx^i, \qquad \omega_0 = dx^i \wedge dp_i, \qquad V = \langle \frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_k} \rangle.$$

Definition 1.3 *[1, 2] A k-symplectic structure on a manifold M of dimension $N = n + kn$ is a family $(\omega_A, V; 1 \leq A \leq k)$, where each ω_A is a closed 2-form and V is an nk-dimensional distribution on M such that*

$$(i) \quad \omega_{A_{\rfloor V\times V}} = 0, \qquad (ii) \quad \cap_{A=1}^k \ker\omega_A = \{0\}.$$

In this case (M, ω_A, V) is called a k-symplectic manifold.

The canonical model of this structure is the k-cotangent bundle $(T^1_k)^*M = J^1(M, \mathbf{R}^k)_0$ of an arbitrary manifold M, that is the vector bundle with total space the manifold of 1-jets of maps with target at $0 \in R^k$, and projection $\tau^*(j^1_{x,o}\sigma) = x$.

The manifold $(T^1_k)^*M$ can be canonically identified with the Whitney sum of k copies of T^*M, say

$$\begin{array}{rcl} (T^1_k)^*M & \equiv & T^*M \oplus \cdots \oplus T^*M, \\ j_{x,0}\sigma & \equiv & (j^1_{x,0}\sigma^1, \ldots, j^k_{x,0}\sigma^k) \end{array}$$

where $\sigma^A = \pi_A \circ \sigma : M \longrightarrow R$ is the A-th component of σ.

The canonical k-symplectic structure $(\omega_0)_A, V; 1 \leq A \leq k)$, on $(T^1_k)^*M$ is defined by

$$(\omega_0)_A = (\tau^*_A)^*(\omega_0)$$
$$V(j^1_{x,0}\sigma) = \ker(\tau^*)_*(j^1_{x,0}\sigma)$$

where $\tau^*_A = (T^1_k)^*M \to T^*M$ is the projection on the A^{th}-copy T^*M of $(T^1_k)^*M$, and ω_0 is the canonical symplectic structure of T^*M.

One can also define the 2-forms ω_A by $\omega_A = -d\theta_A$ where $(\theta_0)^A = (\tau^*_A)^*\theta_0$

If (x^i) are local coordinates on $U \subseteq M$ then the induced local coordinates (x^i, p^A_i), $1 \leq i \leq n$, $1 \leq A \leq k$ on $(T^1_k)^*U = (\tau^*)^{-1}(U)$ are given by

$$x^i(j^1_{x,0}\sigma) = x^i(x), \qquad p^A_i(j^1_{x,0}\sigma) = d_x\sigma^A(\frac{\partial}{\partial x^i}\Big|_x) \,.$$

Then the canonical k-symplectic structure is locally given by

$$(\omega_0)^A = \sum_{i=1}^{n} dx^i \wedge dp_i^A, \qquad V = \langle \frac{\partial}{\partial p_i^1}, \dots, \frac{\partial}{\partial p_i^k} \rangle \quad 1 \leq A \leq k\,. \tag{1.10}$$

and $(\theta_0)^A = p_i^A\, dx^i$.

1.3 Second Order Partial Differential Equations on $T_k^1 M$

Let M be an arbitrary manifold and $\tau : T_k^1 M \longrightarrow M$ its tangent bundle of k^1-velocities.

Definition 1.4 *A section $X : M \longrightarrow T_k^1 M$ of the projection τ will be called a* **k-vector field** *on M.*

Since $T_k^1 M$ can be canonically identified with the Whitney sum $T_k^1 M \equiv TM \oplus \cdots \oplus TM$ of k copies of TM, we deduce that a k-vector field X defines a family of vector fields $X_1, \dots, X_k$ on M. Günther in [3] introduce the following definition.

Definition 1.5 *An* **integral section** *of the k-vector field $X = (X_1, \dots, X_k)$ passing through a point $x \in M$ on M is a map $\phi : U_0 \subset \mathbf{R}^k \to M$, defined on some neighborhood U_0 of $0 \in \mathbf{R}^k$, such that*

$$\phi(0) = x, \quad \phi_*(t)(\frac{\partial}{\partial t^A}) = X_A(\phi(t)) \quad \forall t \in U, \quad 1 \leq A \leq k,$$

or equivalently, ϕ satisfies

$$X \circ \phi = \phi^{(1)}, \tag{1.11}$$

where $\phi^{(1)}$ is the first prolongation of ϕ defined by

$$\begin{array}{rccl} \phi^{(1)}: & U_0 \subset \mathbf{R}^k & \longrightarrow & T_k^1 M \\ & t & \longrightarrow & \phi^{(1)}(t) = j_0^1 \phi_t \end{array}$$

where $\phi_t(s) = \phi(s+t)$ for all $t, s \in \mathbf{R}^k$ such that $s + t \in U_0$.

In local coordinates,

$$\phi^{(1)}(t^1, \dots, t^k) = (\phi^i(t^1, \dots, t^k), \frac{\partial \phi^i}{\partial t^A}(t^1, \dots, t^k)), \qquad 1 \leq A \leq k\,, 1 \leq i \leq n\,. \tag{1.12}$$

We say that a k-vector field $X = (X_1, \dots, X_k)$ on M is **integrable** if there is an integral section passing through each point of M.

Remark If ϕ is an integral section of a k-vector field $(X_1, \dots, X_k)$ then each curve on M defined by $\phi_A = \phi \circ h_A$, where $h_A : \mathbf{R} \to \mathbf{R}^k$ is the natural inclusion $h_A(t) = (0, \dots, t, \dots, 0)$, is an integral curve of the vector field X_A on M, with $1 \leq A \leq k$. We refer to [20, 21] for a discussion on the existence of integral sections.

Definition 1.6 *A k-vector field on T^1_kM, that is, a section $\xi \equiv (\xi_1, \dots, \xi_k) : T^1_kM \to T^1_k(T^1_kM)$ of the projection $\tau_{T^1_kM} : T^1_k(T^1_kM) \to T^1_kM$, is a* **Second Order Partial Differential Equation (SOPDE)** *if and only if it is also a section of the vector bundle $T^1_k(\tau) : T^1_k(T^1_kM) \to T^1_kM$, where $T^1_k(\tau)$ is defined by $T^1_k(\tau)(j^1_0\sigma) = j^1_0(\tau \circ \sigma)$.*

Let (x^i) be a coordinate system on M and (x^i, v^i_A) the induced coordinate system on T^1_kM. From the definition we deduce that the local expression of a SOPDE ξ is

$$\xi_A(x^i, v^i_A) = v^i_A \frac{\partial}{\partial x^i} + (\xi_A)^i_B \frac{\partial}{\partial v^i_B}, \quad 1 \leq A \leq k. \tag{1.13}$$

We recall that the first prolongation $\phi^{(1)}$ of $\phi : U \subset \mathbf{R}^k \to M$ is defined by

$$\begin{array}{rccl} \phi^{(1)}: & U \subset \mathbf{R}^k & \longrightarrow & T^1_kM) \\ & t & \longrightarrow & \phi^{(1)}(t) = j^1_0\phi_t \end{array}$$

where $\phi_t(s) = \phi(s + t)$ for all $t, s \in R$.

Proposition 1.7 *Let ξ an integrable k-vector field on T^1_kM. The necessary and sufficient condition for ξ to be a Second Order Partial Differential Equation (SOPDE) is that its integral sections are first prolongations $\phi^{(1)}$ of maps $\phi : \mathbf{R}^k \to M$. That is*

$$\xi_A(\phi^{(1)}(t)) = (\phi^{(1)})_*(t)(\frac{\partial}{\partial t_A})(t)$$

for all $A = 1, \dots, k$. These maps ϕ will be called solutions of the SOPDE ξ.

From (1.12) and (1.13) we have

Proposition 1.8 *$\phi : \mathbf{R}^k \to M$ is a solution of the SOPDE $\xi = (\xi_1, \dots, \xi_k)$, locally given by (1.13), if and only if*

$$\frac{\partial \phi^i}{\partial t^A}(t) = v^i_A(\phi^{(1)}(t)), \qquad \frac{\partial^2 \phi^i}{\partial t^A \partial t^B}(t) = (\xi_A)^i_B(\phi^{(1)}(t)).$$

If $\xi : T^1_kM \to T^1_kT^1_kM$ is an integrable SOPDE then for all integral sections $\sigma : U \subset \mathbf{R}^k \to T^1_kM$ we have $(\tau_M \circ \sigma)^{(1)} = \sigma$, where $\tau : T^1_kM \to M$ is the canonical projection.

Now we show how to characterize the SOPDEs using the canonical k-tangent structure of T^1_kM.

Definition 1.9 *The Liouville vector field C on T^1_kM is the infinitesimal generator of the one parameter group*

$$\begin{array}{ccc} \mathbf{R} \times (T^1_kM) & \longrightarrow & T^1_kM \\ (s, (x^i, v^i_B)) & \longrightarrow & (x^i, e^s\, v^i_B)\,. \end{array}$$

Thus C is locally expressed as follows:

$$C = \sum_B C_B = \sum_{i,B} v^i_B \frac{\partial}{\partial v^i_B}, \tag{1.14}$$

where each C_B corresponds with the canonical vector field on the B-th copy of TM on T^1_kM.

Let us remark that each vector field C_A on T^1_kM can also be defined using the A-lifts of vectors as follows: $C_A((v_1)_q, \dots , (v_k)_q) = ((v_A)_q)^A(v))$.

From (1.8), (1.13) and (1.14) we deduce the following

Proposition 1.10 *A k-vector field $\xi = (\xi_1, \dots , \xi_k)$ on T^1_kM is a SOPDE if and only if*

$$J^A(\xi_A) = C_A, \qquad \forall\, 1 \leq A \leq k,$$

where $(J^1, \dots , J^k)$ is the canonical k-tangent structure on T^1_kM.

Proposition 1.11 *a) With respect to a local changing of coordinates (1.3) the components $(\xi_A)^i_B$ of the SOPDE ξ change as follows*

$$(\widetilde{\xi}_A)^i_B = (\xi_A)^j_B \frac{\partial x^i}{\partial x^j} + \frac{\partial^2 x^i}{\partial x^j \partial x^k} v^j_A v^k_B \tag{1.15}$$

b) Conversely, if the function $(\xi_A)^i_B(x^j, v^k_C)$ are given on every domain of local chart on T^1_kM so that (1.15) holds for a local change (1.3), then the k-vector field ξ given by (1.13) is a SOPDE.

Next we will denote $\widetilde{T^1_kM} = T^1_kM \setminus \{0\}$, where 0 is the null section of the projection $\tau : T^1_kM \to M$.

Definition 1.12 *A function $f : T^1_kM \to R$ differentiable of class C^∞ on $\widetilde{T^1_kM}$ and continuous on the null section of the projection τ is called homogeneous of degree $r \in \mathbf{Z}$ on the fibres on T^1_kM (briefly r-homogeneous) if*

$$f(x^i, \lambda v^i_A) = \lambda^r f(x^i, v^i_A) \quad , \forall \lambda > 0. \tag{1.16}$$

Equivalently, a function $f : T^1_kM \to R$, differentiable on $\widetilde{T^kM}$ and continuous on the null section of τ is $r-$homogeneous if and only if

$$\mathcal{L}_C f = rf, \tag{1.17}$$

$\mathcal{L}_C f$ being the Lie operator of derivation with respect to the canonical vector field C.

Definition 1.13 *A SOPDE ξ is called homogeneous SOPDE, or spray on T^1_kM if the components $(\xi_A)^i_B$ are 2-homogeneous functions on the fibres of T^1_kM, that is*

$$(\xi_A)^i_B(x^j, \lambda\, v^j_C) = \lambda^2 (\xi_A)^i_B(x^j, v^j_C) \quad \text{for all} \quad \lambda > 0.$$

From (1.16) we deduce the following proposition.

Proposition 1.14 *A SOPDE $\xi \equiv (\xi_1, \dots , \xi_k)$ is a homogeneous SOPDE, or spray, if and only if we have*

$$v^h_C \frac{\partial (\xi_A)^i_B}{\partial v^h_C} = 2(\xi_A)^i_B\,. \tag{1.18}$$

which is equivalent to $[C, \xi_A] = \xi_A$, $A = 1, \dots , k$.

1.4 Hamiltonian and Lagrangian formalisms

The role played by symplectic manifolds in classical mechanics is here played by the k-symplectic manifolds (see Günther, ([3])). Let $(M, \omega_A, V; 1 \leq A \leq k)$ be a k–symplectic manifold. Let us consider the vector bundle morphism defined by Günther ([3]):

$$
\begin{array}{rccl}
\Omega^\sharp : & T^1_k M & \longrightarrow & T^* M \\
 & (X_1, \ldots, X_k) & \longrightarrow & \Omega^\sharp(X_1, \ldots, X_k) = \displaystyle\sum_{A=1}^k X_A \lrcorner\ \omega_A \,.
\end{array}
\tag{1.19}
$$

Definition 1.15 *Let $H : M \longrightarrow \mathbf{R}$ be a function on M. Any k-vector field $(X_1, \ldots, X_k)$ on M such that*

$$\Omega^\sharp(X_1, \ldots, X_k) = dH$$

will be called an evolution k-vector field on M associated with the Hamiltonian function H.

It should be noticed that in general the solution to the above equation is not unique. Nevertheless, it can be proved [9] that there always exists an *evolution k-vector field* associated with a Hamiltonian function H.

Let (x^i, p^A_i) be a local coordinate system on M. Then we have

Proposition 1.16 *If $(X_1, \ldots, X_k)$ is an integrable evolution k-vector field associated to H then its integral sections*

$$
\begin{array}{rccl}
\sigma : & \mathbf{R}^k & \longrightarrow & M \\
 & (t^B) & \longrightarrow & (\sigma^i(t^B), \sigma^A_i(t^B)),
\end{array}
$$

are solutions of the classical local Hamilton equations associated with a regular multiple integral variational problem [22]:

$$\frac{\partial H}{\partial x^i} = -\sum_{A=1}^k \frac{\partial \sigma^A_i}{\partial t^A}, \quad \frac{\partial H}{\partial p^A_i} = \frac{\partial \sigma^i}{\partial t^A}, \quad 1 \leq i \leq n, 1 \leq \ A \leq k \,. \tag{1.20}$$

Given a Lagrangian function of the form $L = L(x^i, v^i_A)$ one obtains, by using a variational principle, the *generalized Euler-Lagrange equations* for L:

$$\sum_{A=1}^k \frac{d}{dt^A}\left(\frac{\partial L}{\partial v^i_A}\right) - \frac{\partial L}{\partial x^i} = 0, \qquad v^i_A = \frac{\partial x^i}{\partial t^A}. \tag{1.21}$$

Following the ideas of Günther [3], we will describe the above equations (1.21) in terms of the geometry of k-tangent structures. In classical mechanics the symplectic structure of Hamiltonian theory and the tangent structure of Lagrangian theory play complementary roles [13, 14, 15, 16, 17]. Also, that the k-symplectic structures and the k-tangent structures play similarly complementary roles.

First of all, we note that such a L can be considered as a function $L : T^1_k M \to R$ with M a manifold with local coordinates (x^i). Next, we construct a k–symplectic structure on the manifold $T^1_k M$, using its canonical k–tangent structure for each $1 \leq A \leq k$.

Let us consider the 1–forms $(\beta_L)_A = dL \circ J^A$, $1 \leq A \leq k$. In a local coordinate system (x^i, v^i_A) we have

$$(\theta_L)_A = \frac{\partial L}{\partial v^i_A} dx^i, \; 1 \leq A \leq k. \tag{1.22}$$

Definition 1.17 *A Lagrangian L is called regular if and only if*

$$det(\frac{\partial^2 L}{\partial v^i_A \partial v^j_B}) \neq 0, \qquad 1 \leq i, j, \leq n, \quad 1 \leq A, B \leq k\,. \tag{1.23}$$

By introducing the following 2–forms $(\omega_L)_A = -d(\theta_L)_A$, $1 \leq A \leq k$, one can easily prove the following.

Proposition 1.18 $L : T^1_k M \longrightarrow \mathbf{R}$ *is a regular Lagrangian if and only if* $((\omega_L)_1, \ldots, (\omega_L)_k, V)$ *is a k-symplectic structure on $T^1_k M$, where V denotes the vertical distribution of $\tau : T^1_k M \to M$.*

Let $L : T^1_k M \longrightarrow \mathbf{R}$ be a regular Lagrangian and let us consider the k–symplectic structure
$((\omega_L)_1, \ldots, (\omega_L)_k, V)$ on $T^1_k M$ defined by L. Let $\Omega^\sharp_L$ be the morphism defined by this k–symplectic structure

$$\Omega^\sharp_L : T^1_k(T^1_k M) \longrightarrow T^*(T^1_k M).$$

Thus, we can set the following equation:

$$\Omega^\sharp_L(X_1, \ldots, X_k) = dE_L, \tag{1.24}$$

where $E_L = C(L) - L$, and where C is the canonical vector field of the vector bundle $\tau : T^1_k M \to M$.

Proposition 1.19 *Let L be a regular Lagrangian. If $\xi = (\xi_1, \cdots, \xi_k)$ is a solution of (1.24) then it is a SOPDE. In addition if ξ is integrable then the solutions of ξ are solutions of the Euler-Lagrange equations (1.21).*

Proof It is a direct computation in local coordinates using (1.13), (1.14) , (1.22) and (1.23). ■.

2 SOPDEs and nonlinear connections on $T^1_k M$

A nonlinear connection on the vector fiber bundle $\tau : T^1_k M \to M$ is a distribution $H : u \to H_u E$ on $E = T^1_k M$ which is supplementary to the vertical distribution $V : u \to V_u E$, where $VE = Ker(\tau)_*$. The horizontal distribution $u \to H_u E$ may be given by n local vector fields

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_{Ai} \frac{\partial}{\partial v^j_A}, \quad i = 1, \ldots, n. \tag{2.1}$$

Functions N^j_{Ai} are called the local coefficients (or components) of the nonlinear connection, which, for this reason, will be denoted by N. $\left\{\frac{\delta}{\delta x^i}(u), \frac{\partial}{\delta v^i_A}(u)\right\}$ is an adapted (local) basis to the decomposition $T_uE = H_uE \oplus V_uE$, called the adapted basis of the nonlinear connection. Its dual basis is

$$\left\{dx^i, \delta v^i_A = dv^i_A + N^i_{A,j}dx^j\right\}. \tag{2.2}$$

Follow [24] we have $[\delta/\delta x^j, \delta/\delta x^k] = R^i_{Ajk}\partial/\partial v^i_A$ and so, the distribution horizontal H is integrable if and only if $R^i_{Ajk} = 0$.

From the general theory we have ([23], [24]):

Theorem 2.1 *a) The transformation rule, with respect to a change of coordinates, of the components $N^i_{Aj}(x^k, v^k_B)$ of a nonlinear connection N are*

$$\frac{\partial \tilde{x}^k}{\partial x^i}\widetilde{N}^j_{Ak} = N^h_{Ai}\frac{\partial \tilde{x}^j}{\partial x^h} - \frac{\partial^2 \tilde{x}^j}{\partial x^i \partial x^h}v^h_A \tag{2.3}$$

b) Conversely, if the functions $N^i_{Aj}(x^k, v^k_B)$ are given on every domain of local chart on T^1_kM, so that (2.3) holds with respect to (1.3), then there exists a unique nonlinear connection N on whose coefficients are the given functions $N^i_{Aj}(x^k, v^k_B)$.

Using this theorem, we obtain the following results:

Theorem 2.2 *Let $\xi \equiv (\xi_1, \dots, \xi_k)$ be a SOPDE. Then the local functions on T^1_kM defined by*

$$N^i_{Aj} = -\frac{1}{k+1}\sum_{B=1}^{k}\frac{\partial(\xi_A)^i_B}{\partial v^j_B} \tag{2.4}$$

are the local components of a nonlinear connection.

Next, the nonlinear connection N given by (2.4) is called *the nonlinear connection associated to the SOPDE* ξ.

Let us denote $\overset{1}{\xi} = \xi$ and

$$\overset{1}{N}{}^i_{Aj} = -\frac{1}{k+1}\sum_{B=1}^{k}\frac{\partial(\overset{1}{\xi}_A)^i_B}{\partial v^j_B}. \tag{2.5}$$

Then $\overset{1}{N}$ is the nonlinear connection associated to the SOPDE $\overset{1}{\xi}$.

Let us consider the functions

$$(\overset{2}{\xi}_A)^i_B = \frac{1}{2}C((\overset{1}{\xi}_A)^i_B = \frac{1}{2}\sum_{C=1}^{k}\sum_{j=1}^{n}\frac{\partial(\overset{1}{\xi}_A)^i_B}{\partial v^j_C}v^j_C \tag{2.6}$$

on every domain of a local chart. These functions change following (1.15) and therefore $\overset{2}{\xi}$ is a SOPDE. Using (2.4) it obtain the nonlinear connection $\overset{2}{N}$ associated to the SOPDE $\overset{2}{\xi}$.

So, starting with a SOPDE on T_k^1M we obtain a sequence of SOPDE $\left(\overset{m}{\xi}\right)_{m\geq 1}$ and a sequence of nonlinear connection $\left(\overset{m}{N}\right)_{m\geq 1}$, for which we obtain the next results.

Theorem 2.3 *The following assertions are equivalent:*

i) the SOPDE $\overset{1}{\xi}$ is homogeneous (spray).

ii) $\overset{1}{\xi}=\overset{2}{\xi}$.

Theorem 2.4 *If $\overset{1}{\xi}$ is a homogeneous SOPDE then the components N^i_{Aj} of the associated nonlinear connection N are 1-homogeneous functions on the fibres of T_k^1M.*

We remark that the converse of this theorem is not true, in generally (see [25] for the case $k=1$).

Theorem 2.5 *$\overset{1}{N}=\overset{2}{N}$ if and only if the components N^i_{Aj} are 1-homogeneous functions on the fibres of T_k^1M.*

Corollary 2.6 *If $\overset{1}{N}{}^i_{Aj}$ are 1-homogeneous functions on the fibres of T_k^1M, then the sequence $\left(\overset{m}{N}\right)_{m\geq 1}$ is constant, but we can not deduce that $\left(\overset{m}{\xi}\right)_{m\geq 1}$ is constant.*

Corollary 2.7 *If $\overset{1}{\xi}$ is a homogeneous SOPDE, then the sequences $\left(\overset{m}{\xi}\right)_{m\geq 1}$ and $\left(\overset{m}{N}\right)_{m\geq 1}$ are constants.*

2.1 SOPDEs and nonlinear connections associated to a regular Lagrangian on T_k^1M

Using the previously notations, if we have a regular Lagrangian on T_k^1M, following the proposition 1.19 and the generalized Euler-Lagrange equations (1.21), we deduce the next results.

Proposition 2.8 *Let L be a regular Lagrangian. The k-vector field $((\xi_L)_1,\dots,(\xi_L)_k)$ locally given by*

$$((\xi_L)_A)^j_B = -\frac{1}{2}g^{ji}_{AB}\left(\frac{\partial^2 L}{\partial v^i_C \partial x^h}v^h_C - \frac{\partial L}{\partial x^i}\right), \quad g^{AB}_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial v^i_A \partial v^j_B}, \tag{2.7}$$

where g^{ji}_{AB} is the inverse of the matrix g^{AB}_{ji}, is a SOPDE and it is a solution of (1.24). Moreover, if ξ_L is integrable, then its solutions are solutions of the generalized Euler-Lagrange equations (1.21).

Definition 2.9 *The SOPDE $((\xi_L)_1,\dots,(\xi_L)_k)$ from the above proposition will be called the canonical SOPDE associated to the regular Lagrangian L.*

From (2.4) it results that there exists a nonlinear connection N_L on T_k^1M associated to $((\xi_L)_1, \ldots, (\xi_L)_k)$ which depends only by L. N_L will be called *the canonical nonlinear connection* of regular Lagrangian L.

Now we consider the particular case of a regular Lagrangian, $L = F^2$, where F is a Finslerian function on T_k^1M which is defined below.

Definition 2.10 *A function $F : T_k^1M \to \mathbf{R}$, $F = F(x^i, v_A^i)$, which is differentiable on $\widetilde{T_k^1M}$ and continuous on null section of the projection τ is called Finslerian on T_k^1M if*

i) $F > 0$ on T_k^1M,

ii) F is 1-homogeneous on the fibres of T_k^1M,

iii) the matrix with the elements

$$g_{ij}^{AB} = \frac{1}{2}\frac{\partial^2 F^2}{\partial v_A^i \partial v_B^j} \tag{2.8}$$

is positively defined on $\widetilde{T_k^1M}$.

From Proposition 1.14, (1.17), (2.7) and ii), iii) we deduce that if $L = F^2$, then the canonical SOPDE ξ_{F^2} is homogeneous (spray).

Theorem 2.11 *If the Lagrangian is the square of a Finslerian on T_k^1M then*

a) the sequence $\left(\overset{m}{\xi}\right)_{m\geq 1}$ is constant, $\overset{1}{\xi} = \xi_{F^2}$,

b) the sequence $\left(\overset{m}{N}\right)_{m\geq 1}$ is constant, $\overset{1}{N} = N_{F^2}$.

Example: Let $L(x^h, v_D^h) = \dfrac{1}{3}\alpha_{ijk}^{ABC}(x^h)v_A^i v_B^j v_C^k$, where $\alpha_{ijk}^{ABC}(x^1, \ldots, x^n)$ is totally symmetric and the matrix with the elements $g_{ij}^{AB} = \dfrac{1}{2}\dfrac{\partial^2 L}{\partial v_A^i \partial v_B^j} = \alpha_{ijk}^{ABC} v_C^k$ is of rank nk on T_k^1M. Then L is a regular Lagrangian with g_{ij}^{AB} 1-homogeneous functions, g_{AB}^{ij} homogeneous functions of degree -1, according with (1.17). Using the relation (2.5) and Theorem 2.4, the components of the canonical nonlinear connection N_L are homogeneous of degree 1. So, the sequence $\left(\overset{m}{N}\right)_{m\geq 1}$ is constant, but we dont tell that the sequence $\left(\overset{m}{\xi}\right)_{m\geq 1}$ has the same property. This is an example of a regular Lagrangian on T_k^1M which in not equal with the square of a Finslerian.

2.2 Other type of relationship between SOPDEs and nonlinear connections on T_k^1M

Let $\xi \equiv (\xi_1, \ldots, \xi_k)$ be a SOPDE on T_k^1M. Then the functions given by

$$N_{Al}^j = -\frac{1}{2}\frac{\partial(\xi_A)_A^j}{\partial v_A^l} \tag{2.9}$$

are the components of a nonlinear connection N on T^1_kM, determined by ξ. Indeed N^j_{Al} verify (2.3), under (1.3).

From (2.9) and (1.15), we deduce that the functions

$$(\overset{*}{\xi}_B)^j_A = -N^j_{Al}v^l_B \tag{2.10}$$

are the components of a SOPDE $\overset{*}{\xi}$ on T^1_kM.

We can continue and we will obtain a nonlinear connection $\overset{*}{N}$ from $\overset{*}{\xi}$ and so on.

By this procedure one can obtain again a sequence of SOPDEs and a sequence of nonlinear connection on T^1_kM, starting with a given SOPDE (or starting with a given nonlinear connection). But, now we have other results about this sequences.

Proposition 2.12 *i)* $\overset{*}{\xi} = \xi$ *if and only if* $C^A_B\left((\xi_A)^j_A\right) = 2(\xi_B)^j_A$ *for all* $j = 1, \dots, n$, $A, B = 1, \dots, k$;

ii) $\overset{*}{N} = N$ *if and only if* $C_A(N^j_{Al}) = N^j_{Al}$ *for all* $j, l = 1, \dots, n$, $A = 1, \dots, k$.

Remark $\overset{*}{\xi} = \xi$ implies $\overset{*}{N} = N$, but the converse is not true. Only $\overset{*}{N} = N$ implies $\overset{*}{\overset{*}{\xi}} = \overset{*}{\xi}$.

Now, if we done the next definition, then we get an interesting result.

Definition 2.13 *A map* $\Phi : \mathbf{R}^k \mapsto M$ *is said to be horizontal for a nonlinear connection* N *on* T^1_kM *if and only if*

$$(\Phi^{(1)})_*(t)\left(\frac{\partial}{\partial t^A}(t)\right) \in H\left(T_{\Phi^{(1)}(t)}(T^1_kM)\right), \ \forall A = 1, \dots, k. \tag{2.11}$$

Theorem 2.14 *If the SOPDE* ξ *associated to the nonlinear connection* N, $(\xi_A)^j_B = -N^j_{Bl}v^l_A$ *is integrable, then the solutions of* ξ *are horizontals with respect to* N.

Proof Since, for all $A = 1, \dots, k$,

$$(\Phi^{(1)})_*(t)\left(\frac{\partial}{\partial t^A}(t)\right) = \frac{\partial \Phi^i}{\partial t^A}(t)\frac{\delta}{\delta x^i} + \left(\frac{\partial^2\Phi^j}{\partial t^A\partial t^B}(t) + N^j_{Bi}\frac{\partial \Phi^i}{\partial t^A}(t)\right)\frac{\partial}{\partial v^j_B}$$

and taking into account the relationship between ξ and N, we obtain that Φ is a solution of ξ iff (2.11) holds, that is Φ is horizontal of N.

Remark Beginning with a SOPDE ξ we can construct directly a sequence of SOPDEs:

$$\left(\overset{*}{\xi}_A\right)^j_B = \frac{1}{2}C^B_A\left((\xi_B)^j_B\right) = \frac{1}{2}\frac{\partial(\xi_B)^j_B}{\partial v^h_B}v^h_A\,. \tag{2.12}$$

After this, we can obtain for each SOPDE of sequence a nonlinear connection by (2.9). It is the same construction as that presented above in (2.9) and (2.10).

Finally, starting with a SOPDE ξ we can construct two sequence of SOPDEs. The first sequence by the method from (2.6) and the second by the method presented above (2.12). For each sequence of SOPDE we can obtain two sequences of nonlinear connections. The first sequence is obtained by the method from (2.4) and the second by the above method (2.9).

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Author's address:

Florian Munteanu
Department of Applied Mathematics, University of Craiova,
Al. I. Cuza st., no. 13, 200585-Craiova, Romania,
email: munteanufm@central.ucv.ro