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Second Order Partial Differential Equations in Hilbert Spaces

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Chapter 1

Gaussian measures

This chapter is devoted to some basic results on Gaussian measures on separable Hilbert spaces, including the Cameron-Martin and Feldman-Hajek formulae. The greater part of the results are presented with complete proofs.

1.1 Introduction and preliminaries

We are given a real separable Hilbert space H (with norm $|\cdot|$ and inner product $\langle \cdot, \cdot \rangle$). The space of all linear bounded operators from H into H, equipped with the operator norm $\|\cdot\|$, will be denoted by $L(H)$. If $T \in L(H)$, then T^* is the adjoint of T. Moreover, by $L^+(H)$ we shall denote the subset of $L(H)$ consisting of all nonnegative symmetric operators. Finally, we shall denote by $\mathcal{B}(H)$ the σ-algebra of all Borel subsets of H.

Before introducing Gaussian measures we need some results about trace class and Hilbert-Schmidt operators.

A linear bounded operator $R \in L(H)$ is said to be of *trace class* if there exist two sequences (a_k), (b_k) in H such that

$$Ry = \sum_{k=1}^{\infty} \langle y, a_k \rangle b_k, \qquad y \in H, \tag{1.1.1}$$

and

$$\sum_{k=1}^{\infty} |a_k|\,|b_k| < +\infty. \tag{1.1.2}$$

Notice that if (1.1.2) holds then the series in (1.1.1) is norm convergent. Moreover, it is not difficult to show that R is compact.

We shall denote by $L_1(H)$ the set of all operators of $L(H)$ of trace class. $L_1(H)$, endowed with the usual linear operations, is a Banach space with the norm

$$\|R\|_{L_1(H)} = \inf \left\{ \sum_{k=1}^{\infty} |a_k|\,|b_k| : \; Ry = \sum_{k=1}^{\infty} \langle y, a_k\rangle b_k, \;\; y \in H, \; (a_k), (b_k) \subset H \right\}.$$

We set $L_1^+(H) = L^+(H) \cap L_1(H)$. If an operator R is of trace class then its trace, Tr R, is defined by the formula

$$\text{Tr } R = \sum_{j=1}^{\infty} \langle Re_j, e_j\rangle,$$

where (e_j) is an orthonormal and complete basis on H. Notice that, if R is given by (1.1.1), we have

$$\text{Tr } R = \sum_{j=1}^{\infty} \langle a_j, b_j\rangle.$$

Thus the definition of the trace is independent on the choice of the basis and

$$|\text{Tr } R| \;\leq \|R\|_{L_1(H)}.$$

Proposition 1.1.1 *Let $S \in L_1(H)$ and $T \in L(H)$. Then*

(i) $ST, TS \in L_1(H)$ *and*

$$\|TS\|_{L_1(H)} \leq \|S\|_{L_1(H)}\|T\|, \; \|ST\|_{L_1(H)} \leq \|S\|_{L_1(H)}\|T\|.$$

(ii) $\text{Tr}(ST) = \text{Tr}(TS)$.

Proof. (i) Assume that $Sy = \sum_{k=1}^{\infty} \langle y, a_k\rangle b_k$, $y \in H$, where $\sum_{k=1}^{\infty} |a_k||b_k| < +\infty$. Then

$$STy = \sum_{k=1}^{\infty} \langle y, T^* a_k\rangle b_k, \; y \in H,$$

and

$$\sum_{k=1}^{\infty} |T^* a_k||b_k| \leq \|T\| \sum_{k=1}^{\infty} |a_k||b_k|.$$

It is therefore clear that $ST \in L_1(H)$ and $\|ST\|_{L_1(H)} \le \|S\|_{L_1(H)}\|T\|$. Similarly we can prove that $\|TS\|_{L_1(H)} \le \|S\|_{L_1(H)}\|T\|$.

(ii) From part (i) it follows that

$$\mathrm{Tr}(ST) = \sum_{k=1}^{\infty}\langle b_k, T^* a_k\rangle = \sum_{k=1}^{\infty}\langle Tb_k, a_k\rangle.$$

In the same way Tr $(TS) = \sum_{k=1}^{\infty}\langle a_k, Tb_k\rangle$, and the conclusion follows. □

We say that $R \in L(H)$ is of Hilbert-Schmidt class if there exists an orthonormal and complete basis (e_k) in H such that

$$\sum_{k,j=1}^{\infty} |\langle Se_k, e_j\rangle|^2 < +\infty. \tag{1.1.3}$$

If (1.1.3) holds then we have

$$\sum_{k=1}^{\infty}|Se_k|^2 = \sum_{k,j=1}^{\infty} |\langle Se_k, e_j\rangle|^2 = \sum_{k,j=1}^{\infty} |\langle e_k, S^* e_j\rangle|^2 = \sum_{j=1}^{\infty}|S^* e_j|^2. \tag{1.1.4}$$

Now if (f_k) is another complete orthonormal basis in H, we have

$$\sum_{m=1}^{\infty} |Sf_m|^2 = \sum_{m,n=1}^{\infty} |\langle Sf_m, e_n\rangle|^2 = \sum_{m,n=1}^{\infty} |\langle f_m, S^* e_n\rangle|^2 = \sum_{n=1}^{\infty} |S^* e_n|^2.$$

Thus, by (1.1.4) we see that the assertion (1.1.3) is independent of the choice of the complete orthonormal basis (e_k). We shall denote by $L_2(H)$ the space of all Hilbert-Schmidt operators on H. $L_2(H)$, endowed with the norm

$$\|S\|^2_{L_2(H)} = \sum_{k,j=1}^{\infty} |\langle Se_k, e_j\rangle|^2 = \sum_{k=1}^{\infty}|Se_k|^2,$$

is a Banach space.

Proposition 1.1.2 *Let $S, T \in L_2(H)$. Then $ST \in L_1(H)$ and*

$$\|ST\|_{L_1(H)} \le \|S\|_{L_2(H)}\|T\|_{L_2(H)}. \tag{1.1.5}$$

Proof. Let (e_k) be a complete and orthonormal basis in H, then

$$\begin{aligned} Ty &= \sum_{k=1}^{\infty}\langle Ty, e_k\rangle e_k = \sum_{k=1}^{\infty}\langle y, T^* e_k\rangle e_k, \\ STy &= \sum_{k=1}^{\infty}\langle y, T^* e_k\rangle S e_k. \end{aligned}$$

Consequently $ST \in L_1(H)$ and

$$\begin{aligned} \|ST\|_{L_1(H)} &\leq \sum_{k=1}^{\infty} |T^* e_k|\,|S e_k| \leq \left(\sum_{k=1}^{\infty}|T^* e_k|^2\right)^{1/2}\left(\sum_{k=1}^{\infty}|S e_k|^2\right)^{1/2} \\ &= \|T\|_{L_2(H)}\|S\|_{L_2(H)}. \end{aligned}$$

Therefore the conclusion follows. □

Warning. If S and T are bounded operators, and ST is of trace class then in general TS is not, as the following example, provided by S. Peszat [183], shows.

Define two linear operators S and T on the product space $H \times H$, by

$$S = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}, \quad T = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$ST = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}, \quad TS = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix},$$

and it is enough to take B of trace class and A not of trace class. □

We have also the following result, see e.g. A. Pietsch [187].

Proposition 1.1.3 *Assume that S is a compact self-adjoint operator, and that (λ_k) are its eigenvalues (repeated according to their multiplicity).*

(i) $S \in L_1(H)$ if and only if $\sum_{k=1}^{\infty}|\lambda_k| < +\infty$. Moreover $\|S\|_{L_1(H)} = \sum_{k=1}^{\infty}|\lambda_k|$, and $\mathrm{Tr}\, S = \sum_{k=1}^{\infty}\lambda_k$.

(ii) $S \in L_2(H)$ if and only if $\sum_{k=1}^{\infty}|\lambda_k|^2 < +\infty$. Moreover

$$\|S\|_{L_2(H)} = \left(\sum_{k=1}^{\infty}|\lambda_k|^2\right)^{1/2}.$$

More generally let S be a compact operator on H. Denote by (λ_k) the sequence of all positive eigenvalues of the operator $(S^*S)^{1/2}$, repeated according to their multiplicity. Denote by $L_p(H)$, $p > 0$, the set of all operators S such that

$$\|S\|_{L_p(H)} = \left(\sum_{k=1}^{\infty} \lambda_k^p\right)^{1/p} < +\infty. \tag{1.1.6}$$

Operators belonging to $L_1(H)$ and $L_2(H)$ are precisely the trace class and the Hilbert-Schmidt operators.

The following result holds, see N. Dunford and J. T. Schwartz [107].

Proposition 1.1.4 *Let $S \in L_p(H)$, $T \in L_q(H)$ with $p > 0, q > 0$. Then $ST \in L_r(H)$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, and*

$$\|TS\|_{L_r(H)} \le 2^{1/r} \|S\|_{L_p(H)} \|T\|_{L_q(H)}. \tag{1.1.7}$$

1.2 Definition and first properties of Gaussian measures

1.2.1 Measures in metric spaces

If E is a metric space, then $\mathcal{B}(E)$ will denote the Borel σ-algebra, that is the smallest σ-algebra of subsets of E which contains all closed (open) subsets of E.

Let metric spaces E_1, E_2 be equipped with σ-fields $\mathcal{E}_1, \mathcal{E}_2$ respectively. Measurable mappings $X : E_1 \to E_2$ will often be called *random variables*. If μ is a measure on $(E_1, \mathcal{E}_1)$, then its image by the transformation X will be denoted by $X \circ \mu$:

$$X \circ \mu(A) = \mu(X^{-1}(A)), \quad A \in \mathcal{E}_2.$$

We call $X \circ \mu$ the *law* or the *distribution* of X, and we set $X \circ \mu = \mathcal{L}(X)$.

If ν and μ are two finite measures on $(E, \mathcal{E})$ such that $\Gamma \in \mathcal{E}$, $\mu(\Gamma) = 0$ implies $\nu(\Gamma) = 0$ then one writes $\nu << \mu$ and one says that ν is *absolutely continuous* with respect to μ. If there exist $A, B \in \mathcal{E}$ such that $A \cap B = \emptyset$, $\mu(A) = \nu(B) = 1$, one says that μ and ν are *singular*.

If $\nu << \mu$ then by the Radon-Nikodým theorem there exists $g \in L^1(E, \mathcal{E}, \mu)$ nonnegative such that

$$\nu(\Gamma) = \int_\Gamma g(x)\mu(dx), \quad \Gamma \in \mathcal{E}.$$

The function g is denoted by $\frac{d\nu}{d\mu}$.

If $\nu << \mu$ and $\mu << \nu$ then one says that μ and ν are *equivalent* and writes $\mu \sim \nu$.

We have the following change of variable formula. If φ is a nonnegative measurable real function on E_2, then

$$\int_{E_1} \varphi(X(x))\mu(dx) = \int_{E_2} \varphi(y) X \circ \mu(dy). \tag{1.2.1}$$

Let μ and ν be two measures on a separable Hilbert space H; if $T \circ \mu = T \circ \nu$ for any linear operator $T : H \to \mathbb{R}^n$, $n \in \mathbb{N}$, then $\mu = \nu$.

Random variables $X_1, \ldots, X_n$ are said to be *independent* if

$$\mathcal{L}(X_1, \ldots, X_n) = \mathcal{L}(X_1) \times \cdots \times \mathcal{L}(X_n).$$

A family of random variables $(X_\alpha)_{\alpha \in A}$ is said to be independent, if any finite subset of the family is independent.

Probability measures on a separable Hilbert space H will always be regarded as defined on $\mathcal{B}(H)$. If μ is a probability measure on H, then its Fourier transform is defined by

$$\hat{\mu}(\lambda) = \int_H e^{i\langle \lambda, x \rangle} \mu(dx), \ \lambda \in H;$$

$\hat{\mu}$ is called the *characteristic function* of μ. One can show that if the characteristic functions of two measures are identical, then the measures are identical as well.

1.2.2 Gaussian measures

We first define Gaussian measures on $\mathbb{R}$. If $a \in \mathbb{R}$ we set

$$N_{a,0}(dx) = \delta_a(dx),$$

where δ_a is the Dirac measure at a. If moreover $\lambda > 0$ we set

$$N_{a,\lambda}(dx) = \frac{1}{\sqrt{2\pi\lambda}}\, e^{-\frac{(x-a)^2}{2\lambda}} dx.$$

The Fourier transform of $N_{a,\lambda}$ is given by

$$\widehat{N_{a,\lambda}}(h) = \int_{\mathbb{R}} e^{ihx} N_{a,\lambda}(dx) = e^{iah - \frac{1}{2}\lambda h^2}, \ h \in \mathbb{R}.$$

More generally we show now that in an arbitrary separable Hilbert space and for arbitrary $Q \in L_1^+(H)$ there exists a unique measure $N_{a,Q}$ such that

$$\widehat{N_{a,\lambda}}(h) = \int_H e^{i\langle h,x\rangle} N_{a,Q}(dx) = e^{i\langle h,x\rangle - \frac{1}{2}\langle Qh,h\rangle},\ h \in H.$$

Let in fact $Q \in L_1^+(H)$. Then there exist a complete orthonormal system (e_k) on H and a sequence of nonnegative numbers (λ_k) such that $Qe_k = \lambda_k e_k,\ k \in \mathbb{N}$. We set $x_h = \langle x, e_h\rangle, h \in \mathbb{N}$, and $P_n x = \sum_{k=1}^n x_k e_k, x \in H,\ n \in \mathbb{N}$. Let us introduce an isomorphism γ from H into ℓ^2: ([1])

$$x \in H \to\ \gamma(x) = (x_k) \in \ell^2.$$

In the following we shall always identify H with ℓ^2. In particular we shall write $P_n x = (x_1, ..., x_n),\ x \in \ell^2$.

A subset I of H of the form $I = \{x \in H :\ (x_1,\ \dots\ , x_n) \in B\}$, where $B \in \mathcal{B}(\mathbb{R}^n)$, is said to be *cylindrical*. It is easy to see that the σ-algebra generated by all cylindrical subsets of H coincides with $\mathcal{B}(H)$.

Theorem 1.2.1 *Let $a \in H$, $Q \in L_1^+(H)$. Then there exists a unique probability measure μ on $(H, \mathcal{B}(H))$ such that*

$$\int_H e^{i\langle h,x\rangle}\mu(dx) = e^{i\langle a,h\rangle} e^{-\frac{1}{2}\langle Qh,h\rangle},\ h \in H. \tag{1.2.2}$$

Moreover μ is the restriction to H (identified with ℓ^2) of the product measure

$$\mathop{\times}_{k=1}^{\infty} \mu_k = \mathop{\times}_{k=1}^{\infty} N_{a_k,\lambda_k},$$

defined on $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$. ([2])

We set $\mu = N_{a,Q}$, and call a the *mean* and Q the *covariance operator* of μ. Moreover $N_{0,Q}$ will be denoted by N_Q.

Proof of Theorem 1.2.1. Since a characteristic function uniquely determines the measure, we have only to prove existence.

Let us consider the sequence of Gaussian measures (μ_k) on $\mathbb{R}$ defined as $\mu_k = N_{a_k,\lambda_k},\ k \in \mathbb{N}$, and the product measure $\mu = \mathop{\times}_{k=1}^{\infty} \mu_k$ in $\mathbb{R}^\infty$, see e.g

[1] For any $p \geq 1$, we denote by ℓ^p the Banach space of all sequences (x_k) of real numbers such that $|x|_p := (\sum_{k=1}^\infty |x_k|^p)^{1/p} < +\infty$.

[2] We shall consider $\mathbb{R}^\infty$ as a metric space with the distance $d(x,y) := \sum_{k=1}^\infty 2^{-k} \frac{|x_k - y_k|}{1+|x_k-y_k|},\ x, y \in \mathbb{R}^\infty$

P. R. Halmos [141, §38.B]. We want to prove that μ is concentrated on ℓ^2, (that it is clearly a Borel subset of $\mathbb{R}^\infty$). For this it is enough to show that

$$\int_{\ell^\infty} |x|_{\ell^2}^2 \; \mu(dx) < +\infty. \tag{1.2.3}$$

We have in fact, by the monotone convergence theorem,

$$\begin{aligned}\int_{\mathbb{R}^\infty} |x|_{\ell^2}^2 \mu(dx) &= \sum_{k=1}^\infty \int_{\mathbb{R}^\infty} x_k^2 \; \mu(dx) = \sum_{k=1}^\infty \left(\int_{\mathbb{R}} (x_k - a_k)^2 \mu_k(dx) + a_k^2 \right) \\ &= \sum_{k=1}^\infty (\lambda_k + a_k^2) = \text{ Tr } Q + |a|^2 < +\infty.\end{aligned}$$

Now we consider the restriction of μ to ℓ^2, which we still denote by μ. We have to prove that (1.2.2) holds. Setting $\nu_n = \prod_{k=1}^n \mu_k$, we have

$$\int_{\ell^2} e^{i\langle x,h\rangle} \mu(dx) = \lim_{n\to\infty} \int_{\ell^2} e^{i\langle P_n h, P_n x\rangle} \mu(dx)$$

$$= \lim_{n\to\infty} \int_{\mathbb{R}^n} e^{i\langle P_n h, P_n x\rangle} \nu_n(dx) = \lim_{n\to\infty} e^{i\langle P_n h, P_n a\rangle - \frac{1}{2}\langle Q P_n h, P_n h\rangle}$$

$$= e^{i\langle h,a\rangle - \frac{1}{2}\langle Qh,h\rangle}. \; \Box$$

If the law of a random variable is a Gaussian measure, then the random variable is called *Gaussian*. It easily follows from Theorem 1.2.1 that a random variable X with values in H is Gaussian if and only if for any $h \in H$ the real valued random variable $\langle h, X\rangle$ is Gaussian.

Remark 1.2.2 From the proof of Theorem 1.2.1 it follows that

$$\int_H |x|^2 N_{a,Q}(dx) = \text{ Tr } Q + |a|^2. \tag{1.2.4}$$

Proposition 1.2.3 *Let $T \in L(H)$, and $a \in H$, and let $\Gamma x = Tx + a$, $x \in H$. Then $\Gamma \circ N_{m,Q} = N_{Tm+a, TQT^*}$.*

Proof. Notice that, by the change of variables formula (1.2.1), we have

$$\int_H e^{i\langle \lambda, y\rangle} \Gamma \circ N_{m,Q}(dy) = \int_H e^{i\langle \lambda, \Gamma x\rangle} N_{m,Q}(dy)$$

$$= \int_H e^{i\langle \lambda, Tx+a\rangle} N_{m,Q}(dy) = e^{i\langle \lambda, a\rangle} e^{i\langle T^*\lambda, m\rangle - \frac{1}{2}\langle QT^*\lambda, T^*\lambda\rangle}.$$

This shows the result. $\Box$

1.2.3 Computation of some Gaussian integrals

We are here given a Gaussian measure $N_{a,Q}$. We set

$$L^2(H, N_{a,Q}) = L^2(H, \mathcal{B}(H), N_{a,Q}).$$

The following identities can be easily proved, using (1.2.2).

Proposition 1.2.4 *We have*

$$\int_H x N_{a,Q}(dx) = a, \tag{1.2.5}$$

$$\int_H \langle x-a, y\rangle\langle x-a, z\rangle N_{a,Q}(dx) = \langle Qy, z\rangle. \tag{1.2.6}$$

$$\int_H |x-a|^2 N_{a,Q}(dx) = \operatorname{Tr} Q. \tag{1.2.7}$$

Proof. We prove as instance (1.2.6). We have

$$\int_H x N_{a,Q}(dx) = \lim_{n\to\infty} \int_H P_n x N_{a,Q}(dx).$$

But

$$\int_H P_n x N_{a,Q}(dx) = (2\pi)^{-n/2} \prod_{k=1}^n \int_{\mathbb{R}} x_k \lambda_k^{-1/2} e^{-\frac{(x_k-a_k)^2}{2\lambda_k}} dx_k = a_k,$$

and the conclusion follows. □

Proposition 1.2.5 *For any $h \in H$, the exponential function E_h, defined as*

$$E_h(x) = e^{\langle h,x\rangle}, \quad x \in H,$$

belongs to $L^p(H, N_{a,Q})$, $p \geq 1$, and

$$\int_H e^{\langle h,x\rangle} N_{a,Q}(dx) = e^{\langle a,h\rangle} e^{\frac{1}{2}\langle Qh,h\rangle}. \tag{1.2.8}$$

Moreover the subspace of $L^2(H, N_{a,Q})$ spanned by all E_h, $h \in H$, is dense on $L^2(H, N_{a,Q})$.

Proof. We have

$$\int_H e^{\langle P_n h, P_n x\rangle} N_{a,Q}(dx) = e^{\langle P_n a, P_n h\rangle} e^{\frac{1}{2}\langle QP_n h, P_n h\rangle}.$$

Letting n tend to 0 this gives (1.2.8).

Let us prove the last statement. Let $\varphi \in L^2(H, N_{a,Q})$ be such that

$$\int_H e^{\langle h,x\rangle}\varphi(x)N_{a,Q}(dx) = 0, \quad h \in H.$$

Denote by φ^+ and φ^- the positive and negative parts of φ. Then

$$\int_H e^{\langle h,x\rangle}\varphi^+(x)N_{a,Q}(dx) = \int_H e^{\langle h,x\rangle}\varphi^-(x)N_{a,Q}(dx), \quad h \in H.$$

Let us define two measures

$$\mu(dx) = \varphi^+(x)N_{a,Q}(dx), \quad \nu(dx) = \varphi^-(x)N_{a,Q}(dx).$$

Then μ and ν are finite measures such that

$$\int_H e^{\langle h,x\rangle}\mu(dx) = \int_H e^{\langle h,x\rangle}\nu(dx), \ h \in H.$$

Let T be any linear transformation from H into $\mathbb{R}^n$, $n \in \mathbb{N}$. Then for any $\lambda \in \mathbb{R}^n$

$$\begin{aligned}\int_{\mathbb{R}^n} e^{\langle\lambda,z\rangle}T\circ\mu(dz) &= \int_H e^{\langle\lambda,Tx\rangle}\mu(dx) = \int_H e^{\langle T^*\lambda,\rangle>}\mu(dx)\\ &= \int_H e^{\langle T^*\lambda,x\rangle}\nu(dx) = \int_{\mathbb{R}^n} e^{\langle\lambda,z\rangle}T\circ\nu(dz).\end{aligned}$$

By a well known finite dimensional result $T\circ\mu = T\circ\nu$. Consequently measures μ and ν are identical and so $\varphi = 0$. □

1.2.4 The reproducing kernel

Here we are given an operator $Q \in L_1^+(H)$. We denote as before by (e_k) a complete orthonormal system in H and by (λ_k) a sequence of positive numbers such that $Qe_k = \lambda_k e_k,\ k \in \mathbb{N}$.

The subspace $Q^{1/2}(H)$ is called the *reproducing kernel* of the measure N_Q. If Ker $Q = \{0\}$, $Q^{1/2}(H)$ is dense on H. In fact, if $x_0 \in H$ is such that $\langle Q^{1/2}h, x_0\rangle = 0$ for all $h \in H$, we have $Q^{1/2}x_0 = 0$ and so $Qx_0 = 0$, which yields $x_0 = 0$.

Let Ker $Q = \{0\}$. We are now going to introduce an isomorphism W from H into $L^2(H, N_Q)$ that will play an important rôle in the following. The isomorphism W is defined by

$$f \in Q^{1/2}(H) \to W_f \in L^2(H, N_Q), \quad W_f(x) = \langle Q^{-1/2}f, x\rangle, \ x \in H.$$

By (1.2.7) it follows that

$$\int_H W_f(x)W_g(x)N_Q(dx) = \langle f, g\rangle,\ f, g \in H.$$

Thus W is an isometry and it can be uniquely extended to all of H. It will be denoted by the same symbol. For any $f \in H$, W_f is a real Gaussian random variable $N_{|f|^2}$.

More generally, for arbitrary elements $f_1, ..., f_n$, $(W_{f_1}, ..., W_{f_n})$ is a Gaussian vector with mean 0 and covariance matrix $((\langle f_i, f_j\rangle))$. If Ker $Q \neq \{0\}$ then the trasformation $f \to W_f$ can be defined in exactly the same way but only for $f \in H_0 = \overline{Q^{1/2}(H)}$. We will write in some cases $\langle Q^{-1/2}y, f\rangle$ instead of $W_f(y)$.

The proof of the following proposition is left as an exercise to the reader.

Proposition 1.2.6 *For any orthonormal sequence (f_n) in H, the family*

$$1,\ W_{f_n},\ W_{f_k}W_{f_l},\ 2^{-1/2}\left(W^2_{f_m} - 1\right),\ m, n, k, l \in \mathbb{N},\ k \neq l,$$

is orthonormal in $L^2(H, N_Q)$.

Next we consider the function $f \to e^{W_f}$.

Proposition 1.2.7 *The transformation $f \to e^{W_f}$ acts continuously from H into $L^2(H, N_Q)$, and*

$$\begin{aligned} \int_H e^{W_f(x)}N_Q(dx) &= e^{\frac{1}{2}|f|^2}, \\ \int_H e^{i\ \lambda W_f(x)}N_Q(dx) &= e^{-\frac{1}{2}\lambda^2|f|^2},\ \lambda \in \mathbb{R}. \end{aligned} \tag{1.2.9}$$

Proof. Since W_f is Gaussian with law $N_{0,|f|^2}$, (1.2.9) follows. Moreover, taking into account (1.2.8) it follows that

$$\int_H \left[e^{W_f} - e^{W_g}\right]^2 dN_Q = \int_H \left[e^{2W_f} - 2e^{W_{f+g}} + e^{2W_g}\right] dN_Q$$

$$= e^{2|f|^2} - 2e^{\frac{1}{2}|f+g|^2} + e^{2|g|^2} = \left[e^{|f|^2} - e^{|g|^2}\right]^2 + 2e^{|f|^2+|g|^2}\left[1 - e^{-\frac{1}{2}|f-g|^2}\right],$$

which shows that W_f is locally uniformly continuous on H. □

Let us define the determinant of $1 + S$ where S is a compact self-adjoint operator in $L_1(H)$:

$$\det\,(1 + S) = \prod_{k=1}^{\infty}(1 + s_k),$$

where (s_k) is the sequence of eigenvalues of S (repeated according to their multiplicity).

Proposition 1.2.8 *Assume that M is a symmetric operator such that $Q^{1/2}MQ^{1/2} < 1$,* (3) *and let $b \in H$. Then*

$$\int_H \exp\left\{\frac{1}{2}\langle My, y\rangle + \langle b, y\rangle\right\} N_Q(dy)$$

$$= \left[\det(1 - Q^{1/2}MQ^{1/2})\right]^{-1/2} \exp\left\{\frac{1}{2}|(1 - Q^{1/2}MQ^{1/2})^{-1/2}Q^{1/2}b|^2\right\}. \tag{1.2.10}$$

Proof. Let (g_n) be an orthonormal basis for the operator $Q^{1/2}MQ^{1/2}$, and let (γ_n) be the sequence of the corresponding eigenvalues.

Claim 1. We have

$$\langle b, x\rangle = \sum_{k=1}^{\infty} \langle Q^{1/2}b, g_n\rangle W_{g_n}(x), \ N_Q\text{-a.e.}$$

Claim 2. We have

$$\langle Mx, x\rangle = \sum_{n=1}^{\infty} \gamma_n |W_{g_n}(x)|^2, \ N_Q\text{-a.e},$$

the series being convergent in $L^1(H, N_Q)$.

We shall only prove the more difficult second claim.

Let $P_N = \sum_{k=1}^{N} e_k \otimes e_k$. (4) Then for any $x \in H$ we have

$$\begin{aligned}
\langle MP_N x, P_N x\rangle &= \langle (Q^{1/2}MQ^{1/2})Q^{-1/2}P_N x, Q^{-1/2}P_N x\rangle \\
&= \sum_{n=1}^{\infty} \langle (Q^{1/2}MQ^{1/2})Q^{-1/2}P_N x, g_n\rangle \langle Q^{-1/2}P_N x, g_n\rangle \\
&= \sum_{n=1}^{\infty} \gamma_n |\langle Q^{-1/2}P_N x, g_n\rangle|^2.
\end{aligned}$$

Consequently, for each fixed x

$$\langle MP_N x, P_N x\rangle = \sum_{n=1}^{\infty} \gamma_n |W_{P_N g_n}|^2, \ N \in \mathbb{N}.$$

[3] This means that $\langle Q^{1/2}MQ^{1/2}x, x\rangle < |x|^2$ for any $x \in H$ different from 0.

[4] We rember that (e_k) is the sequence of eigenvectors of Q.

Moreover for each $L \in \mathbb{N}$

$$\int\limits_H \left| \langle MP_N x, P_N x\rangle - \sum_{n=1}^{L} \gamma_n |W_{P_N g_n}|^2 \right| N_Q(dx)$$

$$\leq \sum_{n=L+1}^{\infty} |\gamma_n| \int_H |W_{P_N g_n}|^2 N_Q(dx)$$

$$= \sum_{n=L+1}^{\infty} |\gamma_n| \, |P_N g_n|^2 \leq \sum_{n=L+1}^{\infty} |\gamma_n|.$$

As $N \to \infty$ then $P_N x \to x$ and $W_{P_N g_n} \to W_{g_n}$ in $L^2(H, N_Q)$. Passing to subsequences if needed, and using the Fatou lemma, we see that

$$\int\limits_H \left| \langle Mx, x\rangle - \sum_{n=1}^{L} \gamma_n |W_{g_n}|^2 \right| N_Q(dx) \leq \sum_{n=L+1}^{\infty} |\gamma_n|.$$

Therefore the claim is proved.

By the claims it follows that

$$\exp\left\{ \frac{1}{2}\langle Mx, x\rangle + \langle b, x\rangle \right\}$$

$$= \lim_{L\to\infty} \exp\left\{ \sum_{n=1}^{L} \frac{1}{2}\gamma_n |W_{g_n}(x)|^2 + \langle Q^{1/2}b, g_n\rangle W g_n(x) \right\},$$

with a.e. convergence with respect to N_Q for a suitable subsequence. Using the fact that $(W g_n)$ are independent Gaussian random variables, we obtain, by a direct calculation, for $p \geq 1$,

$$\int\limits_H \exp\left\{ p \sum_{n=1}^{L} \frac{1}{2}\gamma_n |W_{g_n}(x)|^2 + p\langle Q^{1/2}b, g_n\rangle W g_n(x) \right\} N_Q(dx)$$

$$= \left[\prod_{n=1}^{L} (1 - p\gamma_n) \right]^{-1/2} \exp\left\{ \frac{1}{2} \sum_{n=1}^{\infty} \frac{|\langle Q^{1/2}b, g_n\rangle|^2}{1 - p\gamma_n} \right\}.$$

Since $\gamma_n < 1$, and $\sum_{n=1}^\infty |\gamma_n| < \infty$, there exists $p > 1$ such that $p\gamma_n < 1$, for all $n \in \mathbb{N}$. Therefore

$$\lim_{L\to\infty} \prod_{n=1}^{L} (1 - p\gamma_n)^{-1/2} \exp\left\{ \frac{1}{2} \frac{|\langle Q^{1/2}b, g_n\rangle|^2}{1 - p\gamma_n} \right\}$$

$$= \left[\prod_{n=1}^{\infty} (1 - p\gamma_n) \right]^{-1/2} \exp\left\{ \frac{1}{2} \sum_{n=1}^{\infty} \frac{|\langle Q^{1/2}b, g_n\rangle|^2}{1 - p\gamma_n} \right\}.$$

So the sequence $\left(\exp\left\{ \sum_{n=1}^{L} \left[\frac{1}{2}\gamma_n |W_{g_n}(x)|^2 + \langle Q^{1/2}b, g_n\rangle W_{g_n}(x) \right] \right\} \right)$ is uniformly integrable. Consequently, passing to the limit, we find

$$\int_H \exp\{1/2\ \langle My, y\rangle + \langle b, y\rangle\}\, N_Q(dy)$$

$$= \lim_{L\to\infty} \int_H \exp\left\{ \sum_{n=1}^{L} \left[1/2\ \gamma_n |W_{g_n}(x)|^2 + \langle Q^{1/2}b, g_n\rangle W_{g_n}(x) \right] \right\} N_Q(dx)$$

$$= \lim_{L\to\infty} \prod_{n=1}^{L} (1 - \gamma_n)^{-1/2} \exp\left\{ \frac{1}{2} \frac{|\langle Q^{1/2}b, g_n\rangle|^2}{1 - \gamma_n} \right\}$$

$$= \prod_{n=1}^{\infty} (1 - \gamma_n)^{-1/2} \exp\left\{ \frac{1}{2} \frac{|\langle Q^{1/2}b, g_n\rangle|^2}{1 - \gamma_n} \right\}$$

$$= \left(\det(1 - Q^{1/2}MQ^{1/2}) \right)^{-1/2} \exp\left\{ \frac{1}{2} |(1 - Q^{1/2}MQ^{1/2})^{-1/2} Q^{1/2}b|^2 \right\}. \square$$

Remark 1.2.9 It follows from the proof of the proposition that

$$\langle Mx, x\rangle = \sum_{k=1}^{\infty} \gamma_n W_{g_n}^2(x) = \sqrt{2} \sum_{k=1}^{\infty} \gamma_n \left[2^{-1/2}(W_{g_n}^2(x) - 1) \right] + \sum_{k=1}^{\infty} \gamma_n,$$

and so, by Proposition 1.2.6, we have

$$\begin{aligned}\int_H [\langle Mx,x\rangle]^2 N_Q(dx) &= 2\sum_{k=1}^{\infty}\gamma_n^2 + \left(\sum_{k=1}^{\infty}\gamma_n\right)^2\\ &= 2\|Q^{1/2}MQ^{1/2}\|^2_{L_2(H)} + (\text{Tr}\, Q^{1/2}MQ^{1/2})^2\\ &< +\infty.\end{aligned}$$

Proposition 1.2.10 *Let $T \in L_1(H)$. Then there exists the limit*

$$\langle TQ^{-1/2}y, Q^{-1/2}y\rangle := \lim_{n\to\infty}\langle TQ^{-1/2}P_n y, Q^{-1/2}P_n y\rangle,\ N_Q\text{-a.e.},$$

where $P_n = \sum_{k=1}^n e_k \otimes e_k$.

Moreover we have the following expansion in $L^2(H, N_Q)$:

$$\begin{aligned}\langle TQ^{-1/2}y, Q^{-1/2}y\rangle = \sum_{n=1}^{\infty}\langle Tg_n, g_n\rangle + \sum_{m\neq n=1}^{\infty}\langle Tg_n, g_m\rangle W_{g_n}W_{g_m}\\ \times\sqrt{2}\,\sum_{n=1}^{\infty}\langle Tg_n, g_n\rangle\left[2^{-1/2}\left(W_{g_n}^2 - 1\right)\right].\quad (1.2.11)\end{aligned}$$

The proof of the following result is similar to that of Claim 2 in the proof of Proposition 1.2.8 and it is left to the reader.

Proposition 1.2.11 *Assume that M is a symmetric trace-class operator such that $M < 1$,* ([5]) *and $b \in H$. Then*

$$\begin{aligned}\int_H \exp\left\{1/2\ \langle MQ^{-1/2}y, Q^{-1/2}y\rangle + \langle b, Q^{-1/2}y\rangle\right\} N_Q(dy)\\ = (\det(1-M))^{-1/2}\ e^{\frac{1}{2}|(1-M)^{-1/2}b|^2}.\quad (1.2.12)\end{aligned}$$

1.3 Absolute continuity of Gaussian measures

We consider here two Gaussian measures μ, ν. We want to prove the Feldman-Hajek theorem , that is they are either singular or equivalent.

[5]That is $\langle Mx, x\rangle < |x|^2$ for all $x \neq 0$.

In §1.3.1 we recall some results on equivalence of measures on $\mathbb{R}^\infty$ including the Kakutani theorem. In §1.3.2 we consider the case when $\mu = N_Q$ and $\nu = N_{a,Q}$ with $Q \in L_1^+(H)$ and $a \in H$, proving the Cameron-Martin formula. Finally in §1.3.3 we consider the more difficult case when $\mu = N_Q$ and $\nu = N_R$ with $Q, R \in L_1^+(H)$.

1.3.1 Equivalence of product measures in $\mathbb{R}^\infty$

It is convenient to introduce the notion of *Hellinger* integral.

Let μ, ν be probability measures on a measurable space $(E, \mathcal{E})$. Then $\lambda = \frac{1}{2}(\mu + \nu)$ is also a probability measure on $(E, \mathcal{E})$ and we have obviously

$$\mu << \lambda, \quad \nu << \lambda.$$

We define the *Hellinger integral* by

$$H(\mu, \nu) = \int_E \left[\frac{d\mu}{d\lambda}(x) \frac{d\nu}{d\lambda}(x) \right]^{1/2} \lambda(dx).$$

Instead of $\frac{1}{2}(\mu+\nu)$ one could choose as λ any measure equivalent to $\frac{1}{2}(\mu+\nu)$ without changing the value of $H(\mu, \nu)$.

By using Hölder's inequality we see that

$$|H(\mu, \nu)|^2 \le \int_E \frac{d\mu}{d\lambda}(x)\lambda(dx) \int_E \frac{d\nu}{d\lambda}(x)\lambda(dx) = 1,$$

so that $0 \le H(\mu, \nu) \le 1$.

Exercise 1.3.1 (a) Let $\mu = N_q$ and $\nu = N_{a,q}$, where $a \in \mathbb{R}$ and $q > 0$. Show that we have

$$H(\mu, \nu) = e^{-\frac{a^2}{4q}}. \tag{1.3.1}$$

(b) Let $\mu = N_q$ and $\nu = N_\rho$, where $q, \rho > 0$. Show that we have

$$H(\mu, \nu) = \left[\frac{4q\rho}{(q + \rho)^2} \right]^{1/4}. \tag{1.3.2}$$

Proposition 1.3.2 *Assume that $H(\mu, \nu) = 0$. Then the measures μ and ν are singular.*

Proof. Set $\alpha = \frac{d\mu}{d\lambda}$, $\beta = \frac{d\nu}{d\lambda}$. Since $H(\mu,\nu) = \int_\Omega \sqrt{\alpha\beta}\, d\lambda = 0$, we have $\alpha\beta = 0$, λ-a.e. Consequently, setting

$$A = \{\omega \in \Omega : \ \alpha(\omega) = 0\}, \quad B = \{\omega \in \Omega : \ \beta(\omega) = 0\},$$

we have $\lambda(A \cup B) = 1$. This means that $\lambda(C) = 0$ where $C = \Omega \backslash (A \cup B)$, and hence $\mu(C) = \nu(C) = 0$. Then, as

$$\mu(A) = \int_A \alpha \, d\lambda = 0, \ \ \nu(B) = \int_B \beta \, d\lambda = 0,$$

we have that μ and ν are singular since

$$\mu(A \cup C) = \nu(B) = 0, \ \ (A \cup C) \cap B = \emptyset. \square$$

Proposition 1.3.3 *Let $\mathcal{G} \subset \mathcal{E}$ be a σ-algebra, and let $\mu_{\mathcal{G}}$ and $\nu_{\mathcal{G}}$ be the restrictions of μ and ν to $(E, \mathcal{G})$. Then we have $H(\mu,\nu) \le H(\mu_{\mathcal{G}}, \nu_{\mathcal{G}})$.*

Proof. Let $\lambda_{\mathcal{G}}$ be the restriction of λ to $(E, \mathcal{G})$. It is easy to check that

$$\frac{d\mu_{\mathcal{G}}}{d\lambda_{\mathcal{G}}} = E_\lambda\left(\frac{d\mu}{d\lambda}\Big|\mathcal{G}\right) \ \ \frac{d\nu_{\mathcal{G}}}{d\lambda_{\mathcal{G}}} = E_\lambda\left(\frac{d\nu}{d\lambda}\Big|\mathcal{G}\right), \ \lambda\text{-a.e.}(^6)$$

Consequently we have ([7])

$$H(\mu_{\mathcal{G}}, \nu_{\mathcal{G}}) = \int_E \left[\mathbb{E}_\lambda\left(\frac{d\mu}{d\lambda}\Big|\mathcal{G}\right) \mathbb{E}_\lambda\left(\frac{d\nu}{d\lambda}\Big|\mathcal{G}\right)\right]^{1/2} d\lambda.$$

Since λ-a.e.

$$\frac{\left[\frac{d\mu}{d\lambda}\ \frac{d\nu}{d\lambda}\right]^{1/2}}{\left[\mathbb{E}_\lambda\left(\frac{d\mu}{d\lambda}|\mathcal{G}\right)\ \mathbb{E}_\lambda\left(\frac{d\nu}{d\lambda}|\mathcal{G}\right)\right]^{1/2}} \le \frac{1}{2}\left(\frac{\frac{d\mu}{d\lambda}}{\mathbb{E}_\lambda\left(\frac{d\mu}{d\lambda}|\mathcal{G}\right)} + \frac{\frac{d\nu}{d\lambda}}{\mathbb{E}_\lambda\left(\frac{d\nu}{d\lambda}|\mathcal{G}\right)}\right),$$

taking conditional expectations of both sides one finds, λ-a.e.,

$$\left[\mathbb{E}_\lambda\left(\frac{d\mu}{d\lambda}\Big|\mathcal{G}\right)\ \mathbb{E}_\lambda\left(\frac{d\nu}{d\lambda}\Big|\mathcal{G}\right)\right]^{1/2} \ge \mathbb{E}_\lambda\left(\left(\frac{d\mu}{d\lambda}\right)^{1/2}\ \left(\frac{d\nu}{d\lambda}\right)^{1/2}\Big|\mathcal{G}\right). \quad (1.3.3)$$

[6] $E_\lambda(\eta|\mathcal{G})$ is the conditional expectation of the random variable η with respect to $\mathcal{G}$ and measure λ.

[7] For positive numbers a, b, c, d, $\sqrt{\frac{ab}{cd}} \le \frac{1}{2}\left(\frac{a}{c} + \frac{b}{d}\right)$.

Integrating with respect to λ both sides of (1.3.3), the required result follows. □

Now let us consider two sequences of measures (μ_k) and (ν_k) on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $\nu_k \sim \mu_k$ for all $k \in \mathbb{N}$. We set $\lambda_k = \frac{1}{2}(\mu_k + \nu_k)$, and we consider the Hellinger integral

$$H(\mu_k, \nu_k) = \int_{\mathbb{R}} \left[\frac{d\mu_k}{d\lambda_k}(x)\frac{d\nu_k}{d\lambda_k}(x)\right]^{1/2} \lambda_k(dx), \ k \in \mathbb{N}.$$

Remark 1.3.4 Since (μ_k) and (ν_k) are equivalent, we have

$$\frac{d\mu_k}{d\lambda_k}\,\frac{d\nu_k}{d\lambda_k} = \frac{d\mu_k}{d\lambda_k}\,\frac{d\nu_k}{d\mu_k}\,\frac{d\mu_k}{d\lambda_k} = \frac{d\nu_k}{d\mu_k}\,\left(\frac{d\mu_k}{d\lambda_k}\right)^2.$$

Thus

$$H(\mu_k, \nu_k) = \int_{\mathbb{R}} \left[\frac{d\nu_k}{d\mu_k}(x)\right]^{1/2} \mu_k(dx). \tag{1.3.4}$$

We also consider the product measures on $\mathbb{R}^\infty$

$$\mu = \prod_{k=1}^\infty \mu_k, \quad \nu = \prod_{k=1}^\infty \nu_k,$$

and the corresponding Hellinger integral $H(\mu, \nu)$. As is easily checked we have

$$H(\mu, \nu) = \prod_{k=1}^\infty H(\mu_k, \nu_k).$$

Proposition 1.3.5 (Kakutani) *If $H(\mu, \nu) > 0$ then μ and ν are equivalent. Moreover*

$$f(x) := \frac{d\nu}{d\mu}(x) = \prod_{k=1}^\infty \frac{d\nu_k}{d\mu_k}(x_k), \ x \in \mathbb{R}^\infty, \ \mu\text{-a.e.} \tag{1.3.5}$$

Proof. We set

$$f_n(x) = \prod_{k=1}^n \frac{d\nu_k}{d\mu_k}(x_k), \ x \in \mathbb{R}^\infty, \ n \in \mathbb{N}.$$

We are going to prove that the sequence (f_n) is convergent on $L^1(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty), \mu)$. Let $m, n \in \mathbb{N}$, then we have

$$\int_{\mathbb{R}^\infty} \left| f_{n+m}^{1/2}(x) - f_n^{1/2}(x) \right|^2 \mu(dx)$$

$$= \int_{\mathbb{R}^\infty} \prod_{k=1}^{n} \frac{d\nu_k}{d\mu_k}(x_k) \left| \prod_{k=n+1}^{n+m} \left(\frac{d\nu_k}{d\mu_k}(x_k) \right)^{1/2} - 1 \right|^2 \mu(dx)$$

$$= \prod_{k=1}^{n} \int_{\mathbb{R}^\infty} \frac{d\nu_k}{d\mu_k}(x_k) \mu(dx) \int_{\mathbb{R}^\infty} \left| \prod_{k=n+1}^{n+m} \left(\frac{d\nu_k}{d\mu_k}(x_k) \right)^{1/2} - 1 \right|^2 \mu(dx).$$

Consequently

$$\begin{aligned}
&\int_{\mathbb{R}^\infty} |f_{n+p}^{1/2}(x) - f_n^{1/2}(x)|^2 \mu(dx) \\
&= \int_{\mathbb{R}^\infty} \left[\prod_{k=n+1}^{n+p} \frac{d\nu_k}{d\mu_k}(x_k) - 2 \prod_{k=n+1}^{n+p} \left(\frac{d\nu_k}{d\mu_k}(x_k) \right)^{1/2} + 1 \right] \mu(dx) \\
&= 2 \left(1 - \prod_{k=n+1}^{n+p} \int_{\mathbb{R}} \left(\frac{d\nu_k}{d\mu_k}(x_k) \right)^{1/2} \mu_k(dx_k) \right) \\
&= 2 \left(1 - \prod_{k=n+1}^{n+p} H(\mu_k, \nu_k) \right). \qquad (1.3.6)
\end{aligned}$$

On the other hand we know by assumption that

$$H(\mu, \nu) = \prod_{k=1}^{\infty} H(\mu_k, \nu_k) > 0,$$

or, equivalently, that

$$-\log H(\mu, \nu) = -\sum_{k=1}^{\infty} \log[H(\mu_k, \nu_k)] < +\infty.$$

Consequently, for any $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that if $n > n_\varepsilon$ and $p \in \mathbb{N}$, we have

$$-\sum_{k=n+1}^{n+p} \log[H(\mu_k, \nu_k)] < \varepsilon.$$

By (1.3.6) if $n > n_\varepsilon$ we have

$$\int_{\mathbb{R}^\infty} |\sqrt{f_{n+p}} - \sqrt{f_n}|^2 d\mu \le 2(1 - e^{-\varepsilon}).$$

Thus the sequence $(f_n^{1/2})$ is convergent on $L^2(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty), \mu)$ to some function $f^{1/2}$. Therefore $f_n \to f$ in $L^1(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty), \mu)$.

Finally, we prove that $\nu << \mu$ and $f = \frac{d\nu}{d\mu}$. Let φ be a continuous bounded Borel function on $\mathbb{R}^\infty$, and set $\varphi_n(x) = \varphi(P_n(x))$, $x \in \mathbb{R}^\infty$, where $P_n x = \{x_1, \ldots, x_n, 0, 0, \ldots\}$. Then we have

$$\int_{\mathbb{R}^\infty} \varphi(P_n x)\nu(dx) = \int_{\mathbb{R}^n} \varphi(P_n x)\ \nu_1(dx_1)\ldots\nu_n(dx_n)$$

$$= \int_{\mathbb{R}^n} \varphi(P_n x)\frac{d\nu_1}{d\mu_1}(x_1)\ldots\frac{d\nu_n}{d\mu_n}(x_n)\ \mu_1(dx_1)\ldots\mu_n(dx_n)$$

$$= \int_{\mathbb{R}^\infty} \varphi(P_n x) f_n(x)\mu(dx).$$

Letting n tend to infinity, we find

$$\int_{\mathbb{R}^\infty} \varphi(x)\nu(dx) = \int_{\mathbb{R}^\infty} \varphi(x) f(x)\mu(dx),$$

so that $\nu << \mu$. Finally, by exchanging the rôles of μ and ν, we find $\mu << \nu$. $\square$

1.3.2 The Cameron-Martin formula

We consider here the measures $\mu = N_{a,Q}$ and $\nu = N_Q$, and for any $a \in Q^{1/2}(H)$ we set

$$\rho_a(x) = \exp\left\{-\frac{1}{2}|Q^{-1/2}a|^2 + \langle Q^{-1/2}a, Q^{-1/2}x\rangle\right\},\ x \in H. \tag{1.3.7}$$

Let us recall, see §1.2.4, that $W_f(x) = \langle f, Q^{-1/2}x\rangle$ was defined for all $f \in \overline{Q^{1/2}(H)}$. Since $Q^{-1/2}a \in Q^{1/2}(H)$ the definition (1.3.7) is meaningful.