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Second Order Partial Differential Equations in Hilbert Spaces

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Contents

Preface	x
I THEORY IN SPACES OF CONTINUOUS FUNCTIONS	1
1 Gaussian measures	3
1.1 Introduction and preliminaries	3
1.2 Definition and first properties of Gaussian measures	7
1.2.1 Measures in metric spaces	7
1.2.2 Gaussian measures	8
1.2.3 Computation of some Gaussian integrals	11
1.2.4 The reproducing kernel	12
1.3 Absolute continuity of Gaussian measures	17
1.3.1 Equivalence of product measures in \mathbb{R}^∞	18
1.3.2 The Cameron-Martin formula	22
1.3.3 The Feldman-Hajek theorem	24
1.4 Brownian motion	27
2 Spaces of continuous functions	30
2.1 Preliminary results	30
2.2 Approximation of continuous functions	33
2.3 Interpolation spaces	36
2.3.1 Interpolation between $UC_b(H)$ and $UC_b^1(H)$	36
2.3.2 Interpolatory estimates	39
2.3.3 Additional interpolation results	42
3 The heat equation	44
3.1 Preliminaries	44
3.2 Strict solutions	48

3.3	Regularity of generalized solutions	54
3.3.1	Q -derivatives	54
3.3.2	Q -derivatives of generalized solutions	57
3.4	Comments on the Gross Laplacian	67
3.5	The heat semigroup and its generator	69
4	Poisson's equation	76
4.1	Existence and uniqueness results	76
4.2	Regularity of solutions	78
4.3	The equation $\Delta_Q u = g$	83
4.3.1	The Liouville theorem	87
5	Elliptic equations with variable coefficients	90
5.1	Small perturbations	90
5.2	Large perturbations	93
6	Ornstein-Uhlenbeck equations	99
6.1	Existence and uniqueness of strict solutions	100
6.2	Classical solutions	103
6.3	The Ornstein-Uhlenbeck semigroup	111
6.3.1	π -Convergence	112
6.3.2	Properties of the π -semigroup (R_t)	113
6.3.3	The infinitesimal generator	114
6.4	Elliptic equations	116
6.4.1	Schauder estimates	119
6.4.2	The Liouville theorem	121
6.5	Perturbation results for parabolic equations	122
6.6	Perturbation results for elliptic equations	124
7	General parabolic equations	127
7.1	Implicit function theorems	128
7.2	Wiener processes and stochastic equations	131
7.2.1	Infinite dimensional Wiener processes	131
7.2.2	Stochastic integration	132
7.3	Dependence of the solutions to stochastic equations on initial data	133
7.3.1	Convolution and evaluation maps	133
7.3.2	Solutions of stochastic equations	138
7.4	Space and time regularity of the generalized solutions	139
7.5	Existence	142

7.6	Uniqueness	144
7.6.1	Uniqueness for the heat equation	145
7.6.2	Uniqueness in the general case	146
7.7	Strong Feller property	150
8	Parabolic equations in open sets	156
8.1	Introduction	156
8.2	Regularity of the generalized solution	158
8.3	Existence theorems	165
8.4	Uniqueness of the solutions	178
II	THEORY IN SOBOLEV SPACES	185
9	L^2 and Sobolev spaces	187
9.1	Itô-Wiener decomposition	188
9.1.1	Real Hermite polynomials	188
9.1.2	Chaos expansions	190
9.1.3	The space $L^2(H, \mu; H)$	193
9.2	Sobolev spaces	194
9.2.1	The space $W^{1,2}(H, \mu)$	196
9.2.2	Some additional summability results	197
9.2.3	Compactness of the embedding $W^{1,2}(H, \mu) \subset L^2(H, \mu)$	198
9.2.4	The space $W^{2,2}(H, \mu)$	201
9.3	The Malliavin derivative	203
10	Ornstein-Uhlenbeck semigroups on $L^p(H, \mu)$	205
10.1	Extension of (R_t) to $L^p(H, \mu)$	206
10.1.1	The adjoint of (R_t) in $L^2(H, \mu)$	211
10.2	The infinitesimal generator of (R_t)	212
10.2.1	Characterization of the domain of L_2	215
10.3	The case when (R_t) is strong Feller	217
10.3.1	Additional regularity properties of (R_t)	221
10.3.2	Hypercontractivity of (R_t)	224
10.4	A representation formula for (R_t) in terms of the second quantization operator	228
10.4.1	The second quantization operator	228
10.4.2	The adjoint of (R_t)	230
10.5	Poincaré and log-Sobolev inequalities	230
10.5.1	The case when $M = 1$ and $Q = I$	232

10.5.2	A generalization	235
10.6	Some additional regularity results when Q and A commute	236
11	Perturbations of Ornstein-Uhlenbeck semigroups	238
11.1	Bounded perturbations	239
11.2	Lipschitz perturbations	245
11.2.1	Some additional results on the Ornstein-Uhlenbeck semigroup	251
11.2.2	The semigroup (P_t) in $L^p(H, \nu)$	256
11.2.3	The integration by parts formula	260
11.2.4	Existence of a density	263
12	Gradient systems	267
12.1	General results	268
12.1.1	Assumptions and setting of the problem	268
12.1.2	The Sobolev space $W^{1,2}(H, \nu)$	271
12.1.3	Symmetry of the operator N_0	272
12.1.4	The m -dissipativity of N_1 on $L^1(H, \nu)$	274
12.2	The m -dissipativity of N_2 on $L^2(H, \nu)$	277
12.3	The case when U is convex	281
12.3.1	Poincaré and log-Sobolev inequalities	288
III	APPLICATIONS TO CONTROL THEORY	291
13	Second order Hamilton-Jacobi equations	293
13.1	Assumptions and setting of the problem	296
13.2	Hamilton-Jacobi equations with a Lipschitz Hamiltonian	300
13.2.1	Stationary Hamilton-Jacobi equations	302
13.3	Hamilton-Jacobi equation with a quadratic Hamiltonian	305
13.3.1	Stationary equation	308
13.4	Solution of the control problem	310
13.4.1	Finite horizon	310
13.4.2	Infinite horizon	312
13.4.3	The limit as $\varepsilon \rightarrow 0$	314
14	Hamilton-Jacobi inclusions	316
14.1	Introduction	316
14.2	Excessive weights and an existence result	317
14.3	Weak solutions as value functions	324

14.4 Excessive measures for Wiener processes 328

IV APPENDICES 333

A Interpolation spaces 335

A.1 The interpolation theorem 335

A.2 Interpolation between a Banach space X and the domain of
a linear operator in X 336

B Null controllability 338

B.1 Definition of null controllability 338

B.2 Main results 339

B.3 Minimal energy 340

C Semiconcave functions and Hamilton-Jacobi semigroups 347

C.1 Continuity modulus 347

C.2 Semiconcave and semiconvex functions 348

C.3 The Hamilton-Jacobi semigroups 351

Bibliography 358

Index 376

Chapter 1

Gaussian measures

This chapter is devoted to some basic results on Gaussian measures on separable Hilbert spaces, including the Cameron-Martin and Feldman-Hajek formulae. The greater part of the results are presented with complete proofs.

1.1 Introduction and preliminaries

We are given a real separable Hilbert space H (with norm $|\cdot|$ and inner product $\langle \cdot, \cdot \rangle$). The space of all linear bounded operators from H into H , equipped with the operator norm $\|\cdot\|$, will be denoted by $L(H)$. If $T \in L(H)$, then T^* is the adjoint of T . Moreover, by $L^+(H)$ we shall denote the subset of $L(H)$ consisting of all nonnegative symmetric operators. Finally, we shall denote by $\mathcal{B}(H)$ the σ -algebra of all Borel subsets of H .

Before introducing Gaussian measures we need some results about trace class and Hilbert-Schmidt operators.

A linear bounded operator $R \in L(H)$ is said to be of *trace class* if there exist two sequences $(a_k), (b_k)$ in H such that

$$Ry = \sum_{k=1}^{\infty} \langle y, a_k \rangle b_k, \quad y \in H, \quad (1.1.1)$$

and

$$\sum_{k=1}^{\infty} |a_k| |b_k| < +\infty. \quad (1.1.2)$$

Notice that if (1.1.2) holds then the series in (1.1.1) is norm convergent. Moreover, it is not difficult to show that R is compact.

We shall denote by $L_1(H)$ the set of all operators of $L(H)$ of trace class. $L_1(H)$, endowed with the usual linear operations, is a Banach space with the norm

$$\|R\|_{L_1(H)} = \inf \left\{ \sum_{k=1}^{\infty} |a_k| |b_k| : Ry = \sum_{k=1}^{\infty} \langle y, a_k \rangle b_k, \quad y \in H, (a_k), (b_k) \subset H \right\}.$$

We set $L_1^+(H) = L^+(H) \cap L_1(H)$. If an operator R is of trace class then its trace, $\text{Tr } R$, is defined by the formula

$$\text{Tr } R = \sum_{j=1}^{\infty} \langle Re_j, e_j \rangle,$$

where (e_j) is an orthonormal and complete basis on H . Notice that, if R is given by (1.1.1), we have

$$\text{Tr } R = \sum_{j=1}^{\infty} \langle a_j, b_j \rangle.$$

Thus the definition of the trace is independent on the choice of the basis and

$$|\text{Tr } R| \leq \|R\|_{L_1(H)}.$$

Proposition 1.1.1 *Let $S \in L_1(H)$ and $T \in L(H)$. Then*

(i) $ST, TS \in L_1(H)$ and

$$\|TS\|_{L_1(H)} \leq \|S\|_{L_1(H)} \|T\|, \quad \|ST\|_{L_1(H)} \leq \|S\|_{L_1(H)} \|T\|.$$

(ii) $\text{Tr}(ST) = \text{Tr}(TS)$.

Proof. (i) Assume that $Sy = \sum_{k=1}^{\infty} \langle y, a_k \rangle b_k$, $y \in H$, where $\sum_{k=1}^{\infty} |a_k| |b_k| < +\infty$.

Then

$$STy = \sum_{k=1}^{\infty} \langle y, T^* a_k \rangle b_k, \quad y \in H,$$

and

$$\sum_{k=1}^{\infty} |T^* a_k| |b_k| \leq \|T\| \sum_{k=1}^{\infty} |a_k| |b_k|.$$

It is therefore clear that $ST \in L_1(H)$ and $\|ST\|_{L_1(H)} \leq \|S\|_{L_1(H)}\|T\|$. Similarly we can prove that $\|TS\|_{L_1(H)} \leq \|S\|_{L_1(H)}\|T\|$.

(ii) From part (i) it follows that

$$\operatorname{Tr}(ST) = \sum_{k=1}^{\infty} \langle b_k, T^* a_k \rangle = \sum_{k=1}^{\infty} \langle T b_k, a_k \rangle.$$

In the same way $\operatorname{Tr}(TS) = \sum_{k=1}^{\infty} \langle a_k, T b_k \rangle$, and the conclusion follows. \square

We say that $R \in L(H)$ is of Hilbert-Schmidt class if there exists an orthonormal and complete basis (e_k) in H such that

$$\sum_{k,j=1}^{\infty} |\langle S e_k, e_j \rangle|^2 < +\infty. \quad (1.1.3)$$

If (1.1.3) holds then we have

$$\sum_{k=1}^{\infty} |S e_k|^2 = \sum_{k,j=1}^{\infty} |\langle S e_k, e_j \rangle|^2 = \sum_{k,j=1}^{\infty} |\langle e_k, S^* e_j \rangle|^2 = \sum_{j=1}^{\infty} |S^* e_j|^2. \quad (1.1.4)$$

Now if (f_k) is another complete orthonormal basis in H , we have

$$\sum_{m=1}^{\infty} |S f_m|^2 = \sum_{m,n=1}^{\infty} |\langle S f_m, e_n \rangle|^2 = \sum_{m,n=1}^{\infty} |\langle f_m, S^* e_n \rangle|^2 = \sum_{n=1}^{\infty} |S^* e_n|^2.$$

Thus, by (1.1.4) we see that the assertion (1.1.3) is independent of the choice of the complete orthonormal basis (e_k) . We shall denote by $L_2(H)$ the space of all Hilbert-Schmidt operators on H . $L_2(H)$, endowed with the norm

$$\|S\|_{L_2(H)}^2 = \sum_{k,j=1}^{\infty} |\langle S e_k, e_j \rangle|^2 = \sum_{k=1}^{\infty} |S e_k|^2,$$

is a Banach space.

Proposition 1.1.2 *Let $S, T \in L_2(H)$. Then $ST \in L_1(H)$ and*

$$\|ST\|_{L_1(H)} \leq \|S\|_{L_2(H)}\|T\|_{L_2(H)}. \quad (1.1.5)$$

Proof. Let (e_k) be a complete and orthonormal basis in H , then

$$\begin{aligned} Ty &= \sum_{k=1}^{\infty} \langle Ty, e_k \rangle e_k = \sum_{k=1}^{\infty} \langle y, T^* e_k \rangle e_k, \\ STy &= \sum_{k=1}^{\infty} \langle y, T^* e_k \rangle S e_k. \end{aligned}$$

Consequently $ST \in L_1(H)$ and

$$\begin{aligned} \|ST\|_{L_1(H)} &\leq \sum_{k=1}^{\infty} |T^* e_k| |S e_k| \leq \left(\sum_{k=1}^{\infty} |T^* e_k|^2 \right)^{1/2} \left(\sum_{k=1}^{\infty} |S e_k|^2 \right)^{1/2} \\ &= \|T\|_{L_2(H)} \|S\|_{L_2(H)}. \end{aligned}$$

Therefore the conclusion follows. \square

Warning. If S and T are bounded operators, and ST is of trace class then in general TS is not, as the following example, provided by S. Pezdat [183], shows.

Define two linear operators S and T on the product space $H \times H$, by

$$S = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}, \quad T = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$ST = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}, \quad TS = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix},$$

and it is enough to take B of trace class and A not of trace class. \square

We have also the following result, see e.g. A. Pietsch [187].

Proposition 1.1.3 *Assume that S is a compact self-adjoint operator, and that (λ_k) are its eigenvalues (repeated according to their multiplicity).*

(i) $S \in L_1(H)$ if and only if $\sum_{k=1}^{\infty} |\lambda_k| < +\infty$. Moreover $\|S\|_{L_1(H)} = \sum_{k=1}^{\infty} |\lambda_k|$,

and $\text{Tr } S = \sum_{k=1}^{\infty} \lambda_k$.

(ii) $S \in L_2(H)$ if and only if $\sum_{k=1}^{\infty} |\lambda_k|^2 < +\infty$. Moreover

$$\|S\|_{L_2(H)} = \left(\sum_{k=1}^{\infty} |\lambda_k|^2 \right)^{1/2}.$$

More generally let S be a compact operator on H . Denote by (λ_k) the sequence of all positive eigenvalues of the operator $(S^*S)^{1/2}$, repeated according to their multiplicity. Denote by $L_p(H)$, $p > 0$, the set of all operators S such that

$$\|S\|_{L_p(H)} = \left(\sum_{k=1}^{\infty} \lambda_k^p \right)^{1/p} < +\infty. \quad (1.1.6)$$

Operators belonging to $L_1(H)$ and $L_2(H)$ are precisely the trace class and the Hilbert-Schmidt operators.

The following result holds, see N. Dunford and J. T. Schwartz [107].

Proposition 1.1.4 *Let $S \in L_p(H)$, $T \in L_q(H)$ with $p > 0, q > 0$. Then $ST \in L_r(H)$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, and*

$$\|TS\|_{L_r(H)} \leq 2^{1/r} \|S\|_{L_p(H)} \|T\|_{L_q(H)}. \quad (1.1.7)$$

1.2 Definition and first properties of Gaussian measures

1.2.1 Measures in metric spaces

If E is a metric space, then $\mathcal{B}(E)$ will denote the Borel σ -algebra, that is the smallest σ -algebra of subsets of E which contains all closed (open) subsets of E .

Let metric spaces E_1, E_2 be equipped with σ -fields $\mathcal{E}_1, \mathcal{E}_2$ respectively. Measurable mappings $X : E_1 \rightarrow E_2$ will often be called *random variables*. If μ is a measure on (E_1, \mathcal{E}_1) , then its image by the transformation X will be denoted by $X \circ \mu$:

$$X \circ \mu(A) = \mu(X^{-1}(A)), \quad A \in \mathcal{E}_2.$$

We call $X \circ \mu$ the *law* or the *distribution* of X , and we set $X \circ \mu = \mathcal{L}(X)$.

If ν and μ are two finite measures on (E, \mathcal{E}) such that $\Gamma \in \mathcal{E}$, $\mu(\Gamma) = 0$ implies $\nu(\Gamma) = 0$ then one writes $\nu \ll \mu$ and one says that ν is *absolutely continuous* with respect to μ . If there exist $A, B \in \mathcal{E}$ such that $A \cap B = \emptyset$, $\mu(A) = \nu(B) = 1$, one says that μ and ν are *singular*.

If $\nu \ll \mu$ then by the Radon-Nikodým theorem there exists $g \in L^1(E, \mathcal{E}, \mu)$ nonnegative such that

$$\nu(\Gamma) = \int_{\Gamma} g(x) \mu(dx), \quad \Gamma \in \mathcal{E}.$$

The function g is denoted by $\frac{d\nu}{d\mu}$.

If $\nu \ll \mu$ and $\mu \ll \nu$ then one says that μ and ν are *equivalent* and writes $\mu \sim \nu$.

We have the following change of variable formula. If φ is a nonnegative measurable real function on E_2 , then

$$\int_{E_1} \varphi(X(x))\mu(dx) = \int_{E_2} \varphi(y)X \circ \mu(dy). \quad (1.2.1)$$

Let μ and ν be two measures on a separable Hilbert space H ; if $T \circ \mu = T \circ \nu$ for any linear operator $T : H \rightarrow \mathbb{R}^n$, $n \in \mathbb{N}$, then $\mu = \nu$.

Random variables X_1, \dots, X_n are said to be *independent* if

$$\mathcal{L}(X_1, \dots, X_n) = \mathcal{L}(X_1) \times \dots \times \mathcal{L}(X_n).$$

A family of random variables $(X_\alpha)_{\alpha \in A}$ is said to be independent, if any finite subset of the family is independent.

Probability measures on a separable Hilbert space H will always be regarded as defined on $\mathcal{B}(H)$. If μ is a probability measure on H , then its Fourier transform is defined by

$$\hat{\mu}(\lambda) = \int_H e^{i\langle \lambda, x \rangle} \mu(dx), \quad \lambda \in H;$$

$\hat{\mu}$ is called the *characteristic function* of μ . One can show that if the characteristic functions of two measures are identical, then the measures are identical as well.

1.2.2 Gaussian measures

We first define Gaussian measures on \mathbb{R} . If $a \in \mathbb{R}$ we set

$$N_{a,0}(dx) = \delta_a(dx),$$

where δ_a is the Dirac measure at a . If moreover $\lambda > 0$ we set

$$N_{a,\lambda}(dx) = \frac{1}{\sqrt{2\pi\lambda}} e^{-\frac{(x-a)^2}{2\lambda}} dx.$$

The Fourier transform of $N_{a,\lambda}$ is given by

$$\widehat{N_{a,\lambda}}(h) = \int_{\mathbb{R}} e^{ihx} N_{a,\lambda}(dx) = e^{iah - \frac{1}{2}\lambda h^2}, \quad h \in \mathbb{R}.$$

More generally we show now that in an arbitrary separable Hilbert space and for arbitrary $Q \in L_1^+(H)$ there exists a unique measure $N_{a,Q}$ such that

$$\widehat{N_{a,\lambda}}(h) = \int_H e^{i\langle h,x \rangle} N_{a,Q}(dx) = e^{i\langle h,x \rangle - \frac{1}{2}\langle Qh,h \rangle}, \quad h \in H.$$

Let in fact $Q \in L_1^+(H)$. Then there exist a complete orthonormal system (e_k) on H and a sequence of nonnegative numbers (λ_k) such that $Qe_k = \lambda_k e_k$, $k \in \mathbb{N}$. We set $x_h = \langle x, e_h \rangle$, $h \in \mathbb{N}$, and $P_n x = \sum_{k=1}^n x_k e_k$, $x \in H$, $n \in \mathbb{N}$. Let us introduce an isomorphism γ from H into ℓ^2 :⁽¹⁾

$$x \in H \rightarrow \gamma(x) = (x_k) \in \ell^2.$$

In the following we shall always identify H with ℓ^2 . In particular we shall write $P_n x = (x_1, \dots, x_n)$, $x \in \ell^2$.

A subset I of H of the form $I = \{x \in H : (x_1, \dots, x_n) \in B\}$, where $B \in \mathcal{B}(\mathbb{R}^n)$, is said to be *cylindrical*. It is easy to see that the σ -algebra generated by all cylindrical subsets of H coincides with $\mathcal{B}(H)$.

Theorem 1.2.1 *Let $a \in H$, $Q \in L_1^+(H)$. Then there exists a unique probability measure μ on $(H, \mathcal{B}(H))$ such that*

$$\int_H e^{i\langle h,x \rangle} \mu(dx) = e^{i\langle a,h \rangle} e^{-\frac{1}{2}\langle Qh,h \rangle}, \quad h \in H. \quad (1.2.2)$$

Moreover μ is the restriction to H (identified with ℓ^2) of the product measure

$$\bigotimes_{k=1}^{\infty} \mu_k = \bigotimes_{k=1}^{\infty} N_{a_k, \lambda_k},$$

defined on $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$.⁽²⁾

We set $\mu = N_{a,Q}$, and call a the *mean* and Q the *covariance operator* of μ . Moreover $N_{0,Q}$ will be denoted by N_Q .

Proof of Theorem 1.2.1. Since a characteristic function uniquely determines the measure, we have only to prove existence.

Let us consider the sequence of Gaussian measures (μ_k) on \mathbb{R} defined as $\mu_k = N_{a_k, \lambda_k}$, $k \in \mathbb{N}$, and the product measure $\mu = \bigotimes_{k=1}^{\infty} \mu_k$ in \mathbb{R}^∞ , see e.g.

¹For any $p \geq 1$, we denote by ℓ^p the Banach space of all sequences (x_k) of real numbers such that $|x|_p := (\sum_{k=1}^{\infty} |x_k|^p)^{1/p} < +\infty$.

²We shall consider \mathbb{R}^∞ as a metric space with the distance $d(x,y) := \sum_{k=1}^{\infty} 2^{-k} \frac{|x_k - y_k|}{1 + |x_k - y_k|}$, $x, y \in \mathbb{R}^\infty$

P. R. Halmos [141, §38.B]. We want to prove that μ is concentrated on ℓ^2 , (that it is clearly a Borel subset of \mathbb{R}^∞). For this it is enough to show that

$$\int_{\ell^\infty} |x|_{\ell^2}^2 \mu(dx) < +\infty. \quad (1.2.3)$$

We have in fact, by the monotone convergence theorem,

$$\begin{aligned} \int_{\mathbb{R}^\infty} |x|_{\ell^2}^2 \mu(dx) &= \sum_{k=1}^{\infty} \int_{\mathbb{R}^\infty} x_k^2 \mu(dx) = \sum_{k=1}^{\infty} \left(\int_{\mathbb{R}} (x_k - a_k)^2 \mu_k(dx) + a_k^2 \right) \\ &= \sum_{k=1}^{\infty} (\lambda_k + a_k^2) = \text{Tr } Q + |a|^2 < +\infty. \end{aligned}$$

Now we consider the restriction of μ to ℓ^2 , which we still denote by μ . We have to prove that (1.2.2) holds. Setting $\nu_n = \prod_{k=1}^n \mu_k$, we have

$$\begin{aligned} \int_{\ell^2} e^{i\langle x, h \rangle} \mu(dx) &= \lim_{n \rightarrow \infty} \int_{\ell^2} e^{i\langle P_n h, P_n x \rangle} \mu(dx) \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^n} e^{i\langle P_n h, P_n x \rangle} \nu_n(dx) = \lim_{n \rightarrow \infty} e^{i\langle P_n h, P_n a \rangle - \frac{1}{2} \langle Q P_n h, P_n h \rangle} \\ &= e^{i\langle h, a \rangle - \frac{1}{2} \langle Q h, h \rangle}. \quad \square \end{aligned}$$

If the law of a random variable is a Gaussian measure, then the random variable is called *Gaussian*. It easily follows from Theorem 1.2.1 that a random variable X with values in H is Gaussian if and only if for any $h \in H$ the real valued random variable $\langle h, X \rangle$ is Gaussian.

Remark 1.2.2 From the proof of Theorem 1.2.1 it follows that

$$\int_H |x|^2 N_{a,Q}(dx) = \text{Tr } Q + |a|^2. \quad (1.2.4)$$

Proposition 1.2.3 *Let $T \in L(H)$, and $a \in H$, and let $\Gamma x = Tx + a$, $x \in H$. Then $\Gamma \circ N_{m,Q} = N_{Tm+a, TQT^*}$.*

Proof. Notice that, by the change of variables formula (1.2.1), we have

$$\begin{aligned} \int_H e^{i\langle \lambda, y \rangle} \Gamma \circ N_{m,Q}(dy) &= \int_H e^{i\langle \lambda, \Gamma x \rangle} N_{m,Q}(dy) \\ &= \int_H e^{i\langle \lambda, Tx+a \rangle} N_{m,Q}(dy) = e^{i\langle \lambda, a \rangle} e^{i\langle T^* \lambda, m \rangle - \frac{1}{2} \langle QT^* \lambda, T^* \lambda \rangle}. \end{aligned}$$

This shows the result. \square

1.2.3 Computation of some Gaussian integrals

We are here given a Gaussian measure $N_{a,Q}$. We set

$$L^2(H, N_{a,Q}) = L^2(H, \mathcal{B}(H), N_{a,Q}).$$

The following identities can be easily proved, using (1.2.2).

Proposition 1.2.4 *We have*

$$\int_H x N_{a,Q}(dx) = a, \quad (1.2.5)$$

$$\int_H \langle x - a, y \rangle \langle x - a, z \rangle N_{a,Q}(dx) = \langle Qy, z \rangle. \quad (1.2.6)$$

$$\int_H |x - a|^2 N_{a,Q}(dx) = \text{Tr } Q. \quad (1.2.7)$$

Proof. We prove as instance (1.2.6). We have

$$\int_H x N_{a,Q}(dx) = \lim_{n \rightarrow \infty} \int_H P_n x N_{a,Q}(dx).$$

But

$$\int_H P_n x N_{a,Q}(dx) = (2\pi)^{-n/2} \prod_{k=1}^n \int_{\mathbb{R}} x_k \lambda_k^{-1/2} e^{-\frac{(x_k - a_k)^2}{2\lambda_k}} dx_k = a_k,$$

and the conclusion follows. \square

Proposition 1.2.5 *For any $h \in H$, the exponential function E_h , defined as*

$$E_h(x) = e^{\langle h, x \rangle}, \quad x \in H,$$

belongs to $L^p(H, N_{a,Q})$, $p \geq 1$, and

$$\int_H e^{\langle h, x \rangle} N_{a,Q}(dx) = e^{\langle a, h \rangle} e^{\frac{1}{2} \langle Qh, h \rangle}. \quad (1.2.8)$$

Moreover the subspace of $L^2(H, N_{a,Q})$ spanned by all E_h , $h \in H$, is dense on $L^2(H, N_{a,Q})$.

Proof. We have

$$\int_H e^{\langle P_n h, P_n x \rangle} N_{a,Q}(dx) = e^{\langle P_n a, P_n h \rangle} e^{\frac{1}{2} \langle Q P_n h, P_n h \rangle}.$$

Letting n tend to 0 this gives (1.2.8).

Let us prove the last statement. Let $\varphi \in L^2(H, N_{a,Q})$ be such that

$$\int_H e^{\langle h,x \rangle} \varphi(x) N_{a,Q}(dx) = 0, \quad h \in H.$$

Denote by φ^+ and φ^- the positive and negative parts of φ . Then

$$\int_H e^{\langle h,x \rangle} \varphi^+(x) N_{a,Q}(dx) = \int_H e^{\langle h,x \rangle} \varphi^-(x) N_{a,Q}(dx), \quad h \in H.$$

Let us define two measures

$$\mu(dx) = \varphi^+(x) N_{a,Q}(dx), \quad \nu(dx) = \varphi^-(x) N_{a,Q}(dx).$$

Then μ and ν are finite measures such that

$$\int_H e^{\langle h,x \rangle} \mu(dx) = \int_H e^{\langle h,x \rangle} \nu(dx), \quad h \in H.$$

Let T be any linear transformation from H into \mathbb{R}^n , $n \in \mathbb{N}$. Then for any $\lambda \in \mathbb{R}^n$

$$\begin{aligned} \int_{\mathbb{R}^n} e^{\langle \lambda, z \rangle} T \circ \mu(dz) &= \int_H e^{\langle \lambda, Tx \rangle} \mu(dx) = \int_H e^{\langle T^* \lambda, \cdot \rangle} \mu(dx) \\ &= \int_H e^{\langle T^* \lambda, x \rangle} \nu(dx) = \int_{\mathbb{R}^n} e^{\langle \lambda, z \rangle} T \circ \nu(dz). \end{aligned}$$

By a well known finite dimensional result $T \circ \mu = T \circ \nu$. Consequently measures μ and ν are identical and so $\varphi = 0$. \square

1.2.4 The reproducing kernel

Here we are given an operator $Q \in L_1^+(H)$. We denote as before by (e_k) a complete orthonormal system in H and by (λ_k) a sequence of positive numbers such that $Qe_k = \lambda_k e_k$, $k \in \mathbb{N}$.

The subspace $Q^{1/2}(H)$ is called the *reproducing kernel* of the measure N_Q . If $\text{Ker } Q = \{0\}$, $Q^{1/2}(H)$ is dense on H . In fact, if $x_0 \in H$ is such that $\langle Q^{1/2}h, x_0 \rangle = 0$ for all $h \in H$, we have $Q^{1/2}x_0 = 0$ and so $Qx_0 = 0$, which yields $x_0 = 0$.

Let $\text{Ker } Q = \{0\}$. We are now going to introduce an isomorphism W from H into $L^2(H, N_Q)$ that will play an important rôle in the following. The isomorphism W is defined by

$$f \in Q^{1/2}(H) \rightarrow W_f \in L^2(H, N_Q), \quad W_f(x) = \langle Q^{-1/2}f, x \rangle, \quad x \in H.$$

By (1.2.7) it follows that

$$\int_H W_f(x)W_g(x)N_Q(dx) = \langle f, g \rangle, \quad f, g \in H.$$

Thus W is an isometry and it can be uniquely extended to all of H . It will be denoted by the same symbol. For any $f \in H$, W_f is a real Gaussian random variable $N_{|f|^2}$.

More generally, for arbitrary elements f_1, \dots, f_n , $(W_{f_1}, \dots, W_{f_n})$ is a Gaussian vector with mean 0 and covariance matrix $(\langle f_i, f_j \rangle)$. If $\text{Ker } Q \neq \{0\}$ then the transformation $f \rightarrow W_f$ can be defined in exactly the same way but only for $f \in H_0 = \overline{Q^{1/2}(H)}$. We will write in some cases $\langle Q^{-1/2}y, f \rangle$ instead of $W_f(y)$.

The proof of the following proposition is left as an exercise to the reader.

Proposition 1.2.6 *For any orthonormal sequence (f_n) in H , the family*

$$1, W_{f_n}, W_{f_k}W_{f_l}, 2^{-1/2}(W_{f_m}^2 - 1), \quad m, n, k, l \in \mathbb{N}, \quad k \neq l,$$

is orthonormal in $L^2(H, N_Q)$.

Next we consider the function $f \rightarrow e^{W_f}$.

Proposition 1.2.7 *The transformation $f \rightarrow e^{W_f}$ acts continuously from H into $L^2(H, N_Q)$, and*

$$\int_H e^{W_f(x)} N_Q(dx) = e^{\frac{1}{2}|f|^2}, \tag{1.2.9}$$

$$\int_H e^{i\lambda W_f(x)} N_Q(dx) = e^{-\frac{1}{2}\lambda^2|f|^2}, \quad \lambda \in \mathbb{R}.$$

Proof. Since W_f is Gaussian with law $N_{0,|f|^2}$, (1.2.9) follows. Moreover, taking into account (1.2.8) it follows that

$$\begin{aligned} \int_H [e^{W_f} - e^{W_g}]^2 dN_Q &= \int_H [e^{2W_f} - 2e^{W_f+W_g} + e^{2W_g}] dN_Q \\ &= e^{2|f|^2} - 2e^{\frac{1}{2}|f+g|^2} + e^{2|g|^2} = [e^{|f|^2} - e^{|g|^2}]^2 + 2e^{|f|^2+|g|^2} [1 - e^{-\frac{1}{2}|f-g|^2}], \end{aligned}$$

which shows that W_f is locally uniformly continuous on H . \square

Let us define the determinant of $1 + S$ where S is a compact self-adjoint operator in $L_1(H)$:

$$\det(1 + S) = \prod_{k=1}^{\infty} (1 + s_k),$$

where (s_k) is the sequence of eigenvalues of S (repeated according to their multiplicity).

Proposition 1.2.8 *Assume that M is a symmetric operator such that $Q^{1/2}MQ^{1/2} < 1$, ⁽³⁾ and let $b \in H$. Then*

$$\begin{aligned} & \int_H \exp \left\{ \frac{1}{2} \langle My, y \rangle + \langle b, y \rangle \right\} N_Q(dy) \\ &= \left[\det(1 - Q^{1/2}MQ^{1/2}) \right]^{-1/2} \exp \left\{ \frac{1}{2} |(1 - Q^{1/2}MQ^{1/2})^{-1/2} Q^{1/2}b|^2 \right\}. \end{aligned} \quad (1.2.10)$$

Proof. Let (g_n) be an orthonormal basis for the operator $Q^{1/2}MQ^{1/2}$, and let (γ_n) be the sequence of the corresponding eigenvalues.

Claim 1. We have

$$\langle b, x \rangle = \sum_{k=1}^{\infty} \langle Q^{1/2}b, g_n \rangle W_{g_n}(x), \quad N_Q\text{-a.e.}$$

Claim 2. We have

$$\langle Mx, x \rangle = \sum_{n=1}^{\infty} \gamma_n |W_{g_n}(x)|^2, \quad N_Q\text{-a.e.},$$

the series being convergent in $L^1(H, N_Q)$.

We shall only prove the more difficult second claim.

Let $P_N = \sum_{k=1}^N e_k \otimes e_k$. ⁽⁴⁾ Then for any $x \in H$ we have

$$\begin{aligned} \langle MP_Nx, P_Nx \rangle &= \langle (Q^{1/2}MQ^{1/2})Q^{-1/2}P_Nx, Q^{-1/2}P_Nx \rangle \\ &= \sum_{n=1}^{\infty} \langle (Q^{1/2}MQ^{1/2})Q^{-1/2}P_Nx, g_n \rangle \langle Q^{-1/2}P_Nx, g_n \rangle \\ &= \sum_{n=1}^{\infty} \gamma_n |\langle Q^{-1/2}P_Nx, g_n \rangle|^2. \end{aligned}$$

Consequently, for each fixed x

$$\langle MP_Nx, P_Nx \rangle = \sum_{n=1}^{\infty} \gamma_n |W_{P_Ng_n}|^2, \quad N \in \mathbb{N}.$$

³This means that $\langle Q^{1/2}MQ^{1/2}x, x \rangle < |x|^2$ for any $x \in H$ different from 0.

⁴We remember that (e_k) is the sequence of eigenvectors of Q .

Moreover for each $L \in \mathbb{N}$

$$\begin{aligned}
& \int_H \left| \langle MP_N x, P_N x \rangle - \sum_{n=1}^L \gamma_n |W_{P_N g_n}|^2 \right| N_Q(dx) \\
& \leq \sum_{n=L+1}^{\infty} |\gamma_n| \int_H |W_{P_N g_n}|^2 N_Q(dx) \\
& = \sum_{n=L+1}^{\infty} |\gamma_n| |P_N g_n|^2 \leq \sum_{n=L+1}^{\infty} |\gamma_n|.
\end{aligned}$$

As $N \rightarrow \infty$ then $P_N x \rightarrow x$ and $W_{P_N g_n} \rightarrow W_{g_n}$ in $L^2(H, N_Q)$. Passing to subsequences if needed, and using the Fatou lemma, we see that

$$\int_H \left| \langle Mx, x \rangle - \sum_{n=1}^L \gamma_n |W_{g_n}|^2 \right| N_Q(dx) \leq \sum_{n=L+1}^{\infty} |\gamma_n|.$$

Therefore the claim is proved.

By the claims it follows that

$$\begin{aligned}
& \exp \left\{ \frac{1}{2} \langle Mx, x \rangle + \langle b, x \rangle \right\} \\
& = \lim_{L \rightarrow \infty} \exp \left\{ \sum_{n=1}^L \frac{1}{2} \gamma_n |W_{g_n}(x)|^2 + \langle Q^{1/2} b, g_n \rangle W_{g_n}(x) \right\},
\end{aligned}$$

with a.e. convergence with respect to N_Q for a suitable subsequence. Using the fact that (W_{g_n}) are independent Gaussian random variables, we obtain, by a direct calculation, for $p \geq 1$,

$$\begin{aligned}
& \int_H \exp \left\{ p \sum_{n=1}^L \frac{1}{2} \gamma_n |W_{g_n}(x)|^2 + p \langle Q^{1/2} b, g_n \rangle W_{g_n}(x) \right\} N_Q(dx) \\
& = \left[\prod_{n=1}^L (1 - p\gamma_n) \right]^{-1/2} \exp \left\{ \frac{1}{2} \sum_{n=1}^{\infty} \frac{|\langle Q^{1/2} b, g_n \rangle|^2}{1 - p\gamma_n} \right\}.
\end{aligned}$$

Since $\gamma_n < 1$, and $\sum_{n=1}^{\infty} |\gamma_n| < \infty$, there exists $p > 1$ such that $p\gamma_n < 1$, for all $n \in \mathbb{N}$. Therefore

$$\begin{aligned} & \lim_{L \rightarrow \infty} \prod_{n=1}^L (1 - p\gamma_n)^{-1/2} \exp \left\{ \frac{1}{2} \frac{|\langle Q^{1/2}b, g_n \rangle|^2}{1 - p\gamma_n} \right\} \\ &= \left[\prod_{n=1}^{\infty} (1 - p\gamma_n) \right]^{-1/2} \exp \left\{ \frac{1}{2} \sum_{n=1}^{\infty} \frac{|\langle Q^{1/2}b, g_n \rangle|^2}{1 - p\gamma_n} \right\}. \end{aligned}$$

So the sequence $\left(\exp \left\{ \sum_{n=1}^L \left[\frac{1}{2} \gamma_n |W_{g_n}(x)|^2 + \langle Q^{1/2}b, g_n \rangle W_{g_n}(x) \right] \right\} \right)$ is uniformly integrable. Consequently, passing to the limit, we find

$$\begin{aligned} & \int_H \exp \{ 1/2 \langle My, y \rangle + \langle b, y \rangle \} N_Q(dy) \\ &= \lim_{L \rightarrow \infty} \int_H \exp \left\{ \sum_{n=1}^L \left[1/2 \gamma_n |W_{g_n}(x)|^2 + \langle Q^{1/2}b, g_n \rangle W_{g_n}(x) \right] \right\} N_Q(dx) \\ &= \lim_{L \rightarrow \infty} \prod_{n=1}^L (1 - \gamma_n)^{-1/2} \exp \left\{ \frac{1}{2} \frac{|\langle Q^{1/2}b, g_n \rangle|^2}{1 - \gamma_n} \right\} \\ &= \prod_{n=1}^{\infty} (1 - \gamma_n)^{-1/2} \exp \left\{ \frac{1}{2} \frac{|\langle Q^{1/2}b, g_n \rangle|^2}{1 - \gamma_n} \right\} \\ &= \left(\det(1 - Q^{1/2}MQ^{1/2}) \right)^{-1/2} \exp \left\{ \frac{1}{2} |\langle (1 - Q^{1/2}MQ^{1/2})^{-1/2} Q^{1/2}b \rangle|^2 \right\}. \quad \square \end{aligned}$$

Remark 1.2.9 It follows from the proof of the proposition that

$$\langle Mx, x \rangle = \sum_{k=1}^{\infty} \gamma_k W_{g_k}^2(x) = \sqrt{2} \sum_{k=1}^{\infty} \gamma_k \left[2^{-1/2} (W_{g_k}^2(x) - 1) \right] + \sum_{k=1}^{\infty} \gamma_k,$$

and so, by Proposition 1.2.6, we have

$$\begin{aligned} \int_H [\langle Mx, x \rangle]^2 N_Q(dx) &= 2 \sum_{k=1}^{\infty} \gamma_n^2 + \left(\sum_{k=1}^{\infty} \gamma_n \right)^2 \\ &= 2 \|Q^{1/2} M Q^{1/2}\|_{L_2(H)}^2 + (\text{Tr } Q^{1/2} M Q^{1/2})^2 \\ &< +\infty. \end{aligned}$$

Proposition 1.2.10 *Let $T \in L_1(H)$. Then there exists the limit*

$$\langle TQ^{-1/2}y, Q^{-1/2}y \rangle := \lim_{n \rightarrow \infty} \langle TQ^{-1/2}P_n y, Q^{-1/2}P_n y \rangle, \quad N_Q\text{-a.e.},$$

where $P_n = \sum_{k=1}^n e_k \otimes e_k$.

Moreover we have the following expansion in $L^2(H, N_Q)$:

$$\begin{aligned} \langle TQ^{-1/2}y, Q^{-1/2}y \rangle &= \sum_{n=1}^{\infty} \langle Tg_n, g_n \rangle + \sum_{m \neq n=1}^{\infty} \langle Tg_n, g_m \rangle W_{g_n} W_{g_m} \\ &\quad \times \sqrt{2} \sum_{n=1}^{\infty} \langle Tg_n, g_n \rangle \left[2^{-1/2} (W_{g_n}^2 - 1) \right]. \quad (1.2.11) \end{aligned}$$

The proof of the following result is similar to that of Claim 2 in the proof of Proposition 1.2.8 and it is left to the reader.

Proposition 1.2.11 *Assume that M is a symmetric trace-class operator such that $M < 1$,⁽⁵⁾ and $b \in H$. Then*

$$\begin{aligned} \int_H \exp \left\{ 1/2 \langle M Q^{-1/2}y, Q^{-1/2}y \rangle + \langle b, Q^{-1/2}y \rangle \right\} N_Q(dy) \\ = (\det(1 - M))^{-1/2} e^{\frac{1}{2} \langle (1-M)^{-1/2}b, b \rangle}. \quad (1.2.12) \end{aligned}$$

1.3 Absolute continuity of Gaussian measures

We consider here two Gaussian measures μ, ν . We want to prove the Feldman-Hajek theorem, that is they are either singular or equivalent.

⁵That is $\langle Mx, x \rangle < |x|^2$ for all $x \neq 0$.

In §1.3.1 we recall some results on equivalence of measures on \mathbb{R}^∞ including the Kakutani theorem. In §1.3.2 we consider the case when $\mu = N_Q$ and $\nu = N_{a,Q}$ with $Q \in L_1^+(H)$ and $a \in H$, proving the Cameron-Martin formula. Finally in §1.3.3 we consider the more difficult case when $\mu = N_Q$ and $\nu = N_R$ with $Q, R \in L_1^+(H)$.

1.3.1 Equivalence of product measures in \mathbb{R}^∞

It is convenient to introduce the notion of *Hellinger* integral.

Let μ, ν be probability measures on a measurable space (E, \mathcal{E}) . Then $\lambda = \frac{1}{2}(\mu + \nu)$ is also a probability measure on (E, \mathcal{E}) and we have obviously

$$\mu \ll \lambda, \quad \nu \ll \lambda.$$

We define the *Hellinger integral* by

$$H(\mu, \nu) = \int_E \left[\frac{d\mu}{d\lambda}(x) \frac{d\nu}{d\lambda}(x) \right]^{1/2} \lambda(dx).$$

Instead of $\frac{1}{2}(\mu + \nu)$ one could choose as λ any measure equivalent to $\frac{1}{2}(\mu + \nu)$ without changing the value of $H(\mu, \nu)$.

By using Hölder's inequality we see that

$$|H(\mu, \nu)|^2 \leq \int_E \frac{d\mu}{d\lambda}(x) \lambda(dx) \int_E \frac{d\nu}{d\lambda}(x) \lambda(dx) = 1,$$

so that $0 \leq H(\mu, \nu) \leq 1$.

Exercise 1.3.1 (a) Let $\mu = N_q$ and $\nu = N_{a,q}$, where $a \in \mathbb{R}$ and $q > 0$. Show that we have

$$H(\mu, \nu) = e^{-\frac{a^2}{4q}}. \quad (1.3.1)$$

(b) Let $\mu = N_q$ and $\nu = N_\rho$, where $q, \rho > 0$. Show that we have

$$H(\mu, \nu) = \left[\frac{4q\rho}{(q + \rho)^2} \right]^{1/4}. \quad (1.3.2)$$

Proposition 1.3.2 Assume that $H(\mu, \nu) = 0$. Then the measures μ and ν are singular.

Proof. Set $\alpha = \frac{d\mu}{d\lambda}$, $\beta = \frac{d\nu}{d\lambda}$. Since $H(\mu, \nu) = \int_{\Omega} \sqrt{\alpha\beta} d\lambda = 0$, we have $\alpha\beta = 0$, λ -a.e. Consequently, setting

$$A = \{\omega \in \Omega : \alpha(\omega) = 0\}, \quad B = \{\omega \in \Omega : \beta(\omega) = 0\},$$

we have $\lambda(A \cup B) = 1$. This means that $\lambda(C) = 0$ where $C = \Omega \setminus (A \cup B)$, and hence $\mu(C) = \nu(C) = 0$. Then, as

$$\mu(A) = \int_A \alpha d\lambda = 0, \quad \nu(B) = \int_B \beta d\lambda = 0,$$

we have that μ and ν are singular since

$$\mu(A \cup C) = \nu(B) = 0, \quad (A \cup C) \cap B = \emptyset. \quad \square$$

Proposition 1.3.3 *Let $\mathcal{G} \subset \mathcal{E}$ be a σ -algebra, and let $\mu_{\mathcal{G}}$ and $\nu_{\mathcal{G}}$ be the restrictions of μ and ν to (E, \mathcal{G}) . Then we have $H(\mu, \nu) \leq H(\mu_{\mathcal{G}}, \nu_{\mathcal{G}})$.*

Proof. Let $\lambda_{\mathcal{G}}$ be the restriction of λ to (E, \mathcal{G}) . It is easy to check that

$$\frac{d\mu_{\mathcal{G}}}{d\lambda_{\mathcal{G}}} = E_{\lambda} \left(\frac{d\mu}{d\lambda} \middle| \mathcal{G} \right) \quad \frac{d\nu_{\mathcal{G}}}{d\lambda_{\mathcal{G}}} = E_{\lambda} \left(\frac{d\nu}{d\lambda} \middle| \mathcal{G} \right), \quad \lambda\text{-a.e.}^{(6)}$$

Consequently we have ⁽⁷⁾

$$H(\mu_{\mathcal{G}}, \nu_{\mathcal{G}}) = \int_E \left[E_{\lambda} \left(\frac{d\mu}{d\lambda} \middle| \mathcal{G} \right) E_{\lambda} \left(\frac{d\nu}{d\lambda} \middle| \mathcal{G} \right) \right]^{1/2} d\lambda.$$

Since λ -a.e.

$$\frac{\left[\frac{d\mu}{d\lambda} \frac{d\nu}{d\lambda} \right]^{1/2}}{\left[E_{\lambda} \left(\frac{d\mu}{d\lambda} \middle| \mathcal{G} \right) E_{\lambda} \left(\frac{d\nu}{d\lambda} \middle| \mathcal{G} \right) \right]^{1/2}} \leq \frac{1}{2} \left(\frac{\frac{d\mu}{d\lambda}}{E_{\lambda} \left(\frac{d\mu}{d\lambda} \middle| \mathcal{G} \right)} + \frac{\frac{d\nu}{d\lambda}}{E_{\lambda} \left(\frac{d\nu}{d\lambda} \middle| \mathcal{G} \right)} \right),$$

taking conditional expectations of both sides one finds, λ -a.e.,

$$\left[E_{\lambda} \left(\frac{d\mu}{d\lambda} \middle| \mathcal{G} \right) E_{\lambda} \left(\frac{d\nu}{d\lambda} \middle| \mathcal{G} \right) \right]^{1/2} \geq E_{\lambda} \left(\left(\frac{d\mu}{d\lambda} \right)^{1/2} \left(\frac{d\nu}{d\lambda} \right)^{1/2} \middle| \mathcal{G} \right). \quad (1.3.3)$$

⁶ $E_{\lambda}(\eta|\mathcal{G})$ is the conditional expectation of the random variable η with respect to \mathcal{G} and measure λ .

⁷For positive numbers a, b, c, d , $\sqrt{\frac{ab}{cd}} \leq \frac{1}{2} \left(\frac{a}{c} + \frac{b}{d} \right)$.

Integrating with respect to λ both sides of (1.3.3), the required result follows. \square

Now let us consider two sequences of measures (μ_k) and (ν_k) on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $\nu_k \sim \mu_k$ for all $k \in \mathbb{N}$. We set $\lambda_k = \frac{1}{2}(\mu_k + \nu_k)$, and we consider the Hellinger integral

$$H(\mu_k, \nu_k) = \int_{\mathbb{R}} \left[\frac{d\mu_k}{d\lambda_k}(x) \frac{d\nu_k}{d\lambda_k}(x) \right]^{1/2} \lambda_k(dx), \quad k \in \mathbb{N}.$$

Remark 1.3.4 Since (μ_k) and (ν_k) are equivalent, we have

$$\frac{d\mu_k}{d\lambda_k} \frac{d\nu_k}{d\lambda_k} = \frac{d\mu_k}{d\lambda_k} \frac{d\nu_k}{d\mu_k} \frac{d\mu_k}{d\lambda_k} = \frac{d\nu_k}{d\mu_k} \left(\frac{d\mu_k}{d\lambda_k} \right)^2.$$

Thus

$$H(\mu_k, \nu_k) = \int_{\mathbb{R}} \left[\frac{d\nu_k}{d\mu_k}(x) \right]^{1/2} \mu_k(dx). \quad (1.3.4)$$

We also consider the product measures on \mathbb{R}^∞

$$\mu = \prod_{k=1}^{\infty} \mu_k, \quad \nu = \prod_{k=1}^{\infty} \nu_k,$$

and the corresponding Hellinger integral $H(\mu, \nu)$. As is easily checked we have

$$H(\mu, \nu) = \prod_{k=1}^{\infty} H(\mu_k, \nu_k).$$

Proposition 1.3.5 (Kakutani) *If $H(\mu, \nu) > 0$ then μ and ν are equivalent. Moreover*

$$f(x) := \frac{d\nu}{d\mu}(x) = \prod_{k=1}^{\infty} \frac{d\nu_k}{d\mu_k}(x_k), \quad x \in \mathbb{R}^\infty, \quad \mu\text{-a.e.} \quad (1.3.5)$$

Proof. We set

$$f_n(x) = \prod_{k=1}^n \frac{d\nu_k}{d\mu_k}(x_k), \quad x \in \mathbb{R}^\infty, \quad n \in \mathbb{N}.$$

We are going to prove that the sequence (f_n) is convergent on $L^1(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty), \mu)$. Let $m, n \in \mathbb{N}$, then we have

$$\begin{aligned}
& \int_{\mathbb{R}^\infty} \left| f_{n+m}^{1/2}(x) - f_n^{1/2}(x) \right|^2 \mu(dx) \\
&= \int_{\mathbb{R}^\infty} \prod_{k=1}^n \frac{d\nu_k}{d\mu_k}(x_k) \left| \prod_{k=n+1}^{n+m} \left(\frac{d\nu_k}{d\mu_k}(x_k) \right)^{1/2} - 1 \right|^2 \mu(dx) \\
&= \prod_{k=1}^n \int_{\mathbb{R}^\infty} \frac{d\nu_k}{d\mu_k}(x_k) \mu(dx) \int_{\mathbb{R}^\infty} \left| \prod_{k=n+1}^{n+m} \left(\frac{d\nu_k}{d\mu_k}(x_k) \right)^{1/2} - 1 \right|^2 \mu(dx).
\end{aligned}$$

Consequently

$$\begin{aligned}
& \int_{\mathbb{R}^\infty} |f_{n+p}^{1/2}(x) - f_n^{1/2}(x)|^2 \mu(dx) \\
&= \int_{\mathbb{R}^\infty} \left[\prod_{k=n+1}^{n+p} \frac{d\nu_k}{d\mu_k}(x_k) - 2 \prod_{k=n+1}^{n+p} \left(\frac{d\nu_k}{d\mu_k}(x_k) \right)^{1/2} + 1 \right] \mu(dx) \\
&= 2 \left(1 - \prod_{k=n+1}^{n+p} \int_{\mathbb{R}} \left(\frac{d\nu_k}{d\mu_k}(x_k) \right)^{1/2} \mu_k(dx_k) \right) \\
&= 2 \left(1 - \prod_{k=n+1}^{n+p} H(\mu_k, \nu_k) \right). \tag{1.3.6}
\end{aligned}$$

On the other hand we know by assumption that

$$H(\mu, \nu) = \prod_{k=1}^{\infty} H(\mu_k, \nu_k) > 0,$$

or, equivalently, that

$$-\log H(\mu, \nu) = -\sum_{k=1}^{\infty} \log[H(\mu_k, \nu_k)] < +\infty.$$

Consequently, for any $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that if $n > n_\varepsilon$ and $p \in \mathbb{N}$, we have

$$- \sum_{k=n+1}^{n+p} \log[H(\mu_k, \nu_k)] < \varepsilon.$$

By (1.3.6) if $n > n_\varepsilon$ we have

$$\int_{\mathbb{R}^\infty} |\sqrt{f_{n+p}} - \sqrt{f_n}|^2 d\mu \leq 2(1 - e^{-\varepsilon}).$$

Thus the sequence $(f_n^{1/2})$ is convergent on $L^2(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty), \mu)$ to some function $f^{1/2}$. Therefore $f_n \rightarrow f$ in $L^1(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty), \mu)$.

Finally, we prove that $\nu \ll \mu$ and $f = \frac{d\nu}{d\mu}$. Let φ be a continuous bounded Borel function on \mathbb{R}^∞ , and set $\varphi_n(x) = \varphi(P_n(x))$, $x \in \mathbb{R}^\infty$, where $P_n x = \{x_1, \dots, x_n, 0, 0, \dots\}$. Then we have

$$\begin{aligned} \int_{\mathbb{R}^\infty} \varphi(P_n x) \nu(dx) &= \int_{\mathbb{R}^n} \varphi(P_n x) \nu_1(dx_1) \dots \nu_n(dx_n) \\ &= \int_{\mathbb{R}^n} \varphi(P_n x) \frac{d\nu_1}{d\mu_1}(x_1) \dots \frac{d\nu_n}{d\mu_n}(x_n) \mu_1(dx_1) \dots \mu_n(dx_n) \\ &= \int_{\mathbb{R}^\infty} \varphi(P_n x) f_n(x) \mu(dx). \end{aligned}$$

Letting n tend to infinity, we find

$$\int_{\mathbb{R}^\infty} \varphi(x) \nu(dx) = \int_{\mathbb{R}^\infty} \varphi(x) f(x) \mu(dx),$$

so that $\nu \ll \mu$. Finally, by exchanging the rôles of μ and ν , we find $\mu \ll \nu$. \square

1.3.2 The Cameron-Martin formula

We consider here the measures $\mu = N_{a,Q}$ and $\nu = N_Q$, and for any $a \in Q^{1/2}(H)$ we set

$$\rho_a(x) = \exp \left\{ -\frac{1}{2} |Q^{-1/2} a|^2 + \langle Q^{-1/2} a, Q^{-1/2} x \rangle \right\}, \quad x \in H. \quad (1.3.7)$$

Let us recall, see §1.2.4, that $W_f(x) = \langle f, Q^{-1/2} x \rangle$ was defined for all $f \in \overline{Q^{1/2}(H)}$. Since $Q^{-1/2} a \in Q^{1/2}(H)$ the definition (1.3.7) is meaningful.