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Second-Order SM Approach to SISO Time-Delay System Output Tracking

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***Abstract*—A fully linearizable single-input–single-output relative-degree n system with an output time delay is considered in this paper. Using the approach of Padé approximation, system center approach, and second-order sliding-mode (SM) control, we have obtained good output tracking results. The Smith predictor is used to compensate the difference between the actual delayed output and its approximation. A second-order supertwisting SM observer observes the disturbance in the plant. A nonlinear example is studied to show the effect of this methodology.**

***Index Terms*—Output delay, Padé approximation, second-order sliding-mode (SM) control, stable system center.**

I. Introduction

OUTPUT time delay is a common feature in many systems and must be taken into account when designing a controller. The output tracking of a real-time reference profile in nonlinear systems with output delay by sliding-mode (SM) control was addressed in this paper [22]. In addition to the first-order Padé approximation, the more precise second- and third-order Padé approximations have been used to replace the output delay element [12], [17].

n the literature, second-order SM control has been widely used and yields better accuracy than standard SM control [2], [13]–[16]. In this paper, we use the second-order SM control to study the fully linearizable single-input–single-output (SISO) time-delay-system output tracking problem. We use a new transformation to transfer the relative-degree n system into a relative-degree two system. With first-, second-, and third-order Padé approximations, we transfer the time-delay-system tracking problem into a nonminimum-phase-system output tracking problem. A stable system center approach and second-order SM control have been used to get good output tracking results.

In real life, we need to feed back the actual delayed output rather than its approximation, and this yields limit cycles. We use the Smith predictor (SP) [23] to compensate the difference between the approximate and the actual delayed output and obtain greatly improved output tracking results. When disturbances are present, the output tracking accuracy is lost. We use a second-order supertwisting SM observer to observe the disturbances and obtain good output tracking results. The one-link robot arm [11] is a fourth-order nonlinear system. We have used our methodology on this example and got good output tracking results for this output-delay problem.

This paper is organized as follows. Section II is dedicated to a new transformation and the output tracking-problem formulation. Section III uses the Padé approximation to approximate the delayed system. Section IV presents the system center and second-order SM control to solve the output tracking problem. A numerical example demonstrating the various aspects of different Padé approximations is given in Section V. In Section VI, we present results when feeding back the actual delayed output and compensation results using the SP and a second-order supertwisting SM observer. A one-link-robot-arm example output delay tracking problem is considered using the approach in this paper, and good output tracking results are obtained in Section VII. The conclusions are summarized in Section VIII.

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II. Problem Formulation

Consider a controllable fully feedback linearizable nonlinear SISO dynamic system without time delay

$$\dot{x} = f(x,t) + g(x,t)u \qquad y = h(x) \tag{1}$$

where $x(t) \in R^n$ is a state vector, $y(t) \in R^1$ is a controlled output, and $u(t) \in R^1$ is a control input.

As a fully linearizable relative-degree n system, it can be transformed [11] to

$$y^{(n)} = \phi(\xi,t) + b(\xi,t)u \tag{2}$$

where $\xi = [y, \dot{y}, \ldots, y^{(n-1)}]^T \in R^n$.

A. Output Redefinition

Gopalswamy and Hedrick [10] define a coordinate transformation

$$\begin{bmatrix} z \\ q_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 \\ a_0 & a_1 & \cdots & a_{n-2} & 1 \end{bmatrix} \xi \tag{3}$$

where

$$q_1 = y^{(n-1)} + a_{n-2}y^{(n-2)} + \cdots + a_1\dot{y} + a_0 y \tag{4}$$

is a new output to get a relative-degree one system and a_i is selected to be the coefficient of a Hurwitz polynomial, $z(t) \in R^{n-1}$ and $q(t) \in R^1$.

Note that, by Gopalswamy and Hedrick's transformation [10], the system is transferred from a relative-degree n system to a relative-degree one system by the redefinition of output q. The output tracking problem of the original system can be equivalently solved by the new output with suitable choices of a_i. Kosiba *et al.* [12], Liu *et al.* [17], and Shtessel *et al.* [22] have used this transformation to get good output tracking results.

Following this approach, we let the first $n-1$ rows of ξ be ϑ, i.e., $\vartheta = [y, \dot{y}, \ldots, y^{(n-2)}]^T \in R^{n-1}$, and define a different new coordinate transformation

$$\begin{bmatrix} z \\ q_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 \\ a_0 & a_1 & \cdots & a_{n-3} & 1 \end{bmatrix} \vartheta \tag{5}$$

where

$$q_1 = y^{(n-2)} + a_{n-3}y^{(n-3)} + \cdots + a_1\dot{y} + a_0 y \tag{6}$$

is a new output. We get a relative-degree two system, and the a_i's are selected to be the coefficients of a Hurwitz polynomial, $z(t) \in R^{n-2}$ and $q_1(t) \in R^1$.

Note that the difference between our transformation and that of Gopalswamy and Hedrick [10] is that we transfer the original system to a relative-degree two system by the redefinition of output q_1.

The system (1) is rewritten in a new basis $[z, q_1, \dot{q}_1]^T$ as

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = z_3 \\ \cdots\ \cdots \\ \dot{z}_{n-2} = q_1 - a_{n-3}z_{n-2} - \cdots - a_1 z_2 - a_0 z_1 \\ \dot{q}_1 = q_2 \\ \dot{q}_2 = \hat{\phi}(z, q, t) + \hat{b}(z, q, t)u \end{cases} \tag{7}$$

where

$$\begin{aligned} \hat{\phi}(z, q, t) =& a_{n-3}q_2 - a_{n-3} \\ & \times [a_{n-3}(q_1 - a_{n-3}z_{n-3} - \cdots - a_1 z_1) \\ & \quad + a_{n-4}z_{n-2} + \cdots + a_0 z_2] \\ & + a_{n-4}q_1 - a_{n-4}(a_{n-3}z_{n-3} + \cdots + a_0 z_0) \\ & + a_{n-5}z_{n-3} + \cdots + a_0 y^2 + \phi(z, q_1, q_2, t) \\ \hat{b} =& b(z, q_1, q_2, t). \end{aligned} \tag{8}$$

Note that the internal dynamics of the system (7) is stable and can be disregarded when solving the output tracking problem as time increases. Therefore, output tracking in system (1) can be transformed to the output tracking of the scalar system

$$\begin{aligned} \dot{q}_1 &= q_2 \\ \dot{q}_2 &= \hat{\phi}(z, q_1, q_2, t) + \hat{b}(z, q_1, q_2, t)u. \end{aligned} \tag{9}$$

B. Problem Formulation

Assume here that the desired command output profile is given in the form

$$y_c = \bar{A} + \bar{B}\sin\omega_n t. \tag{10}$$

In the following discussion, we will abuse notation by setting $q_1 = q$. Then, we can assume the output tracking (command) profile q_c (corresponding to q) as follows:

$$q_c = A + B\sin\omega_n t + C\cos\omega_n t \tag{11}$$

where A, B, C, and ω_n are piecewise constants and A, B, and C can be calculated using the output redefinition function (6). The signal (11) can be described by a linear system of exogenous differential equations with the following characteristic polynomial:

$$\lambda^3 + 0\lambda^2 + \omega_n^2\lambda + 0. \tag{12}$$

Now, we restrict $\hat{\phi}(\cdot)$ to be bounded. $\hat{b}(\cdot)$ is nonsingular. Assume that the output of y is accessible with a fixed time delay τ. This is equivalent to the modified output q with a time delay τ; we define

$$\hat{y}(t) = q(t - \tau). \tag{13}$$

The problem is to design SM control $u(t)$ that forces the output variable $\hat{y}$ to track asymptotically the command profile q_c or the delayed output $y(t-\tau)$ to track asymptotically y_c.

Note that the fixed time delay τ should not be assumed large since the Padé approximation works only for smaller time delays. The output tracking results for different time delays are discussed in [18].

III. Padé Approximations and Time-Delay Systems

In this section, we use the Padé approximation to approximate the time delay. The Padé approximation [1] uses the quotient of two polynomials to estimate a power series. We will use the direct solutions of the Padé equations

$$[L/M] = P_L(x)/Q_M(x)$$

where $P_L(x)$ is a polynomial of degree L and $Q_M(x)$ is a polynomial of degree M. When we approximate a formal power series $f(x)$, the explicit equation is

$$\lim_{x\to 0} \frac{Q_M(x)f(x) - P_L(x)}{x^{L+M}} = 0.$$

The corresponding Laplace transform of (13) $\hat{y}(s)/q(s) = e^{-s\tau}$ can always be approximated by first-, second- and third-order Padé approximations as

$$\begin{aligned} e^{-s\tau} &\approx \frac{1 - \frac{s\tau}{2}}{1 + \frac{s\tau}{2}} \\ e^{-s\tau} &\approx \frac{1 - \frac{s\tau}{2} + \frac{s^2\tau^2}{12}}{1 + \frac{s\tau}{2} + \frac{s^2\tau^2}{12}} \\ e^{-s\tau} &\approx \frac{1 - \frac{s\tau}{2} + \frac{s^2\tau^2}{10} - \frac{s^3\tau^3}{120}}{1 + \frac{s\tau}{2} + \frac{s^2\tau^2}{10} + \frac{s^3\tau^3}{120}} \end{aligned} \tag{14}$$

where s is the Laplace variable.

We introduce a new output variable $\tilde{y}$ to make the approximation exact. By introducing a new variable $\eta = \tilde{y} + (-1)^j q$, the system (13) can be approximated by

$$\begin{cases} \dot{\zeta} = \tilde{Q}_1\zeta + \tilde{Q}_2\tilde{y} \\ \dot{\tilde{y}} = \dot{\eta} - (-1)^j q_2 \\ \dot{q}_2 = \hat{\phi}(\tilde{y}, q_2, \zeta, t) + \hat{b}(\tilde{y}, q_2, \zeta)u \end{cases} \tag{15}$$

where η is the first row of ζ and j is the Padé order. For the first-order Padé approximation

$$\tilde{Q}_1 = \frac{2}{\tau} \qquad \tilde{Q}_2 = -\frac{4}{\tau} \tag{16}$$

for the second order

$$\tilde{Q}_1 = \begin{bmatrix} 0 & 1 \\ -\frac{12}{\tau^2} & \frac{6}{\tau} \end{bmatrix} \qquad \tilde{Q}_2 = -\frac{12}{\tau}\begin{bmatrix} 1 \\ \frac{6}{\tau} \end{bmatrix} \tag{17}$$

and for the third order

$$\tilde{Q}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{120}{\tau^3} & -\frac{60}{\tau_2} & \frac{12}{\tau} \end{bmatrix} \qquad \tilde{Q}_2 = -\begin{bmatrix} \frac{24}{\tau} \\ \frac{288}{\tau^2} \\ \frac{2256}{\tau^3} \end{bmatrix}. \tag{18}$$

For (16)–(18), respectively, system (15) is nonminimum phase because $\tilde{Q}_1$ has at least one non-Hurwitz eigenvalue.

Note that the output delay system tracking problem has been transformed into a nonminimum-phase-system output tracking problem (with no delay) by the Padé approximation. However, there is an important difference between $\hat{y}$ and $\tilde{y}$. We will investigate this in Section VI.

IV. System Center Method and Second-Order SM Control

The nonminimum-phase-system tracking problem is solved in this section by the system center method and SM control.

The equation of the system center (ideal internal dynamics) [21] that defines a command (tracking) profile $\zeta_c(t)$ for the internal state vector for the system with delay approximated by the nonminimum-phase system (13) is

$$\dot{\zeta}_c = \tilde{Q}_1\zeta_c + \tilde{Q}_2 q_c \tag{19}$$

which is unstable.

According to the theorem of system center [21], the stable system center $\tilde{\zeta}_c(t)$ that converges asymptotically to the bounded particular solution $\zeta_c(t)$ of the unstable equation of system center, i.e., $\tilde{\zeta}_c(t) \to \zeta_c(t)$, is given by

$$\tilde{\zeta}_c^{(3)} + c_2\tilde{\zeta}_c^{(2)} + c_1\tilde{\zeta}_c^{(1)} + c_0\tilde{\zeta}_c = -\left(P_2\theta_c^{(2)} + P_1\theta_c^{(1)} + P_0\theta_c\right) \tag{20}$$

where $\theta_c = \tilde{Q}_2 q_c$ and the coefficients of P_0, P_1, and P_2 are computed as follows:

$$\begin{cases} P_0 = c_0\tilde{Q}_1^{-1} \\ P_1 = \frac{\left(c_1-\omega_n^2\right)\tilde{Q}_1^{-1}+\left(c_0-c_2\omega_n^2\right)\tilde{Q}_1^{-2}}{I+\omega_n^2\tilde{Q}_1^{-2}} \\ P_2 = \frac{c_2\tilde{Q}_1^{-1}+\left(c_1-\omega_n^2\right)\tilde{Q}_1^{-2}+c_0\tilde{Q}_1^{-3}}{I+\omega_n^2\tilde{Q}_1^{-2}}. \end{cases} \tag{21}$$

The coefficients c_0, c_1, and c_2 are chosen to provide the specified eigenvalues of the homogeneous differential equation

$$\tilde{\zeta}_c^{(3)} + c_2\tilde{\zeta}_c^{(2)} + c_1\tilde{\zeta}_c^{(1)} + c_0\tilde{\zeta}_c = 0. \tag{22}$$

Next, we will use the second SM control technique to the output delay system tracking problem.

The sliding function is defined as

$$\sigma = e_q + C_1\tilde{e}_\zeta \tag{23}$$

where $e_q = q_c - \tilde{y}$ and $\tilde{e_\zeta} = \tilde{\zeta}_c - \zeta$. Therefore

$$\begin{aligned} \ddot{\sigma} &= \ddot{q}_c - \ddot{\tilde{y}} + C_1\left(\ddot{\tilde{\zeta}}_c - \ddot{\zeta}\right) \\ &= (-1)^{j+1}(\hat{\phi} + \hat{b}u) + \ddot{q}_c - \ddot{\eta} + C_1\left(\ddot{\tilde{\zeta}}_c - \ddot{\zeta}\right). \end{aligned}$$

Using higher order SM control [13] and quasi-continuous second-order SM control [14], [16], we can design suitable second-order SM controllers, respectively, as follows:

$$u = (-1)^j\hat{b}^{-1}\rho\,\mathrm{sign}\left(\dot{\sigma} + \lambda|\sigma|^{1/2}\mathrm{sign}(\sigma)\right) \tag{24}$$

$$u = (-1)^j\hat{b}^{-1}\rho\frac{\dot{\sigma} + |\sigma|^{1/2}\mathrm{sign}(\sigma)}{|\dot{\sigma}| + |\sigma|^{1/2}} \tag{25}$$

where ρ is a sufficiently large positive gains and λ is a positive constant.

The SM existence condition for the second-order SM controller (23) can be achieved by having a sufficiently large gain ρ. When we choose a variable $\psi = \dot{\sigma} + \lambda|\sigma|^{1/2}\mathrm{sign}(\sigma)$, $\dot{\psi}\psi = -\rho|\psi| + \Psi(\cdot)$, where $\Psi(\cdot)$ can be calculated by (23), (24), and the plant. Since $\Psi(\cdot)$ is bounded, we can choose a ρ that is sufficiently large so that ψ goes to zero after finite time, and then, $\dot{\sigma}$ and σ go to zero. The quasi-continuous SM controller (24) SM existence condition can be attributed to the existence condition of a second-order SM controller by the following inequality:

$$\begin{aligned} \frac{\dot{\sigma} + |\sigma|^{1/2}\mathrm{sign}\sigma}{|\dot{\sigma}| + |\sigma|^{1/2}} &\leq \frac{\dot{\sigma} + |\sigma|^{1/2}\mathrm{sign}\sigma}{\left|\dot{\sigma} + |\sigma|^{1/2}\mathrm{sign}\sigma\right|} \\ &= \mathrm{sign}\left(\dot{\sigma} + |\sigma|^{1/2}\mathrm{sign}\sigma\right). \end{aligned} \tag{26}$$

Assume that the SM exists on the sliding surface $\sigma = 0$; then, $e_q = -C_1\tilde{e}_\zeta$. Therefore

$$\begin{aligned} \dot{\tilde{e}}_\zeta &= \left(\tilde{Q}_1 - \tilde{Q}_2C_1\right)\tilde{e}_\zeta + \left(\dot{\tilde{\zeta}}_c - \tilde{Q}_1\tilde{\zeta}_c - \tilde{Q}_2q_c\right), \\ e_q &= -C_1\tilde{e}_\zeta. \end{aligned} \tag{27}$$

Since $\tilde{\zeta}_c(t) \to \zeta_c(t)$ with increasing time, then $\dot{\tilde{\zeta}}_c - \tilde{Q}_1\tilde{\zeta}_c - \tilde{Q}_2q_c \to \dot{\zeta}_c - \tilde{Q}_1\zeta_c - \tilde{Q}_2q_c = 0$. Therefore, the SM dynamics (27) asymptotically approaches the homogeneous differential equation

$$\dot{\tilde{e}}_\zeta = \left(\tilde{Q}_1 - \tilde{Q}_2C_1\right)\tilde{e}_\zeta. \tag{28}$$

C_1 is selected to provide asymptotic tracking error dynamics (27) to zero.

V. Numerical Example and Simulations

Consider the relative-degree two second-order system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_2 + u \\ y = x_1. \end{cases} \tag{29}$$

The desired command output profile is given in Section II and has the characteristic equation (11).

Now, assume that the system output y is accessible with a time delay $\underline{y} = y(t - \tau)$. This is equivalent to the modified output q with time delay τ, i.e., $\hat{y} = q(t - \tau)$. The problem is to design the SMC that provides asymptotic tracking $\hat{y} \to q_c$ $(\underline{y} \to y_c)$ as time increases. Since the system is already a relative-degree two system, $\hat{y} = \underline{y}$ and $q_c = y_c$.

Replacing the time-delay function by the respective first-, second-, and third-order Padé approximations, system (29) is approximately represented by the nonminimum-phase system without delay

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = -z_2 + u \\ \dot{\eta} = \frac{2}{\tau}\eta - \frac{4}{\tau}\tilde{y} \\ \dot{\tilde{y}} = -z_2 + \frac{2}{\tau}\eta - \frac{4}{\tau}\tilde{y} \end{cases} \tag{30}$$

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = -z_2 + u \\ \dot{\zeta}_1 = \zeta_2 - \frac{12}{\tau}\tilde{y} \\ \dot{\zeta}_2 = \frac{6}{\tau}\zeta_2 - \frac{12}{\tau^2} - \frac{72}{\tau^2}\tilde{y} \\ \dot{\tilde{y}} = z_2 - \frac{12}{\tau}\tilde{y} + \zeta_2 \end{cases} \tag{31}$$

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = -z_2 + u \\ \dot{\zeta}_1 = \zeta_2 - \frac{24}{\tau}\tilde{y} \\ \dot{\zeta}_2 = \zeta_3 - \frac{288}{\tau^2}\tilde{y} \\ \dot{\zeta}_3 = \frac{12}{\tau}\zeta_3 - \frac{60}{\tau^2}\zeta_2 + \frac{120}{\tau^3}\zeta_1 - \frac{2256}{\tau^3}\tilde{y} \\ \dot{\tilde{y}} = -z_2 + \zeta_2 - \frac{24}{\tau}\tilde{y} \end{cases} \tag{32}$$

where the output $\tilde{y}$ is an approximation to the original output $\hat{y}$.

Now, we select parameters as follows: $c_0 = 1000$, $c_1 = 300$, $c_2 = 30$, $\omega_n = 2$, $a_0 = 20$, $A = 1$, and $B = 2$. Furthermore, we choose $C_1 = -0.75$ for the first-order Padé approximation, $[-0.75,\ 0]$ for the second-order Padé approximation, and $[-0.75,\ 0,\ 0]$ for the third-order Padé approximation. The control is

$$u = -25\text{sign}\left(\dot{\sigma} + |\sigma|^{1/2}\text{sign}(\sigma)\right)$$

for the first- and third-order Padé approximations and

$$u = 25\text{sign}\left(\dot{\sigma} + |\sigma|^{1/2}\text{sign}(\sigma)\right)$$

for the second-order Padé approximation. This is consistent with the results in [17].

We have used MATLAB, Simulink, and Scilab to get good tracking results for $\tau = 0.2$ (see Fig. 1 for the first-order Padé results and Fig. 2 for the second-order results). For further details relating to the choices of design parameters and output tracking comparative results, consult [18]. Note that we have used $\tilde{y}$, the approximation of the actual output, in the feedback loop.

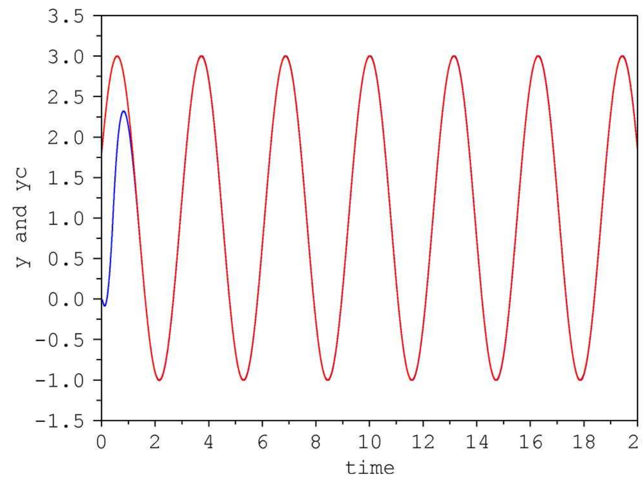

Fig. 1. Padé 1: Second-order SM control output tracking.

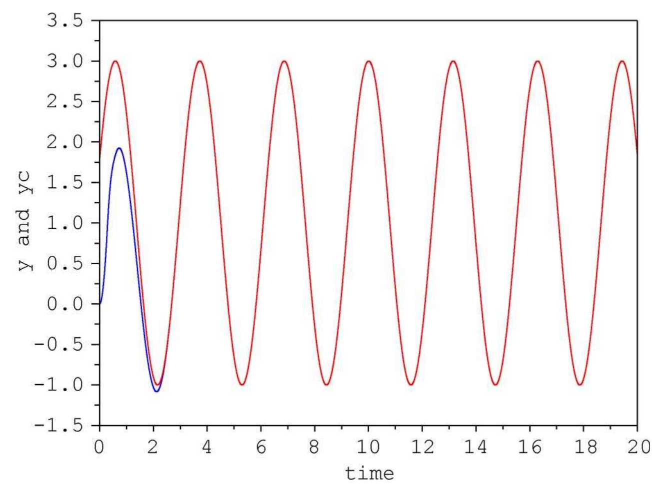

Fig. 2. Padé 2: Second-order SM control output tracking.

VI. Limit Cycles and the SP

In real-life situations, the feedback should be the *actual* delayed output $\hat{y}$ and not $\tilde{y}$, i.e., $\hat{y}$ should be included in the SM surface design. We consider the first-order Padé model to show the effect. Using the same example as in Section V, we use $\hat{y}$ in the feedback loop instead of $\tilde{y}$ and get output tracking errors as in Fig. 3. This limit cycle should be avoided in our design.

Next, we use the SP [24] for $\hat{y} - \tilde{y}$ to compensate the difference between $\hat{y}$ and $\tilde{y}$, i.e., $\hat{y} - (\hat{y} - \tilde{y}) = \tilde{y}$ is implemented in the feedback loop. Some preliminary ideas regarding the SP and SM are available (using the Gopalswamy and Hedrick [10] transformation) in a private memorandum [23].

The sliding surface including SP is calculated as

$$\sigma^{\text{SP}} = e_q^{\text{SP}} + C_1\tilde{e}_\eta$$

with $e_q^{\text{SP}} = q_c - [\hat{y} - (\hat{y} - \tilde{y})] = q_c - \tilde{y}$ and $\tilde{e}_\eta = \tilde{\eta} - \eta$.

Stabilizing σ^{SP} by means of the control u (used in the Padé model), the output tracking control of the original casual time-delay system is equivalent to controlling the system with the Padé model.

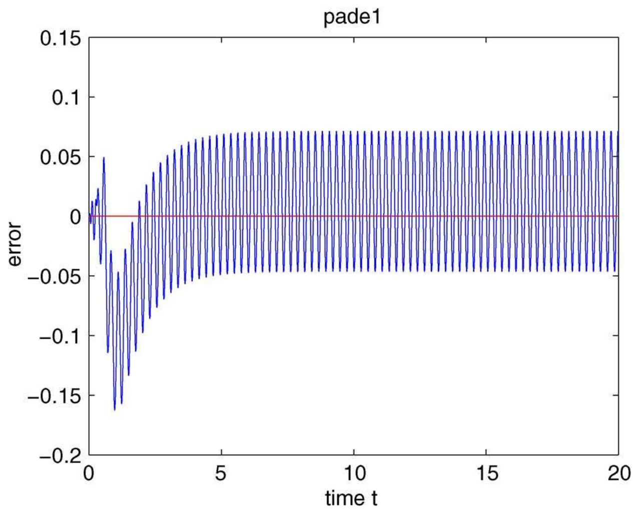

Fig. 3. Padé 1: $\hat{y}$ feedback output tracking error.

We use the example in Section V to test the effects of the system center approach and SP: the second-order SM control with $\hat{q}$ in the feedback loop (Section VI-A), stable system center approach and SM control with $\hat{q}$ in the feedback loop (Section VI-B), and the stable system center approach and SM control with the SP included in the feedback loop (Section VI-C). When unmatched disturbance is included in the system, the output tracking results become poor. We have used a second-order supertwisting SM observer [9] to observe the disturbance in the plant model and obtain almost perfect output tracking results in Section VI-D.

A. Padé Model With Actual Delayed Output in the Feedback Loop

We simulate the unperturbed system with actual delayed output in the feedback loop

$$\begin{aligned}\hat{y} &= q(t-0.2)\\ \hat{e}_q &= q_c - \hat{y}\\ \tilde{e}_\eta &= \tilde{\eta}_c - \eta\\ \sigma &= \hat{e}_q - 0.75\tilde{e}_\eta\\ u &= -25\mathrm{sign}\left(\dot{\sigma} + |\sigma|^{0.5}\mathrm{sign}(\sigma)\right)\end{aligned}$$

where -0.75 is the C_1 parameter we used in the Padé model and $\tilde{e}_\eta$ is from the system center approach.

The simulation results are shown in Fig. 4. Clearly, the tracking accuracy is not good enough; therefore, we will study the SP next to gain an improvement.

B. Padé Model With SP

The unperturbed system that incorporates the Padé model with the SP included in the feedback loop is simulated by the following controller:

$$\begin{aligned}\tilde{e}_\eta &= \tilde{\eta}_c - \eta\\ \hat{e}_q^{\mathrm{SP}} &= q_c - \tilde{y}\\ \sigma^{\mathrm{SP}} &= \hat{e}_q^{\mathrm{SP}} - 0.75\tilde{e}_\eta\\ u &= -25\mathrm{sign}\left(\dot{\sigma}^{\mathrm{SP}} + |\sigma^{\mathrm{SP}}|^{0.5}\mathrm{sign}(\sigma^{\mathrm{SP}})\right).\end{aligned}$$

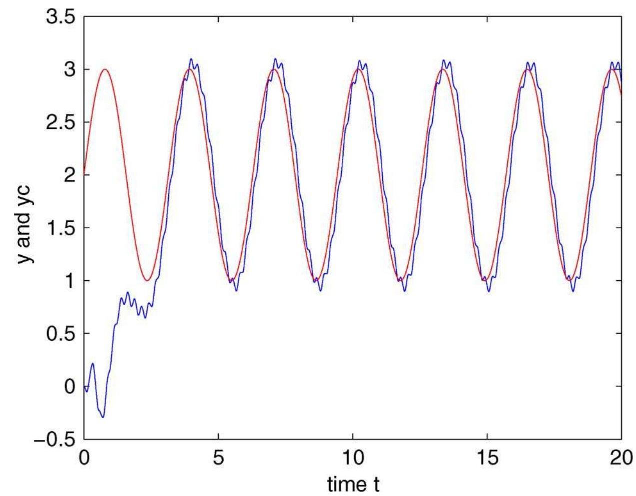

Fig. 4. y_c and y with feedback $q(t-0.2)$.

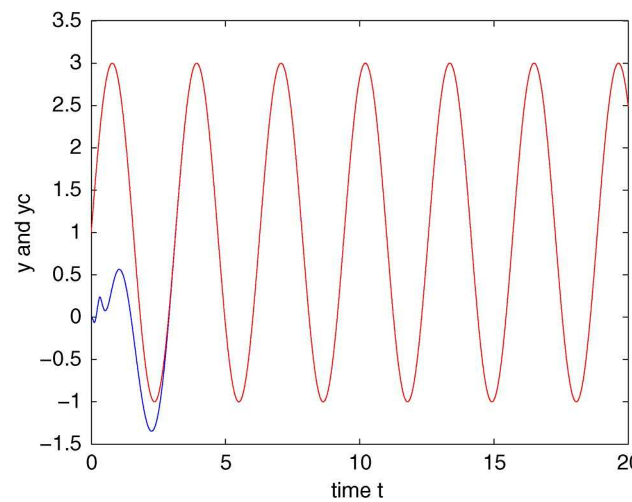

Fig. 5. y_c and y for SP incorporated approach.

Fig. 5 shows that the output tracking accuracy is greatly improved. Note that, by using an SP, we have replaced $\hat{q}$ by its approximation $\tilde{y}$ in the feedback loop. This example shows that the SP yields good results for the output tracking problem.

C. Disturbance Included in System

We now intentionally add a disturbance in the plant but we ignore it in the Padé model design since it is assumed *a priori* unknown.

The perturbed system can be expressed as

$$\dot{x}_1 = x_2 \tag{33}$$
$$\dot{x}_2 = -x_2 + \Delta\varphi(z,q,t) + (1+\Delta b(z,q,t))\,u \tag{34}$$
$$\hat{y} = y(t-0.2) \tag{35}$$
$$\dot{z}_1 = z_2 \tag{36}$$
$$\dot{z}_2 = -z_2 + u \tag{37}$$
$$\dot{\eta} = \frac{2}{\tau}\eta - \frac{4}{\tau}\tilde{y} \tag{38}$$
$$\dot{\tilde{y}} = -z_2 + \frac{2}{\tau}\eta - \frac{4}{\tau}\tilde{y} \tag{39}$$

with $\Delta b(z,q,t) = 0.5\sin 5t$ and

$$\Delta\varphi(z,q,t) = 0.2.$$

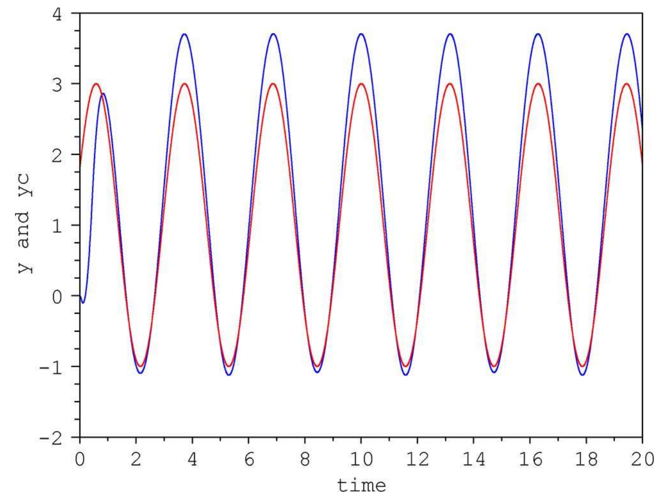

Fig. 6. y_c and y for system with disturbance.

The controller and all the other parameters such as C_1, ρ, and σ are selected as in Section V. From Fig. 6, we can clearly see that the tracking accuracy is lost because the disturbance affects the plant. Note that the Padé model with SP does not contain the disturbance terms $\Delta\varphi(z, q, t)$ and $\Delta b(z, q, t)$, since they are *a priori* unknown. This result shows that the SP is not good for disturbance avoidance.

D. Second-Order Supertwisting SM Observer

We can consider the Padé model (30) as an observer system for the original system (29). When disturbance is included, the unperturbed Padé model (36)–(39) needs the correction variables ν_1 and ν_2 as output injections [9]

$$\nu_1 = \lambda|x_1 - z_1|^{1/2}\text{sign}(x_1 - z_1) \tag{40}$$
$$\nu_2 = \alpha\text{sign}(x_1 - z_1) \tag{41}$$

where $\alpha = 20$ and $\lambda = 5$ in our design.

The new Padé model will be

$$\dot{z}_1 = z_2 + \nu_1 \tag{42}$$
$$\dot{z}_2 = -z_2 + u + \nu_2 \tag{43}$$
$$\dot{\eta} = \frac{2}{\tau}\eta - \frac{4}{\tau}\tilde{y} \tag{44}$$
$$\dot{\tilde{y}} = -z_2 + \frac{2}{\tau}\eta - \frac{4}{\tau}\tilde{y}. \tag{45}$$

These equations (42)–(45) are the observer system for (33)–(35). Using our second-order SM control design with SP, we get good output tracking results as shown in Fig. 7 because the second-order supertwisting SM observer is used in observing the disturbance.

VII. Nonlinear Example

We consider a simple nonlinear one-link robot arm [11] whose rotary motion about one end is controlled by means of an elastically coupled actuator. Elastic coupling between actuators and links is a phenomenon that cannot be neglected in many practical situations, and experience has shown that robot arms in which the motion is transmitted by means of long shafts or transmission belts or in which the actuator is a harmonic drive show a resonant behavior in the same range of frequencies used for control. The effect of elastic coupling between actuators and links, commonly referred to as joint elasticity, can be modeled by inserting a linear torsional spring at each joint, between the shaft of the actuator and the end about which the link is rotating. In the case of a simple one-link arm, the model thus obtained is shown in Fig. 8. The system can be described by means of two second-order differential equations, one characterizing the mechanical balance of the link. Using q_1 and q_2 to denote the angular positions of the actuator shaft and the link, with respect to a fixed reference frame, the actuator equation can be written in the form

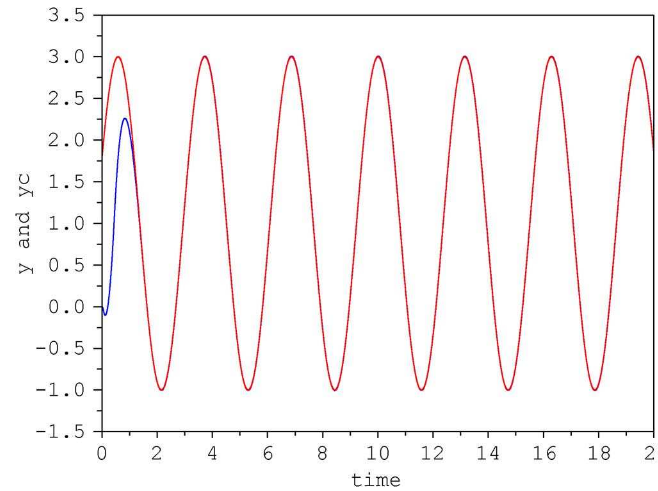

Fig. 7. y_c and y for system with disturbance and observer.

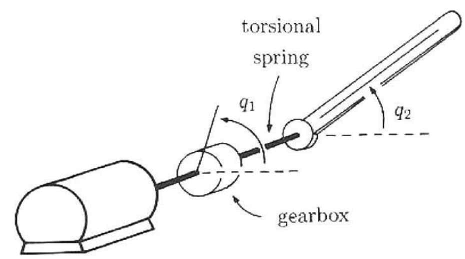

Fig. 8. One-link robot arm.

$$J_1\ddot{q}_1 + F_1\dot{q}_1 + \frac{K}{N}\left(q_2 - \frac{q_1}{N}\right) = T \tag{46}$$

where J_1 and F_1 represent inertia and viscous friction constants, K is the elasticity constant of the spring which represents the elastic coupling with the joint, and N is the transmission gear ratio. T is the torque produced at the actuator axis. The link equation can be written in a similar form

$$J_2\ddot{q}_2 + F_2\dot{q}_2 + K\left(q_2 - \frac{q_1}{N}\right) + mgd\cos(q_2) = 0 \tag{47}$$

in which m and d represent the mass and the position of the center of gravity of the link.

Choose the state vector

$$x = (q_1, q_2, \dot{q}_1, \dot{q}_2)'.$$

The system can be represented in the input–output form

$$\begin{aligned}\dot{x} &= f(x) + g(x)u \\ y &= h(x)\end{aligned} \tag{48}$$

with the input $u = T$, and

$$f(x) = \begin{bmatrix} x_3 \\ x_4 \\ -\frac{K}{J_1 N^2}x_1 + \frac{K}{J_1 N}x_2 - \frac{F_1}{J_1}x_3 \\ \frac{K}{J_2 N}x_1 - \frac{K}{J_2}x_2 - \frac{mgd}{J_2}\cos(x_2) - \frac{F_2}{J_2}x_4 \end{bmatrix}$$
$$g(x) = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{J_1} \\ 0 \end{bmatrix}. \tag{49}$$

Note that this is a fourth-order nonlinear system.

As the output of this system, it is natural to choose the angular position q_2 of the link with respect to the fixed reference

$$y = h(x) = x_2. \tag{50}$$

Note that the system has relative degree $r = n = 4$ at each point of the state space. Thus, this system can be exactly linearized via state feedback and coordinate transformation around any point of the state space. The linearizing feedback is

$$u = \frac{-L_f^4 h(x) + \nu}{L_g L_f^3 h(x)} \tag{51}$$

and the system in the normal form is

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= \frac{-L_f^4 h(x) + \nu}{L_g L_f^3 h(x)}\end{aligned} \tag{52}$$

where

$$L_f^4 h(x) = \frac{\partial f_4}{\partial x_4}\frac{\partial f_4}{\partial x_1}x_3 + \frac{\partial f_4}{\partial x_4}\frac{\partial f_4}{\partial x_2}x_4 + \left(\frac{\partial f_4}{\partial x_4}\frac{\partial f_4}{\partial x_3} + \frac{\partial f_4}{\partial x_1}\right) f_3 + \left(\frac{\partial f_4}{\partial x_2} + \left(\frac{\partial f_4}{\partial x_4}\right)^2\right) f_4$$

where f_3 and f_4 are the third and fourth rows of f, respectively, and $L_g L_f^3 h(x) = K/(J_1 J_2 N)$.

For simplicity, we select the parameters as follows:

$$\begin{aligned}K = J_1 = N = J_2 + F_1 = F_2 = J_2 = 1 \\ mgd = 1.\end{aligned} \tag{53}$$

Next, we want to use our second-order SM control approach to this example. We define a transformation

$$q = z_3 + 2a_0 z_2 + a_0^2 z_1 \tag{54}$$

and then, system (52) is

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= q - 2a_0 z_2 - a_0^2 z_1 \\ \dot{q} &= z_4 + 2a_0\left(q - a_0 z_2 - a_0^2 z_1\right) + a_0^2 z_2 \\ \dot{z}_4 &= \frac{-L_f^4 h(x) + \nu}{L_g L_f^3 h(x)}\end{aligned} \tag{55}$$

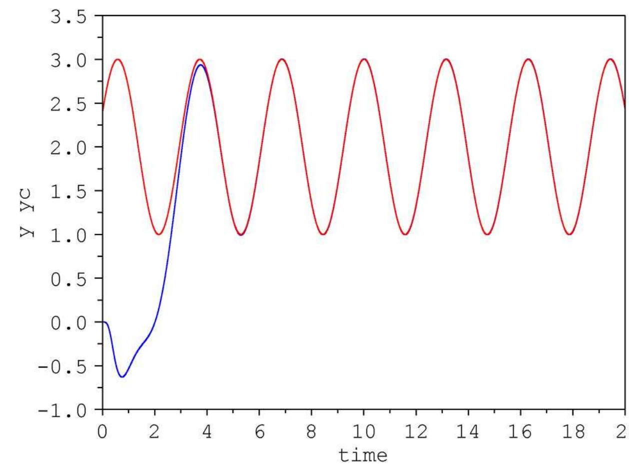

Fig. 9. Nonlinear SM output delay system tracking results for first-order Padé approximation.

where a_0 is a positive constant and is selected to make sure that, when q goes to zero, x_1 goes to zero. In fact, by the output redefinition (54), system (52) is changed to a relative-degree two system.

Corresponding to q, we choose

$$q_c = \ddot{y}_c + 2a_0\dot{y}_c + a_0^2 y_c. \tag{56}$$

Now, we assume that the system output (z_1 or x_2) has a fixed time delay τ, which is equivalent to the relative-degree two output function q having a fixed time delay τ. Suppose

$$\hat{y} = q(t - \tau). \tag{57}$$

We have used the first-order Padé approximation, stable system center approach, SM control, and SP technique to investigate the output tracking results for the system and obtained good output tracking results, as shown in Fig. 9.

Note that, in this example, we have chosen $a_0 = 2$, $c_0 = 1000$, $c_1 = 300$, $c_2 = 30$, $\tau = 0.2$, $C_1 = -0.75$, $\lambda = 10$, $\rho = 600$, $A = 1$, and $B = 2$. Clearly, the second- and third-order Padé approximation results can be similarly obtained. We omit them here.

VIII. Conclusion

A SISO output delay system tracking problem has been considered in this paper. By a transformation, the relative-degree n system is transformed into a relative-degree two system. The Padé approximation, stable system center approach, and second-order SM control are used to get good output tracking results. When we feed back the actual delayed output, limit cycles appear. We next use an SP to obtain good output tracking results for the system without disturbances. When disturbances are included in the system, the output tracking results are bad. We have used the second-order supertwisting SM observer to estimate the disturbances and obtain almost-perfect output tracking results. A nonlinear one-link-robot-arm example is also used to show the effect of this methodology. For more details regarding parameter choices, chattering phenomenon discussions, and different Padé order comparisons, please consult [18].

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REFERENCES

[1] G. A. Baker, Jr., *Recursive Calculation of Padé Approximation, Padé Approximants and Their Applications*, P. R. Graves-Morris, Ed. New York: Academic, 1973, pp. 83–91.

[2] G. Bartolini, E. Punta, and T. Zolezzi, "Approximability properties for second-order sliding mode control systems," *IEEE Trans. Autom. Control*, vol. 52, no. 10, pp. 1813–1825, Oct. 2007.

[3] F. J. Bejarano, A. Poznyak, and L. Fridman, "Observer for linear time invariant systems with unknown inputs based on the hierarchical super-twisting concept," in *Proc. Int. Workshop Variable Struct. Syst.*, 2006, pp. 208–213.

[4] D. Brambilla, L. M. Capisani, A. Ferrara, and P. Pisu, "Fault detection for robot manipulators via second-order sliding modes," *IEEE Trans. Ind. Electron.*, vol. 55, no. 11, pp. 3954–3963, Nov. 2008.

[5] C. I. Byrnes and A. Isidori, "Output regulation for nonlinear systems: An overview," *Int. J. Robust Nonlinear Control*, vol. 10, no. 5, pp. 323–337, Apr. 2000.

[6] J.-L. Chang, "Robust discrete-time model reference sliding-mode controller design with state and disturbance estimation," *IEEE Trans. Ind. Electron.*, vol. 55, no. 11, pp. 4065–4074, Nov. 2008.

[7] W. Chen and M. Saif, "Output feedback controller design for a class of MIMO nonlinear systems using high-order sliding-mode differentiators with application to a laboratory 3-D crane," *IEEE Trans. Ind. Electron.*, vol. 55, no. 11, pp. 3985–3997, Nov. 2008.

[8] H. H. Choi, "Sliding-mode output feedback control design," *IEEE Trans. Ind. Electron.*, vol. 55, no. 11, pp. 4047–4054, Nov. 2008.

[9] J. Davila, L. Fridman, and A. Levant, "Second-order sliding-mode observer for mechanical systems," *IEEE Trans. Autom. Control*, vol. 50, no. 11, pp. 1785–1789, Nov. 2005.

[10] S. Gopalswamy and J. K. Hedrick, "Tracking nonlinear non-minimum phase systems using sliding control," *Int. J. Control*, vol. 57, no. 5, pp. 1141–4458, 1993.

[11] A. Isidori, *Nonlinear Control Systems*, 3rd ed. Berlin, Germany: Springer-Verlag, 1995.

[12] E. A. Kosiba, G. Liu, Y. B. Shtessel, and A. Zinober, "Output tracking via sliding modes in causal system with time delay modeled by higher order Padé approximations," in *Proc. Int. Workshop Variable Struct. Syst.*, Alghero, Sardinia, Italy, 2006, pp. 250–255.

[13] A. Levant, "Higher-order sliding modes, differentiation and output-feedback control," *Int. J. Control*, vol. 76, no. 9/10, pp. 924–941, Jun. 2003.

[14] A. Levant, "Quasi-continuous high-order sliding-mode controllers," *IEEE Trans. Autom. Control*, vol. 50, no. 11, pp. 1812–1816, Nov. 2005.

[15] A. Levant, "Sliding order and sliding accuracy in sliding mode," *Int. J. Control*, vol. 58, no. 6, pp. 1247–1263, 1993.

[16] A. Levant and Y. Pavlov, "Generalized homogeneous quasi-continuous controllers," *Int. J. Robust Nonlinear Control*, vol. 18, no. 4/5, pp. 385–398, Apr. 2007.

[17] G. Liu, A. S. I. Zinober, and Y. B. Shtessel, "Output delay systems tracking using system centre approach and sliding mode control," in *Taming Heterogeneity and Complexity of Embedded Control*. London, U.K.: ISTE, 2006.

[18] G. Liu, "Sliding mode output tracking control of nonlinear non-minimum phase and time-delay systems," Ph.D. dissertation, Univ. Sheffield, Sheffield, U.K., 2007.

[19] Z. J. Palmor, "Time-delay compensation-Smith predictor and its modification," in *The Control Handbook*, S. Levine, Ed. Boca Raton, FL: CRC, 1996, ch. 10.

[20] A. Pisano, A. Davila, L. Fridman, and E. Usai, "Cascade control of PM DC drives via second-order sliding-mode technique," *IEEE Trans. Ind. Electron.*, vol. 55, no. 11, pp. 3846–3854, Nov. 2008.

[21] I. A. Shkolnikov and Y. B. Shtessel, "Tracking in a class of nonminimum-phase systems with nonlinear internal dynamics via sliding mode control using method of system center," *Automatica*, vol. 38, no. 5, pp. 837–842, 2002.

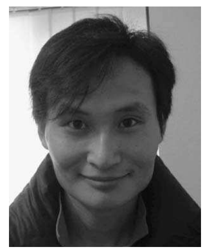
[22] Y. B. Shtessel, A. S. I. Zinober, and I. A. Shkolnikov, "Sliding mode control for nonlinear systems with output delay via method of stable system center," *Trans. ASME, J. Dyn. Syst. Meas. Control*, vol. 125, no. 2, pp. 253–256, Jun. 2003.

[23] Y. B. Shtessel, L. Mirkin, and Z. Palmor, *Sliding mode control of nonlinear systems with time-delayed control using Padé approximation and Smith predictor*, Jul. 29, 2003. Private Memorandum.

[24] O. J. M. Smith, "A controller to overcome dead time," *ISA J.*, vol. 6, pp. 28–33, 1957.

[25] V. I. Utkin, *Sliding Modes in Control Optimization*. Berlin, Germany: Springer-Verlag, 1999.

[26] B. Wang, X. Yu, and X. Li, "ZOH discretization effect on higher-order sliding-mode control systems," *IEEE Trans. Ind. Electron.*, vol. 55, no. 11, pp. 4055–4064, Nov. 2008.



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