

Second Order Stochastic Dominance, Reward-Risk Portfolio Selection and the CAPM*

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Abstract

Starting from the reward-risk model for portfolio selection introduced in De Giorgi (2004), we derive the reward-risk Capital Asset Pricing Model (CAPM) analogously to the mean-variance CAPM. The reward-risk portfolio selection arises from an axiomatic definition of reward and risk measures based on few properties, one of them being the isotonicity with respect to the second order stochastic dominance. We show that at any financial market equilibrium, investors' optimal allocation are comonotonic and the pricing kernel is an increasing function of the market portfolio, that corresponds to the increments of the distortion function characterizing investors' risk measures. Finally, we test the reward-risk CAPM on market data for several choices of the distortion function.

Keywords: stochastic dominance, mean-risk models, portfolio optimization, CAPM.

JEL Classification: G11, D81.

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1 Introduction

The Modern Portfolio Theory of Markowitz (1952) evaluates investments in term of their mean and variance: The portfolio choice problem is very intuitive and consists in minimizing the variance (risk) over the set of feasible portfolios' payoffs, given that the mean (reward) is greater than a target value (see for an overview De Giorgi 2002). In the mean-variance model, if investors agree on the assets' distributions and the risk free asset exists, the Security Market Line Theorem is satisfied at any financial market equilibrium, i.e. assets' excess expected returns are proportional to the market excess expected return. This is the main conclusion of the Capital Asset Pricing Model (CAPM) of Sharpe (1964), Lintner (1965) and Mossin (1966). It suggests to evaluate assets' returns looking at the proportional factors in the Security Market Line equation, the so called β -factors. As stated by Jagannathan and Wang (1996) the CAPM "is widely viewed as one of the two or three major contributions of academic research to financial managers during the post-war era".

Nevertheless, from a theoretical point of view, the variance as measure of risk has been hardly criticized. First, it is a symmetric measure on the space of random variables and treats positive and negative deviations from the mean in the same way, while the formers are obviously welcome. Second, it is inappropriate to describe the risk of low probability events, that are typical when dealing with credit portfolio losses. Moreover, the empirical validity of the mean-variance CAPM has been debated in several papers, as discussed by Jagannathan and McGrattan (1995).

Many authors address the issue of finding measure of risk that are able to better describe the characteristics of assets' distributions with respect to the investors' perception of risk. Several alternative risk measures as been proposed in the literature: semivariance (Markowitz 1959, Ogryczak and Ruszczyński 1997), general lower partial moments (Jean 1975, Bawa 1975, Unser 2000), value-at-risk (Jorion 1997), expected shortfall (Acerbi and Tasche 2002, Rockafellar and Uryasev 2002), among others. Consequently, replacing the variance with more sophisticated measures of risk, the corresponding equilibrium capital-market models has been considered, e.g. by Hogan and Warren (1974), Bawa and Lindenberg (1977), Harlow and Rao (1989), De Giorgi (2002) and Post and Van Vilet (2004). Still, the choice of one particular measure is under debate and strongly depends on investors' perception of risk.

Artzner, Delbaen, Eber, and Heath (1997, 1999), being concerned with *banking regulation*, address their attention to understanding how measure of risk should behave, instead of arguing in favour of one particular measure. They proposed an axiomatic definition of risk measures throughout four properties (positive sub-additivity, monotonicity, translation invariance and homogeneity) and introduced the concept of coherent measure of risk, that strongly influenced the way of thinking at risk measurement and risk management. The same approach of characterizing measures of risk by a set of principles or axioms has been extended to *measures of reward* in De Giorgi (2004), where we define measures of reward and measures of risk for *portfolio selection*, imposing for both the isotonicity with respect to

the second order stochastic dominance, besides other properties, in order to avoid paradoxical preferences when compared to the Expected Utility paradigm. We also give a unique characterization of reward measures (the mean) and we suggest a class of risk measures for portfolio selection, that are related to the Choquet Expected Utility Theory. In this setup, a risk measure arises from a convex distortion of the physical survival distribution function. Moreover, the portfolio decision problem based on the Choquet expectation can be formulated as linear quintile regression, as showed by Bassett, Koenker, and Kordas (2004). A large class of risk measures proposed in the literature can be represented in this way, as discussed in De Giorgi (2004).

In this paper we consider the reward-risk model for portfolio selection introduced in De Giorgi (2004) and we study the financial market equilibria, in order to obtain a possible extension of the CAPM resulting from the mean-variance analysis, to a similar equilibrium model in the reward-risk setup. We show that if markets are complete, then investors' optimal allocations are comonotonic, analogously to the Tobin separation in the classical mean-variance approach. Moreover, at any financial market equilibrium, the Capital Asset Pricing Model holds, where the pricing kernel is not the market portfolio as in the mean-variance CAPM, but a strictly decreasing function of it, that corresponds to the increments of the investors' distortion function at the optimal allocations. The comonotonicity of investors' optimal allocations implies that at equilibrium the market is also comonotonic to every investor's consumption. Moreover, it follows that financial market equilibria exist only if all investors possess the same distortion function, i.e. the same reward-risk setup.

Finally, we test the market efficiency with respect to the reward-risk setup and the reward-risk CAPM, for a large class of strictly convex distortion functions characterized by a unique parameter. We select the distortion that minimizes the statistics associated to the J -test for over-identifying restrictions. It follows that the parameter of the distortion which minimize the test statistics can be interpreted as the investor's degree of pessimism, as introduced in Bassett, Koenker, and Kordas (2004). The empirical analysis based on market data from 1932 to 2002 shows that investors are lightly pessimistic. Moreover, the estimated parameter and the associated risk measure provide a pricing kernel near to that obtained for the mean-semivariance CAPM.

Recently, Post and Van Vilet (2004) also provide empirical evidence in favor of the mean-semivariance CAPM. However, the analysis of Post and Van Vilet (2004) is restricted to the mean-semivariance CAPM versus the mean-variance CAPM, imposing the corresponding model for the pricing kernel and without any equilibrium consideration. In this paper, we start with a general model for investors' reward-risk preferences and obtain the reward-risk CAPM from the study of financial market equilibria.

The remainder of the paper is organized as follows: in Section 2 we present the reward-risk portfolio selection problem and we derive the reward-risk Capital Asset Pricing Model. In Section 3, we test empirically the market efficiency based on the reward-risk CAPM. Section 4 concludes.

2 The model

We consider a two-periods exchange economy. The model setup follows Duffie (1988). Let $\Omega = \{1, \dots, S\}$ denote the state of nature at the final period T . $\mathcal{F} = 2^\Omega$ is the power algebra on Ω , i.e. the set of all possible events arising from Ω . Uncertainty is modeled by the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where the probability measure \mathbb{P} on Ω satisfies $p_s = \mathbb{P}[\{s\}] > 0$ for all $s = 1, \dots, S$, i.e. every state of the world have strictly positive probability to occur. The space \mathcal{G} of real-valued measurable functions is endowed with the scalar product $X \cdot Y = \sum_{s=1}^S X(s)Y(s)p_s = \mathbb{E}_{\mathbb{P}}[XY]$.

There are $K+1$ assets with payoffs A_k . The asset 0 is the risk-free asset with payoff $A_0 = \mathbf{1}$. The supply of risky assets is exogenously given and denoted by $\bar{\theta}_k > 0$ ($k = 1, \dots, K$), while the risk-free asset is in elastic supply with exogenously given price $\frac{1}{1+r}$, where $r > 0$ is the risk-free rate of return. The marketed subspace \mathcal{X} is the span of $(A_k)_{k=0,1,\dots,K}$. Without loss of generality, we assume that no-redundant assets exist, i.e. $\dim(\mathcal{X}) = K+1$, where obviously $K+1 \leq S$. The market portfolio is the sum of all available risky assets, i.e. $\tilde{\omega} = \sum_{k=1}^K A_k \bar{\theta}_k$.

There are $i = 1, \dots, I$ investors, initially endowed with wealth $w^i > 0$. The numbers θ_k^i denote the amount of security k held by agent i , q_k denotes the k -th security price. Thus, when trading these securities, the agent can attain the consumption plan $X = \sum_{k=0}^K A_k \theta_k^i \in \mathcal{X}$ where θ^i satisfies the budget restriction (i.e. $q(X) = \sum_{k=0}^K q_k \theta_k^i \leq w^i$). We denote by \mathcal{B}^i the subset of \mathcal{X} , such that $X \in \mathcal{B}^i$ is budget-feasible for investor i , i.e. $\mathcal{B}^i = \{X \in \mathcal{X} \mid q(X) \leq w^i\}$. Note that \mathcal{B}^i is a convex set.

Agents evaluate consumption plans according to a risk-reward pair (μ, ρ^i) , where $\mu(X) = \mathbb{E}[X]$ and $\rho^i : \mathcal{G} \rightarrow \mathbb{R}$ is a risk measure as defined in De Giorgi (2004, Theorem 4.2). The measure ρ^i satisfies the following four properties: convexity, the risk of the zero payoff is zero, invariance with respect to adding risk-free positions and, finally, the isotonicity with respect to the second order stochastic dominance. The measure ρ^i arises from an axiomatic definition of risk measures based on the four properties listed above. Note that we do not impose the same measure of risk for all investors. In fact, investors' perception of risk can differ and thus also the way of measuring it (Weber and Milliman 1997), as long as the four properties above are satisfied.

The consumption plan $X = \sum_{k=0}^K A_k \theta_k^i \in \mathcal{X}$ for investor i is said to be (μ, ρ^i) -efficient iff

- (i) $q(X) \leq w^i$ (budget feasible), and
- (ii) $\nexists Y \in \mathcal{X}$ such that $q(Y) \leq w^i$ and one of the following two statements is satisfied
 - (a) $\rho^i(X) > \rho^i(Y)$ and $\mu(X) = \mu(Y)$ or,
 - (b) $\rho^i(X) = \rho^i(Y)$ and $\mu(X) < \mu(Y)$.

From De Giorgi (2004, Theorem 2.1), $X \in \mathcal{X}$ is (μ, ρ^i) -efficient iff X is budget feasible and uniquely minimizes the function $\mathcal{R}^i = \xi^i \rho^i - \mu^i$ over \mathcal{B}^i , for some $\xi^i > 0$. Moreover, there

exists a convex, non-decreasing function g^i on $[0, 1]$, with $g^i(0) = 0$ and $g^i(1) = 1$ such that

$$\mathcal{R}^i(X) = - \int_{-\infty}^0 (g^i(\bar{F}_X(x)) - 1) dx - \int_0^{\infty} g^i(\bar{F}_X(x)) dx \quad (1)$$

and \bar{F}_X is the cumulative distribution function of X under \mathbb{P} . The convex function g^i is called distortion and uniquely characterizes investors' risk preferences. Moreover, the function \mathcal{R}^i corresponds to a Choquet integral and is related to the Non-Expected Utility Theory of Schmeidler (1989). Therefore, the investor's portfolio choice problem is:

$$\min_{X \in \mathcal{B}^i} \mathcal{R}^i(X), \quad (2)$$

or equivalently

$$\max_{X \in \mathcal{B}^i} -\mathcal{R}^i(X). \quad (3)$$

We introduce the following definition:

Definition 2.1 (Financial market equilibrium). *Given a risk-free rate r , a financial market equilibrium consists of a price vector $\hat{q} \in \mathbb{R}^{K+1}$ with $\hat{q}_0 = \frac{1}{1+r}$ and allocations $\hat{X}^i \in \mathcal{X}$ for $i = 1, \dots, I$, such that*

- (i) \hat{X}^i maximizes $-\mathcal{R}^i$ over \mathcal{B}^i (investors' portfolio choice), and
- (ii) $\exists \alpha_0 \in \mathbb{R}$ such that $\alpha_0 \mathbf{1} + \sum_{i=1}^I \hat{X}^i = \tilde{\omega}$ (markets clear).

The property (ii) in Definition 2.1, says that the sum of investors' optimal allocations is the market portfolio plus the exogenously supplied number of risk-free assets. In fact, the market portfolio has been defined as just the sum of supplied risky assets.

Instead of using Definition 2.1 directly, we first impose some restrictions on the equilibrium prices \hat{q} . Note that here the goal function \mathcal{R}^i is strictly monotone and therefore a necessary condition for the portfolio decision problem given above to have a solution (and thus, in order property (i) of the previous definition to be satisfied) is that consumers cannot exploit an arbitrage opportunity. Therefore, a necessary condition for the existence of market equilibria is that the following equation holds:

$$\mathcal{G}_+ \cap \{X \in \mathcal{X} \mid q(X) \leq 0\} = \{0\}, \quad (4)$$

where \mathcal{G}_+ is the subset of elements in \mathcal{G} with non-negative outcomes in all states of nature. Equation (4) means that non-negative payoffs must have strictly positive price, if not the zero payoff. In fact, if equation (4) fails to be true for some price vector q , then investors can freely obtain a positive payoff in all state of nature, that is strictly positive in at least one state of nature. Therefore, they could infinitely increase their objective function and no optimal solution to their investment problem would exist. A price vector $q \in \mathbb{R}^{K+1}$ such that equation (4) is satisfied, is said to be *arbitrage free for the marketed subspace \mathcal{X}* . The following Lemma holds.

Lemma 2.1 (Existence of the pricing portfolio). *Let $q \in \mathbb{R}^{K+1}$ be an arbitrage free price vector for the marketed subspace \mathcal{X} . Then it exists $\ell \in \mathcal{X}$, $\mathbb{E}_{\mathbb{P}}[\ell] = 1$ such that*

$$q(X) = \frac{1}{1+r} \ell \cdot X \quad (5)$$

for all $X \in \mathcal{X}$ and $q(X) \geq 0$.

Proof. The arbitrage free equation (4) implies that

$$\{X \in \mathcal{X} | q(X) \leq 0\} \cap \left\{ Y \in \mathcal{G} | Y \geq 0, \sum_{s \in \Omega} Y(s) = 1 \right\} = \emptyset.$$

Let define the convex subspace of \mathcal{G} by $\mathcal{K} = \{X \in \mathcal{X} | q(X) \leq 0\}$, i.e. the set of marketed allocation with negative price. Moreover, let $\mathcal{P} = \{Y \in \mathcal{G} | Y \geq 0, \sum_{s \in \Omega} Y(s) = 1\}$. Since $\mathcal{K} \cap \mathcal{P} = \emptyset$, then by the Farka's Lemma we find a linear functional Ψ on \mathcal{G} with $\Psi(X) = 0$ for $X \in \mathcal{K}$ and $\Psi(Y) > 0$ for $Y \in \mathcal{P}$. Moreover, by the Riesz Representation Theorem (see Duffie 1988, Chapter I.6) we find $\psi \in \mathcal{G}$ with $\Psi(Z) = \psi \cdot Z$ for all $Z \in \mathcal{G}$. Let $s \in \Omega$ and define Y_s by $Y_s(s') = 1$ if $s' = s$ and $Y_s(s') = 0$ else. Y_s is the Arrow security for state s . Obviously $Y_s \in \mathcal{P}$ and $0 < \Psi(Y_s) = \psi(s) p_s$ for all $s \in \Omega$. Since $p_s > 0$ then $\psi(s) > 0$. We define $\ell = \frac{\psi}{\mu(\psi)}$ and a probability measure π on (Ω, \mathcal{F}) by $\pi(s) = \ell(s) p_s$. We have

$$\mu(\psi)^{-1} \Psi(Z) = \sum_{s \in \Omega} Z(s) \pi(s) = \mathbb{E}_{\pi}[Z].$$

Consider the following investment: Borrow $\theta_0 = -1$ units of the risk-free asset, to finance $\theta_i = \frac{q_0}{q_i}$ units of asset $k \in \{1, \dots, K\}$ (moreover, $\theta_i = 0$ for $i \neq 0, k$). Then, $X = \sum_{k=0}^K \theta_k A_k \in \mathcal{K}$ since $q(X) = 0$, by construction. Therefore, $\mu(\psi)^{-1} \Psi(X) = \mathbb{E}_{\pi}[X] = 0$. It follows:

$$q_k = \frac{q_0}{\mathbb{E}_{\pi}[A_0]} \mathbb{E}_{\pi}[A_k] = \frac{1}{1+r} \mathbb{E}_{\pi}[A_k].$$

From the computation above, we see that for an arbitrage free price vector $q \in \mathbb{R}^{K+1}$, we find $\ell \in \mathcal{G}$ such that

$$q_k = \frac{1}{1+r} \ell \cdot A_k$$

for $k = 1, \dots, K$ and thus for $X = \sum_{k=0}^K \theta_k A_k \in \mathcal{X}$,

$$q(X) = \sum_{k=0}^K \theta_k q_k = \frac{1}{1+r} \sum_{k=0}^K \theta_k (\ell \cdot A_k) = \frac{1}{1+r} \ell \cdot \left(\sum_{k=0}^K \theta_k A_k \right) = \frac{1}{1+r} \ell \cdot X.$$

By construction $\ell > 0$ and $\mathbb{E}_{\mathbb{P}}[\ell] = 1$. Note that ℓ may not be an element of \mathcal{X} , but since we restrict the pricing rule just described to \mathcal{X} , we can assume without loss of generality¹ that $\ell \in \mathcal{X}$. In fact, if $\ell \notin \mathcal{X}$, we can decompose ℓ into one part ℓ_{\parallel} in \mathcal{X} and one part ℓ_{\perp} orthogonal to \mathcal{X} . Since for all $X \in \mathcal{X}$, $\ell_{\perp} \cdot X = 0$, the pricing rule can be rewritten as $\ell_{\parallel} \cdot X$. Moreover, since $\mathbf{1} \in \mathcal{X}$, $0 = \ell_{\perp} \cdot \mathbf{1} = \mathbb{E}_{\mathbb{P}}[\ell_{\perp}]$. Thus, we assume $\ell \in \mathcal{X}$ and $\mathbb{E}_{\mathbb{P}}[\ell] = 1$. \square

ℓ is called the pricing portfolio (Duffie 1988) or ideal security (Magill and Quinzii 1996). Using the pricing portfolio ℓ we can rewrite the budget set as $\mathcal{B}^i = \{X \in \mathcal{X} \mid \ell \cdot X \leq (1+r)w^i\}$ and the no-arbitrage decision problem of investor i is given by

$$\max_{X \in \mathcal{X}} -\mathcal{R}^i(X), \ell \cdot X \leq (1+r)w^i. \quad (6)$$

An equivalent definition of financial market equilibria is now the following:

Definition 2.2. *Given a risk-free rate r , a financial market equilibrium consists of a price vector $\hat{\ell} \in \mathcal{X}$ and allocations $\hat{X}^i \in \mathcal{X}$ for $i = 1, \dots, I$, such that*

- (i) \hat{X}^i maximizes $-\mathcal{R}^i$ subject to $\hat{\ell} \cdot X \leq (1+r)w^i$ for $i = 1, \dots, I$, and
- (ii) $\exists \alpha_0 \in \mathbb{R}$ such that $\alpha_0 \mathbf{1} + \sum_{i=1}^I \hat{X}^i = \tilde{\omega}$.

We now come back to the individual portfolio choice of equations (2) and (3). We rewrite the function \mathcal{R}^i using its integral representation (1). Let consider $X \in \mathcal{G}$ and take a permutation ζ of $\Omega = \{1, \dots, S\}$ such that $X(\zeta(1)) \leq X(\zeta(2)) \leq \dots \leq X(\zeta(S))$. Then

$$\mathcal{R}^i(X) = -X(\zeta(1)) - \sum_{s=1}^{S-1} g^i \left(1 - \sum_{l=1}^s p_{\zeta(l)} \right) [X(\zeta(s+1)) - X(\zeta(s))].$$

Let

$$\begin{aligned} q_{\zeta(1)}^i &= 1 - g^i(1 - p_{\zeta(1)}), \\ q_{\zeta(s)}^i &= g^i \left(1 - \sum_{l=1}^{s-1} p_{\zeta(l)} \right) - g^i \left(1 - \sum_{l=1}^s p_{\zeta(l)} \right), \quad \text{for } s = 2, \dots, S. \end{aligned}$$

Note that $q_{\zeta(s)}^i \geq 0$ since g is non-decreasing, $\sum_{s=1}^S q_{\zeta(s)}^i = 1$, and $q_{\zeta(1)}^i \geq q_{\zeta(2)}^i \geq \dots \geq q_{\zeta(S)}^i$ since g^i is convex. Moreover,

$$\mathcal{R}^i(X) = - \sum_{s=1}^S q_{\zeta(s)}^i X(\zeta(s)) = - \sum_{s=1}^S \frac{q_{\zeta(s)}^i}{\sum_{l: X(\zeta(l))=X(\zeta(s))} p_{\zeta(l)}} p_{\zeta(s)} X(\zeta(s)).$$

¹This assumption just refers to the pricing rule $\ell \cdot X$ and not to the way ℓ is obtained. It might occur that the new ℓ cannot be written as Radon-Nikodym Derivative with respect to some equivalent probability measure.

Note that

$$\frac{\sum_{l: X(\zeta(l))=X(\zeta(s))} q_{\zeta(l)}^i}{\sum_{l: X(\zeta(l))=X(\zeta(s))} p_{\zeta(l)}} = \frac{g^i(\mathbb{P}[X \geq X(\zeta(s))]) - g^i(\mathbb{P}[X > X(\zeta(s))])}{\mathbb{P}[X = X(\zeta(s))]} = f_X^i(X(\zeta(s)))$$

where

$$f_X^i(x) = \frac{g^i(\mathbb{P}[X \geq x]) - g^i(\mathbb{P}[X > x])}{\mathbb{P}[X = x]}$$

is a positive, non-increasing function of x , since g^i is non-decreasing and convex. Moreover, by definition, $f_X^i(X) \in \mathcal{G}$ with

$$\mathbb{E}_{\mathbb{P}}[f_X^i(X)] = 1, \quad f_X^i(X) \geq 0 \quad \text{and} \quad \mathcal{R}^i(X) = -\mathbb{E}_{\mathbb{P}}[f_X^i(X) X] = -f_X^i(X) \cdot X.$$

Thus, the vector $f_X^i(X) \in \mathcal{G}$ is a probability measure on (Ω, \mathcal{F}) and the functional \mathcal{R}^i is the negative expectation with respect to $f_X^i(X) \in \mathcal{G}$. Similar results are given by Carlier and Dana (2003) for non atomic spaces. The optimization problem (6) can be rewritten as

$$\max_{X \in \mathcal{X}, \lambda^i} f_X^i(X) \cdot X - \lambda^i (\ell \cdot X - (1+r)w^i) = \max_{X \in \mathcal{X}, \lambda^i} (f_X^i(X) - \lambda^i \ell) \cdot X + \lambda^i (1+r)w^i. \quad (7)$$

where λ^i is the Lagrange multiplier. Let $\mathcal{L}^i(X, \lambda^i) = (f_X^i(X) - \lambda^i \ell) \cdot X + \lambda^i (1+r)w^i$ be the Lagrange function. The following relationship between any efficient allocation \hat{X}^i and the pricing portfolio is satisfied.

Theorem 2.1. *Let $\hat{X}^i \in \arg \max_{X \in \mathcal{X}} -\mathcal{R}^i(X)$, s.t. $\ell \cdot X \leq (1+r)w^i$, then*

$$f_{\hat{X}^i}^i(\hat{X}^i)_{\parallel} = \ell$$

and $\ell \cdot \hat{X}^i = (1+r)w^i$ for all $i = 1, \dots, I$, where for $Y \in \mathcal{G}$, $Y = Y_{\perp} + Y_{\parallel}$ is the unique orthogonal decomposition of Y with respect to \mathcal{X} , i.e. $Y_{\perp} \perp \mathcal{X}$ and $Y_{\parallel} \in \mathcal{X}$.

Proof. (i) We prove: $\ell \cdot \hat{X}^i = (1+r)w^i$.

Let $\hat{X}^i \in \arg \max_{X \in \mathcal{X}} -\mathcal{R}^i(X)$, s.t. $\ell \cdot X \leq (1+r)w^i$. Since the function $-\mathcal{R}^i(X)$ is strictly monotone and the risk-less asset exists, \hat{X}^i must satisfy the budget restriction with equality, i.e. $\ell \cdot \hat{X}^i = (1+r)w^i$.

(ii) We prove: $f_{\hat{X}^i}^i(\hat{X}^i)_{\parallel} = \ell$.

Let $Z \in \mathcal{X}$ such that $\ell \cdot Z = 0$ (i.e. $Z \in \text{span}(\ell)^{\perp} \cap \mathcal{X}$) and $Y^i = \hat{X}^i + \epsilon Z$ for $\epsilon > 0$. Then $Y_i \in \mathcal{X} \cap \mathcal{B}^i$ and

$$\begin{aligned} \mathcal{L}^i(Y^i, \lambda^i) &= (f_{Y^i}^i(Y^i) - \lambda^i \ell) \cdot Y^i + \lambda^i (1+r)w^i \\ &= f_{Y^i}^i(Y^i) \cdot Y^i - \lambda^i \ell \cdot \hat{X}^i + \lambda^i (1+r)w^i \\ &= \left(f_{Y^i}^i(Y^i) - f_{\hat{X}^i}^i(\hat{X}^i) \right) \cdot Y^i + \epsilon f_{\hat{X}^i}^i(\hat{X}^i) \cdot Z + \mathcal{L}^i(\hat{X}^i, \lambda^i). \end{aligned}$$

Let ζ be a permutation of Ω such that $\hat{X}^i(\zeta(1)) \leq \hat{X}^i(\zeta(2)) \leq \dots \leq \hat{X}^i(\zeta(S))$. Without loss of generality, for $\epsilon > 0$ small enough, $Y^i(\zeta(1)) \leq Y^i(\zeta(2)) \leq \dots \leq Y^i(\zeta(S))$. In fact, if for some $s \in \{1, \dots, S-1\}$, $\hat{X}^i(\zeta(s)) = \hat{X}^i(\zeta(s+1))$ and $\hat{Y}^i(\zeta(s)) > \hat{Y}^i(\zeta(s+1))$, then we take the permutation $\tilde{\zeta}$ of Ω such that $\tilde{\zeta}(l) = \zeta(l)$ for all $l \neq s, s+1$ and $\tilde{\zeta}(s) = \zeta(s+1)$, $\tilde{\zeta}(s+1) = \zeta(s)$. Then $(f_{Y^i}^i(Y^i) - f_{\hat{X}^i}^i(\hat{X}^i)) \cdot Y_i = -\sum_{s=1}^S (q_{\zeta(s)} - q_{\tilde{\zeta}(s)}) Y_i(\zeta(s)) = 0$ and thus

$$\mathcal{L}^i(Y^i, \lambda^i) = \epsilon f_{\hat{X}^i}^i(\hat{X}^i) \cdot Z + \mathcal{L}^i(\hat{X}^i, \lambda^i).$$

Therefore $f_{\hat{X}^i}^i(\hat{X}^i) \cdot Z = 0$, else either $Y_i = \hat{X}^i + \epsilon Z$ or $Y_i = \hat{X}^i - \epsilon Z$ contradicts the optimality of \hat{X}^i .

Let now decompose $f_{\hat{X}^i}^i(X^i)$ as $f_{\hat{X}^i}^i(X^i) = f_{\hat{X}^i}^i(X^i)_{\parallel} + f_{\hat{X}^i}^i(X^i)_{\perp}$, where $f_{\hat{X}^i}^i(X^i)_{\parallel} \in \mathcal{X}$ and $f_{\hat{X}^i}^i(X^i)_{\perp} \perp \mathcal{X}$. Let $Z \in \text{span}(\ell)^\perp$. Then $0 = \ell \cdot Z = \ell \cdot (Z_{\perp} + Z_{\parallel}) = \ell \cdot Z_{\parallel}$, therefore $Z_{\parallel} \in \text{span}(\ell)^\perp \cap \mathcal{X}$. From the previous result it follows

$$0 = f_{\hat{X}^i}^i(X^i) \cdot Z_{\parallel} = f_{\hat{X}^i}^i(X^i)_{\parallel} \cdot Z_{\parallel} = f_{\hat{X}^i}^i(X^i)_{\parallel} \cdot Z.$$

Since this is true for all $Z \in \text{span}(\ell)^\perp$, it follows that $f_{\hat{X}^i}^i(X^i)_{\parallel} \in \text{span}(\ell)$ and therefore it exists $\hat{\alpha}^i \in \mathbb{R}$ such that $f_{\hat{X}^i}^i(\hat{X}^i)_{\parallel} = \hat{\alpha}^i \ell$. Since $\mathbf{1} \in \mathcal{X}$, $0 = f_{\hat{X}^i}^i(\hat{X}^i)_{\perp} \cdot \mathbf{1} = \mathbb{E}_{\mathbb{P}}[f_{\hat{X}^i}^i(\hat{X}^i)_{\perp}]$ and thus

$$1 = \mathbb{E}_{\mathbb{P}}[f_{\hat{X}^i}^i(\hat{X}^i)] = \mathbb{E}_{\mathbb{P}}[f_{\hat{X}^i}^i(\hat{X}^i)_{\perp}] + \mathbb{E}_{\mathbb{P}}[f_{\hat{X}^i}^i(\hat{X}^i)_{\parallel}] = \mathbb{E}_{\mathbb{P}}[f_{\hat{X}^i}^i(\hat{X}^i)_{\parallel}] = \hat{\alpha}^i \mathbb{E}_{\mathbb{P}}[\ell] = \hat{\alpha}^i.$$

This completes the proof. □

Now suppose that markets are complete, i.e. $\mathcal{X} = \mathcal{G}$. Then $f_{\hat{X}^i}^i(\hat{X}^i) \in \mathcal{X}$ and thus $f_{\hat{X}^i}^i(\hat{X}^i)_{\perp} = 0$ for all i . Therefore, $f_{\hat{X}^i}^i(\hat{X}^i) = f_{\hat{X}^i}^i(\hat{X}^i)_{\parallel}$. From the previous Theorem, we immediately obtain the following result on the optimal allocations \hat{X}^i , $i = 1, \dots, I$.

Corollary 2.1. *Let $\hat{X}^i \in \arg \max_{X \in \mathcal{X}} -\mathcal{R}^i(X)$, s.t. $\ell \cdot X \leq (1+r)w^i$ for $i = 1, \dots, I$. Suppose that the corresponding distortions g^i are strictly convex for all $i = 1, \dots, I$. Then if $K+1 = S$, i.e. markets are complete, the optimal payoffs $\hat{X}^1, \dots, \hat{X}^I$ are comonotonic, i.e. for all $s, s' \in \Omega$ and $i, j \in \{1, \dots, I\}$ we have $(\hat{X}^i(s) - \hat{X}^i(s'))(\hat{X}^j(s) - \hat{X}^j(s')) \geq 0$ and the inequality is strict if $\hat{X}^i(s) \neq \hat{X}^i(s')$ for some i .*

Proof. From Theorem 2.1, $f_{\hat{X}^i}^i(\hat{X}^i) = \ell$ for $i = 1, \dots, I$. The functions $f_{\hat{X}^i}^i$ are strictly decreasing, since the g^i 's are strictly convex. Suppose now that for $s, s' \in \Omega$, $\hat{X}^i(s) \geq \hat{X}^i(s')$. Then $f_{\hat{X}^i}^i(\hat{X}^i(s)) \leq f_{\hat{X}^i}^i(\hat{X}^i(s'))$, i.e. $\ell(s) \leq \ell(s')$. Thus, $f_{\hat{X}^j}^j(\hat{X}^j(s)) \leq f_{\hat{X}^j}^j(\hat{X}^j(s'))$ and therefore also $\hat{X}^j(s) \geq \hat{X}^j(s')$ for any $j \in \{1, \dots, I\}$. Moreover, the inequality is strict for \hat{X}^j if it is for \hat{X}^i . □

This last Corollary states that investors' optimal allocations are comonotonic, i.e. cannot be used as hedge of each other. This is a well-known result from the classical mean-variance model, that is directly implied by the Tobin Separation. The comonotonicity of investors' optimal payoffs immediately implies the following property of the pricing portfolio $\hat{\ell}$ at any financial market equilibrium.

Corollary 2.2. *Let $(\hat{\ell}, \hat{X}^1, \dots, \hat{X}^I)$ be a financial market equilibrium. Suppose that the corresponding distortions g^i are strictly convex for all $i = 1, \dots, I$. Then if $K + 1 = S$, i.e. markets are complete, there exists a strictly decreasing function f such that $f(\tilde{\omega}) = \ell$ and $f(\tilde{\omega}) = f_{\tilde{\omega}}^i(\tilde{\omega})$ for all $i = 1, \dots, I$.*

Proof. From the previous Corollary, we have that all optimal payoffs \hat{X}^i ($i = 1, \dots, I$) are comonotonic and therefore also the sums $\sum_{i=1}^I \hat{X}^i$ and $\sum_{i=1}^I \hat{X}^i + \alpha \mathbf{1}$ for all $\alpha \in \mathbb{R}$. By definition of financial market equilibrium, we find $\alpha_0 \in \mathbb{R}$ such that $\tilde{\omega} = \sum_{i=1}^I \hat{X}^i + \alpha_0 \mathbf{1}$. Therefore, $\tilde{\omega}$ is also comonotonic to $\hat{X}^1, \dots, \hat{X}^I$ and thus $f_{\tilde{\omega}}^i(\tilde{\omega}) = f_{\hat{X}^i}^i(\hat{X}^i) = \ell$. Take $f = f_{\tilde{\omega}}^i$ for some $i = 1, \dots, I$. \square

The last Corollary also implies a necessary condition for the existence of financial market equilibria. In fact, since $f_{\hat{X}^i}^i(\hat{X}^i) = f_{\hat{X}^j}^j(\hat{X}^j)$ for all $i, j \in \{1, \dots, I\}$, it follows that investors' distortions g^i must correspond at the survival probabilities $\bar{F}_{\hat{X}^i}(\hat{X}^i(s)) = \bar{F}_{\hat{X}^j}(\hat{X}^j(s))$. Obviously, this last equation is true because of the strict comonotonicity stated in Corollary 2.1. We are now able to prove the Security Market Line Theorem.

Theorem 2.2. *Let $(\hat{\ell}, \hat{X}^1, \dots, \hat{X}^I)$ be a financial market equilibrium and $q(\tilde{\omega}) > 0$. Suppose that the corresponding distortions g^i are strictly convex for all $i = 1, \dots, I$. Then if $K + 1 = S$, i.e. markets are complete, for all $X \in \mathcal{X}$:*

$$\mathbb{E}_{\mathbb{P}}[f(R_{\tilde{\omega}})(R_X - r)] = 0, \quad (8)$$

where $R_X = \frac{X - q(X)}{q(X)}$ and $R_{\tilde{\omega}} = \frac{\tilde{\omega} - q(\tilde{\omega})}{q(\tilde{\omega})}$. Therefore For $X \in \mathcal{X}$

$$\mathbb{E}_{\mathbb{P}}[R_X] - r = \frac{\text{cov}_{\mathbb{P}}[f(R_{\tilde{\omega}}), R_X]}{\text{cov}_{\mathbb{P}}[f(R_{\tilde{\omega}}), R_{\tilde{\omega}}]} (\mathbb{E}_{\mathbb{P}}[R_{\tilde{\omega}}] - r). \quad (9)$$

Proof. (i) $\tilde{\omega}$ and $R_{\tilde{\omega}}$ are comonotonic.

Since $q(\tilde{\omega}) > 0$, then if for $s, s' \in \Omega$, $\tilde{\omega}(s) \geq \tilde{\omega}(s')$ then $R_{\tilde{\omega}}(s) \geq R_{\tilde{\omega}}(s')$. Thus, $\tilde{\omega}$ and $R_{\tilde{\omega}}$ are comonotonic.

(ii) Since $\tilde{\omega}$ and $R_{\tilde{\omega}}$ are comonotonic, then $f(\tilde{\omega}) = f(R_{\tilde{\omega}})$, where f is defined as in the proof of the previous Corollary. Moreover, for $X \in \mathcal{X}$

$$\mathbb{E}_{\mathbb{P}}[f(R_{\tilde{\omega}}) R_X] = f(\tilde{\omega}) \cdot \left(\frac{X - q(X)}{q(X)} \right) = \frac{1}{q(X)} f(\tilde{\omega}) \cdot X - 1 = (1 + r) - 1 = r.$$

Therefore, for $X \in \mathcal{X}$

$$\begin{aligned}
& (r - \mathbb{E}_{\mathbb{P}}[R_{\tilde{\omega}}]) (\mathbb{E}_{\mathbb{P}}[R_X] - r) = (r - \mathbb{E}_{\mathbb{P}}[R_X]) (\mathbb{E}_{\mathbb{P}}[R_{\tilde{\omega}}] - r) \\
\Rightarrow & (\mathbb{E}_{\mathbb{P}}[f(R_{\tilde{\omega}}) R_{\tilde{\omega}}] - \mathbb{E}_{\mathbb{P}}[R_{\tilde{\omega}}]) (\mathbb{E}_{\mathbb{P}}[R_X] - r) = (\mathbb{E}_{\mathbb{P}}[f(R_{\tilde{\omega}}) R_X] - \mathbb{E}_{\mathbb{P}}[R_X]) (\mathbb{E}_{\mathbb{P}}[R_{\tilde{\omega}}] - r) \\
\Rightarrow & \text{cov}_{\mathbb{P}}[f(R_{\tilde{\omega}}), R_{\tilde{\omega}}] (\mathbb{E}_{\mathbb{P}}[R_X] - r) = \text{cov}_{\mathbb{P}}[f(R_{\tilde{\omega}}), R_X] (\mathbb{E}_{\mathbb{P}}[R_{\tilde{\omega}}] - r) \\
\Rightarrow & \mathbb{E}_{\mathbb{P}}[R_X] - r = \frac{\text{cov}_{\mathbb{P}}[f(R_{\tilde{\omega}}), R_X]}{\text{cov}_{\mathbb{P}}[f(R_{\tilde{\omega}}), R_{\tilde{\omega}}]} (\mathbb{E}_{\mathbb{P}}[R_{\tilde{\omega}}] - r).
\end{aligned}$$

□

We call the factor $\frac{\text{cov}_{\mathbb{P}}[f(R_{\tilde{\omega}}), R_X]}{\text{cov}_{\mathbb{P}}[f(R_{\tilde{\omega}}), R_{\tilde{\omega}}]}$, the $f - \beta$ -factor.

3 Empirical analysis

Let $R_k = \frac{A_k - q_k}{q_k}$ be the k -th asset return for $k = 1, \dots, K$ and $\mathbf{r}_t = (r_{1,t}, \dots, r_{K,t})'$ be the observation of the risky assets' returns $(R_1, \dots, R_K)'$ at time $t = 1, \dots, \tau$. The function $F : \mathbb{R}^K \rightarrow [0, 1]^K$ is the empirical multivariate cumulative distribution function of $(R_1, \dots, R_K)'$ for the observations $(\mathbf{r}_t)_{t=1, \dots, \tau}$, i.e. for $\mathbf{r} \in \mathbb{R}^K$

$$F(\mathbf{r}) = \frac{1}{\tau} \sum_{t=1}^{\tau} \mathbf{1}_{\mathbf{r}_t \leq \mathbf{r}},$$

where $\mathbf{1}_{\mathbf{r}_t \leq \mathbf{r}}$ is the vector $(\mathbf{1}_{r_{1,t} \leq r_1}, \dots, \mathbf{1}_{r_{K,t} \leq r_K})'$. We suppose that assets' returns are independent and identically distributed with multivariate cumulative distribution F . Let $r_{\tilde{\omega}, t}$ be the observation at time $t = 1, \dots, \tau$ of the market portfolio return $R_{\tilde{\omega}} = \frac{\tilde{\omega} - q(\tilde{\omega})}{q(\tilde{\omega})}$ and G be its empirical distribution function, i.e. for $\kappa \in \mathbb{R}$

$$G(\kappa) = \frac{1}{\tau} \sum_{t=1}^{\tau} \mathbf{1}_{r_{\tilde{\omega}, t} \leq \kappa}.$$

We consider several increasing, strictly convex distortions g on $[0, 1]$ with $g(0) = 0$ and $g(1) = 1$, parameterized by a one-dimensional parameter γ , which can be interpreted as the degree of risk aversion or pessimism (Bassett, Koenker, and Kordas 2004). We compute the corresponding increments with respect to the empirical distribution function G , i.e. for $\kappa \in \{r_{\tilde{\omega}, 1}, \dots, r_{\tilde{\omega}, \tau}\}$:

$$f_{R_{\tilde{\omega}}}(\kappa) = \frac{g\left(\frac{1}{T} \sum_{t=1}^T \mathbf{1}_{r_{\tilde{\omega}, t} \geq \kappa}\right) - g\left(\frac{1}{T} \sum_{t=1}^T \mathbf{1}_{r_{\tilde{\omega}, t} > \kappa}\right)}{\frac{1}{T} \sum_{t=1}^T \mathbf{1}_{r_{\tilde{\omega}, t} = \kappa}}. \quad (10)$$

We numerically compute the parameter γ in order to solve:

$$\sum_{t=1}^{\tau} f_{R_{\tilde{\omega}}}(r_{\tilde{\omega}, t}) (r_{\tilde{\omega}, t} - r) = 0$$

Distortion function	Parameter estimate	JT
$g(x) = x^\gamma$	$\hat{\gamma} = 1.157241$	12.99802 (0.16)
$g(x) = \Phi(\Phi^{-1}(x) + \gamma)$	$\hat{\gamma} = 0.1266901$	10.08102 (0.34)
$g(x) = \frac{1 - \exp(\gamma x)}{1 - \exp(\gamma)}$	$\hat{\gamma} = 0.4738711$	9.18521 (0.42)
$g(x) = -\frac{1}{\gamma} \log(-(1 - \exp(-\gamma))x + 1)$	$\hat{\gamma} = 0.4704917$	9.02273 (0.43)
$g(x) = 1 - (1 - x)^\gamma$	$\hat{\gamma} = 0.8779927$	8.640267 (0.47)

Table 1: Parameter estimates.

where r is as before the risk-free rate of return. The Table 3 reports the estimated parameters for the several choices of the distortion g . According to the Theorem 2.2 and equation (8), the null hypothesis to be tested is that $\mathbb{E}[\hat{\epsilon}_k] = 0$ for $k = 1, \dots, K$, where

$$\hat{\epsilon}_k = \frac{1}{\tau} \sum_{t=1}^{\tau} \hat{\epsilon}_{k,t}$$

and $\hat{\epsilon}_{k,t} = f_{R_{\bar{\omega}}}(r_{\bar{\omega},t})(r_{k,t} - r)$. As in Post and Van Vilet (2004) we use the Hansen's (1982) J-test for over-identifying restrictions based on the Generalized Method of Moments (for an overview see also Davidson and Mackinnon 1993, Chapter 17.6). The test statistic is given by

$$JT = \tau \hat{\epsilon}' W_\tau^{-1} \hat{\epsilon},$$

where

$$W_\tau = \frac{1}{\tau} \sum_{t=1}^{\tau} \hat{\epsilon}_t \hat{\epsilon}_t'$$

Under the null hypothesis the test statistic is $\chi^2(K - 1)$ asymptotically distributed. The value of the test statistic JT for the several choices of the one-parametric distortion g are reported in Table 3 together with the corresponding significance levels.

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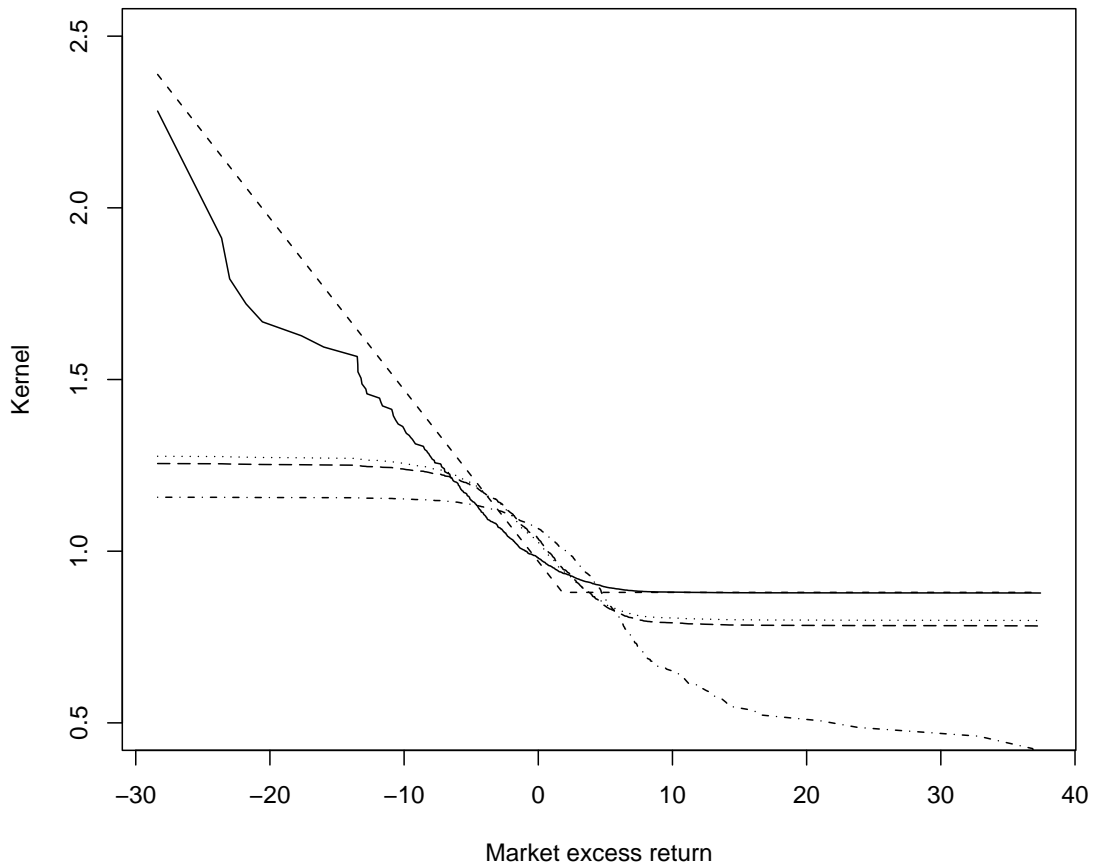


Figure 1: Estimated pricing kernel as function of the observed market excess returns, for several choices of the distortion function g : $g(x) = 1 - (1 - x)^\gamma$ (full line), $g(x) = -\frac{1}{\gamma} \log(-1 - \exp(-\gamma))x + 1$ (dotted line), $g(x) = x^\gamma$ (dotted-dashed line) and $g(x) = \frac{1 - \exp(\gamma x)}{1 - \exp(\gamma)}$ (long dashed line). The test statistics are reported in Table 3. The short dashed line corresponds to the estimate of the pricing kernel based on the third order stochastic dominance provided by Post and van Vilet (2004).

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