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ИНСТИТУТ ПО МАТЕМАТИКА С ИЗЧИСЛИТЕЛЕН ЦЕНТЪР
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Second-order subdifferentials of $C^{1,1}$ functions and optimality conditions

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Second-order subdifferentials of $C^{1,1}$ functions and optimality conditions

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Abstract

We present second-order subdifferentials of Clarke's type of $C^{1,1}$ functions, defined in separable Banach spaces with separable duals, i.e. of functions whose gradient mapping is locally Lipschitz. One of them is an extension of the generalized Hessian matrix of such functions in $\mathbf{R}^n$, considered by J.B. H.-Urruty, J.J. Strodiot and V.H. Nguyen. Various properties of these subdifferentials are proved. Second order optimality conditions (necessary, sufficient) for constrained minimization problems with $C^{1,1}$ data are obtained.

0 Introduction.

One of the motivations of this paper is the article of J.B. Hiriart-Urruty, J.J. Strodiot and V.H. Nguyen [6]. They defined so called generalized Hessian matrix of $C^{1,1}$ functions in $\mathbf{R}^n$ and investigated many of its properties, including necessary second-order optimality conditions for minimization problems with $C^{1,1}$ data.

Our goal in this paper is to define similar notions in infinite dimensional Banach spaces and to investigate their properties.

The Rademacher theorem plays a crutual role to define in $\mathbf{R}^n$ the generalized Hessian matrix to $C^{1,1}$ functions. In infinite dimensional Banach spaces we use a weaker generalization of it, due to D. Yost [5], but for $C^{1,1}$ functions defined on separable Banach spaces with separable duals. So we define an extension of the generalized Hessian matrix, called here second-order subdifferential. Many of the properties of the generalized Hessian matrix are valid also in our case.

We obtain a necessary condition and sufficient conditions for constrained minimization problems with $C^{1,1}$ data. The necessary condition can not be proved by the method in [6], which does not work in infinite dimensional Banach spaces.

Our approach is based on the method, described by V.M. Alexeev, V.M. Tykhomirov and S.V. Fomin in [1] for obtaining necessary and sufficient conditions for constrained minimization problems, defined by twice Frechet differentiable functions.

1 Basic definitions and properties.

Let E be a separable Banach space with separable dual E^* .

Consider the class $C^{1,1}(G)$ of all functions $f : G \to \mathbf{R}$, G is open in E, whose first derivatives are locally Lipschitz (then f is strictly differentiable, and hence, Frechet differentiable, see [2, p.32]). By a theorem of D. Yost [5, Theorem 22] it follows that for every $f \in C^{1,1}(G)$ f' is Gateaux differentiable on a dense subset $G(f)$ of G (we shall say that f is twice Gateaux differentiable on $G(f)$). For simplicity we shall write $C^{1,1}$ instead $C^{1,1}(G)$.

Denote by $\mathcal{L}(E \times E)$ the Banach space of all bilinear continuous functionals on $E \times E$ with the norm $\|.\|$: $\|L\| = \sup_{\substack{\|h_1\|=1 \\ \|h_2\|=1}} \mid L[h_1' h_2] \mid$, where $L \in \mathcal{L}(E \times E)$, $h_1, h_2 \in E$. Let $\mathcal{L}(E, E^*)$ be the Banach space of all linear continuous mappings $L : E \to E^*$ with the norm $\|.\|$: $\|L\| = \sup_{\|h\|=1} \|L(h)\|^*$, where $\|.\|^*$ is the norm in E^*, $L \in \mathcal{L}(E, E^*)$, $h \in E$.

It is well-known that $\mathcal{L}(E \times E)$ and $\mathcal{L}(E, E^*)$ are linear isomorphic (see [1], section 2.2.5). So, in the sequel we shall identify $\mathcal{L}(E \times E)$ and $\mathcal{L}(E, E^*)$.

Definition 1.1 *For every $x \in G$, h_1, $h_2 \in E$ we define*

$$f^{00}(x;h_1,h_2) := \limsup_{\substack{y\to x\\ t\downarrow 0}} \frac{f'(y+th_1)[h_2]-f'(y)[h_2]}{t},$$

$$d^2f(x;h_1,h_2) := \limsup_{G(f)\ni z\to x} f''(z)[h_1,h_2].$$

Proposition 1.2 *For every $x \in G$, h_1, $h_2 \in E$ we have*

$$f^{00}(x;h_1,h_2) = d^2f(x;h_1,h_2).$$

Proof. For every $\epsilon > 0$ there exists $\delta > 0$ such that

$$f''(y)[h_1,h_2] \ < \ d^2f(x;h_1,h_2)+\epsilon$$

whenever $y \in G(f)\cap B(x;\delta)$.

By the proof of D. Yost's theorem ([5], Theorem 22) we have that if we take A_i be i-dimentional subspaces, $A_1 \subset A_2 \subset A_3 \subset ...$ and $\bigcup_{i=1}^{\infty} A_i$ is dense in E,then $G(f) = \bigcap_{i=1}^{\infty} D_i$, where $E\setminus D_i$ is A_i-null (i.e., by definition, $meas((x+A_i)\setminus(D_i\cap(x+A_i)) = 0$ for every $x \in E$). Denote $D_{i,x} = D_i\cap(x+A_i)$. By Fubini's theorem there exists $Y_{i,x}\subset D_{i,x}\cap B(x;\delta/2)$, such that $meas(D_{i,x}\cap B(x;\delta/2)\setminus Y_{i,x}) = 0$ (i-dimentional measure in $x+A_i$) and for every $y\in Y_{i,x}$ $meas(L_y\setminus(L_y\cap D_{i,x})) = 0$, where $L_y = \{y+th_1 \ : \ t \in (0, \frac{\delta}{2\|h_1\|})\}$.

By [5], Theorem 19 it follows that for $Y_i := \bigcup_{x\in E} Y_{i,x}$, $Y := \bigcap_{i=1}^{\infty} Y_i$ is dense in $B(x,\delta/2)$.

Let $y\in Y$. Then for every i=1,2,... $y\in Y_{i,x_i}$ for some $x_i\in E$ and $meas(L_y\setminus(L_y\cap D_{i,x_i})) = 0$ (one-dimentinal measure). Hence $meas \bigcup_{i=1}^{\infty}(L_y\setminus(L_y\cap D_{i,x_i})) = 0$. Then $meas(L_y\setminus(L_y\cap G(f)) = 0$ and:

$$f'(y+th_1)[h_2]-f'(y)[h_2] = \int_0^t f''(y+sh_1)[h_1,h_2]ds,$$

because $f''(.)[h_1, h_2]$ exists almost everywhere in L_y. Since $\|y + sh_1 - x\| < \delta$ for $s \in (0, t)$, we have

$$f'(y + th_1)[h_2] - f'(y)[h_2] = \int_0^t f''(y + sh_1)[h_1, h_2]ds \leq$$

$$(1.1) \qquad \leq \int_0^t (d^2 f(x; h_1, h_2) + \epsilon)ds = t(d^2 f(x; h_1, h_2) + \epsilon).$$

Since f' is continuous and Y is dense in $B(x; \delta/2)$, we have that (1.1) is fulfilled for $y \in B(x; \delta/2)$ and $t \in (0, \frac{\delta}{2\|h_1\|})$. Therefore $f^{00}(x; h_1, h_2) \leq d^2 f(x; h_1, h_2) + \epsilon$. The proof is completed, since the inequality in the other direction is trivial. ∎

Proposition 1.3 *The function $f^{00}(x; h_1, h_2)$ is upper semicontinuous as functions of x and*

$$| f^{00}(x; h_1, h_2) | \leq l_x \|h_1\|.\|h_2\|,$$

where l_x is a Lipschitz constant of f' on a neighbourhood of x.

Proof. Let $\{x_i\} \to x$ as $i \to \infty$. For each $i > 0$ by the definition of lim sup there exist $y_i \in E$ and $t_i > 0$ such that $\|y_i - x_i\| + t_i < \frac{1}{i}$ and

$$f^{00}(x_i; h_1, h_2) - \frac{1}{i} \leq \frac{f'(y_i + t_i h_1)[h_2] - f'(y_i)[h_2]}{t_i}.$$

Then

$$\limsup_{i \to \infty} f^{00}(x_i; h_1, h_2) \leq f^{00}(x; h_1, h_2).$$

The inequality follows immediately from the locally Lipschitz property of f' near the point x. ∎

Having in mind Proposition 1.2 we claim that $d^2 f(x; h_1, h_2)$ has the same properties.

$\mathcal{L}(E, E^*)$ is a conjugate space (see Holmes [4, Chapter 23B]); the w^*-topology of $\mathcal{L}(E, E^*)$ is called weak*-operator topology. The predual of $\mathcal{L}(E, E^*)$ is the linear hull V of all functionals $l_{h_1,h_2} \in (\mathcal{L}(E, E^*))^*$, $h_1, h_2 \in E$ of the form $l_{h_1,h_2}(L) := L[h_1, h_2]$ with the norm in $(\mathcal{L}(E, E^*))^*$. Then V is a separable normed space, therefore we can note the following.

Remark 1.4 *Every w^*- compact subset in $\mathcal{L}(E\times E)$ is metrizable (see Holmes [4, Chapter 12F]).*

We have that the sequence $\{L_n\} \subset \mathcal{L}(E\times E)$ converges in the w^*-topology to some $L_0 \in \mathcal{L}(E\times E)$ iff $L_n[h_1,h_2] \to L_0[h_1,h_2]$ for every $h_1,\ h_2\ \in E$.

There are two natural ways to define second subdifferentials of $f \in C^{1,1}$ (by analogy with the first order Clarke's subdifferential, see [2], p.27 and Theorem 2.8.6).

Definition 1.5

$\partial_c^2 f(x) := \{\ L \in \mathcal{L}(E\times E)\ :\ L[h_1,h_2]\ \le\ f^{00}(x;h_1,h_2),\ \ \forall (h_1,h_2) \in E\times E\}.$

Definition 1.6

$$\partial^2 f(x) := \overline{co}^*\{L \in \mathcal{L}(E\times E)\ :\ L = w^* - \lim_{G(f)\ni z\to x} f''(z)\ \}.$$

It's easy to see that $\partial^2 f(x) \subset \partial_c^2 f(x)$, for every $x \in G$. Indeed, let $L = w^*-\lim_{G(f)\ni z_n\to x} f''(z_n)$. Since $f''(z_n)[h_1,h_2] \le f^{00}(z_n;h_1,h_2)$ for every $h_1,\ h_2 \in E$, by Proposition 1.3 we have

$$L[h_1,h_2] = \lim_{n\to\infty} f''(z_n)[h_1,h_2] \le \limsup_{n\to\infty} f^{00}(z_n;h_1,h_2) \le f^{00}(x;h_1,h_2).$$

Hence $L \in \partial_c^2 f(x)$ and since $\partial_c^2 f(x)$ is obviously convex and w^*-closed, we obtain $\partial^2 f(x) \subset \partial_c^2 f(x)$.

Also, it is easy to see that

$$d^2 f(x;h_1,h_2) = \sup\{L[h_1,h_2]\ ;\ L \in \partial^2 f(x)\}. \tag{1.2}$$

In the case when $E = \mathbf{R}^n$ Definition 1.6 was considered and used by J.B.H. Urruty, J.J. Strodiot and V.H. Nguyen [6] ($\partial^2 f(x)$ was called there generalized Hessian matrix). They used Rademacher's theorem (instead Yost's one); then f is twice Frechet differentiable almost everywhere. Therefore $f''(z)$ (when it exists) is a symmetric matrix (see [1], Setion 2.2.5), and hence $\partial^2 f(x)$ consists of symmetric matrices. In our infinite dimensional case we cannot say that $\partial^2 f(x)$ consists of symmetric bilinear functionals.

Note that in $\mathbf{R}^n$ $\partial_c^2 f(x)$ is in fact the *plenary hull* of $\partial^2 f(x)$ in the terminology of [7].

Proposition 1.7 *$\partial^2 f(x)$ and $\partial_c^2 f(x)$ are non-empty convex and w^*-compact sets. $\partial^2 f$ and $\partial_c^2 f$ are locally norm bounded in $\mathcal{L}(E \times E)$.*

Proof. Let $x_n \in E(f)$ and $x_n \to x$, as $n \to \infty$. Since the set $D := \{f''(x_n) \ : \ n > \nu\}$ is norm bounded in $\mathcal{L}(E \times E)$ for some ν (by the Lipschitz constant of f' at x), we can apply the Alaoglou-Bourbaki theorem. Thus the limit of a w^*-convergent subsequence of D belongs to $\partial^2 f(x)$. Since $\partial^2 f(x) \subset \partial_c^2 f(x)$, then $\partial_c^2 f(x) \neq \emptyset$ too. The locally boundedness and the w^*-closedness of $\partial^2 f$ and $\partial_c^2 f$ follows by the fact that f' is locally Lipschitz.

Again by Alaouglou-Bourbaki theorem we obtain that $\partial^2 f(x)$ and $\partial_c^2 f(x)$ are w^*-compact. The convexity of the sets is obvious. ∎

Definition 1.8 *The function $f : G \to \mathbf{R}$ is said to be twice strictly differentiable at $x \in G$ if there exists $D_s^2 f(x) \in \mathcal{L}(E \times E)$, such that:*

$$\lim_{\substack{y \to x \\ t \downarrow 0}} \frac{f'(y + th_1)[h_2] - f'(y)[h_2]}{t} = D_s^2 f(x)[h_1, h_2].$$

Proposition 1.9 *If f is twice strictly differentiable at $x \in G$ then $\partial_c^2 f(x) = \{D_s^2 f(x)\}$.*

Proof. By definition it follows that $f^{00}(x; h_1, h_2) = D_s^2 f(x)[h_1, h_2]$ and then $L[h_1, h_2] \leq D_s^2 f(x)[h_1, h_2]$ for all $h_1, \ h_2 \in E$ and $L \in \partial_c^2 f(x)$. Hence $L[h_1, h_2] = D_s^2 f(x)[h_1, h_2]$ and therefore $L = D_s^2 f(x)$. ∎

Proposition 1.10 *$\partial^2 f(x)$ is a singleton (i.e. $\partial^2 f(x) = \{L\}$) iff f is twice strictly differentiable at $x \in G$ (then $\partial_c^2 f(x) = \{L\}$).*

Proof. Let $\partial^2 f(x) = \{L\}$. Then $L = w^* - \lim_{G(f) \ni z \to x} f''(z)$ and by Proposition 1.2

$$f^{00}(x; h_1, h_2) = \limsup_{G(f) \ni z \to x} f''(z)[h_1, h_2] = L[h_1, h_2].$$

We can write:

$$\liminf_{\substack{x' \to x \\ t \downarrow 0}} \frac{(f'(x' + th_1) - f'(x'))[h_2]}{t} = -\limsup_{\substack{x' \to x \\ t \downarrow 0}} \frac{(f'(x') - f'(x' + th_1))[h_2]}{t} =$$

$$= -\limsup_{\substack{x' \to x \\ t \downarrow 0}} \frac{(f'(x' + th_1 - th_1) - f'(x' + th_1))[h_2]}{t} = -f^{00}(x; -h_1, h_2) =$$

$$= -L[-h_1, h_2] = L[h_1, h_2] = f^{00}(x; h_1, h_2),$$

whence f is twice strictly differentiable.

The other direction follows by the inclusion $\partial_c^2 f(x) \supset \partial^2 f(x)$ and by Proposition 1.9. ∎

By this propositions we obtain immediately the following.

Corollary 1.11 *$\partial_c^2 f(x)$ is a singleton iff $\partial^2 f(x)$ is a singleton.*

Proposition 1.12 *The multivalued mapping $\partial_c^2 f : (E, \|.\|) \to (\mathcal{L}(E \times E), w^*)$ is upper-semicontinuous.*

Proof. *Claim:* The multivalued mapping $\partial_c^2 f : (E, \|.\|) \to (\mathcal{L}(E \times E), w^*)$ has closed graph.

Let $x_n \to x$ as $n \to \infty$, $L_n \in \partial_c^2 f(x_n)$ and let $L_n \xrightarrow{w^*} L$. We'll show that $L \in \partial_c^2 f(x)$.

Let us fix $(h_1, h_2) \in (E \times E)$. Then $L_n[h_1, h_2] \to L[h_1, h_2]$. We have $f^{00}(x_n; h_1, h_2) \ \geq \ L_n[h_1, h_2]$ for all n and from the upper-semicontinuity of f^{00} as a function of x (Propostion 1.3) we obtain that

$$f^{00}(x; h_1, h_2) \ \geq \ L[h_1, h_2],$$

whence $L \in \partial_c^2 f(x)$.

Now we are in position to prove the proposition.

Assume the contrary. Then there exists a neighborhood U^* of $\partial_c^2 f(x)$ in the w^*-topology, such that for every $n \in \mathbf{N}$ there exists $x_n \in B(x, 1/n)$ such that $\partial_c^2 f(x_n) \not\subset U^*$, i.e. there exists $L_n \in \partial_c^2 f(x_n) \setminus U^*$.

From L_n we can chose w^*-converging subsequence (since $\partial_c^2 f$ is locally bounded by Propostion 1.7), whose w^*-limit L_0 belongs to $E \setminus U^*$. Since $\partial_c^2 f$ has w^*-closed graph (by the *Claim*), L_0 is also an element of $\partial_c^2 f(x)$, which is a contradiction. ∎

Proposition 1.13 *The multivalued mapping $\partial^2 f : (E, \|.\|) \to (\mathcal{L}(E \times E), w^*)$ is upper-semicontinuous.*

Proof. Denote $D^2f(x) = \{L \in \mathcal{L}(E \times E) \,:\, L = w^*- \lim_{G(f)\ni z\to x} f''(z)\}$. Then by definition $\partial^2 f(x) = \overline{co}^* D^2 f(x)$.

Claim: $D^2 f$ has closed graph.

Let $x_n \to x$ and $L_n \in D^2 f(x_n)$, $L_n \xrightarrow{w^*} L_0$. Then

$$L_n = w^* - \lim_{\substack{G(f)\ni z_{n,m}\to x_n \\ m\to\infty}} f''(z_{n,m}).$$

Since $\partial^2 f$ is locally bounded (by Proposition 1.7), there exists a neighbourhood U of x such that the set $\cup\{\partial^2 f(x) \,:\, x \in U\}$ is contained in $\lambda B_{\mathcal{L}}$ for some $\lambda > 0$, where $B_{\mathcal{L}}$ is the closed unit ball in $\mathcal{L}(E \times E)$. By Remark 1.4 the set $2\lambda B_{\mathcal{L}}$ with the w^*-topology is metrizable. Denote this metric by d; then $L_n = d - \lim_{m\to\infty} f''(z_{n,m})$, $L_0 = d - \lim_{n\to\infty} L_n$. Hence we can find indeces $m(n)$ such that $z_{n,m(n)} \to x$, as $n \to \infty$ and $L_0 = d - \lim_{n\to\infty} f''(z_{n,m(n)})$, therefore $L_0 \in D^2 f(x)$ and the claim is proved.

Let $V \supset \partial^2 f(x)$ be an w^*-open set in $\mathcal{L}(E \times E)$. Since $\partial^2 f$ is w^*-compact, there exists $\epsilon > 0$ such that

$$V_\epsilon := \{L \in \mathcal{L}(E \times E) \,:\, d(L, \partial^2 f(x)) \le \epsilon\} \subset V \cap 2\lambda B_{\mathcal{L}}.$$

As in the proof of Proposition 1.12 we obtain (by assuming the contrary and using the *Claim*) that there exists an open set $W \ni x$, $W \subset U$ for which $D^2 f(y) \subset V_\epsilon$ for all $x \in W$. But V_ϵ is convex, therefore $coD^2 f(y) \subset V_\epsilon$ for all $y \in W$, hence

$$\partial^2 f(y) = \overline{co}^* D^2 f(y) \subset V_\epsilon \subset V, \;\; \forall y \in W$$

and the proof is completed. ∎

Let $\psi : \mathbf{R} \to E$ be an affine function, $\phi : G \to \mathbf{R}$ be a $C^{1,1}$ function . It is clear that $\phi \circ \psi \in C^{1,1}(G)$.

It is easy to derive in the same way like in [6, Theorem 2.2] :

Proposition 1.14 *For all* x_0, u, v $\in \mathbf{R}$

$$(\phi \circ \psi)^{00}(x_0; u, v) = \phi^{00}(\psi(x_0), \psi'(x_0)u, \psi'(x_0)v).$$

Hence by Proposition 1.2 $d^2(\phi \circ \psi)(x_0; u, v) = d^2\phi(\psi(x_0), \psi'(x_0)u, \psi'(x_0)v)$ and since $d^2 f(x_0, h_1, h_2)$ is by (1.2) the support function of $\partial^2 f(x_0)h_1$ in the h_2 direction, we deduce the following from the proposition above:

$$\cup\{L \; : \; L \in \partial^2(\phi \circ \psi)(x_0)\} = \cup\{L[\psi'(x_0), \psi'(x_0)] \; : \; L \in \partial^2\phi(\psi(x_0))\}$$

The following Proposition is stated in [6] without proof. Here we include the proof for completness.

Proposition 1.15 *Let I be an open interval containing [0,1] and let $\phi \in C^{1,1}(I)$. Then*

$$\phi(1) - \phi(0) - \phi'(0) \;\in\; \frac{1}{2}\partial^2\phi(t)$$

for some $t \in (0,1)$.

Proof. Define:

$$h(t) = \phi(1) - \phi(t) - \phi'(t)(1-t) - (1-t)^2\lambda, \quad t \in [0,1],$$

where $\lambda = \phi(1) - \phi(0) - \phi'(0)$. So we have $h(1) = h(0) = 0$. Obviously h is locally Lipschitz on [0,1]. There exists $\xi \in (0,1)$ such that either

1) ξ is a minimum of h over [0,1], or

2) ξ is a maximum of h over [0,1].

Let 1) be fulfilled. Then by the necessary condition for a local minimum (see Clarke [2])

$$0 \in \partial h(\xi) = -\partial\phi'(\xi)(1-\xi) + 2(1-\xi)\lambda.$$

Hence $\lambda \in \frac{1}{2}\partial\phi'(\xi)$. But by Clarke [2], Theorem 2.5.1 and by definition of $\partial^2\phi$ we have $\partial\phi'(\xi) = \partial^2\phi(\xi)$.

Let 2) be fulfilled. Then ξ is a minimum point of the function $-h$ over [0,1] and since $\partial(-h)(\xi) = -\partial h(\xi)$ (by Clarke [2]) we have $0 \in \partial(-h)(\xi) = -\partial h(\xi)$, so $0 \in \partial h(\xi)$ and as in case 1) we obtain $\lambda \in \frac{1}{2}\partial^2\phi(\xi)$. ∎

Using the Propositions 1.14 and 1.15 we immediately obtain the second-order expansion by the type of those in [6]:

Proposition 1.16 *Let $f \in C^{1,1}(G)$. Then for every a, $b \in G$ there exists $c \in (a,b)$ and $L_c \in \partial^2 f(c)$ such that*

$$f(b) = f(a) + f'(a)[b-a] + 1/2L_c[b-a, b-a].$$

2 Necessary and sufficient optimality conditions.

We consider now the following constrained minimization problem:

$$\text{P(E)}\qquad \begin{array}{l} f_0(x) \to \min \\ x \in E \\ f_i(x) \le 0\ ,\ 1 \le i \le m \\ F(x) = 0, \end{array}$$

where $F(x) = (g_1(x), .., g_k(x))^T$ and the functions $f_0,\ f_i\ ,\ 1 \le i \le m,\ g_j,\ 1 \le j \le k$ are $C^{1,1}(G)$ functions. The Lagrangian function for P(E) is:

$$\mathcal{L}(x; \lambda, \mu) = \sum_{i=0}^{m} \lambda_i f_i(x) + \sum_{j=1}^{k} \mu_j g_j(x),$$

where $(\lambda, \mu) := (\lambda_0, ..., \lambda_m, \mu_1, ..., \mu_k) \in \mathbf{R}^{m+1} \times \mathbf{R}^k$ are the Lagrange multipliers.

Denote by $locminP(E)$ the set of all points of local minimum of P(E) and

$$\Lambda(x) = \{(\lambda, \mu) \in \mathbf{R}^{m+1} \times \mathbf{R}^k :\ \sum_{i=0}^{m} \lambda_i f_i'(x) + \sum_{j=1}^{k} \mu_j g_j'(x) = 0,$$

$$\lambda_i f_i(x) = 0,\ \ 1 \le i \le m\ \ \lambda_i \ge 0,\ \ 0 \le i \le m,\ \ \sum_{i=0}^{m} \lambda_i = 1\}.$$

$$K(x) = \{h \in E\ :\ f_i'(x)[h] \le 0,\ \ 0 \le i \le m,\ \ g_j'(x)[h] = 0\ \ 1 \le j \le k\}.$$

Proposition 2.1 *(Necessary condition) ([1], Section 3.4.2) Let $x_0 \in locminP(E)$ and $Im\ F'(x) = \mathbf{R}^k$. Then $\Lambda(x_0)$ is non-empty compact set.*

Lemma 2.2 *(min-max)([1], Section 3.3.4) Let $A : E \to \mathbf{R}^k$ be linear continuous surgective operator, $AE = \mathbf{R}^k$, $x_i^* \in E^*$, $0 \le i \le m$, $y \in \mathbf{R}^k$, $a \in \mathbf{R}^{m+1}$,*

$$\max_{0 \le i \le m} x_i^*[x] \ge 0\ \ \forall x \in Ker\ A \tag{2.1}$$

Denote

$$(2.2) \qquad S(a,y) = \inf_{Ax+y=0} \max_{0\le i\le m} (a_i + x_i^*[x]).$$

Then

$$(2.3) \qquad a) \;\; S(a,y) = \sup_{(\lambda,\mu)\in\Lambda} (\sum_{i=0}^{m} \lambda_i a_i + \sum_{j=1}^{k} \mu_j y_j),$$

where

$$\Lambda = \{(\lambda,\mu) \in \mathrm{R}^{m+1}\times\mathrm{R}^k \; : \; \lambda_i \ge 0, \; \sum_{i=0}^{m} \lambda_i = 1, \; \sum_{i=0}^{m} \lambda_i x_i^* + A^*\mu = 0\}.$$

b) inf in (2.2) and sup in (2.3) are attained.

Let $x_0 \in locminP(E)$. If $f_0(x_0) \neq 0$, we replace f_0 by the function $\tilde{f}_0(x) = f_0(x) - f_0(x_0)$. If $f_i(x_0) \neq 0$ for some i, we remove this constraint, because for a local minimum the constraints $f_i(x) < 0$ are unessential. So, without loss of generality we assume that $f_i(x_0) = 0$, $0 \le i \le m$. Consider the problem

(P') $\quad f(x) := \max\{f_0(x), ..., f_m(x)\} \to \min \;\; ; \; F(x) = 0.$

It is easy to see the validity of the following.

Lemma 2.3 *([1], Section 3.4.2)* $x_0 \in locmin(P')$.

Now we can state the necessary optimality condition for P(E).

Theorem 2.4 *Let in P(E) Im* $F'(x_0) = \mathrm{R}^k$. *If* $x_0 \in locminP(E)$, *then for every* $h \in K(x_0)$ *there exist* $L_i \in \partial^2 f_i(x_0)$, $0 \le i \le m$, $M_j \in \partial^2 g_j(x_0)$ $1 \le j \le k$ *such that*

$$\max_{(\lambda,\mu)\in\Lambda(x_0)} (\sum_{i=0}^{m} \lambda_i L_i[h,h] + \sum_{j=1}^{k} \mu_j M_j[h,h]) \;\ge\; 0.$$

Proof. By Proposition 2.1 the set $\Lambda(x_0)$ is non-empty.

Let us assume the contrary, i.e. there exists $h \in K(x_0)$ such that for every $L_i \in \partial^2 f_i(x_0)$, $0 \leq i \leq m$, and every $M_j \in \partial^2 g_j(x_0)$, $1 \leq j \leq k$ we have

$$(2.4) \qquad \max_{(\lambda,\mu)\in\Lambda(x_0)} \left(\sum_{i=0}^{m} \lambda_i L_i[h,h] + \sum_{j=1}^{k} \mu_j M_j[h,h]\right) < 0.$$

It is clear that $\|h\| \neq 0$. Let $t_n \to 0$ as $n \to \infty$. By Proposition 1.16 we have:

$$g_j(x_0 + t_n h) = g_j(x_0) + t_n g_j'(x_0)[h] + \frac{t_n^2}{2} M_{j,n}[h,h] =$$

$$= \frac{t_n^2}{2} M_{j,n}[h,h],$$

and $M_{j,n} \in \partial^2 g_j(x_0 + \gamma_{j,n} t_n h)$, where $\gamma_{j,n} \in (0,1)$.

Since $x_0 + \gamma_{j,n} t_n h \to x_0$, as $n \to \infty$ and $\partial^2 g_j$ are locally bounded and have w^*-closed graphs we can chose w^*-convergent subsequences from $\{M_{j,n}\}_{n\geq 1})$, whose w^* limits M_j are in $\partial^2 g_j(x_0)$ for $1 \leq j \leq k$.

Analogously, having in mind that $f_i(x_0) = 0$ (see the remark before Lemma 2.3):

$$f_i(x_0 + t_n h) = f_i(x_0) + t_n f_i'(x_0)[h] + \frac{t_n^2}{2} L_{i,n}[h,h] =$$

$$= t_n f_i'(x_0)[h] + \frac{t_n^2}{2} L_{i,n}[h,h],$$

where $L_{i,n} \in \partial^2 f_i(x_0 + \eta_{i,n} t_n h)$, $\eta_{i,n} \in (0,1)$ and choose w^*-converging subsequences from $\{L_{i,n}\}_{n\geq 1}$, whose w^*-limits L_i are in $\partial^2 f_i(x_0)$ for $0 \leq i \leq m$.

For every $\epsilon > 0$ there exists an integer N_1 such that for every $n \geq N_1$ the following inequalities are fulfilled:

$$| M_{j,n}[h,h] - M_j[h,h] | < 2\epsilon$$

$$| L_{i,n}[h,h] - L_i[h,h] | < 2\epsilon$$

Hence for $n \geq N_1$ we have:

$$(2.5) \qquad g_j(x_0 + t_n h) = \frac{t_n^2}{2} M_j[h,h] + \phi_{j,n}(\epsilon),$$

where

$$(2.6) \qquad |\ \phi_{j,n}(\epsilon)\ | < \ t_n^2\epsilon \quad for \quad 1 \le j \le k, \ \ n \ge N_1.$$

and

$$(2.7) \qquad f_i(x_0 + t_n h) = t_n f_i'(x_0)[h] + \frac{t_n^2}{2} L_i[h,h] + \psi_{i,n}(\epsilon),$$

where

$$(2.8) \qquad |\ \psi_{i,n}(\epsilon)\ | < \ t_n^2\epsilon \quad for \quad 0 \le i \le m, \ \ n \ge N_1.$$

Denote $x_i^* = f_i'(x_0)$, $a_i = \frac{1}{2}L_i[h,h]$, $0 \le i \le m$, $y_j = \frac{1}{2}M_j[h,h]$, $1 \le j \le k$, $y = (y_1, ..., y_k)$, $A = F'(x_0)$. By [1], Section 3.2.4, Lemma 1, the condition (2.1) of Lemma 2.2 is fulfilled. Applying Lemma 2.2 we can find $\xi = \xi(h) \in E$, such that

$$(2.9) \qquad F'(x_0)\xi + y = 0$$

and

$$(2.10) \qquad \max_{0 \le i \le m}(a_i + f_i'(x_0)[\xi]) = \max_{(\lambda,\mu)\in\Lambda(x_0)} (\sum_{i=0}^{m} \lambda_i a_i + \sum_{j=1}^{k} \mu_j y_j) =: \Psi(h),$$

where $\Psi(h) < 0$ from (2.4).

Let l be the maximal of the Lipschtz constants for g_j' and f_i' in a neighborhood U_1 of x_0. There exists a neibourhood $U_2 \subset U_1$ of x_0 and a constant s such that g_j' and f_i' are norm bounded by s there.

Using the mean value theorem, (2.5) and (2.9) we have:

$$\begin{aligned} g_j(x_0 + t_n h + t_n^2\xi) &= g_j(x_0 + t_n h + t_n^2\xi) - g_j(x_0 + t_n h) + g_j(x_0 + t_n h) = \\ &= t_n^2 g_j'(x_0 + t_n h + \kappa_{j,n} t_n^2 \xi)[\xi] + \frac{t_n^2}{2} M_j[h,h] + \phi_{j,n}(\epsilon) = \\ &= t_n^2 g_j'(x_0 + t_n h + \kappa_{j,n} t_n^2 \xi)[\xi] - t_n^2 g_j'(x_0)[\xi] + \phi_{j,n}(\epsilon) \ \le \\ &\le \ t_n^2 l \|t_n h + \kappa_{j,n} t_n^2 \xi\| \|\xi\| + \phi_{j,n}(\epsilon) \ \le \\ &\le \ t_n^3 l(\|h\| + t_n\|\xi\|)\|\xi\| + |\ \phi_{j,n}(\epsilon)\ | =: \theta_{j,n}(\epsilon), \end{aligned}$$

where $\kappa_{j,n} \in (0,1)$.

Using (2.6) we obtain

$$\theta_{j,n}(\epsilon) \leq o(t_n^2) + t_n^2 \epsilon, \tag{2.11}$$

where $\frac{o(t_n^2)}{t_n^2} \to 0$.

By the proof of the Ljusternik theorem ([1], Section 2.3.5) there exists a map Φ, defined on a neighbourhood $U \subset U_2$ of the point x_0, such that:

$$F(x + \Phi(x)) = 0, \quad \|\Phi(x)\| \leq q\|F(x)\|, \quad \forall x \in U,$$

where q is a constant (see also [1], Section 3.4.2).

Substitute $r(t_n) = \Phi(x_0 + t_n h + t_n^2 \xi)$.

Then for n sufficiently large $x_0+t_n h+t_n^2\xi \in U$, $g_j(x_0+t_n h+t_n^2\xi+r(t_n)) = 0$ for $1 \leq j \leq k$ and by (2.11)

$$\|r(t_n)\| \leq q\|F(x_0 + t_n h + t_n^2 \xi)\| =$$

$$= q(\sum_{j=1}^{k} g_j^2(x_0 + t_n h + t_n^2\xi))^{1/2} \ \leq \ qk\theta_n(\epsilon) \leq qk(o(t_n^2) + t_n^2\epsilon). \tag{2.12}$$

Again by the mean-value theorem and (2.7) we obtain:

$$f_i(x_0+t_n h+t_n^2\xi+r(t_n)) = f_i(x_0+t_n h+t_n^2\xi+r(t_n)) - f_i(x_0+t_n h) + f_i(x_0+t_n h) =$$

$$= f_i'(x_0 + t_n h + \nu_{i,n}(t_n^2\xi + r(t_n))[t_n^2\xi + r(t_n)] - t_n^2 f_i'(x_0)[\xi] + t_n^2 f_i'(x_0)[\xi] +$$

$$+ t_n f_i'(x_0)[h] + \frac{t_n^2}{2} L_i[h,h] + \psi_{i,n}(\epsilon) \leq \ t_n^2 l\|t_n h + \nu_{i,n}(t_n^2\xi + r(t_n))\|.\|\xi\| +$$

$$+ s\|r(t_n)\| + t_n^2 f_i'(x_0)[\xi] + \frac{t_n^2}{2} L_i[h,h] + \psi_{i,n}(\epsilon),$$

where $\nu_{i,n} \in (0,1)$. Hence by 2.8, 2.10 and 2.12 we obtain:

$$f(x_0 + t_n h + t_n^2\xi + r(t_n)) \leq$$

$$\leq \max_{0 \leq i \leq m} (t_n^2 l\|t_n h + \nu_{i,n}(t_n^2\xi + r(t_n))\|.\|\xi\| +$$

$$+ s\|r(t_n)\| + t_n^2 f_i'(x_0)[\xi] + \frac{t_n^2}{2} L_i[h,h] + \psi_{i,n}(\epsilon)) \leq$$

$$\leq o_1(t_n^2) + (t_n^2 l\|\xi\| + s)\|r(t_n)\| + t_n^2\Psi(h) + \mid \psi_{i,n}(\epsilon) \mid \leq$$
$$\leq o_1(t_n^2) + (t_n^2 l\|\xi\| + s)qk\theta_{j,n}(\epsilon) + t_n^2\Psi(h) + t_n^2\epsilon =$$
$$= t_n^2[\frac{o_1(t_n^2)}{t_n^2} + (t_n^2 l\|\xi\| + s)qk\epsilon + \Psi(h) + \epsilon],$$

where $\frac{o_1(t_n^2)}{t_n^2} \to 0$.

Now it is clear that if we chose ϵ to be sufficiently small then for sufficiently large n, since $\Psi(h) < 0$, we have

$$f(x_0 + t_n h + t_n^2\xi + r(t_n)) < 0,$$

which is in a contradiction with Lemma 2.2. ∎

Recall that the abstract minimization problem [f,X] (minimize f over X) is Tychonov well-posed if it has an unique solution x_0 and every minimizing sequence (i.e. $\{x_n\} \subset X,\ f(x_n) \to f(x_0)$) is convergent to x_0. The reader can consult with [8] for different notions of well posedness and properties of well posed optimization problems.

The second-order sufficient condition for the problem P(E) is the following.

Theorem 2.5 *Let in P(E)* $f_i(x_0) = 0,\ 0 \leq i \leq m,\ Im\ F'(x_0) = \mathbf{R}^k$, $\Lambda(x_0) \neq \emptyset$ *and there exists a constant* $\alpha > 0$ *such that for every* $L_i \in \partial^2 f_i(x_0)$, $0 \leq i \leq m$, $M_j \in \partial^2 g_j(x_0)$, $1 \leq j \leq k$ *we have*

$$\max_{(\lambda,\mu)\in\Lambda(x_0)} (\sum_{i=0}^{m} \lambda_i L_i[h,h] + \sum_{j=1}^{k} \mu_j M_j[h,h]) \geq 2\alpha\|h\|^2 \quad \forall h \in K(x_0).$$

Then x_0 is a strict local minimum of the problem P(X) for every finite-dimensional subspace $X \ni x_0$ of E. If, in addition, the functions f_i, $0 \leq i \leq m$ are convex and the functions g_j, $1 \leq j \leq k$ are affine, then the problem P(X) has unique solution x_0 and hence it is Tykhonov well posed. Also in this case the proplem P(E) has unique solution x_0.

Proof. We'll show that for every finite-dimentional subspace $X \ni x_0$ there exists $\delta > 0$ such that the conditions: $h \in \delta B \cap X$ and

$$f_i(x_0 + h) \leq 0,\ \ 0 \leq i \leq m, \quad F(x_0 + h) = 0, \tag{2.13}$$

where B is the unit ball in E, are inconsistent. From this we obtain immediately the desired conclusion.

Let us assume the contrary: there exists a finite-dimentional subspace $X \ni x_0$, such that for every $\delta > 0$ there exists $h_\delta \in \delta B \cap X$ such that the conditions (2.13) are fulfilled.

Denote

$$H = \bigcup_{\delta > 0} h_\delta.$$

Let $h \in H$. By Proposition 1.16 we have:

$$f_i(x_0 + h) = f_i'(x_0)[h] + \frac{1}{2} L_i(x_0 + \eta_i h)[h, h],$$

where $\eta_i \in (0, 1)$, $L_i(x_0 + \eta_i h) \in \partial^2 f_i(x_0 + \eta_i h)$, $0 \leq i \leq m$.

$$g_j(x_0 + h) = g_i'(x_0)[h] + \frac{1}{2} M_j(x_0 + \gamma_j h)[h, h],$$

where

$$\gamma_j \in (0, 1), \quad M_j(x_0 + \gamma_j h) \in \partial^2 g_j(x_0 + \gamma_j h), \quad 1 \leq j \leq k.$$

Since the set $\Lambda(x_0)$ is compact (by Proposition 2.1) there exists a constant c_1, such that for every $(\lambda, \mu) \in \Lambda(x_0)$ $\|\mu\| \leq c_1$.

Substitute

$x_i^* = f_i'(x_0)$

$a_i = \frac{1}{2} L_i(x_0 + \eta_i h)[h, h], \quad 0 \leq i \leq m,$

$A = F'(x_0)$

$y_j = \frac{1}{2} M_j(x_0 + \gamma_j h)[h, h], \quad 1 \leq j \leq k, \quad y = (y_1, ..., y_k),$

$f(x) = \max_{0 \leq i \leq m} f_i(x)$

and obtain that (2.13) is equivalent to

$$(2.14) \qquad x_i^*[h] + a_i = f_i(x_0 + h) \leq 0, \ 0 \leq i \leq m, \ Ah + y = 0.$$

Since $\partial^2 f_i$ and $\partial^2 g_j$ are locally bounded, we have

$$(2.15) \qquad \begin{aligned} &\exists c_2 > 0, \ \exists \delta_1 > 0 \ : \ x \in B(x_0, \delta_1), \ L_i \in \partial^2 f_i(x), \ 0 \leq i \leq m, \\ &M_j \in \partial^2 g_j(x), \ 1 \leq j \leq k \ \Rightarrow \|L_i\| < c_2, \ \ \|M_j\| < c_2 \end{aligned}$$

and if we define $x_i^*[h]_+ := \max\{x_i^*[h], 0\}$ then

$$\|h\| < \delta_1 \Rightarrow x_i^*[h]_+ \leq |\, a_i \,| \leq c_2\|h\|^2,\ \|Ah\| = \|y\| \leq c_2\|h\|^2. \tag{2.16}$$

By Proposition 2.1 for $(\lambda, \mu) \in \Lambda(x_0)$ we have

$$\sum_{i=0}^{m} \lambda_i x_i^*[x] + \sum_{j=1}^{k} \mu_j g_j'(x_0)[x] = 0 \quad \forall x \in E.$$

Therefore $\sum_{i=0}^{m} \lambda_i x_i^*[x] = 0, \quad \forall x \in Ker\ A.$

Hence

$$\max_{0\leq i\leq m} x_i^*[x] \geq 0, \quad \forall x \in Ker\ A,$$

which is the condition (2.1) of Lemma 2.2. By Lemma 2.2 we obtain that

$$\max_{0\leq i\leq m} f_i(x_0 + h) = \max_{0\leq i\leq m} (x_i^*[h] + a_i) \geq$$

$$\geq \min_{Ax+y=0} \max_{0\leq i\leq m} (x_i^*[x] + a_i) =$$

$$= \max_{(\lambda,\mu)\in\Lambda(x_0)} \frac{1}{2}\Big(\sum_{i=0}^{m} \lambda_i L_i(x_0 + \eta_i h)[h,h] + \sum_{j=1}^{k} \mu_j M_j(x_0 + \gamma_j h)[h,h]\Big).$$

The distance from h to the cone $K(x_0)$ we estimate by a Hophman's lemma ([1], Section 3.3.4) and after that by (2.16):

$$d(h,K) := \inf\{\|h - y\|,\ y \in K\} \leq c\Big(\sum_{i=0}^{m} x_i^*[h]_+ + \|Ah\|\Big) < c_3\|h\|^2$$

for some constant c, which does not depend on h; $c_3 = c.c_2$.

Hence h can be represented in the type $h = h' + h''$, where

$$h' \in K(x_0),\ \ \|h''\| < c_3\|h\|^2. \tag{2.17}$$

If $\|h\| < \frac{1}{c_3}$, then

$$(1 - c_3\|h\|)\|h\| \leq \|h'\| \leq \|h\|(1 + c_3\|h\|) \leq 2\|h\|. \tag{2.18}$$

For every $L_i \in \partial^2 f_i(x_0)$, $M_j \in \partial^2 g_j(x_0)$ we have

$$\max_{(\lambda,\mu)\in\Lambda(x_0)} \|\sum_{i=0}^{m} \lambda_i L_i + \sum_{j=1}^{k} \mu_j M_j\| \leq$$

$$(2.19) \qquad \leq \max_{(\lambda,\mu)\in\Lambda(x_0)} (\sum_{i=0}^{m} \lambda_i \|L_i\| + \sum_{j=1}^{k} \mu_j \|M_j\|) \leq (1+kc_1)c_2 =: c_4,$$

where c_2 is from (2.15).

Since $\partial^2 f_i$ and $\partial^2 g_j$ are w^*-u.s.c. and locally bounded, we can find (applying Alaoglou-Bourbaki theorem) a sequence $\{h_n\} \subset H$, $\|h_n\| \to 0$ and elements $L_i \in \partial^2 f_i(x_0)$, $M_j \in \partial^2 g_j(x_0)$ such that

$$L_i(x_0 + \eta_i h_n) \xrightarrow{w^*} L_i,$$

$$M_j(x_0 + \gamma_j h_n) \xrightarrow{w^*} M_j$$

and for every $h \in \{h_n\}$

$$| L_i(x_0 + \eta_i h)[h,h] - L_i[h,h] |< \alpha\|h\|^2/2$$

$$| M_j(x_0 + \gamma_j h)[h,h] - M_j[h,h] |< \alpha\|h\|^2/2c_1$$

(here we use the fact that X is finite-dimensional i.e. the restrictions of $L_i(x_0 + \eta_i h_n)$ and $M_j(x_0 + \gamma_j h_n)$ to $X \times X$ converge in the norm topology respectively to the restrictions of L_i and M_j to $X \times X$, when $n \to \infty$).

Then for every $h \in \{h_n\}$, $\|h\| < \min\{\frac{1}{c_3}, \delta_1\}$, having in mind that $h = h' + h''$, where h' and h'' satisfy (2.17) and (2.18) and using (2.19), we obtain:

$$0 \geq f(x_0 + h) \geq$$

$$\geq \max_{(\lambda,\mu)\in\Lambda(x_0)} \frac{1}{2}(\sum_{i=0}^{m} \lambda_i L_i(x_0 + \eta_i h)[h,h] + \sum_{j=1}^{k} \mu_j M_j(x_0 + \gamma_j h)[h,h]) =$$

$$= \max_{(\lambda,\mu)\in\Lambda(x_0)} \frac{1}{2}(\sum_{i=0}^{m} \lambda_i L_i(x_0 + \eta_i h)[h,h] - \sum_{i=0}^{m} \lambda_i L_i[h,h] +$$

$$+\sum_{i=0}^{m}\lambda_i L_i[h,h]+\sum_{j=1}^{k}\mu_j M_j[h,h]-\sum_{j=1}^{k}\mu_j M_j[h,h]+\sum_{j=1}^{k}\mu_j M_j(x_0+\gamma_j h)[h,h])\geq$$

$$\geq\frac{1}{2}(\frac{-\alpha\|h\|^2}{2}-\frac{\alpha\|h\|^2c_1}{2c_1})-c_4\|h'\|.\|h''\|-\frac{c_4}{2}\|h''\|^2+$$

$$+\max_{(\lambda,\mu)\in\Lambda(x_0)}\frac{1}{2}(\sum_{i=0}^{m}\lambda_i L_i[h',h']+\sum_{j=1}^{k}\mu_j M_j[h',h'])\geq$$

$$\geq\frac{-\alpha}{2}\|h\|^2-2c_3c_4\|h\|^3-\frac{c_4}{2}c_3^2\|h\|^4+\alpha\|h'\|^2\geq$$

$$\geq\|h\|^2(\frac{\alpha}{2}-2c_3\|h\|+c_3^2\|h\|^2-2c_3c_4\|h\|-\frac{c_4}{2}c_3^2\|h\|^2),$$

which is a contradiction.

Therefore x_0 is a strict local minimum of P(X). When f_i, $0\leq 1\leq m$ are convex and g_j, $1\leq j\leq k$ are affine functions, then the Lagrange function is convex and the admissible set is convex, therefore the local minimum is global. It is a routine matter to prove that in a finite dimensional subspace X the uniqueness of the solution implies Tykhonov well-posedness. ■

The following sufficient condition is a modification of that one in [1, Section 3.4.3]. The regularity condition $ImF'(x_0)=\mathrm{R}^k$ is removed, but the conditions $\lambda_0=1$, $\lambda_i>0$, $1\leq i\leq m$ are imposed.

Theorem 2.6 *Let in P(E) $f_i(x_0)=0$, $1\leq i\leq m$, there exists a number $\alpha>0$ and Lagrange's multipliers $(\lambda,\mu)\in\mathrm{R}^{m+1}\times\mathrm{R}^k$, such that $\lambda_0=1$, $\lambda_i>0$, $1\leq i\leq m$,*

$$\mathcal{L}_x(x_0;\lambda,\mu)=f_0'(x_0)+\sum_{i=1}^{m}\lambda_i f_i'(x_0)+\sum_{j=1}^{k}\mu_j g_j'(x_0)=0 \tag{2.20}$$

and for all $L\in\partial^2\mathcal{L}(x_0;\lambda,\mu)$

$$L[h,h]\geq 2\alpha\|h\|^2,\quad\forall h\in C(x_0), \tag{2.21}$$

where $C(x_0)=\{h\in E:\ f_i'(x_0)[h]=0, 1\leq i\leq m, F'(x_0)[h]=0\}$.

Then x_0 is a local minimum of the problem P(X) for every finite dimensional subspace $X\ni x_0$.

Proof. We shall follow the proof of [1, Theorem, Section 3.4.3]. The only difference is the using of Proposition 1.16, instead the usual Taylor expansion.

. To do this, assume the contrary, i.e. there exists a finite dimentional subspace $X \ni x_0$, such that for every $\delta > 0$ there exists $h_\delta \in X \cap \delta B$ (B is the closed unit ball in E) such that

$$f_i(x_0 + h_\delta) \le 0, \quad 1 \le i \le m, \quad F(x_0 + h_\delta) = 0, \quad and \quad f_0(x_0 + h_\delta) < f_0(x_0).$$

Denote $H := \bigcup_{\delta > 0} h_\delta$ and let $h \in H$.

Since $\sum_{i=1}^{m} \lambda_i f_i(x_0 + h) \le 0$ by Proposition 1.16 we have:

$$f_0(x_0 + h) \ge \mathcal{L}(x_0 + h; \lambda, \mu) + \frac{1}{2} L(x_0 + \eta_h h)[h, h],$$

where $L(x_0 + \eta_h h) \in \partial^2 \mathcal{L}(x_0 + \eta_h h; \lambda, \mu)$, $\eta_h \in (0, 1)$. The locally boundedness of $\partial^2 \mathcal{L}(.; \lambda, \mu)$ allows us to find a sequence $\{h_n\} \subset H$, $\|h_n\| \to 0$, such that $L_n := L(x_0 + \eta_{h_n} h_n)$ is w^*-convergent to some $L \in \partial^2 \mathcal{L}(x_0; \lambda, \mu)$ (by the w^*-upper-semicontinuity of $\partial^2 \mathcal{L}(.; \lambda, \mu)$). Since X is finite dimensional, the restrictions of L_n to $X \times X$ converge to the restriction of L to $X \times X$ in the norm topology.

So we have

$$f_0(x_0 + h_n) \ge f_0(x_0) + \frac{1}{2} L[h_n, h_n] + r(h_n), \quad \forall n > \nu$$

for some ν, where $\|r(h)\| \le \frac{\alpha}{2} \|h\|^2$.

Further, following the proof of the theorem in [1, Section 3.4.3] we have, that for large n, $f(x_0 + h_n) \ge f(x_0)$, which is a contradiction. ∎

Theorem 2.7 *Let in P(E) f_i, $0 \le i \le m$ be convex and twice Frechet differentiable at x_0, $f_i(x_0) = 0$, $1 \le i \le m$, g_j be affine functions $g_j(x) = l_j(x) + b_j$, $l_j \in E^*$, l_j be linearly independent $1 \le j \le k$ and for some $\alpha > 0$ there exists $(\lambda, \mu) \in \Lambda(x_0)$ such that*

$$\mathcal{L}''(x_0; \lambda, \mu)[h, h] \ge \alpha \|h\|^2 \quad \forall h \in K_1(x_0),$$

where $K_1(x_0) = \{h \in E \ : \ f_i'(x_0)[h] \le 0, \ 1 \le i \le m, \ l_j[h] = 0, \ 1 \le j \le k\}$. Then the problem $P(E)$ is Tykhonov well-posed with solution x_0.

Proof. Since $K_1(x_0) \supset K(x_0)$ and $ImF'(x_0) = \mathbf{R}^k$ (by the linearly independence of l_j), by [3, Section 10.1.1] we obtain that x_0 is a local minimum of the problem P(E). Let $\{x_n\}$ be a minimizing sequence of P(E), i.e. x_n be an admisible points and $f(x_n) \to f(x_0)$. Since $l_j(x_n) = l_j(x_0)$ for $1 \le j \le k$, $f_i(x_0) = 0$ and $0 \ge f_i(x_n) - f_i(x_0) \ge f_i'(x_0)[x_n - x_0]$ for $1 \le i \le m$, we have that $x_n - x_0 \in K_1(x_0)$ for every n.

By the Taylor expansion of $\mathcal{L}$, having in mind that $\mathcal{L}_x(x_0; \lambda, \mu) = 0$ (by the condition $(\lambda, \mu) \in \Lambda(x_0)$), for every $z \in E$ we have

$$\mathcal{L}(z; \lambda, \mu) - \mathcal{L}(x_0; \lambda, \mu) = \frac{1}{2}\mathcal{L}''(x_0; \lambda, \mu)[z - x_0, z - x_0] + o(\|z - x_0\|^2).$$

There exists $\delta > 0$ such that if $\|z - x_0\| < \delta$ then $| \frac{o(\|z-x_0\|^2)}{\|z-x_0\|^2} | < \alpha/4$.

Let $\delta' \le \delta$. If we suppose that $\|x_n - x_0\| > \delta'$ for infinitely many n, then for such n there exists $\eta_n \in (0,1)$ such that $\eta_n\|x_n - x_0\| = \delta'$ and since $\mathcal{L}(., \lambda, \mu)$ is convex,

$$\lambda_0(f_0(x_n) - f_0(x_0)) \ge \mathcal{L}(x_n; \lambda, \mu) - \mathcal{L}(x_0; \lambda, \mu) \ge$$

$$\ge \frac{1}{\eta_n}(\mathcal{L}(x_0 + \eta_n(x_n - x_0); \lambda, \mu) - \mathcal{L}(x_0\lambda, \mu)) =$$

$$= \frac{1}{\eta_n}(\frac{1}{2}\mathcal{L}''(x_0; \lambda, \mu)[\eta_n(x_n - x_0), \eta_n(x_n - x_0)] + o(\eta_n^2\|(x_n - x_0\|^2)) \ge$$

$$\ge \eta_n\|x_n - x_0\|^2\frac{\alpha}{4} = \frac{\delta'\alpha}{4}\|x_n - x_0\| > \frac{\delta'^2\alpha}{4}.$$

Hence, for large n we obtain a contradiction, therefore $\delta' \le \|x_n - x_0\|$ only for finitely many n, which completes the proof. ∎

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