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Second-order superintegrable systems in conformally flat spaces. I. Two-dimensional classical structure theory

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This paper is the first in a series that lays the groundwork for a structure and classification theory of second-order superintegrable systems, both classical and quantum, in conformally flat spaces. Many examples of such systems are known, and lists of possible systems have been determined for constant curvature spaces in two and three dimensions, as well as few other spaces. Observed features of these systems are multiseparability, closure of the quadratic algebra of second-order symmetries at order 6, use of representation theory of the quadratic algebra to derive spectral properties of the quantum Schrödinger operator, and a close relationship with exactly solvable and quasi-exactly solvable systems. Our approach is, rather than focus on particular spaces and systems, to use a general theoretical method based on integrability conditions to derive structure common to all systems. In this first paper we consider classical superintegrable systems on a general two-dimensional Riemannian manifold and uncover their common structure. We show that for superintegrable systems with nondegenerate potentials there exists a standard structure based on the algebra of 2×2 symmetric matrices, that such systems are necessarily multiseparable and that the quadratic algebra closes at level 6. Superintegrable systems with degenerate potentials are also analyzed. This is all done without making use of lists of systems, so that generalization to higher dimensions, where relatively few examples are known, is much easier. © *2005 American Institute of Physics.* [DOI: 10.1063/1.1897183]

I. INTRODUCTION AND EXAMPLES

The goal of this series of papers is a structure and classification theory of second-order superintegrable systems, both classical and quantum, in conformally flat spaces. A classical superintegrable system $\mathcal{H}=\Sigma_{ij}g^{ij}p_ip_j+V(\mathbf{x})$ on an n-dimensional local Riemannian manifold is one that admits $2n-1$ functionally independent symmetries (i.e., constants of the motion) $\mathcal{S}_k$, $k=1,\ldots,2n-1$ with $\mathcal{S}_1=\mathcal{H}$. That is, $\{\mathcal{H},\mathcal{S}_k\}=0$ where

$$\{f,g\}=\sum_{j=1}^{n}\left(\partial_{x_j}f\partial_{p_j}g-\partial_{p_j}f\partial_{x_j}g\right)$$

is the Poisson bracket for functions $f(\mathbf{x},\mathbf{p}),g(\mathbf{x},\mathbf{p})$ on phase space.[1–8] Note that $2n-1$ is the maximum possible number of functionally independent symmetries and, locally, such symmetries always exist. The main interest is in symmetries that are polynomials in the p_k and are globally

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defined, except for lower dimensional singularities such as poles and branch points. Many tools in the theory of Hamiltonian systems have been brought to bear on superintegrable systems, such as R-matrix theory, Lax pairs, exact solvability, quasi-exact solvability, and the Jacobi metric.[9–13] However, the most detailed and complete results are obtained from separation of variables methods in those cases where they are applicable. Standard orthogonal separation of variables techniques are associated with second-order symmetries, e.g., Refs. 14–20 and multiseparable Hamiltonian systems provide numerous examples of superintegrability. In these papers we shall concentrate on second-order superintegrable systems, that is, those in which the symmetries take the form $\mathcal{S}=\Sigma a^{ij}(\mathbf{x})p_ip_j+W(\mathbf{x})$, quadratic in the momenta.

There is an analogous definition for second-order quantum superintegrable systems with Schrödinger operator

$$H=\Delta+V(\mathbf{x}),\quad \Delta=\frac{1}{\sqrt{g}}\sum_{ij}\partial_{x_i}(\sqrt{g}g^{ij})\partial_{x_j},$$

where Δ is the Laplace–Beltrami operator on a Riemannian manifold, expressed in local coordinates x_j.[15] Here there are $2n-1$ second-order symmetry operators

$$S_k=\frac{1}{\sqrt{g}}\sum_{ij}\partial_{x_i}(\sqrt{g}a^{ij}_{(k)})\partial_{x_j},\quad k=1,\ldots,2n-1$$

with $S_1=H$ and $[H,S_k]\equiv HS_k-S_kH=0$. Again multiseparable systems yield many examples of superintegrability. However, as we shall show, not all multiseparable systems are superintegrable and not all second-order superintegrable systems are multiseparable. There is also a quantization problem in extending the results for classical systems to operator systems. This problem turns out to be very easily solved in two dimensions and not difficult in higher dimensions for nondegenerate systems.

Superintegrable systems can (1) be solved explicitly, and (2) they can be solved in multiple ways. It is the information gleaned from comparing the distinct solutions and expressing one solution set in terms of another that is a primary reason for their interest.

To illustrate some of the main features of superintegrable systems we give a simple example in real Euclidean space. (To make clearer the connection with quantum theory and Hilbert space methods we shall, for this example alone, adopt standard physical normalizations, such as using the factor $-\frac{1}{2}$ in front of the free Hamiltonian.) Consider the Schrödinger eigenvalue equation $H\Psi=E\Psi$ or

$$-\frac{1}{2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Psi+\frac{1}{2}\left(\omega^2(x^2+y^2)+\frac{k_1^2-\frac{1}{4}}{x^2}+\frac{k_2^2-\frac{1}{4}}{y^2}\right)\Psi=E\Psi. \tag{1}$$

This equation separates in three systems: *Cartesian* coordinates (x,y); *polar* coordinates $x=r\cos\theta$, $y=r\sin\theta$, and *elliptical* coordinates

$$x^2=c^2\frac{(u_1-e_1)(u_2-e_1)}{(e_1-e_2)},\quad y^2=c^2\frac{(u_1-e_2)(u_2-e_2)}{(e_2-e_1)}.$$

The bound states are degenerate with energies given by $E_n=\omega(2n+2+k_1+k_2)$ for integer n. The corresponding wave functions are (1) *Cartesian:*

$$\Psi_{n_1,n_2}(x,y)=2\omega^{(1/2)(k_1+k_2+2)}\sqrt{\frac{n_1!n_2!}{\Gamma(n_1+k_1+1)\Gamma(n_2+k_2+1)}}x^{(k_1+1/2)}y^{(k_2+1/2)}e^{-(\omega/2)(x^2+y^2)}L^{k_1}_{n_1}(\omega x^2)\times L^{k_2}_{n_2}(\omega y^2),\quad n=n_1+n_2, \tag{2}$$

and the $L^k_n(x)$ are Laguerre polynomials.[21] (2) *Polar:*

$$\Psi(r,\theta) = \Phi_q^{(k_1,k_2)} \times(\theta)\omega^{(1/2)(2q+k_1+k_2+1)}\sqrt{\frac{2m!}{\Gamma(m+2q+k_1+k_2+1)}}e^{(-\omega r^2/2)}r^{(2q+k_1+k_2+1)}L_m^{2q+k_1+k_2+1}(\omega r^2),$$
$$n = m+q, \tag{3}$$

$$\Phi_q^{(k_1,k_2)}(\theta) = \sqrt{2(2q+k_1+k_2+1)\frac{q!\Gamma(k_1+k_2+q+1)}{\Gamma(k_2+q+1)\Gamma(k_1+q+1)}}(\cos\theta)^{k_1+(1/2)}(\sin\theta)^{k_2+(1/2)}P_q^{(k_1,k_2)} \times(\cos 2\theta), \tag{4}$$

and the $P_q^{(k_1,k_2)}(\cos 2\theta)$ are Jacobi polynomials.[21] (3) *Elliptical:*

$$\Psi = e^{-\omega(x^2+y^2)}x^{k_1+1/2}y^{k_2+1/2}\prod_{m=1}^{n}\left(\frac{x^2}{\theta_m-e_1}+\frac{y^2}{\theta_m-e_2}-c^2\right)$$

where

$$\frac{x^2}{\theta-e_1}+\frac{y^2}{\theta-e_2}-c^2 = -c^2\frac{(u_1-\theta)(u_2-\theta)}{(\theta-e_1)(\theta-e_2)} \tag{5}$$

are ellipsoidal wave functions.[22,23] A basis for the second-order symmetry operators is

$$L_1 = \partial_x^2 + \frac{\left(\frac{1}{4}-k_1^2\right)}{x^2} - \omega^2x^2, \quad L_2 = \partial_y^2 + \frac{\left(\frac{1}{4}-k_2^2\right)}{y^2} - \omega^2y^2,$$
$$L_3 = (x\partial_y - y\partial_x)^2 + \left(\frac{1}{4}-k_1^2\right)\frac{y^2}{x^2} + \left(\frac{1}{4}-k_2^2\right)\frac{x^2}{y^2} - \frac{1}{2}. \tag{6}$$

(Note that $-2H=L_1+L_2$.) The separable solutions are eigenfunctions of the symmetry operators L_1, L_3 and $L_3+e_2L_1+e_1L_2$ with eigenvalues

$$\lambda_c = -\omega(2n_1+k_1+1), \quad \lambda_p = (2q+k_1+k_2+1)^2 + (1+k_1^2+k_2^2),$$

$$\lambda_e = 2(1-k_1)(1-k_2) - 2e_2\omega(k_1+1) - 2e_1\omega(k_2+1) - \omega^2e_1e_2 - 4\sum_{m=1}^{q}\left[e_2\frac{k_1+1}{\theta_m-e_1}+e_1\frac{k_2+1}{\theta_m-e_2}\right],$$

respectively. The algebra constructed by repeated commutators is

$$[L_1,L_3] = [L_3,L_2] \equiv R, \quad [L_i,R] = -4\{L_i,L_j\} + 16\omega^2L_3, \quad i\neq j, \quad i,j=1,2,$$

$$[L_3,R] = 4\{L_1,L_3\} - 4\{L_2,L_3\} + 8(1-k_2^2)L_1 - 8(1-k_1^2)L_2,$$

$$R^2 = \tfrac{8}{3}\{L_1,L_2,L_3\} + \tfrac{64}{3}\{L_1,L_2\} + 16\omega^2L_3^2 - 16(1-k_2^2)L_1^2 - 16(1-k_1^2)L_2^2 - \tfrac{128}{3}\omega^2L_3 - 64\omega^2(1-k_1^2)(1-k_2^2). \tag{7}$$

Note that these relations are quadratic. Here $\{A,B\}=AB+BA$, is a double symmetrizer and there is a corresponding definition for the triple symmetrizer. The important fact to observe about the algebra generated by L_1, L_2, L_3, R is that it is *closed under commutation*.[24,25] This is a remarkable fact, but typical of superintegrable systems with nondegenerate potentials, as we shall show. Indeed the closure is at level 6, since we have to express the square of the third-order operator R

in terms of the L_j basis of second-order operators. Note that the degeneracy of the energy eigenspace is broken by the alternate separated bases of eigenfunctions. The eigenfunctions of one separable system can be expanded in terms of the eigenfunctions of another, and this is the source of nontrivial special function expansion theorems.[26] The symmetry operators are in formal self-adjoint form and suitable for spectral analysis. Also, the quadratic algebra identities allow us to relate eigenbases and eigenvalues of one symmetry operator to those of another. Indeed the representation theory of the abstract quadratic algebra can be used to derive spectral properties of the generators L_j, in a manner analogous to the use of Lie algebra representation theory to derive spectral properties of quantum systems that admit Lie symmetry algebras.[26–29] (Note however that for superintegrable systems with nondegenerate potential, there is no first-order Lie symmetry.)

A common feature of quantum superintegrable systems, exhibited in the above-given example, is that after splitting off a multiplicative functional factor, $x^{(k_1+1/2)}y^{(k_2+1/2)}e^{-(\omega/2)(x^2+y^2)}$ in the example, the Schrödinger and symmetry operators are acting on a space of polynomials.[30] This is closely related to the theory of exactly and quasi-exactly solvable systems.[11,31,32] In the example the one-dimensional ordinary differential equations (ODEs) obtained by separation in the Cartesian and polar systems are exactly solvable, in terms of hypergeometric functions, i.e., there is an infinite set of nested invariant subspaces under the Cartesian or polar separated ODEs, and the energy eigenvalues are easily obtained. The elliptic system separated equations are quasi-exactly solvable, i.e., there is a single invariant finite dimensional subspace of a separated ODE and only for certain parameter choices, and polynomial solutions are obtained for only particular values of E. However, these values are just the energy eigenvalues obtained in the Cartesian and polar systems. This characterization of quasi-exactly solvable systems as embedded in PDE superintegrable systems provides insight into the nature of these phenomena.

The classical analog of the above-given example is obtained by the replacements $\partial_x \to p_x$, $\partial_y \to p_y$. Commutators go over to Poisson brackets. The operator symmetries become second-order constants of the motion. Symmetrized operators become products of functions. The quadratic algebra relations simplify: the highest order terms agree with the operator case but there are fewer nonzero lower order terms. Indeed, the classical algebra has basis

$$\mathcal{S}_1 = p_x^2 + \frac{\frac{1}{4}-k_1^2}{x^2} - \omega^2 x^2, \quad \mathcal{S}_2 = p_y^2 + \frac{\frac{1}{4}-k_2^2}{y^2} - \omega^2 y^2,$$
$$\mathcal{S}_3 = (xp_y - yp_x)^2 + \left(\frac{1}{4}-k_1^2\right)\frac{y^2}{x^2} + \left(\frac{1}{4}-k_2^2\right)\frac{x^2}{y^2}, \quad -2\mathcal{H} = \mathcal{S}_1 + \mathcal{S}_2. \tag{8}$$

The classical quadratic algebra relations are

$$\{\mathcal{S}_1,\mathcal{S}_3\} = \{\mathcal{S}_3,\mathcal{S}_2\} \equiv \mathcal{R}, \quad \{\mathcal{S}_i,\mathcal{R}\} = 8\mathcal{S}_i\mathcal{S}_j + 16\omega^2\mathcal{S}_3, \quad i \neq j, \quad i,j = 1,2,$$
$$\{\mathcal{S}_3,\mathcal{R}\} = 8\mathcal{S}_1\mathcal{S}_3 - 8\mathcal{S}_2\mathcal{S}_3 + (4-16k_2^2)\mathcal{S}_1 - (4-16k_1^2)\mathcal{S}_2, \tag{9}$$
$$\mathcal{R}^2 = 16\mathcal{S}_1\mathcal{S}_2\mathcal{S}_3 - 16\omega^2\mathcal{S}_3^2 + (4-16k_2^2)\mathcal{S}_1^2 - (4-16k_1^2)\mathcal{S}_2^2 + 4\omega^2(1-4k_1^2)(1-4k_2^2).$$

In the example the potential

$$V(x,y) = \frac{1}{2}\left(\omega^2(x^2+y^2) + \frac{k_1^2 - \frac{1}{4}}{x^2} + \frac{k_2^2 - \frac{1}{4}}{y^2}\right)$$

is *nondegenerate* in the sense that at any point x_0, y_0 where the potential is defined and analytic and the $\mathcal{S}_k$ are functionally independent, we can prescribe the values of $V_1(x_0,y_0), V_2(x_0,y_0), V_{11}(x_0,y_0)$ arbitrarily by choosing appropriate values for the parameters

ω, k_1, k_2. Here, $V_1 = \partial V/\partial x$, $V_2 = \partial V/\partial y$, etc. [Another way to look at this is to say that

$$V_1(x_0,y_0), V_2(x_0,y_0), V_{11}(x_0,y_0)$$

are the parameters.] This is in addition to the trivial constant that we can always add to a potential. As we shall show, this requirement for a superintegrable system implies that the potential is any solution of a system of coupled PDEs of the form

$$V_{22} = V_{11} + A^{22}(x,y)V_1 + B^{22}(x,y)V_2, \quad V_{12} = A^{12}(x,y)V_1 + B^{12}(x,y)V_2,$$

where the functions A^{ij}, B^{ij} are subject to certain compatibility conditions, so that the solution space is of dimension four. In $n \geqslant 2$ dimensions the analogous nondegenerate potentials depend on $n+2$ parameters. Systems with nondegenerate potentials have the most beautiful properties but there are also superintegrable systems with degenerate potentials depending on $< n+2$ parameters. For $n=2$ we will show that all of these systems depending on two or three parameters are in a certain sense specializations of the nondegenerate systems. (For degenerate systems, first-order symmetries may exist.) However, superintegrable systems with one parameter (i.e., constant) potentials are in general not restrictions of systems with nondegenerate potentials. [Note that in the classical case the symmetries corresponding to a constant potential are just Killing tensors.[15]] Indeed superintegrable systems with constant potential do not necessarily have a closed quadratic algebra. See Ref. 44 for a counterexample.

Many examples of such systems are known, and lists of possible systems have been determined for constant curvature spaces in two and three dimensions, as well as a few other spaces.[33–38] Here, rather than focus on particular spaces and systems, we employ a theoretical method based on integrability conditions to derive structure common to all such systems. In this paper we consider classical superintegrable systems on a general two-dimensional (2D) Riemannian manifold, real or complex, and uncover their common structure. We show that for superintegrable systems with nondegenerate potentials there exists a standard structure based on the algebra of 2×2 symmetric matrices, that such systems are necessarily multiseparable, and that the quadratic algebra closes at level 6. Superintegrable systems with degenerate potentials are also analyzed. This is all done without making use of lists of such systems, so that generalization to higher dimensions, where relatively few examples are known,[38] is much easier.

In the next paper in this series we will study the Stäckel transform, or coupling constant metamorphosis,[39,40] for 2D classical superintegrable systems. This is a conformal transformation of a superintegrable system on one space to a superintegrable system on another space. We will prove that all nondegenerate 2D superintegrable systems are Stäckel transforms of constant curvature systems and give a complete classification of all 2D superintegrable systems. The following papers will extend these results to three-dimensional (3D) systems and the quantum analogs of 2D and 3D classical systems.

II. SECOND-ORDER KILLING TENSORS FOR 2D COMPLEX RIEMANNIAN MANIFOLDS

Before proceeding to the study of superintegrable systems with potential, we review some basic facts about second-order symmetries (without potential) of the underlying 2D complex Riemannian spaces, i.e., second-order Killing tensors.[15] These were worked out by Koenigs,[41] though here we make an alternate presentation suggested by Refs. 42, 43, and 17. It is always possible to find a local coordinate system $(x,y) \equiv (x_1,x_2)$ defined in a neighborhood of (0,0) on the manifold such that the metric is

$$ds^2 = \lambda(x,y)(dx^2 + dy^2) = \lambda dz\, d\overline{z}, \quad z = x + iy, \quad \overline{z} = x - iy,$$

and the Hamiltonian is $H_0 = (p_1^2 + p_2^2)/\lambda$. We can consider a second-order Killing tensor (symmetry) as a quadratic form $\mathcal{L} = \Sigma_{i,j=1}^2 a^{ij}(x,y)p_i p_j$, $a^{ij} = a^{ji}$, that is in involution with the free Hamiltonian H_0: $\{H_0, \mathcal{L}\} = 0$. The conditions are

$$a_i^{ii} = -\frac{\lambda_1}{\lambda}a^{i1} - \frac{\lambda_2}{\lambda}a^{i2}, \quad i = 1,2,$$
$$2a_i^{ij} + a_j^{ii} = -\frac{\lambda_1}{\lambda}a^{j1} - \frac{\lambda_2}{\lambda}a^{j2}, \quad i,j = 1,2, \quad i \neq j. \tag{10}$$

From these conditions we easily obtain the requirements

$$2a_1^{12} = -(a^{11} - a^{22})_2, \quad 2a_2^{12} = (a^{11} - a^{22})_1.$$

From the integrability conditions for these last equations we see that

$$\Delta a^{12} = 0, \quad \Delta(a^{11} - a^{22}) = 0, \quad \Delta = \partial_x^2 + \partial_y^2$$

and that there exist analytic functions $f(z)$, $g(\overline{z})$ such that

$$2a^{12} = f(z) + g(\overline{z}), \quad a^{11} - a^{22} = i(f(z) - g(\overline{z})).$$

Substituting these results in the remaining equations we find

$$(a^{11}\lambda)_1 = -\tfrac{1}{2}\lambda_2(f+g), \quad (a^{22}\lambda)_2 = -\tfrac{1}{2}\lambda_1(f+g).$$

The integrability condition for these last equations is

$$\partial_{12}((\lambda(f-g)) + \frac{i}{2}\partial_1((\lambda_1(f+g)) - \frac{i}{2}\partial_2((\lambda_2(f+g)) = 0$$

or

$$f'' + 3f'\frac{\lambda_z}{\lambda} + 2f\frac{\lambda_{zz}}{\lambda} = -g'' - 3g'\frac{\lambda_{\overline{z}}}{\lambda} - 2g\frac{\lambda_{\overline{z}\overline{z}}}{\lambda}. \tag{11}$$

If the space admits at least one Killing tensor independent of the Hamiltonian, then we can always assume that it is of the form $(f,g)=(1,1)$, i.e., we can make the change of coordinates $Z=\int dz/\sqrt{f(z)}$, $\overline{Z}=\int d\overline{z}/\sqrt{g(\overline{z})}$ so that (11) implies

$$\lambda_{zz} = \lambda_{\overline{z}\overline{z}}.$$

Prescribing the values of $g(0), g'(0), g''(0), f(0), f'(0)$, we can use (11) to compute $f''(0)$. Differentiating this equation successively with respect to z and $\overline{z}$ we can compute all derivatives of f and g. Thus any solution (f,g) of the integrability conditions is uniquely determined by the five prescribed values. Once f and g are given, the Killing tensor a^{ij} is determined to within addition of an arbitrary multiple of the Hamiltonian H_0. Thus the maximum dimension of the space of second-order Killing tensors is six. As is very well known, this maximum is actually achieved for flat space and spaces of nonzero constant curvature. Recall that a 2D manifold is of constant curvature if and only if $k=(\partial_{z\overline{z}}\ln\lambda)/\lambda$ is a constant. The space is flat if and only if $k\equiv 0$.

Note that the maximum dimension of six is achieved if and only if the integrability conditions for (11) are themselves satisfied identically. Applying the operator $\partial_{z\overline{z}}$ to both sides of this expression we find

$$3\partial_{\overline{z}}\left(\frac{\lambda_z}{\lambda}\right)f'' + \left(2\partial_{\overline{z}}\left(\frac{\lambda_{zz}}{\lambda}\right) + 3\partial_{z\overline{z}}\left(\frac{\lambda_z}{\lambda}\right)\right)f' + 2\partial_{z\overline{z}}\left(\frac{\lambda_{zz}}{\lambda}\right)f$$
$$= -3\partial_z\left(\frac{\lambda_{\overline{z}}}{\lambda}\right)g'' - \left(2\partial_z\left(\frac{\lambda_{\overline{z}\overline{z}}}{\lambda}\right) + 3\partial_{z\overline{z}}\left(\frac{\lambda_{\overline{z}}}{\lambda}\right)\right)g' - 2\partial_{z\overline{z}}\left(\frac{\lambda_{\overline{z}\overline{z}}}{\lambda}\right)g. \tag{12}$$

If dimension six is achieved then this last condition on f and g cannot be independent of (11). Hence, either the coefficients of f'', f', f, g'', g', g all vanish identically, in which case $\partial_{z\bar{z}}\ln\lambda \equiv 0$ and the space is flat, or $\partial_{z\bar{z}}\ln\lambda \neq 0$ and (12) is obtained from (11) through multiplication by $\partial_{z\bar{z}}\ln\lambda$. In the second case one can easily see that

$$\partial_z\left(\frac{\partial_{z\bar{z}}\ln\lambda}{\lambda}\right) = \partial_{\bar{z}}\left(\frac{\partial_{z\bar{z}}\ln\lambda}{\lambda}\right) = 0$$

so the space is of nonzero constant curvature.

If the dimension of the space of symmetries is less than six then (12) is independent of (11). In this case we can eliminate f'' and g'' between these two equations and obtain a condition relating only f', f, g', g:

$$\left[2\left(\frac{\lambda_{zz}}{\lambda}\right)_{\bar{z}} - 9k\lambda_z + 3\left(\frac{\lambda_z}{\lambda}\right)_{z\bar{z}}\right]f' + \left[2\left(\frac{\lambda_{zz}}{\lambda}\right)_{z\bar{z}} - 6k\lambda_{zz}\right]f$$
$$= -\left[2\left(\frac{\lambda_{zz}}{\lambda}\right)_{z} - 9k\lambda_z + 3\left(\frac{\lambda_z}{\lambda}\right)_{z\bar{z}}\right]g' - \left[2\left(\frac{\lambda_{\bar{z}\bar{z}}}{\lambda}\right)_{z\bar{z}} - 6k\lambda_{zz}\right]g. \quad (13)$$

Thus the remaining systems have spaces of symmetries of dimensions $\leqslant 4$. A straightforward computation shows that this last equation can be rewritten as

$$[5\lambda k_z]f + [2\lambda k_{zz} + 8\lambda_z k_z]f' = -[5\lambda k_{\bar{z}}]g' - [2\lambda k_{\bar{z}\bar{z}} + 8\lambda_{\bar{z}}k_{\bar{z}}]g \quad (14)$$

where $2\lambda k_{zz}+8\lambda_z k_z = 2\lambda k_{\bar{z}\bar{z}}+8\lambda_{\bar{z}}k_{\bar{z}}$. If the space of symmetries is of dimension four then the integrability conditions for this last equation are satisfied identically. The systems with dimension four (which we call the Darboux spaces) were classified by Koenigs and are four in number.[41] If the equations are not satisfied identically, then we can repeat this procedure and find integrability conditions for the spaces of symmetries of dimension three. These spaces were also classified by Koenigs. In the next paper in this series we will find an alternate, much simpler derivation of these spaces that shows that they all admit superintegrable systems with nondegenerate potentials.

Functional independence and functional linear independence of superintegrable systems. Suppose we have a Hamiltonian $H=H_0+V=\Sigma_{i,j=1}^2 g^{ij}p_ip_j+V(x,y)$ and constants of the motion $L_k = \mathcal{L}_k + W^{(k)} = \Sigma_{i,j=1}^2 a_{(k)}^{ij}p_ip_j + W^{(k)}(x,y)$, for $k=1,2$. We say that such a system is *superintegrable* provided the two functions L_h together with H are functionally independent in the four-dimensional phase space. (Here the possible V will always be assumed to form a vector space and we require functional independence for each such V and the associated $W^{(k)}$. This means that we require that the three quadratic forms $\mathcal{L}_k, H_0$ are functionally independent.)

In the work to follow it will be important that the functionally independent symmetries also be functionally linearly independent. It is clear that there are no constants α, β, γ not all 0 such that $\alpha\mathcal{L}_1+\beta\mathcal{L}_2+\gamma H_0 \equiv 0$. However such a relation is possible if α, β, γ are functions. Indeed we have the example

$$H_0 = p_z p_{\bar{z}} + V(\bar{z}), \quad \mathcal{L}_1 = p_z^2, \quad \mathcal{L}_2 = p_z(zp_z - \bar{z}p_{\bar{z}}),$$

$$W^{(1)} = 0, \quad W^{(2)} = W^{(2)}(\bar{z}), \quad (15)$$

where $-\bar{z}V_{\bar{z}} = W_{\bar{z}}^{(2)}$. Here $\mathcal{L}_2 = z\mathcal{L}_1 - \bar{z}H_0$. (This superintegrable system is in Lie form.[41] It is not multiseparable.) The following result shows that this example is unique.

Theorem 1: *The flat space system* (15) *is the only superintegrable system in a 2D complex Riemannian space such that the functionally independent symmetries are functionally linearly dependent*.

Proof: Suppose L_1, L_2, H are functionally independent symmetries that are functionally linearly dependent. Without loss of generality we can assume that

$$\mathcal{L}_2 = f(x,y)\mathcal{L}_1 + g(x,y)H, \quad df \neq 0, \quad dg \neq 0.$$

Since L_2 is a symmetry we have the condition $\{f,H_0\}\mathcal{L}_1+\{g,H_0\}H_0=0$ or

$$\begin{aligned} f_x a^{11} + g_x/\lambda &= 0, \\ f_y a^{22} + g_y/\lambda &= 0, \\ f_y a^{11} + 2f_x a^{12} + g_y/\lambda &= 0, \\ f_x a^{22} + 2f_y a^{12} + g_x/\lambda &= 0. \end{aligned} \tag{16}$$

Thus $g_x=-\lambda f_x a^{11}$, $g_y=-\lambda f_y a^{22}$ and the remaining conditions take the form

$$\begin{pmatrix} 2a^{12} & a^{11}-a^{22} \\ a^{22}-a^{11} & 2a^{12} \end{pmatrix}\begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $df\neq 0$ the determinant of the 2×2 matrix must be zero:

$$4(a^{12})^2 + (a^{22}-a^{11})^2 = 0.$$

We consider the case $a^{22}-a^{11}=-2ia^{12}$. Then $f_x=-if_y$, so $f=f(z)$. From the Killing equations (10) we see that $a_2^{12}=ia_1^{12}$ so, $a^{12}=a^{12}(z)$. The symmetry conditions for $V,W^{(1)},W^2$ are $W_k^{(j)}=a^{k1}V_1+a^{k2}V_2$, $j,k=1,2$ and the integrability conditions for these equations are the Bertrand–Darboux (BD) conditions $(W_1^{(j)})_2=(W_2^{(j)})_1$, $j=1,2$, which in this case simplify to

$$V_{22} - V_{11} + 2iV_{12} = -\left[3\frac{a_2^{12}}{a^{12}} + 2i\frac{\lambda_1}{\lambda} + 2\frac{\lambda_2}{\lambda}\right](V_2 + iV_1),$$

$$V_{22} - V_{11} + 2iV_{12} = -\left[3\frac{a_2^{12}}{a^{12}} + 2i\frac{\lambda_1}{\lambda} + 2\frac{\lambda_2}{\lambda}\right](V_2 + iV_1) - 3\frac{f_2}{f}(V_2 + iV_1).$$

Subtracting the second BD equation from the first, we find $V_1-iV_2=0$ or $V=V(\overline{z})$. The remaining Killing tensor equations are

$$(a^{11}\lambda)_1 = -\lambda_2 a^{12}, \quad (a^{11}\lambda)_2 = 2i(a^{12}\lambda)_2 - \lambda_1 a^{12},$$

with integrability condition

$$-(\lambda_2 a^{12})_2 = 2i(\lambda a^{12})_{12} - (\lambda_1 a^{12})_1.$$

At this point it is useful to write all equations in terms of the variables $z,\overline{z}$. Then the Killing tensor equations become

$$\lambda_{zz} = 0, \quad \left(\lambda\left[a^{11} - \frac{i}{2}\right]\right)_z = 0 \tag{17}$$

and the previous integrability condition becomes

$$s'' + 3s'\frac{\lambda_{zz}}{\lambda} + 2\frac{\lambda_{zz}}{\lambda} = 0,$$

where

$$2a^{12} = s(z), \ a^{11} - a^{22} = is(z).$$

We can change to new variables $Z(z),\overline{Z}(\overline{z})=\overline{z}$ such that this last equation becomes $s(Z)\equiv 1$, $\lambda_{ZZ}=0$. From now on, we assume that the original coordinates $z,\overline{z}$ were chosen so that $\lambda_{zz}=0$, $2a^{12}=1$, $a^{11}-a^{22}=i$. In the new coordinates we have

$$H_0 = 2\frac{p_z p_{\overline{z}}}{\lambda}, \quad \mathcal{L}_1 = 2ip_z^2 + 4\left(a^{11} - \frac{i}{2}\right)p_z p_{\overline{z}}, \quad g_{\overline{z}} = - i\lambda f'(z),$$

$$g_z = - 2\lambda\left(a^{11} - \frac{i}{2}\right)f'(z),$$

so the integrability equation for g is

$$- i\lambda_z f' - i\lambda f'' = - 2\lambda_{\overline{z}}f\left(a^{11} - \frac{i}{2}\right)f' - 2\lambda a^{11}_{\overline{z}} f',$$

which simplifies to $(\lambda f'(z))_{\overline{z}}=0$. From (17) we see that there are functions $M(\overline{z}),N(\overline{z})$ such that

$$\lambda = izM'(\overline{z}) + N(\overline{z}), \quad a^{11} - \frac{i}{2} = \frac{M(\overline{z})}{izM'(\overline{z}) + N(\overline{z})}.$$

If $M'(\overline{z})\neq 0$ then we can choose a new variable $\overline{Z}(\overline{z})$ such that $M'(\overline{Z})=-i$. Assume that we have made this choice for $\overline{z}$ from the beginning. Then the equation $(\lambda f'(z))_{\overline{z}}=0$ implies $zf'(z)+N(\overline{z})f'(z)=Q(\overline{z})$, so $N'(\overline{z})f''(z)=0$. If $f''(z)=0$ then f is linear in z and this is impossible unless f is constant, a contradiction. Thus $L'=0$ and we can take $\lambda=z$, $f(z)=\ln(z)$, $a^{11}-i/2=-i\overline{z}/z$ which implies that the space is flat. Further we can introduce a new variable $Z(z)$ such that in the new variables $\lambda=1$. If on the other hand $M'(\overline{z})\equiv 0$, then again the space is flat and we can introduce a new variable $\overline{Z}(\overline{z})$ such that $\lambda=1$ with respect to the new variables. In the case that $a^{22}-a^{11}=+2ia^{12}$ the argument is the same, but with the roles of z and $\overline{z}$ interchanged. Q.E.D.

III. MAXIMUM DIMENSIONS OF THE SPACES OF POLYNOMIAL CONSTANTS IN 2D FOR TWO-PARAMETER POTENTIALS

In order to demonstrate the existence and structure of quadratic algebras for 2D superintegrable systems, it is important to compute the dimensions of the spaces of symmetries of these systems that are of orders 2, 3, 4, and 6. These symmetries are necessarily of a special type. The highest order terms in the momenta are independent of the parameters in the potential, while the terms of order 2 less in the momenta are linear in these parameters, those of order 4 less are quadratic, and those of order 6 less are cubic. We will obtain these dimensions exactly, but first we need to establish sharp upper bounds.

Consider a Hamiltonian in a general two-dimensional space of the form

$$H = \frac{p_1^2 + p_2^2}{\lambda} + \alpha_1 V^1 + \alpha_2 V^2. \tag{18}$$

Here, λ and the terms in the potential V^i depend on the coordinates x_i while the α_i are arbitrary parameters. We say that V is a *two-parameter potential* if the gradients of V^1 and V^2 are linearly independent, that is $V^1_1V^2_2-V^1_2V^2_1\neq 0$. We are free to redefine V^1 and V^2 by taking linear combinations and so we will also assume that $V^1_1\neq 0$ and $V^2_2\neq 0$.

A. Quadratic constants

We wish to determine how large the space of second-order constants of the motion can be when V is a two-parameter superintegrable potential, i.e., it admits three functionally independent constants of the motion. The general constant of second order in the momenta is

$$L = a^{11}p_1^2 + a^{22}p_2^2 + 2a^{12}p_1p_2 + \alpha_1 b^1 + \alpha_2 b^2 \tag{19}$$

with a^{ij} and b^i being functions of the coordinates alone.

Since $\{H,L\}$ is polynomial in p_1, p_2, α_1 and α_2, and the a^{ij}, b^i and V^i depend only on the coordinates x_1 and x_2, the vanishing of $\{H,L\}$ gives eight equations for the derivatives of a^{ij} and b^i. Introducing two new symbols, $c^1=a_1^{12}$ and $c^2=a_2^{12}$, we solve these equations to obtain

$$a_1^{11} = -\frac{\lambda_1}{\lambda}a^{11} - \frac{\lambda_2}{\lambda}a^{12}, \quad b_1^1 = V_1^1\lambda a^{11} + V_2^1\lambda a^{12},$$

$$a_2^{11} = -2c^1 - \frac{\lambda_2}{\lambda}a^{22} - \frac{\lambda_1}{\lambda}a^{12}, \quad b_2^1 = V_1^1\lambda a^{12} + V_2^1\lambda a^{22},$$

$$a_1^{22} = -2c^2 - \frac{\lambda_1}{\lambda}a^{11} - \frac{\lambda_2}{\lambda}a^{12}, \quad b_1^2 = V_1^2\lambda a^{11} + V_2^2\lambda a^{12},$$

$$a_2^{22} = -\frac{\lambda_2}{\lambda}a^{22} - \frac{\lambda_1}{\lambda}a^{12}, \quad b_2^2 = V_1^2\lambda a^{12} + V_2^2\lambda a^{22}. \tag{20}$$

Without expressions for the derivatives of c^1 and c^2 the system is not involutive. However, the integrability conditions for b^1 and b^2 give equations (the Bertrand–Darboux equations) that can be used to express c^1 and c^2 entirely in terms of the a^{ij}. Calculating each of b_{12}^1 and b_{12}^2 in two different ways and replacing derivatives of the form a_j^{ii} with the above-given expressions leads to two equations for c^1 and c^2,

$$\begin{aligned} -3V_1^1\lambda c^1 + 3V_2^1\lambda c^2 &= (V_1^1\lambda_2 + V_2^1\lambda_1 + \lambda V_{12}^1)(a^{22} - a^{11}) + (V_1^1\lambda_1 + V_{11}^1\lambda - V_2^1\lambda_2 - V_{22}^1\lambda)a^{12}, \\ -3V_1^2\lambda c^1 + 3V_2^2\lambda c^2 &= (V_1^2\lambda_2 + V_2^2\lambda_1 + \lambda V_{12}^2)(a^{22} - a^{11}) + (V_1^2\lambda_1 + V_{11}^2\lambda - V_2^2\lambda_2 - V_{22}^2\lambda)a^{12}. \end{aligned} \tag{21}$$

These can be solved for c^1 and c^2 since the gradients of V^1 and V^2 are linearly independent.

Since all of the derivatives of a^{11}, a^{22}, and a^{12} can be expressed in terms of the a^{ij}, at any regular point, the second-order part of L is determined by three numbers.

Theorem 2: *The space of second-order constants for a 2D superintegrable potential with two parameters is exactly three-dimensional.*

B. Cubic constants

Theorem 3: *The space of third-order constants for a 2D superintegrable potential with two parameters is at most one-dimensional.*

Proof: The general constant of third order in the momenta has the form

$$L = a^{111}p_1^3 + a^{222}p_2^3 + 3a^{112}p_1^2p_2 + 3a^{122}p_1p_2^2 + (\alpha_1 b^{11} + \alpha_2 b^{12})p_1 + (\alpha_1 b^{21} + \alpha_2 b^{22})p_2. \tag{22}$$

As for the second-order constants, we demand that the coefficients of p_1, p_2, α_1, and α_2 vanish. The terms of zeroth order in the momenta lead to

$$V_1^1 b^{11} + V_2^1 b^{21} = 0, \quad V_1^2 b^{12} + V_2^2 b^{22} = 0, \tag{23}$$

$$V_1^1 b^{12} + V_2^1 b^{22} + V_1^2 b^{11} + V_2^2 b^{21} = 0.$$

Since we have chosen V^1 and V^2 such that $V_2^2 \neq 0$, we can solve for b^{11}, b^{22}, and b^{21} in terms of b^{12}, and find

$$b^{11} = -\frac{V_2^1}{V_2^2} b^{12}, \quad b^{22} = \frac{V_2^1}{V_2^2} b^{12}, \quad b^{21} = -\frac{V_1^1}{V_2^2} b^{12}. \tag{24}$$

The coefficients in $\{H, L\}$ that are first order in the momenta give the six equations,

$$\begin{aligned}
3V_1^1 a^{111} + 3V_2^1 a^{112} &= \frac{2}{\lambda} b_1^{11} + \frac{\lambda_1}{\lambda^2} b^{11} + \frac{\lambda_2}{\lambda^2} b^{21},\\
3V_1^2 a^{111} + 3V_2^2 a^{112} &= \frac{2}{\lambda} b_1^{12} + \frac{\lambda_1}{\lambda^2} b^{12} + \frac{\lambda_2}{\lambda^2} b^{22},\\
3V_1^2 a^{122} + 3V_2^2 a^{222} &= \frac{2}{\lambda} b_2^{22} + \frac{\lambda_1}{\lambda^2} b^{12} + \frac{\lambda_2}{\lambda^2} b^{22},\\
3V_1^1 a^{122} + 3V_2^1 a^{222} &= \frac{2}{\lambda} b_2^{21} + \frac{\lambda_1}{\lambda^2} b^{11} + \frac{\lambda_2}{\lambda^2} b^{21},\\
3V_1^1 a^{112} + 3V_2^1 a^{122} &= \frac{2}{\lambda} b_1^{21} + \frac{2}{\lambda} b_2^{11},\\
3V_1^2 a^{112} + 3V_2^2 a^{122} &= \frac{2}{\lambda} b_1^{22} + \frac{2}{\lambda} b^{12}.
\end{aligned} \tag{25}$$

The first four of these, together with (24), allow a^{111}, a^{222}, a^{112}, and a^{122} to be expressed in terms of b^{12} and its derivatives, provided that, as assumed, $V_1^1 V_2^2 - V_2^1 V_1^2 \neq 0$. Then, substituting these expressions and (24) into the last two equations we obtain two equations for b_1^{12} and b_2^{12} of the form

$$-V_1^1 b_1^{12} + V_2^1 b_2^{12} = \frac{f_1(\lambda, \lambda_i, V_k^j)}{\lambda(V_1^1 V_2^2 - V_2^1 V_1^2) V_2^2} b^{12}, \tag{26}$$

$$-V_1^2 b_1^{12} + V_2^2 b_2^{12} = \frac{f_2(\lambda, \lambda_i, V_k^j)}{\lambda(V_1^1 V_2^2 - V_2^1 V_1^2) V_2^2} b^{12}, \tag{27}$$

where the two functions $f_m(\lambda, \lambda_i, V_k^j)$ are polynomial in their arguments. So the derivatives of b^{12} are multiples of b^{12} provided $V_1^1 V_2^2 - V_2^1 V_1^2 \neq 0$ and $V_2^2 \neq 0$. Hence at any regular point, all of the a^{ijk} and b^{ij} are determined by one number and so the space of third-order constants is at most one-dimensional. Q.E.D.

C. Fourth- and sixth-order constants

Theorem 4: *The space of fourth-order constants for a 2D superintegrable potential with two parameters is at most six-dimensional.*

Proof: The general constant of fourth order has the form

$$L = \sum_{i,j,k,l=1,2} a^{ijkl} p_i p_j p_k p_l + \sum_{i,j,k=1,2} b^{ij,k} \alpha_k p_i p_j + \sum_{i,j=1,2} c^{,ij} \alpha_i \alpha_j. \tag{28}$$

The vanishing of the coefficients of p_i in $\{H,L\}$ allow all of the derivative of the $c^{,ij}$ to be expressed in terms of the $b^{ij,k}$,

$$\begin{aligned}
c_1^{,11} &= \lambda V_1^1 b^{11,1} + \lambda V_2^1 b^{12,1},\\
c_2^{,11} &= \lambda V_1^1 b^{12,1} + \lambda V_2^1 b^{22,1},\\
c_1^{,12} &= \lambda V_1^1 b^{11,2} + \lambda V_2^1 b^{12,2} + \lambda V_1^2 b^{11,1} + \lambda V_2^2 b^{12,1},\\
c_2^{,12} &= \lambda V_1^1 b^{12,2} + \lambda V_2^1 b^{12,2} + \lambda V_1^2 b^{11,1} + \lambda V_2^2 b^{12,1},\\
c_1^{,22} &= \lambda V_1^2 b^{11,2} + \lambda V_2^2 b^{12,2},\\
c_2^{,22} &= \lambda V_1^2 b^{12,2} + \lambda V_1^2 b^{22,2}.
\end{aligned} \tag{29}$$

The integrability conditions of these equations, that is, equations of the form $c_{12}^{,ij}=c_{21}^{,ij}$, along with terms from $\{H,L\}$ that are cubic in the momenta, provide eleven equations for the twelve derivatives of the $b^{ij,k}$. If we define $\mathbf{b}=(b_1^{11,1},b_2^{11,1},b_1^{11,2},b_2^{11,2},b_1^{12,1},b_1^{12,2},b_2^{12,2},b_1^{22,1},b_2^{22,1},b_1^{22,2},b_2^{22,2})$, i.e., all of the derivatives of the b's excluding $b_2^{12,1}$, then when these equations are written in matrix form as $\mathbf{Ab}=\mathbf{B}$, the coefficient matrix $\mathbf{A}$ has determinant that is a constant multiple of $\lambda^{-5}V_1^1(V_1^1V_2^2-V_2^1V_1^2)$. Hence all of the derivative of the b's except $b_2^{12,1}$ can be expressed in terms of the $b^{ij,k}$ and the a^{ijkl} provided that $V_1^1\neq 0$ and $V_1^1V_2^2-V_2^1V_1^2\neq 0$. For the remaining derivative, we define $d^1=b_2^{12,1}$.

Now, the integrability conditions for the $b^{ij,k}$ and the equations obtained from the terms of $\{H,L\}$ that are of fifth order in the momenta give twelve equations for the ten derivatives of $a^{ijkl,m}$ and the two derivatives d_1^1 and d_2^1. The coefficient matrix of these terms in the equations has determinant that is a constant multiple of $(V_1^1V_2^2-V_2^1V_1^2)^3(\lambda V_1^1)^{-2}$, hence these equations can be solved provided $V_1^1V_2^2-V_2^1V_1^2\neq 0$ and $V_1^1\neq 0$.

So, the 5 a^{ijkl}, 6 $b^{ij,k}$, 3 $c^{,ij}$ and d^1 form an involutive system. Each of these symbols can be specified arbitrarily at a point. The three $c^{,ij}$ give rise to three zeroth-order constants, the six $b^{ij,k}$ give rise to six quadratic constants (three multiplied by α_1 and three multiplied by α_2), and so there are at most $5+1=6$ genuinely fourth-order constants. Q.E.D.

For the general sixth-order constant

$$L = \sum_{i,j,k,l,m,n=1,2} a^{ijklmn} p_i p_j p_k p_l p_m p_n + \sum_{i,j,k,l,m=1,2} b^{ijkl,m} \alpha_m p_i p_j p_k p_l + \sum_{i,j,k,l=1,2} c^{ij,kl} \alpha_k \alpha_l p_i p_j + \sum_{i,j,k=1,2} d^{,ijk} \alpha_i \alpha_j \alpha_k \tag{30}$$

the argument proceeds similarly.

Theorem 5: *The space of sixth-order constants for a 2D superintegrable potential with two parameters is at most ten-dimensional.*

We will show that the space is exactly ten-dimensional.

IV. NONDEGENERATE SUPERINTEGRABLE SYSTEMS IN TWO DIMENSIONS

Now we take up our main topic: a nondegenerate superintegrable system on a two-dimensional manifold. In earlier work we have classified the possible superintegrable systems on

2D complex flat space, the two-sphere, and on Darboux spaces.[44,45,34–36]. The theory we present here applies to all 2D spaces and adds greater understanding of the structure of these systems. The Hamiltonian system is

$$H = \frac{p_1^2 + p_2^2}{\lambda(x,y)} + V(x,y) \tag{31}$$

in local orthogonal coordinates. We say that the system is *second-order superintegrable* with *nondegenerate potential* if it admits three functionally independent second-order symmetries and the potential is three-parameter (in addition to the usual additive parameter). That is, at each point where the potential is defined and analytic (a regular point), we can prescribe the value of V_1, V_2 and V_{11} for some unique choice of parameters. Using the two Bertrand–Darboux equations satisfied by the potential (coming from the two symmetries other than the Hamiltonian) we can solve for $V_{22}-V_{11}$ and V_{12} in terms of the first derivatives of V.

Thus a nondegenerate potential $V(x,y)$ obeys

$$\begin{aligned} V_{22} &= V_{11} + A^{22}V_1 + B^{22}V_2, \\ V_{12} &= A^{12}V_1 + B^{12}V_2. \end{aligned} \tag{32}$$

Here, V_1,V_2,V_{11} can be prescribed arbitrarily at a fixed regular point.

A seemingly weaker requirement for a superintegrable system is that, as usual, it admits three functionally independent constants of the motion, but only for a two-parameter family of potentials $V(x,y)=\alpha V^{(1)}(x,y)+\beta V^{(2)}(x,y)$, where the gradients of $V^{(1)},V^{(2)}$ are linearly independent.

Lemma 1: "Two implies three." If the system (31) admits three functionally independent constants of the motion and a two-parameter family of potentials, then it admits a three-parameter family (32).

Proof: The system admits a symmetry $\Sigma a^{ij}p_ip_j+W$ if and only if the Bertrand–Darboux equation is satisfied. This is $\partial_j W_i=\partial_i W_j$ or

$$(V_{22}-V_{11})a^{12} + V_{12}(a^{11}-a^{22}) = \left[\frac{(\lambda a^{12})_1 - (\lambda a^{11})_2}{\lambda}\right]V_1 + \left[\frac{(\lambda a^{22})_1 - (\lambda a^{12})_2}{\lambda}\right]V_2.$$

We can always find a symmetry such that a^{11},a^{12},a^{22} take on any prescribed values at a regular point $\mathbf{x}_0$. Thus we can solve the three Bertrand–Darboux equations for the potential to obtain the system

$$V_{22} = V_{11} + A^{22}V_1 + B^{22}V_2,$$

$$V_{12} = A^{12}V_1 + B^{12}V_2,$$

$$0 = A^3V_1 + B^3V_2.$$

Case 1. $A^3\equiv B^3\equiv 0$. Then the equations are (32) and the system admits a three-parameter family of potentials.

Case 2. $A^3\not\equiv 0$. Then $V_1=D^4V_2$ so we find $V_{11}=D^5V_2$, $V_{22}=D^6V_2$, $V_{12}=D^7V_2$. Thus V depends on only one parameter. Impossible!

Case 3. $B^3\not\equiv 0$. Then $V_2=E^4V_1$ so we find $V_{11}=E^5V_1$, $V_{22}=E^6V_1$, $V_{12}=E^7V_1$. Thus V depends on only one parameter. Impossible! Q.E.D.

[Note added in proof. There is a fourth case to consider. It could be that V satisfies (32) but that the integrability conditions are not satisfied indentically, and this yields a further condition $V_{11}=A^{11}V_1+B^{11}V_2$. The lemma still holds but the proof for this case requires the Stäckel transform and will be given later in this series.]

To obtain the integrability conditions for Eq. (32) we introduce the dependent variables $W^{(1)}=V_1$, $W^{(2)}=V_2$, $W^{(3)}=V_{11}$, the vector

$$\mathbf{w} = \begin{pmatrix} W^{(1)} \\ W^{(2)} \\ W^{(3)} \end{pmatrix}, \tag{33}$$

and the matrices

$$\mathbf{A}^{(1)} = \begin{pmatrix} 0 & 0 & 1 \\ A^{12} & B^{12} & 0 \\ A^{13} & B^{13} & B^{12} - A^{22} \end{pmatrix}, \tag{34}$$

$$\mathbf{A}^{(2)} = \begin{pmatrix} A^{12} & B^{12} & 0 \\ A^{22} & B^{22} & 1 \\ A^{23} & B^{23} & A^{12} \end{pmatrix}, \tag{35}$$

where

$$A^{13} = A_2^{12} - A_1^{22} + B^{12}A^{22} + A^{12}A^{12} - B^{22}A^{12},$$

$$B^{13} = B_2^{12} - B_1^{22} + A^{12}B^{12}, \tag{36}$$

$$A^{23} = A_1^{12} + B^{12}A^{12}, \quad B^{23} = B_1^{12} + B^{12}B^{12}. \tag{37}$$

Then the integrability conditions for the system

$$\partial_{x_j}\mathbf{w} = \mathbf{A}^{(j)}\mathbf{w}, \quad j = 1,2, \tag{38}$$

must hold. These conditions are

$$A_i^{(j)} - A_j^{(i)} = A^{(i)}A^{(j)} - A^{(j)}A^{(i)} \equiv [A^{(i)},A^{(j)}]. \tag{39}$$

If and only if these conditions hold, the system has a solution V depending on three parameters.

From the conditions that

$$L = \sum_{k,j=1}^{2} a^{kj}(x,y)p_k p_j + W(x,y), \quad a^{kj} = a^{jk},$$

be a symmetry of the Hamiltonian and relations (32) we can solve for all of the first partial derivatives $\partial_i(\lambda a^{jk})$ to obtain

$$\partial_1(\lambda a^{11}) = -\frac{\lambda_2}{\lambda}(\lambda a^{12}), \quad \partial_2(\lambda a^{22}) = -\frac{\lambda_1}{\lambda}(\lambda a^{12}),$$

$$3\partial_2(\lambda a^{12}) = (\lambda a^{11} - \lambda a^{22})\left(-B^{12} - \frac{\lambda_1}{\lambda}\right) + (\lambda a^{12})\left(-B^{22} + \frac{\lambda_2}{\lambda}\right),$$

$$3\partial_1(\lambda a^{22}) = (\lambda a^{11} - \lambda a^{22})\left(2B^{12} - \frac{\lambda_1}{\lambda}\right) + (\lambda a^{12})\left(2B^{22} + \frac{\lambda_2}{\lambda}\right), \tag{40}$$

$$3\partial_1(\lambda a^{12}) = (\lambda a^{11} - \lambda a^{22})\left(A^{12} + \frac{\lambda_2}{\lambda}\right) + (\lambda a^{12})\left(A^{22} + \frac{\lambda_1}{\lambda}\right),$$

$$3\partial_2(\lambda a^{11}) = (\lambda a^{11} - \lambda a^{22})\left(-2A^{12} + \frac{\lambda_2}{\lambda}\right) + (\lambda a^{12})\left(-2A^{22} + \frac{\lambda_1}{\lambda}\right).$$

This system closes, so the space of solutions is at most three dimensional. However, by the assumption of superintegrability there are at least three functionally independent symmetries. Hence the space of second-order symmetries is exactly three dimensional. A symmetry is uniquely determined by the 2×2 symmetric matrix $(A^{ij}(\mathbf{x}_0))$ of its values at a regular point $\mathbf{x}_0$, and any such matrix corresponds to a symmetry.

To determine the integrability conditions for system (40) we define the vector-valued function

$$\mathbf{h}(x,y,z) = \begin{pmatrix} a^{11} \\ a^{12} \\ a^{22} \end{pmatrix}$$

and directly compute the 3×3 matrix functions $\mathcal{A}(j)$ to get the first-order system

$$\partial_{x_j}\mathbf{h} = \mathcal{A}^{(j)}\mathbf{h}, \quad j = 1,2. \tag{41}$$

The integrability conditions for this system are

$$\mathcal{A}_1^{(2)} - \mathcal{A}_2^{(1)} = \mathcal{A}^{(1)}\mathcal{A}^{(2)} - \mathcal{A}^{(2)}\mathcal{A}^{(1)} \equiv [\mathcal{A}^{(1)},\mathcal{A}^{(2)}]. \tag{42}$$

Now we investigate the space of third-order constants of the motion:

$$K = \sum_{k,j,i=1}^{2} a^{kji}(x_1,x_2)p_k p_j p_i + b^{\ell}(x_1,x_2)p_{\ell}, \tag{43}$$

which must satisfy $\{H,K\}=0$. Here a^{kji} is symmetric in the indices k,j,i.

The conditions are

$$2\frac{\partial a^{iii}}{\partial x_i} = -3\left(\frac{\partial \ln \lambda}{\partial_{x_i}}a^{iii} + \frac{\partial \ln \lambda}{\partial_{x_j}}a^{jii}\right), \quad i \neq j,$$

$$3\frac{\partial a^{jii}}{\partial x_i} + \frac{\partial a^{iii}}{\partial x_j} = 3\left(-\frac{\partial \ln \lambda}{\partial_{x_i}}a^{iij} - \frac{\partial \ln \lambda}{\partial_{x_j}}a^{ijj}\right), \quad i \neq j,$$

$$2\left(\frac{\partial a^{122}}{\partial x_1} + \frac{\partial a^{112}}{\partial x_2}\right) = -\frac{\partial \ln \lambda}{\partial_{x_1}}a^{122} - \frac{\partial \ln \lambda}{\partial_{x_1}}a^{111} - \frac{\partial \ln \lambda}{\partial_{x_2}}a^{222} - \frac{\partial \ln \lambda}{\partial_{x_2}}a^{112}, \tag{44}$$

$$\frac{\partial b^1}{\partial x_2}+\frac{\partial b^2}{\partial x_1}=3\sum_{s=1}^{2}\lambda a^{s21}\frac{\partial V}{\partial x_s},$$

$$\frac{\partial b^j}{\partial x_j}=\frac{3}{2}\sum_{s=1}^{2}a^{sjj}\frac{\partial V}{\partial x_s}-\frac{1}{2}\sum_{s=1}^{2}\frac{\partial \ln\lambda}{\partial x_s}b^s, \quad j=1,2, \tag{45}$$

and

$$\sum_{s=1}^{2}b^s\frac{\partial V}{\partial x_s}=0. \tag{46}$$

The general solution for the terms third order in the p_j is a sum of third-order monomials in the p_j and $J_3=x_1p_2-x_2p_1$. The a^{kji} is just a third-order Killing tensor. We require the potential V to be superintegrable and nondegenerate, and that the highest order terms, the a^{kji} in the constant of the motion, be independent of the three parameters in V. The b^ℓ must depend on these parameters linearly. We set

$$b^\ell(x_1,x_2)=\sum_{j=1}^{2}f^{\ell,j}(x_1,x_2)\frac{\partial V}{\partial x_j}(x_1,x_2).$$

(We are excluding the purely first order symmetries.) Substituting this expression into (46) we see that

$$f^{\ell,j}+f^{j,\ell}=0, \quad 1\leqslant \ell,j\leqslant 2.$$

Further

$$b_1^1=f_1^{1,2}V_2+f^{1,2}V_{12}, \quad b_2^1=f_2^{1,2}V_2+f^{1,2}V_{22},$$

$$b_1^2=f_1^{2,1}V_1+f^{2,1}V_{11}, \quad b_2^2=f_2^{2,1}V_1+f^{2,1}V_{12},$$

where the subscript j denotes the partial derivative with respect to x_j. Substituting these results and expressions (32) into the defining equations (45) and equating coefficients of V_1, V_2, V_{11}, respectively, we obtain the independent conditions:

$$\begin{aligned}\lambda a^{111}&=\tfrac{1}{3}f^{1,2}(2A^{12}-(\ln\lambda)_2),\\ \lambda a^{222}&=\tfrac{1}{3}f^{1,2}(-2B^{12}+(\ln\lambda)_1),\\ \lambda a^{112}&=\tfrac{1}{9}f^{1,2}(2A^{22}+2B^{12}+(\ln\lambda)_1),\\ \lambda a^{122}&=\tfrac{1}{9}f^{1,2}(-2A^{12}+2B^{22}-(\ln\lambda)_2),\end{aligned} \tag{47}$$

$$\begin{aligned}f_1^{1,2}&=\tfrac{1}{3}f^{1,2}(A^{22}-2B^{12}-(\ln\lambda)_1),\\ f_2^{1,2}&=\tfrac{1}{3}f^{1,2}(-2A^{12}-B^{22}+(\ln\lambda)_2),\end{aligned} \tag{48}$$

Note that (47) yields expressions for all a^{ijk} in terms of $f^{1,2}$ and the A^{ij},B^{ij}, functions. Similarly (48) yields expressions for $f_k^{1,2}$ in terms of $f^{1,2}$ and the $A^{k\ell}$,$B^{k\ell}$ functions. Thus we have an involutive system for $f^{1,2}$, possibly subject to additional conditions from (45). Thus any third-order constant of the motion defined by $f^{1,2}(x,y)$ is uniquely determined by its value $f^{1,2}(x_0,y_0)$ at some regular point (x_0,y_0). This means that the space of third-order constants of the motion is at most one-dimensional.

There are two cases to consider.

Case 1: $2A^{12}=B^{22}=(\ln\lambda)_2$, $2B^{12}=-A^{22}=(\ln\lambda)_1$. Then it follows from (47) that all $a^{ijk}\equiv 0$. The integrability conditions require $(\ln\lambda)_{11}+(\ln\lambda)_{22}=0$, which is the condition for flat space, Thus by an appropriate orthogonal change of coordinates we can assume that $\lambda\equiv 1$. In these new coordinates we see that $A^{ij}=B^{ij}\equiv 0$ for all i,j. The general solution is

$$f^{1,2}=c_1,$$

where c_1, is a constant. This is the *homogeneous isotropic oscillator:*

$$V(x,y)=\alpha x+\beta y+\gamma(x^2+y^2). \tag{49}$$

Note that for this very special case a nonzero Poisson bracket of two second-order constants of the motion must be first order.

Case 2: The conditions for Case 1 do not hold for all A^{ij},B^{ij}. Now (47) yield expressions for all a^{ijk} in terms of f^{12} and the A^{ij},B^{ij} functions and not all a^{ijk} vanish. Similarly (48) yields expressions for all $f_i^{1,2}$ in terms of $f^{1,2}$ and the $A^{k\ell},B^{k\ell}$ functions. We will show that the space of symmetries is exactly one dimensional.

Theorem 6: *Let K be a third-order constant of the motion for a superintegrable system with nondegenerate potential V*:

$$K=\sum_{k,j,i=1}^{2}a^{kji}(x,y)p_kp_jp_i+\sum_{\ell=1}^{2}b^{\ell}(x,y)p_\ell.$$

Then

$$b^{\ell}(x,y)=\sum_{j=1}^{2}f^{\ell,j}(x,y)\frac{\partial V}{\partial x_j}(x,y) \tag{50}$$

with

$$f^{\ell,j}+f^{j,\ell}=0,\quad 1\leqslant \ell,j\leqslant 2,$$

and the a^{ijk}, b^{ℓ} *are uniquely determined by the number* $f^{1,2}(x_0,y_0)$ *at some regular point* (x_0,y_0) *of* V.

Let

$$L_1=\sum a^{kj}_{(1)}p_kp_j+W_{(1)},\quad L_2=\sum a^{kj}_{(2)}p_kp_j+W_{(2)}$$

be second-order constants of the the motion for a superintegrable system with nondegenerate potential and let $\mathcal{A}_{(i)}(x,y)=\{a^{kj}_{(i)}(x,y)\}$, $i=1,2$ be 2×2 matrix functions. Then the Poisson bracket of these symmetries is given by

$$\{L_1,L_2\}=\sum_{k,j,i=1}^{2}a^{kji}(x,y)p_kp_jp_i+b^{\ell}(x,y)p_\ell \tag{51}$$

where

$$f^{k,\ell}=2\lambda\sum_j\left(a^{kj}_{(2)}a^{j\ell}_{(1)}-a^{kj}_{(1)}a^{j\ell}_{(2)}\right). \tag{52}$$

Thus $\{L_1,L_2\}$ is uniquely determined by the skew-symmetric matrix

$$[\mathcal{A}_{(2)},\mathcal{A}_{(1)}] \equiv \mathcal{A}_{(2)}\mathcal{A}_{(1)} - \mathcal{A}_{(1)}\mathcal{A}_{(2)}, \tag{53}$$

hence by the constant matrix $[\mathcal{A}_{(2)}(x_0,y_0),\mathcal{A}_{(1)}(x_0,y_0)]$ evaluated at a regular point.

Corollary 1: Let V be a superintegrable nondegenerate potential. The space of third-order constants of the motion is one-dimensional and is spanned by Poisson brackets of the second-order constants of the motion.

Corollary 2: Let V be a superintegrable nondegenerate potential and L_1,L_2 be second-order constants of the motion with matrices $\mathcal{A}_{(1)}$,$\mathcal{A}_{(2)}$, respectively. Then

$$\{L_1,L_2\} \equiv 0 \Leftrightarrow [\mathcal{A}_{(1)},\mathcal{A}_{(2)}] \equiv 0 \Leftrightarrow [\mathcal{A}_{(1)}(\mathbf{x}_0),\mathcal{A}_{(2)}(\mathbf{x}_0)] = 0$$

at a regular point $\mathbf{x}_0$.

A. A standard form for 2D superintegrable systems

For superintegrable nondegenerate potentials there is a standard structure allowing the identification of the space of second-order constants of the motion with the space of 2×2 symmetric matrices, as well as identification of the space of third-order constants of the motion with the space of 2×2 skew-symmetric matrices. Indeed, if $\mathbf{x}_0$ is a regular point then there is a $1-1$ linear correspondence between second-order operators L and their associated symmetric matrices $\mathcal{A}(\mathbf{x}_0)$. Let $\{L_1,L_2\}' = \{L_2,L_1\}$ be the reversed Poisson bracket. Then the map

$$\{L_1,L_2\}' \Leftrightarrow [\mathcal{A}_{(1)}(\mathbf{x}_0),\mathcal{A}_{(2)}(\mathbf{x}_0)]$$

is an algebraic isomorphism. Here, L_1,L_2 are in involution if and only if matrices $\mathcal{A}_{(1)}(\mathbf{x}_0)$, $\mathcal{A}_{(2)}\times(\mathbf{x}_0)$ commute. If $\{L_1,L_2\}\neq 0$ then it is a third-order symmetry and can be uniquely associated with the skew-symmetric matrix $[\mathcal{A}_{(1)}(\mathbf{x}_0),\mathcal{A}_{(2)}(\mathbf{x}_0)]$. Since commutators of second-order constants of the motion span the space of third-order constants, we can identify these $1-1$ with 2×2 skew-symmetric matrices. Let $\mathcal{E}^{ij}$ be the 2×2 matrix with a 1 in row i, column j and 0 for every other matrix element. Then the symmetric matrices

$$\mathcal{A}^{(ij)} = \tfrac{1}{2}(\mathcal{E}^{ij} + \mathcal{E}^{ji}) = \mathcal{A}^{(ji)}, \quad i,j = 1,2 \tag{54}$$

form a basis for the three-dimensional space of symmetric matrices. Moreover,

$$[\mathcal{A}^{(ij)},\mathcal{A}^{(k\ell)}] = \tfrac{1}{2}(\delta_{jk}\mathcal{B}^{(i\ell)} + \delta_{j\ell}\mathcal{B}^{(ik)} + \delta_{ik}\mathcal{B}^{(j\ell)} + \delta_{i\ell}\mathcal{B}^{(jk)}) \tag{55}$$

where

$$\mathcal{B}^{(ij)} = \tfrac{1}{2}(\mathcal{E}^{ij} - \mathcal{E}^{ji}) = -\mathcal{B}^{(ji)}, \quad i,j = 1,2.$$

Here $\mathcal{B}^{(ii)}=0$ and $\mathcal{B}^{(12)}$ forms a basis for the space of skew-symmetric matrices. Thus (55) gives the commutation relations for the second-order symmetries. If V is the isotropic oscillator then there is no truly third-order symmetry. For any other nondegenerate potential, the space of symmetries is exactly one dimensional.

To gain a deeper understanding of this structure, it is useful to reformulate the problem of determining the second-order symmetries of (31). We set

$$W(x) = f^1V_1 + f^2V_2 + f^{11}V_{11}$$

and substitute this result into $W_i=\lambda\Sigma_{j=1}^{2} a^{ij}V_j$. Additionally we must impose the Killing tensor conditions

$$a_i^{ii} = -(\ln\lambda)_1 a^{1i} - (\ln\lambda)_2 a^{2i}, \quad 2a_i^{ij} + a_j^{ii} = -(\ln\lambda)_1 a^{1j} - (\ln\lambda)_2 a^{2j}, \quad i \neq j.$$

From the expressions for W_i we obtain the equations for the a^{ij}:

$$\lambda a^{11} = f_1^1 + f^2 A^{12} + f^{11} A^{13},$$

$$\lambda a^{12} = f_2^1 + f^1 A^{12} + f^2 A^{22}, \tag{56}$$

$$\lambda a^{22} = f_2^2 + f^1 B^{12} + f^2 B^{22}$$

and the condition on the first derivatives of the f^i:

$$f_2^1 - f_1^2 = - f^1 A^{12} + f^2 (A^{22} - B^{12}) - f^{11} B^{13}. \tag{57}$$

Note the expressions for f_1^{11} and f_2^{11} in terms of f^1, f^2, f^{11}:

$$f_1^{11} + f^1 + f^{11}(B^{12} - A^{22}) = 0, \quad f_2^{11} + f^2 + f^{11} A^{12} = 0.$$

Differentiating (57) with respect to each of x_1 and x_2 and substituting (56) into the Killing equations we see that we can express each of the second derivatives of f^1, f^2 in terms of lower order derivatives of f^1, f^2, f^{11}. Thus the system is in involution at the second derivative level, but not at the first derivative level because we have only one condition for the six derivatives $f_1^1, f_2^1, f_1^2, f_2^2$. We can uniquely determine a symmetry at a regular point by choosing the six parameters $(f^1, f^2, f^{11}, f_1^1, f_2^1, f_2^2)$. The values of f^1, f^2, f^{11} at the regular point are analogous to the three parameters that we can add to the potentials in the three parameter family. For our standard basis, we fix $(f^1, f^2, f^{11})_{\mathbf{x}_0} = (0,0,0)$. Then from (56) and (57) we have

$$\begin{pmatrix} f_1^1 & f_2^1 \\ f_1^2 & f_2^2 \end{pmatrix} = \lambda \begin{pmatrix} a^{11} & a^{12} \\ a^{21} & a^{22} \end{pmatrix}.$$

Thus we can define a standard set of basis symmetries $\mathcal{S}^{(jk)} = \Sigma_{i,h} a_{(jk)}^{ih}(\mathbf{x}) p_i p_h + W^{(jk)}(\mathbf{x})$ corresponding to a regular point $\mathbf{x}_0$ by

$$\begin{pmatrix} f_1^1 & f_2^1 \\ f_1^2 & f_2^2 \end{pmatrix}_{\mathbf{x}_0} = \lambda(\mathbf{x}_0) \begin{pmatrix} a^{11} & a^{12} \\ a^{21} & a^{22} \end{pmatrix}_{\mathbf{x}_0} = \lambda(\mathbf{x}_0) \mathcal{A}^{(jk)}, \quad W^{(jk)}(\mathbf{x}_0) = 0.$$

The condition on $W^{(jk)}$ is actually three conditions since $W^{(jk)}$ depends on three parameters.

B. Multiseparability of 2D systems

From the general theory of variable separation for Hamilton–Jacobi equations[19,20] we know that a second-order symmetry L defines a separable system for

$$H = \frac{p_x^2 + p_y^2}{\lambda(x,y)} + V(x,y) = E$$

if and only if

1. The symmetries H, L form a linearly independent set as quadratic forms.
2. The two quadratic forms have a common eigenbasis of differential forms.

This last requirement means that, expressed in Cartesian coordinates, the matrix $\mathcal{A}(\mathbf{x})$ can be diagonalized by conjugacy transforms in a neighborhood of a regular point.

Corollary 3: Let V be a superintegrable nondegenerate potential and L be a second-order constant of the motion with matrix function $\mathcal{A}(\mathbf{x})$. *If at some regular point* $\mathbf{x}_0$ *the matrix* $\mathcal{A}(\mathbf{x}_0)$ *has two distinct eigenvalues, then H,L characterize an orthogonal separable coordinate system.*

Note: Since a generic 2×2 symmetric matrix has distinct roots, it follows that any superintegrable nondegenerate potential is multiseparable.

C. The quadratic algebra

Next we investigate the space of fourth-order constants of the motion for 2D systems in some detail. We already know that the dimension of this space is at most 6. Here a constant of the motion

$$F = \sum_{\ell,k,j,i=1}^{2} a^{\ell kji}(x,y,z)p_\ell p_k p_j p_i + \sum_{m,q=1}^{2} b^{mq}(x,y,z)p_m p_q + W(x,y,z), \tag{58}$$

must satisfy $\{H,F\}=0$. Again $a^{\ell kji}$, b^{mq} are symmetric in all indices.

The conditions are

$$\frac{\partial a^{iiii}}{\partial x_i} = -2\sum_{s=1}^{2} a^{siii}\frac{\partial \ln \lambda}{\partial x_s}, \tag{59}$$

$$4\frac{\partial a^{jiii}}{\partial x_i} + \frac{\partial a^{iiii}}{\partial x_j} = -6\sum_{s=1}^{2} a^{siij}\frac{\partial \ln \lambda}{\partial x_s}, \quad i \neq j, \tag{60}$$

$$3\frac{\partial a^{jjii}}{\partial x_i} + 2\frac{\partial a^{iiij}}{\partial x_j} = -\sum_{s=1}^{2} a^{siii}\frac{\partial \ln \lambda}{\partial x_s} - 3\sum_{s=1}^{2} a^{sijj}\frac{\partial \ln \lambda}{\partial x_s}, \quad i \neq j,$$

$$2\frac{\partial b^{ij}}{\partial x_i} + \frac{\partial b^{ii}}{\partial x_j} = 6\lambda\sum_{s=1}^{3} a^{sjii}\frac{\partial V}{\partial x_s} - \sum_{s=1}^{2} b^{sj}\frac{\partial \ln \lambda}{\partial x_s}, \quad i \neq j,$$

$$\frac{\partial b^{ii}}{\partial x_i} = 2\lambda\sum_{s=1}^{3} a^{siii}\frac{\partial V}{\partial x_s} - \sum_{s=1}^{2} b^{sj}\frac{\partial \ln \lambda}{\partial x_s}, \tag{61}$$

and

$$\lambda\sum_{s=1}^{3} b^{si}\frac{\partial V}{\partial x_s} = \frac{\partial W}{\partial x_i}. \tag{62}$$

Note that the $a^{\ell kji}$ is a fourth-order Killing tensor. We require the potential V to be superintegrable and nondegenerate and that the highest order terms, the $a^{\ell kji}$ in the constant of the motion, be independent of the three parameters in V. The b^{mq} must depend linearly and W quadratically on these parameters.

We set

$$b^{jk} = \sum_{\alpha=1}^{3} f^{jk,\alpha}W^{(\alpha)}, \quad f^{jk,\alpha} = f^{kj,\alpha},$$

where $W^{(\alpha)}$ is defined by

$$\begin{pmatrix} W^{(1)} \\ W^{(2)} \\ W^{(3)} \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \\ V_{11} \end{pmatrix}.$$

Then conditions (61) become

$$\frac{\partial}{\partial_{x_h}}f^{jk,\alpha}+\frac{\partial}{\partial_{x_k}}f^{hj,\alpha}+\frac{\partial}{\partial_{x_j}}f^{kh,\alpha}-2\lambda a^{\alpha hjk}=-\sum_{\gamma=1}^{3}(f^{jk,\gamma}A^{(h)}_{\gamma\alpha}+f^{hj,\gamma}A^{(k)}_{\gamma\alpha}+f^{kh,\gamma}A^{(j)}_{\gamma\alpha})$$
$$-\sum_{s=1}^{2}(f^{sk,\alpha}\delta_{hk}+f^{sj,\alpha}\delta_{kh}+f^{sh,\alpha}\delta_{jk})\frac{\partial}{\partial_{x_s}}\ln\lambda, \qquad (63)$$

where $1\leq j,k,h\leq 2$ and we set $a^{3hjk}\equiv 0$. From the integrability conditions $(\partial/\partial_{x_j})(\partial W/\partial x_i)=(\partial/\partial_{x_i})(\partial W/\partial x_j)$, $i\neq j$ for Eq. (62) we obtain the conditions

$$\partial_{x_j}f^{\beta k,\alpha}+\partial_{x_j}f^{\alpha k,\beta}-\partial_{x_k}f^{\beta j,\alpha}-\partial_{x_k}f^{\alpha j,\beta}=\sum_{s=1}^{2}(A^{(k)}_{\beta s}f^{sj,\alpha}+A^{(k)}_{\alpha s}f^{sj,\beta}-A^{(j)}_{\beta s}f^{sk,\alpha}-A^{(j)}_{\alpha s}f^{sk,\beta})$$
$$+\sum_{\gamma=1}^{3}(f^{\beta j,\gamma}A^{(k)}_{\gamma\alpha}+f^{\alpha j,\gamma}A^{(k)}_{\gamma\beta}-f^{\beta k,\gamma}A^{(j)}_{\gamma\alpha}-f^{\alpha k,\gamma}A^{(j)}_{\gamma\beta})$$
$$-(f^{\beta k,\alpha}+f^{\alpha k,\beta})\frac{\partial}{\partial_{x_j}}\ln\lambda+(f^{\beta j,\alpha}+f^{\alpha j,\beta})\frac{\partial}{\partial_{x_k}}\ln\lambda, \qquad (64)$$

where $j\neq k$, $1\leq\alpha,\ \beta\leq 3$ and we set $f^{3j,\alpha}\equiv 0$.

There are eight independent equations (63) with $\alpha\neq 3$ and we use five of these to define the five components a^{ihjk} as linear combinations of $(\partial/\partial_{x_h})f^{jk,\alpha}$ and $f^{jk,\alpha}$. We can then eliminate the a^{ihjk} from the remaining three equations to obtain three conditions relating $(\partial/\partial_{x_h})f^{jk,\alpha}$ and $f^{jk,\alpha}$. There are six terms of the form $(\partial/\partial_{x_h})f^{jk,3}$. Equation (64) with $\alpha=\beta=3$ is satisfied identically. There are two equations (64) with $\beta=3$, $1\leq\alpha\leq 2$ and four equations (63) with $\alpha=3$. Thus all six terms of the form $(\partial/\partial_{x_h})f^{jk,3}$ can be expressed as linear combinations of $f^{jk,\alpha}$. There are a total of twelve distinct terms of the form $(\partial/\partial_{x_h})f^{jk,m}$, $1\leq h,j,k,m\leq 2$. We have seen that there are three conditions on these terms remaining from (63); there are an additional three such conditions from (64) with $\alpha,\beta\neq 3$. Thus there is a shortfall of six conditions on the first derivatives $(\partial/\partial_{x_h})f^{jk,m}$.

There are a total of eighteen distinct terms of the form $(\partial^2/\partial_{x_h}\partial_{x_\ell})f^{jk,m}$ with $1\leq h,j,k,\ell,m\leq 2$. Differentiating with respect to x_1,x_2 the three first-order conditions of (63), from which the a^{ihjk} have been eliminated, we obtain six independent conditions on these second derivatives. Differentiating each of our expressions for the a^{ihjk} and substituting into Eq. (59) we find six additional conditions on the second derivatives. Also, we can differentiate the three equations from (62) with $\alpha,\beta\neq 3$ to obtain six additional conditions on the second derivatives. This allows us to express each second-order derivative as a linear combination of lower order derivatives, Thus the system is in involution.

We conclude that any fourth-order symmetry is uniquely determined by the values $f^{jk,\alpha}(\mathbf{x}_0)$ and a subset of six of the values $(\partial/\partial_{x_h})f^{jk,m}(\mathbf{x}_0)$ at a regular point $\mathbf{x}_0$. Note that by adding an appropriate linear combination of purely second-order symmetries to the fourth-order symmetry we can achieve $f^{jk,\alpha}(\mathbf{x}_0)=0$ for all j,k,α, so the maximum possible dimension of the space of purely fourth-order symmetries is six. However any second-order polynomial in the second-order symmetries is a fourth-order symmetry, and the subspace of polynomial symmetries is at least five and at most six. We show that it is exactly six.

Theorem 7: *The six distinct monomials*

$$(\mathcal{S}^{(11)})^2,\quad(\mathcal{S}^{(22)})^2,\quad(\mathcal{S}^{(12)})^2,\quad\mathcal{S}^{(11)}\mathcal{S}^{(22)},\quad\mathcal{S}^{(11)}\mathcal{S}^{(12)},\quad\mathcal{S}^{(12)}\mathcal{S}^{(22)},$$

form a basis for the space of fourth order symmetries.

Proof: Since the three symmetries $\mathcal{S}^{(11)}$, $\mathcal{S}^{(22)}$, $\mathcal{S}^{(12)}$ are functionally independent, the six monomials listed above are linearly independent. Hence they form a basis. Q.E.D.

We can use this result to explicitly expand a general fourth-order symmetry

$$F = \sum_{\ell,k,j,i=1}^{2} a^{\ell kji}(x,y,z)p_\ell p_k p_j p_i + \sum_{m,q=1}^{2} b^{mq}(x,y,z)p_m p_q + W(x,y,z)$$

in terms of the standard basis. Without loss of generality we can assume that $(0,0)=\mathbf{0}$ is a regular point. Then F is uniquely determined by the data $a^{\ell kji}(\mathbf{0})$, $\partial_m a^{\ell kji}(\mathbf{0})$, $b^{mq}(\mathbf{0})$, $W(\mathbf{0})$. We can uniquely match the data $a^{\ell kji}(\mathbf{0})$ by taking a linear combination of the basis symmetries

$$(\mathcal{S}^{(11)})^2, \quad (\mathcal{S}^{(22)})^2, \quad (\mathcal{S}^{(12)})^2, \quad \mathcal{S}^{(11)}\mathcal{S}^{(12)}, \quad \mathcal{S}^{(12)}\mathcal{S}^{(22)}.$$

This leaves the symmetry $\mathcal{S}^{(11)}\mathcal{S}^{(22)}-(\mathcal{S}^{(12)})^2$, whose leading order terms vanish at the regular point. The expansion coefficient for this term is obtained uniquely from the derivative data $\partial_m a^{\ell kji}(\mathbf{0})$. Now we have matched all of the fourth order terms in F with an expansion of the form $\mathcal{F}=\Sigma\xi_{ijk\ell}\mathcal{S}^{(ij)}\mathcal{S}^{(k\ell)}$. The difference $F-\mathcal{F}$ is a second-order symmetry. It is uniquely determined by the data $b^{mq}(\mathbf{0}),W(\mathbf{0})$, which has not changed since $W^{(ij)}(\mathbf{0})=0$ for all terms in the standard basis, Thus $F-\mathcal{F}=\Sigma b^{mq}(\mathbf{0})\mathcal{S}^{(mq)}+W(\mathbf{0})$ and we have expanded the original symmetry in terms of second-order polynomials in the standard basis.

Similarly we see that the maximal dimension of ten sixth-order symmetries is achieved by monomials in the second order symmetries.

Theorem 8: *The ten distinct monomials*

$$(\mathcal{S}^{(ii)})^3, \quad (\mathcal{S}^{(ij)})^3, \quad (\mathcal{S}^{(ii)})^2\mathcal{S}^{(jj)}, \quad (\mathcal{S}^{(ii)})^2\mathcal{S}^{(ij)}, \quad (\mathcal{S}^{(ij)})^2\mathcal{S}^{(ii)}, \quad \mathcal{S}^{(11)}\mathcal{S}^{(12)}\mathcal{S}^{(22)},$$

for $i,j=1,2,\ i\neq j$ *form a basis for the space of sixth-order symmetries.*

Proof: Since the three symmetries $\mathcal{S}^{(11)}$, $\mathcal{S}^{(22)}$, $\mathcal{S}^{(12)}$ are functionally independent, the ten monomials listed above are linearly independent. Hence they form a basis. Q.E.D.

These theorems establish the closure of the quadratic algebra for 2D superintegrable potentials: All fourth-order and sixth-order symmetries can be expressed as polynomials in the second-order symmetries.

Again, we can use these results to explicitly expand a general sixth-order symmetry

$$G = \sum_{i,j,k,l,m,n=1,2} a^{ijklmn}p_i p_j p_k p_l p_m p_n + \sum_{i,j,k,l=1,2} b^{ijkl}p_i p_j p_k p_l + \sum_{i,j=1,2} c^{ij}p_i p_j + W \tag{65}$$

in terms of the standard basis. Without loss of generality we can assume that $(0,0)=\mathbf{0}$ is a regular point. Then G is uniquely determined by the data $a^{ijklmn}(\mathbf{0})$, $\partial_q a^{ijklmn}(\mathbf{0})$, $b^{ijkl}(\mathbf{0})$, $\partial_m b^{ijkl}(\mathbf{0})$, $W(\mathbf{0})$. We can uniquely match the data $a^{ijklmn}(\mathbf{0})$ by taking a linear combination of the seven symmetries

$$(\mathcal{S}^{(ii)})^3, \quad (\mathcal{S}^{(ij)})^3, \quad (\mathcal{S}^{(ii)})^2\mathcal{S}^{(jj)}, \quad (\mathcal{S}^{(ii)})^2\mathcal{S}^{(ij)},$$

for $i,j=1,2,\ i\neq j$. This leaves the three symmetries

$$\mathcal{S}^{(11)}(\mathcal{S}^{(11)}\mathcal{S}^{(22)}-(\mathcal{S}^{(12)})^2), \quad \mathcal{S}^{(12)}(\mathcal{S}^{(11)}\mathcal{S}^{(22)}-(\mathcal{S}^{(12)})^2), \quad \mathcal{S}^{(22)}(\mathcal{S}^{(11)}\mathcal{S}^{(22)}-(\mathcal{S}^{(12)})^2)$$

whose leading order terms vanish at the regular point. The expansion coefficients for these three terms are obtained uniquely from the derivative data $\partial_q a^{ijklmn}$. Now we have matched all of the sixth order terms in G with an expansion of the form $\mathcal{G}=\Sigma\xi_{ijklmn}\mathcal{S}^{(ij)}\mathcal{S}^{(kl)}\mathcal{S}^{(mn)}$. The difference $G-\mathcal{G}$ is a fourth-order symmetry. It is uniquely determined by the data $b^{ijkl}(\mathbf{0})$, $W(\mathbf{0})$ $b^{mq}(\mathbf{0})$, $W(\mathbf{0})$ [which has not changed since $W^{(ij)}(\mathbf{0})=0$ for all terms in the standard basis], and the data $\partial_m \tilde{b}^{ijkl}(\mathbf{0})$ which has changed. Now we can use the argument presented above to expand this fourth-order symmetry in terms of polynomials in the standard basis.

Example: We indicate, briefly, how the example that we started with, (1), fits into the present structure. In the example the potential is

$$V(x,y) = \frac{1}{2}\left(\omega^2(x^2+y^2) + \frac{k_1^2 - \frac{1}{4}}{x^2} + \frac{k_2^2 - \frac{1}{4}}{y^2}\right),$$

where ω, k_1 and k_2 are arbitrary parameters. It is easy to verify that, apart from an additve constant, this is the general solution of the system

$$V_{22} - V_{11} = \frac{3}{x}V_1 - \frac{3}{y}V_2, \quad V_{12} = 0.$$

Hence we have a nondegenerate potential with

$$A^{22} = \frac{3}{x}, \quad B^{22} = -\frac{3}{y}, \quad A^{12} = B^{12} = 0.$$

A natural basis of functionally independent second-order symmetries is $\{\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3\}$, (8). To apply the above results directly, we need to choose a standard basis at a regular point. We choose the regular point $\mathbf{x}_0 = (1,1)$. Then the standard second-order symmetries $\mathcal{S}^{(11)}$, $\mathcal{S}^{(22)}$, $\mathcal{S}^{(12)}$ are the unique symmetries that restrict to p_x^2, $p_y^2, p_x p_y$, respectively, at $\mathbf{x}_0$. Thus

$$\mathcal{S}^{(11)} = p_x^2 + \left(\frac{1}{4} - k_1^2\right)\left(\frac{1}{x^2} - 1\right) + \omega^2(1-x^2),$$

$$\mathcal{S}^{(22)} = p_y^2 + \left(\frac{1}{4} - k_2^2\right)\left(\frac{1}{y^2} - 1\right) + \omega^2(1-y^2),$$

$$\mathcal{S}^{(12)} = \frac{1}{2}(p_x^2 + p_y^2 - (xp_y - yp_x)^2) + \frac{1}{2}\left(\left(\frac{1}{4} - k_1^2\right)\frac{1-y^2}{x^2} + \left(\frac{1}{4} - k_2^2\right)\left(\frac{1-x^2}{y^2}\right)\right) + \omega^2(2 - x^2 - y^2).$$

The bases are related by

$$\mathcal{S}_1 = \mathcal{S}^{(11)} + \omega^2 + k_1^2 - \tfrac{1}{4}, \quad \mathcal{S}_2 = \mathcal{S}^{(22)} + \omega^2 + k_2^2 - \tfrac{1}{4},$$

$$\mathcal{S}_3 = \mathcal{S}^{(11)} + \mathcal{S}^{(22)} - 2\mathcal{S}^{(12)} + \tfrac{1}{2} - k_1^2 - k_2^2.$$

Using these relations and our theory we can verify the quadratic algebra structure for (8).

V. FINE STRUCTURE FOR 2D SUPERINTEGRABLE SYSTEMS: A ONE-PARAMETER POTENTIAL

Here we consider a superintegrable system that admits three functionally independent constants of the motion, but only for a one-parameter family of potentials $V(x,y) = \alpha V^{(0)}(x,y)$, where the gradient of $V^{(0)}$ is nonzero. If the one-parameter family of potentials cannot be extended to a two-parameter family, then by the proof of Lemma 1 the system must admit a four-dimensional family of symmetries $\Sigma a_{(k)}^{ij} p_i p_j + W^{(k)}$, $k = 1, \ldots, 4$. The Bertrand–Darboux equations for the potential are equivalent to a single first-order equation that, without loss of generality, we can write as

$$V_1 + DV_2 = 0. \tag{66}$$

We change variables to a new orthogonal coordinate system $\{u, v\}$ so that (66) transforms to $\partial_u V = 0$. In these coordinates the Bertrand–Darboux equation for a symmetry becomes

$$-V_{vv}\lambda a^{12} = [(\lambda a^{12})_v - (\lambda a^{22})_u]V_v \tag{67}$$

where

$$-\lambda a^{12}A(v) = (\lambda a^{12})_v - (\lambda a^{22})_u$$

for all symmetries a^{ij}. Thus the equation for the potential $V(v)$ becomes $V_{vv}=A(v)V_v$.

The equations for a second-order symmetry are now

$$(\lambda a^{11})_u = -\lambda_v a^{12},$$

$$(\lambda a^{22})_u = \tfrac{2}{3}A(v)\lambda a^{12} + \tfrac{1}{3}\lambda_v a^{12} + \tfrac{1}{3}\lambda_u(a^{22}-a^{11}),$$

$$(\lambda a^{22})_v = -\lambda_u a^{12}, \tag{68}$$

$$(\lambda a^{12})_v = -\tfrac{1}{3}A(v)\lambda a^{12} + \tfrac{1}{3}\lambda_v a^{12} + \tfrac{1}{3}\lambda_u(a^{22}-a^{11}),$$

$$2(\lambda a^{12})_u + (\lambda a^{11})_v = \lambda_u a^{12} + \lambda_v(a^{11}-a^{22}).$$

From the integrability condition $\partial_u(\lambda a^{22})_v=\partial_v(\lambda a^{22})_u$ and (68) we can derive an equation of the form $\lambda_u a_v^{22}=\cdots$ where the right-hand side does not depend on the derivatives of the a^{ij}. If $\lambda_u\neq 0$ then we have an involutory system $a_u^{ij}=\cdots$, $a_v^{ij}=\cdots$ at the first derivative level. Hence the space of symmetries would be at most three-dimensional. This is a contradiction, so we must have $\lambda_u=0$. This implies that the system admits the first-order symmetry $L=p_u$ as well as a second-order symmetry p_u^2.

Introducing these simplifications into (68) and setting $a_v^{11}=s$ we obtain the involutive system

$$a_u^{11} = -\frac{\lambda'}{\lambda}a^{12},$$

$$a_u^{22} = \frac{2}{3}A(v)a^{12} + \frac{1}{3}\frac{\lambda'}{\lambda}a^{12},$$

$$a_v^{22} = -\frac{\lambda'}{\lambda}a^{22},$$

$$a_v^{12} = -\left(\frac{1}{3}A(v) + \frac{2}{3}\frac{\lambda'}{\lambda}\right)a^{12}, \tag{69}$$

$$a_u^{12} = -\frac{1}{2}\left(\frac{\lambda'}{\lambda}a^{22} + s\right),$$

$$s_u = \left(-\left(\frac{\lambda'}{\lambda}\right)' + \frac{1}{3}\frac{\lambda'}{\lambda}\left(2\frac{\lambda'}{\lambda} + A(v)\right)\right)a^{12},$$

$$s_v = \frac{1}{3}\left(2\frac{\lambda'}{\lambda} + A(v)\right)\left(-\frac{\lambda'}{\lambda}a^{22} - s\right) + \left(\left(\frac{\lambda'}{\lambda}\right)^2 - \left(\frac{\lambda'}{\lambda}\right)'\right)a^{22},$$

where $\lambda=\lambda(v)$. This system can depend on at most four constants a^{11}, a^{12}, a^{22}, s at a regular point. Since the system is at least four-dimensional, we see that it is *exactly* four-dimensional and that the integrability conditions must be satisfied. (Thus the system corresponds to a Darboux space.[41,35,36]) The only nontrivial integrability condition is $\partial_u a_v^{22}=\partial_v a_u^{22}$ or

$$2A' - \frac{2}{3}A^2 + \frac{1}{3}A\frac{\lambda'}{\lambda} + \frac{1}{3}\left(\frac{\lambda'}{\lambda}\right)^2 + \left(\frac{\lambda'}{\lambda}\right)' = 0. \tag{70}$$

In terms of the potential function V, this condition can be expressed as

$$\frac{\lambda'}{\lambda} + 2\frac{V''}{V'} = \alpha\lambda^{-1/3}(V')^{1/3}$$

for α a constant.

Theorem 9: *Every system with a one-parameter potential and three functionally independent second-order symmetries is the restriction of some three-parameter potential to a single parameter, such that the restricted potential is annihilated by some first-order symmetry of the Darboux space*.

Proof: From the discussion above, we can pass to coordinates u,v such that the system takes the form

$$H = \frac{p_u^2 + p_v^2}{\lambda(v)} + \gamma V(v).$$

The Poisson bracket $\{p_u, S\}$ for any second-order symmetry $S=\Sigma a^{ij}p_ip_j+W$ of our system is also a second-order symmetry $\Sigma a_u^{ij}p_ip_j+W_u$. Thus the linear operation of differentiating with respect to u leaves the four-dimensional space of second-order symmetries invariant. We can get more detailed information about this space by choosing a basis in which ∂_u is in Jordan canonical form. A two-dimensional subspace of the symmetries is spanned by H and p_u^2, which are in the null space of ∂_u. Thus the possible Jordan forms for ∂_u are

$$(i):\begin{pmatrix} \xi_1 & 0 & 0 & 0 \\ 0 & \xi_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (ii):\begin{pmatrix} \xi & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$(iii):\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (iv):\begin{pmatrix} \xi & 1 & 0 & 0 \\ 0 & \xi & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$(v):\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where ξ and ξ_1 are nonzero.

We will use these canonical forms to show that there always exists a three-dimensional subspace of the four parameter subspace of second-order symmetries and a nondegenerate potential $\tilde{V}$, containing, V as a special case, such that the subspace is spanned by $\tilde{H}=(p_u^2+p_v^2)/\lambda+\tilde{V}$, $p_u^2+\tilde{W}_1$, and $\Sigma a^{ij}p_ip_j+\tilde{W}_2$ where $\Sigma a^{ij}p_ip_j+W_k$ is one of the symmetries of the one-parameter system. First note from the Bertrand–Darboux equations and Eq. (69) that the defining equations for the nondegenerate potential associated to these three symmetries must be

$$\widetilde{V}_{vv} = \widetilde{V}_{uu} + 3(\ln a^{12})_u \widetilde{V}_u + A(v)\widetilde{V}_v,$$
$$\widetilde{V}_{uv} = -\frac{\lambda'}{\lambda}\widetilde{V}_u. \tag{71}$$

Here $(\ln a^{12})_{uv}=0$ and $\Sigma a^{ij}p_ip_j+W_k$ is the third symmetry. The integrability conditions for Eq. (71) reduce to the single requirement

$$A\left(\frac{\lambda'}{\lambda}\right)' + A'\left(\frac{\lambda'}{\lambda}\right) + 2\left(\frac{\lambda'}{\lambda}\right)\left(\frac{\lambda'}{\lambda}\right)' - \left(\frac{\lambda'}{\lambda}\right)'' = 0. \tag{72}$$

Consider the case where ∂_u acting on the space of second-order symmetries has an eigenvector S with eigenvalue $\xi \neq 0$. Then this symmetry must have the form

$$a^{11} = b^{11}(v)e^{\xi u}, \quad a^{12} = b^{12}(v)e^{\xi u}, \quad a^{22} = b^{22}(v)e^{\xi u}.$$

Substituting these expressions into Eq. (69) we obtain the conditions

$$2\xi^2 - \left(\frac{\lambda'}{\lambda}\right)' + \left(\frac{\lambda'}{\lambda}\right)^2 + A\frac{\lambda'}{\lambda} = 0, \quad (\ln a^{12})_u = \xi \tag{73}$$

which, together with (70), implies (72). Further, the integrability conditions for the three symmetries $\widetilde{H}$, $p_u^2+\widetilde{W}$, S to correspond to a nondegenerate potential are

$$18\xi^2 = 12\left(\frac{\lambda'}{\lambda}\right)' - 8\left(\frac{\lambda'}{\lambda}\right)^2 - 8A\frac{\lambda'}{\lambda} + 6A' - 2A^2, \quad 2\xi^2 = \left(\frac{\lambda'}{\lambda}\right)' - \left(\frac{\lambda'}{\lambda}\right)^2 - A\frac{\lambda'}{\lambda}, \tag{74}$$

and these are also implied by (73) and (70).

For the remaining systems there is a second-order symmetry whose quadratic terms are $S_2 = \Sigma a^{ij}p_ip_j$ such that the quadratic terms in $S_1 = \partial_u S_2$ also correspond to a symmetry, and $\partial_u S_1 \equiv 0$. Clearly, there are constants α,β with $|\alpha|^2+|\beta|^2>0$ and

$$S_1 = \alpha\frac{p_u^2+p_v^2}{\lambda} + \beta p_u^2, \quad S_2 = uS_1 + T_2(v),$$

where T_2 is a quadratic form in p_u,p_v that depends only on v. From conditions (69) it is straightforward to compute that

$$a^{12} = b^{12}(v), \quad a^{22} = \frac{\alpha u + \beta}{\lambda}, \quad a^{11} = -\frac{\lambda'}{\lambda}b^{12}(v)u + c^{12}(v),$$

and, finally, that

$$-\left(\frac{\lambda'}{\lambda}\right)' + \left(\frac{\lambda'}{\lambda}\right)^2 + A\frac{\lambda'}{\lambda} = 0, \quad (\ln a^{12})_u = \xi. \tag{75}$$

The integrability conditions for the three symmetries $\widetilde{H}$, $p_u^2+\widetilde{W}$, $S_2+\widetilde{W}_2$ to correspond to a nondegenerate potential are

$$0 = 12\left(\frac{\lambda'}{\lambda}\right)' - 8\left(\frac{\lambda'}{\lambda}\right)^2 - 8A\frac{\lambda'}{\lambda} + 6A' - 2A^2, \quad 0 = \left(\frac{\lambda'}{\lambda}\right)' - \left(\frac{\lambda'}{\lambda}\right)^2 - A\frac{\lambda'}{\lambda}, \tag{76}$$

and these, as well as (72) are implied by (73) and (70). Q.E.D.

Remark: It is easy to show using conditions (69) that the Jordan form (iv) does not, in fact, occur.

VI. CONCLUSIONS AND FURTHER WORK

In this paper we have uncovered the structure of 2D classical superintegrable systems with nondegenerate potential and verified the existence of a quadratic algebra of symmetries for all such systems. We have shown how to compute the quadratic algebra relations in general. We have shown that superintegrable systems with degenerate one and two parameter potentials (in addition to the trivial added constant) can be considered as restrictions of nondegenerate systems. We have verified that, with one exception, all nondegenerate superintegrable 2D systems are multiseparable. In the next paper in this series we will develop the properties of the Stäckel transform between superintegrable systems and verify that all nondegenerate 2D systems are Stäckel transforms of 2D constant curvature systems (already classified[44,45]). This will lead to a simple classification of all 2D nondegenerate superintegrable systems. Koenigs[41] in a remarkable paper has already classified all 2D (zero potential) spaces that admit three second-order Killing tensors. Our classification, considerably simpler than Koenigs', will show that all of his spaces also admit nondegenerate potentials. The next papers will extend these results to the case $n=3$, a prelude to a treatment for general n. The case $n=2$ is very special and new techniques have to be developed for higher n. However the basic conclusions and structure theorems can be generalized. We will also show how to solve the quantization problem and carry over the structure theory to the operator case.

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