GEOPHYSICAL RESEARCH LETTERS, VOL. 26, NO. 5, PAGES 557-560, MARCH 1, 1999

Secular gravitational instability of a compressible viscoelastic sphere

Ladislav Hanyk and Ctirad Matyska

Department of Geophysics, Charles University, Prague, Czech Republic

David A. Yuen

Minnesota Supercomputer Institute and Department of Geology and Geophysics, University of Minnesota, Minneapolis

Abstract. For a self-gravitating viscoelastic compressible sphere we have shown that unstable modes can exist by means of the linear viscoelastic theory by both initial-value and normal-mode approaches. For a uniform sphere we have derived analytical expressions for the roots of the secular determinant based on the asymptotic expansion of the spherical Bessel functions. From the two expressions, both the destabilizing nature of gravitational forces and the stabilizing influences of increasing elastic strength are revealed. Fastest growth times on the order of ten thousand years are developed for the longest wavelength. In contrast, a self-gravitating incompressible viscoelastic model is found to be stable. This result of linear approximation suggests that a more general approach, e.g., non-Maxwellian rheology or a non-linear finite-amplitude theory, should be considered in global geodynamics.

Introduction

It is commonly assumed that the Maxwellian viscoelastic responses of the Earth models to surface loads with the Heaviside time-dependence reach a static equilibrium after sufficiently long times [e.g., *Wu and Peltier,* 1982]. There has been a long debate as to the conditions required for static stability of this fluid [see *Fang,* 1998]. It was shown [*Longman,* 1963] that the compressible fluid can be stable, if the density distribution satisfies the Adams-Williamson equation.

The density distribution of realistic Earth models, e.g. PREM [*Dziewonski and Anderson,* 1981], does not satisfy the Adams-Williamson equation, neither do simplified compressible models consisting of a finite number of homogeneous layers. The question arises as to the long-time asymptotic behaviour of such models. *Plag and Jüttner* [1995] investigated the instabilities of the PREM model. They found unstable modes with the characteristic times comparable to those of the stable modes. On the other hand, *Wolf* [1985] considered a uniform compressible half-space model and did not find any instabilities in the analytical solution, based on field equations with neglected internal buoyancy.

Here we will present results for one-, two- and three-layered self-gravitating spherical compressible models. In particular, we will demonstrate the secular instability of a homogeneous compressible sphere analytically. Although the considered models are rather simple to represent a sufficient approximation of the real Earth, we can gain valuable insights into the fundamental physics and be in a better position to learn more when a more complicated model will be investigated.

Analysis of the Gravitational Instability

Responses of the self-gravitating viscoelastic compressible spherical models have already been studied by means of normal-mode expansion [e.g., *Yuen and Peltier,* 1982, *Wu and Peltier,* 1982]. We have employed our initial-value technique [*Hanyk et al.,* 1996, 1998] for examining the behaviour of the response for the homogeneous model, both elastically compressible and elastically incompressible, with parameters listed in Table 1 (i.e., density ρ, shear modulus μ, bulk modulus K and viscosity η). Much to our surprise, we have found that a secular instability with a growing trend appears in the compressible model. Both the vertical displacement and the gravitional perturbation load Love numbers, $h_n(t)$ and $k_n(t)$, are shown for angular order $n = 2$ in Fig. 1, where a growing unstable timescale of around 20 kyr is obtained. On the contrary, an equilibrium state is attained for the incompressible model. We will now employ the normal-mode theory for verifying this result.

Table 1. Physical Parameters of the Models

model 1:	homogeneous sphere	
	radius	6371 km
	density	$5517\ \mathrm{kg\,m^{-3}}$
	shear modulus	1.4519×10^{11} Pa
	bulk modulus	4.4967×10^{11} Pa
	viscosity	10^{21} Pa s
model 2:	core-mantle sphere	
	core radius	3480 km
	core density	$10926\ \mathrm{kg\,m^{-3}}$
	mantle density	$4314\ \mathrm{kg\,m^{-3}}$
	core bulk modulus	3.5288×10^{11} Pa
	core shear modulus	0 Pa
	core viscosity	0 Pa s
	otherwise see model 1	
model 3:	core-mantle sphere with a lithosphere	
	lithospheric thickness	120 km
	lithospheric viscosity	$\to \infty$ Pa s
	otherwise see model 2	

Paper number 1999GL900024.
0094-8276/99/1999GL900024$05.00

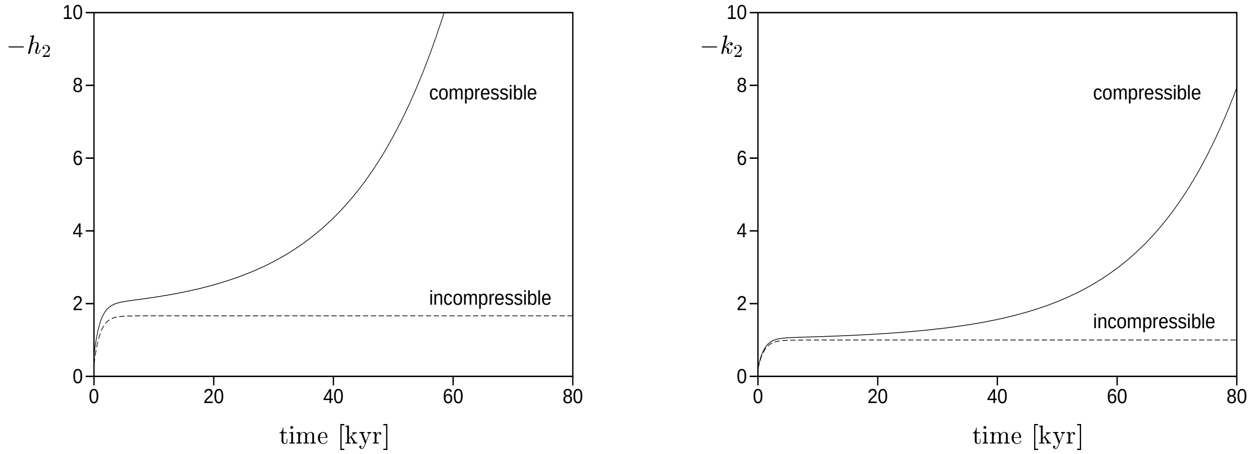

Figure 1. Temporal responses of the vertical displacement and the gravitational perturbation load Love numbers $h_2(t)$ and $k_2(t)$ for both compressible (solid curves) and incompressible (dashed curves) homogeneous spheres (model 1). A Heaviside function in time has been used. Angular order $n = 2$ has been considered.

The analytical solutions for the homogeneous compressible sphere in the Laplace domain were given by *Wu and Peltier* [1982] and recently by *Vermeersen et al.* [1996]. Our purpose is to search for roots of the secular determinant det $M(s)$ of this model for the positive values of the Laplace variable, $s > 0$. In the upper panel of Fig. 2 we show a set of roots which appear in the region of $s \to 0_+$. Following the notation by *Plag and Jüttner* [1995], we will refer to the RT (Rayleigh-Taylor) modes for the unstable responses corresponding to these positive roots of the secular determinant.

A simple calculation clarifies the presence of these roots: the secular determinant is expressed in terms of spherical Bessel functions of the argument $k(s)r$, where r denotes the

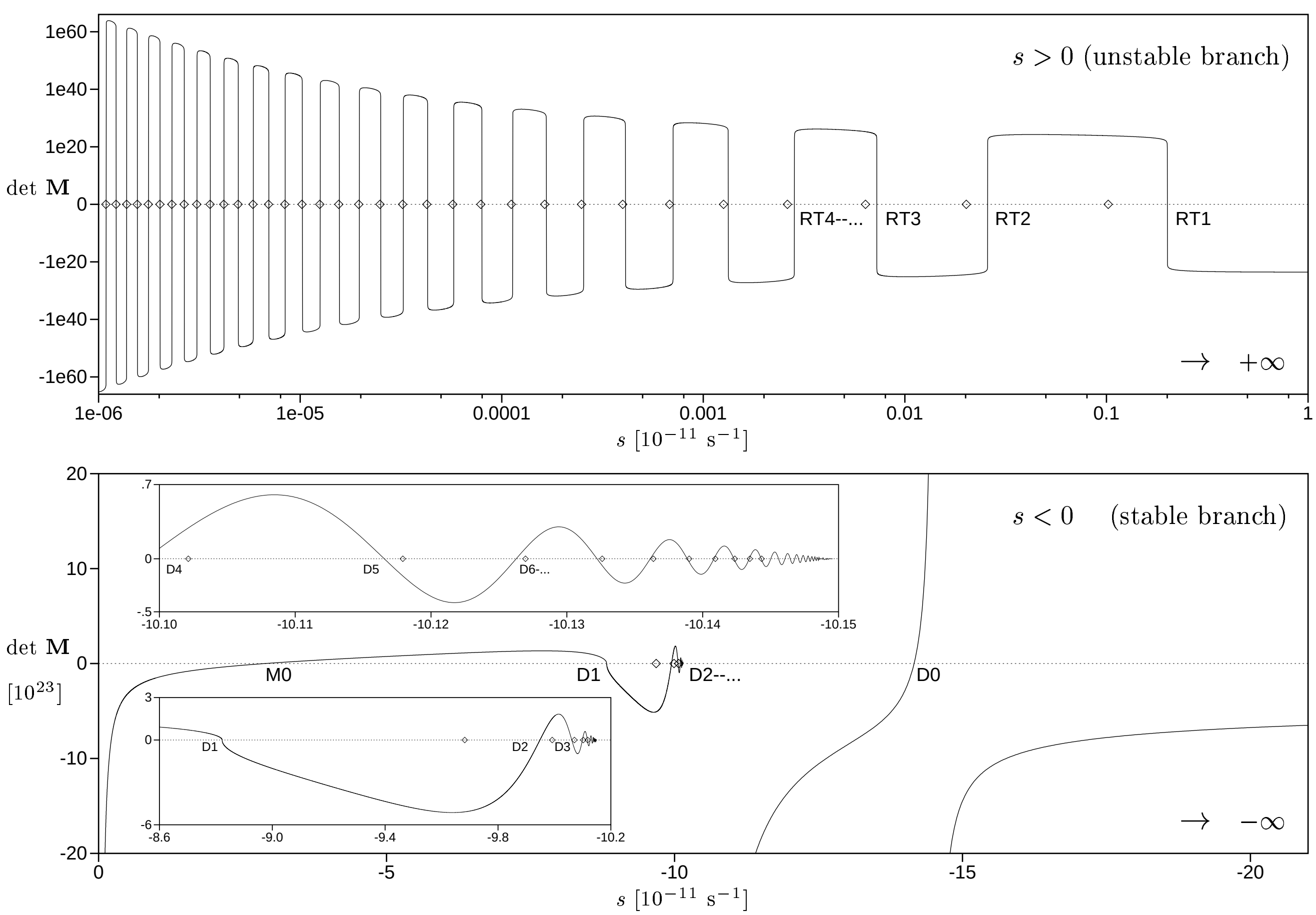

Figure 2. Secular determinant as a function of s (solid lines), inverse relaxation times (diamonds) according to eqn. (1) and (2) for both unstable (upper panel) and stable (lower panel) modes. Model 1 for angular order $n = 2$ is considered. Zoom-in views are shown for the so-called dilatation modes (Dm).

Table 2. Responses of the Model 1, $n = 2$

mode i	s_2^i $[10^{-11}\ \mathrm{s}^{-1}]$	$1/s_2^i$ [kyr]	h_2^i/s_2^i	k_2^i/s_2^i
M0	−2.891240	−1.096	−1.40088	−0.83084
D0	−14.158778	−0.224	−0.00585	−0.00033
D1	−8.935232	−0.355	0.00000	0.00000
D2	−9.948630	−0.319	−0.00012	+0.00005
RT1	+0.200408	+15.81	+0.20535	+0.04479
RT2	+0.025711	+123.25	+0.04034	+0.00431
RT3	+0.007258	+436.57	+0.01852	+0.00113
h_2^e, k_2^e			−0.58151	−0.22005
$\sum_{e,\mathrm{M0,D0-D10}}$			−1.98840	−1.05116
$\sum_{e,\mathrm{M0,D0-D10,RT1-RT100}}$			−1.66921	−1.00000

The analytical values of the isostatic load Love numbers are $h_n^{\mathrm{is}} = -(2n+1)/3$, $k_n^{\mathrm{is}} = -1$.

radius and for $k(s)$ see *Wu and Peltier* [1982]. In the limit of $s \to 0_+$, the value of $k(s)$ goes to $+\infty$, hence the number of roots of spherical Bessel functions goes to $+\infty$ and the infinite number of roots of the secular determinant can be anticipated.

We have substantiated this preliminary calculation by derivation of the asymptotic expansion of the secular equation $\det M(s) = 0$, using the asymptotic forms of the spherical Bessel functions involving large arguments. In the $s \to 0_+$ limit, the secular equation becomes $\sin(k(s)r - n\pi/2) = 0$, from which the analytical formula for the roots of the RT modes follows,

$$s_n^{\mathrm{RTm}} = \frac{n(n+1)}{K\eta} \frac{\left(r^2\rho\xi\right)^2}{\left[(2m+n)\frac{\pi}{2}\right]^4} . \tag{1}$$

In (1), n is the angular order, m the overtone number and $\xi = \frac{4}{3}\pi G\rho$. The asymptotic validity of this formula is demonstrated in Fig. 2, where the roots according to (1) are denoted by diamonds lying on the zero line.

It follows from (1) that the s_n^{RTm} are positive (and the RT modes unstable) for all finite and positive values of the physical parameters. On the other hand, the values s_n^{RTm} go to the stable limit with a value of 0 in both incompressible ($K \to \infty$) and elastic limits ($\eta \to \infty$). The lowest growth time $1/s_n^{\mathrm{RT1}}$, which dominates the response, must be found numerically by root-finding procedures, as the asymptotic formula (1) is not accurate for low m. However, formula (1) can be considered as the analytical proof of existence of the unstable modes and of the initial-value results shown in Fig. 1.

For the sake of completeness, we also discuss the previous analysis of the stable branch of the secular determinant, $s < 0$, by *Vermeersen et al.* [1996]. In that paper, both the mistakingly used expression (33) and sign confusions in the solution vector (27)–(32) caused the incorrect evaluation of the secular determinant, most noticeable in locations of the roots of the dilatation modes (D modes) with low overtone numbers. The plot of the corrected analytical secular determinant for negative values of the Laplace variable s and the angular order $n = 2$ is shown in the lower panel of Fig. 2.

From the asymptotic expansion of spherical Bessel functions in the region of $s \to -(K/(K+\frac{4}{3}\mu))(\mu/\eta)_+$, where the argument $k(s)$ goes to $+\infty$ again, we obtained the secular equation in the asymptotical form $\cos(k(s)r - n\pi/2) = 0$, in contrast to eq. (44) by *Vermeersen et al.* [1996]. Our analytical formula for the roots of the dilatation modes reads

$$s_n^{\mathrm{Dm}} = -\frac{\mu}{\eta} \frac{\left[(2m+n-1)\frac{\pi}{2}\right]^2 K - 4r^2\rho\xi}{\left[(2m+n-1)\frac{\pi}{2}\right]^2 (K+\frac{4}{3}\mu) - 4r^2\rho\xi} , \tag{2}$$

with n being the angular order and m the overtone number. The dilatation modes can become also unstable, as was already shown by *Vermeersen and Mitrovica* [1998]. There is a transition from the stable to the unstable modes with decreasing bulk modulus K from the numerator in (2), while μ is fixed. From the denominator one sees that with a further decrease in K the modes can become stable again.

We give the numerical roots s_n^i of the M0 mode and the first D and RT modes, corresponding growth times $1/s_n^i$, the elastic parts h_n^e, k_n^e of the load Love numbers and their modal amplitudes h_n^i, k_n^i for angular order $n = 2$ in Table 2. An interesting observation can be made from the last two columns of Table 2: the so-called isostatic limits, h_n^{is} and k_n^{is}, calculated independently as the response of a fluid incompressible sphere, cannot be reached by the values h_n^∞ and k_n^∞ from the criterion of the completeness of the modal decomposition,

$$h_n^\infty = h_n^e + \sum_i^\infty h_n^i/s_n^i , \quad k_n^\infty = k_n^e + \sum_i^\infty k_n^i/s_n^i , \tag{3}$$

when the M0 mode and the D modes only are included in summation. However, the expressions (3) do fit the isostatic limits accurately when the RT modes are taken into account.

In Fig. 3 we show a collection of growth times for unstable modes of the homogeneous compressible sphere (model 1) for several angular orders n. We note that the fastest growth times are obtained for the lowest n and that for $n < 16$ the growth times are less than 100 kyr.

In Fig. 4 we show the influences of increasing complexity in the model. Table 1 gives the physical parameters for the additional models 2 and 3 to the homogeneous model 1,

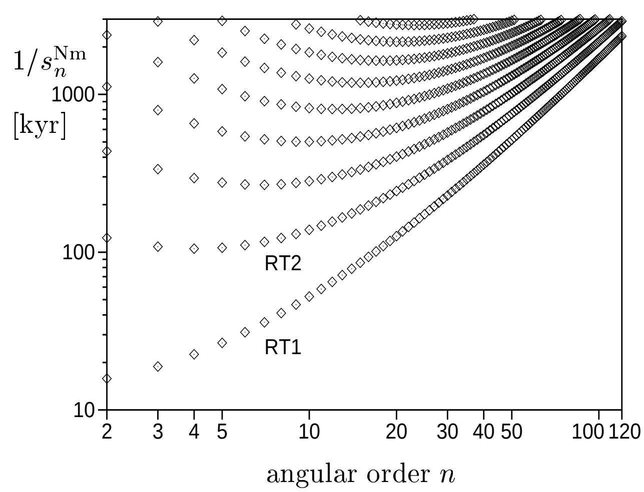

Figure 3. Growth times $1/s_n^{\mathrm{RTm}}$ of the unstable RT modes as a function of angular order n of the model 1. The fundamental (RT1) and the overtone branches (RT2–...) are shown.

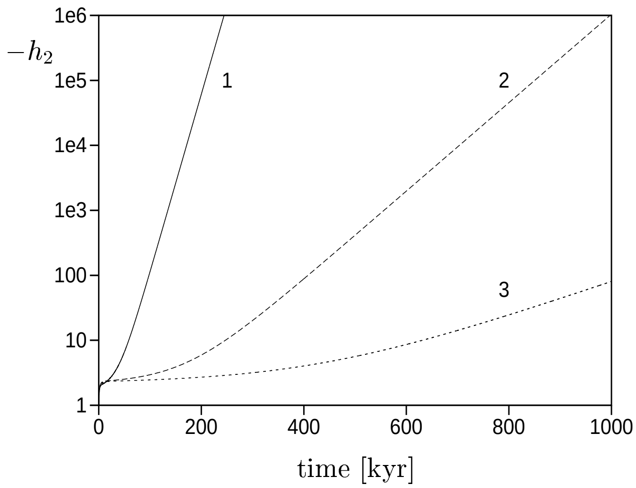

Figure 4. Temporal history of the vertical displacement load Love number $h_n(t)$ for angular order $n = 2$. Results for three different models with increasing number of layers (see Table 1 for their physical parameters) are shown.

discussed up to now. The responses in the vertical displacement or the load Love number $h_n(t)$ for angular order $n = 2$ are shown. The increase in the layering serves to increase the growth time. This means that more complicated Earth models would also have longer growth times than the simpler models.

Concluding Remarks

In this study we have uncovered a set of unstable modes for a homogeneous Maxwellian viscoelastic sphere. From analytical expressions for the secular determinant, these modes can be shown to have origins arising from the gravitational Rayleigh-Taylor instability of a compressible viscoelastic layer. Gravitation in combination with compresibility is a crucial destabilizing factor; higher viscosity plays a stabilizing role. The fact of existence of gravitational instabilities in linear viscoelastic models raises the tantalizing question of the influences played by finite-amplitude viscoelasticity in geodynamics, as has already been discussed by several authors [*Harder,* 1991, *Bercovici et al.,* 1992, *Moser et al.,* 1993, *Plag and Jüttner,* 1995]. In the linear theory, the growth times of the unstable modes are the shortest for the lowest angular orders, which suggests that also in the finite-amplitude viscoelastic regime they may play a decisive role in secular rotational instabilities [*Moser et al.,* 1993]. A recent citation of a large-amplitude rotational instability with a deep geobiological consequence [*Kirschvink et al.,* 1997] has been invoked on the basis of paleomagnetic data.

Acknowledgments. We thank D. Wolf and P. Johnston for thorough and constructive reviews and Z. Martinec, J. Matas and Y. Ricard for discussions. This research has been supported by the Grant Agency of the Czech Republic under Nos. 205/96/0212 and 205/97/1015, the Charles University grants 1/97/B-GEO/MFF and 170/98/B-GEO/MFF and the Geosciences Program of the Department of Energy.

References

Bercovici, D. B., H. B. Dick, and T. Wagner, Non-linear viscoelasticity and the formation of transverse ridges, *J. Geophys. Res., 97,* 14495–14506, 1992.

Dziewonski, A. M., and D. L. Anderson, Preliminary reference Earth model, *Phys. Earth Planet. Inter., 25,* 297–356, 1981.

Fang, M., Static deformation of the outer core, in *Dynamics of the Ice Age Earth: A Modern Perspective,* edited by P. Wu, pp. 155–190, Trans Tech Publ., Switzerland, 1998.

Hanyk, L., D. A. Yuen, and C. Matyska, Initial-value and modal approaches for transient viscoelastic responses with complex viscosity profiles, *Geophys. J. Int., 127,* 348–362, 1996.

Hanyk, L., C. Matyska, and D. A. Yuen, Initial-value approach for viscoelastic responses of the Earth's mantle, in *Dynamics of the Ice Age Earth: A Modern Perspective*, edited by P. Wu, pp. 135–154, Trans Tech Publ., Switzerland, 1998.

Harder, H., Numerical simulation of thermal convection with Maxwellian viscoelasticity, *J. Non-Newton. Fluid Mech., 39,* 67–88, 1991.

Kirschvink, J. L., R. L. Ripperdan, and D. A. Evans, Evidence for a large-scale reorganization of early Cambrian continental masses by inertial interchange true polar wander, *Science, 277,* 541–545, 1997.

Longman, I. M., A Green's function for determining the deformation of the Earth under surface mass loads, 2. Computations and numerical results, *J. Geophys. Res., 68,* 485–496, 1963.

Moser, J., C. Matyska, D. A. Yuen, A. V. Malevsky, and H. Harder, Mantle rheology, convection and rotational dynamics, *Phys. Earth Planet. Inter., 79,* 367–381, 1993.

Plag, H.-P., and H.-U. Jüttner, Rayleigh-Taylor instabilities of a self-gravitating Earth, *J. Geodynamics, 20,* 267–288, 1995.

Vermeersen, L. L. A., R. Sabadini, and G. Spada, Compressible rotational deformation, *Geophys. J. Int., 126,* 735–761, 1996.

Vermeersen, L. L. A., and J. X. Mitrovica, Gravitational stability of viscoelastic relaxation models (abstract), *Annales Geophys. Suppl., 16/I,* C 55, 1998.

Wolf, D., The normal modes of a uniform, compressible Maxwell half-space, *J. Geophys., 56,* 100–105, 1985.

Wu, P., and W. R. Peltier, Viscous gravitational relaxation, *Geophys. J. R. Astron. Soc., 70,* 435–485, 1982.

Yuen, D. A., and W. R. Peltier, Normal modes of the viscoelastic earth, *Geophys. J. R. Astron. Soc., 69,* 495–526, 1982.

L. Hanyk and C. Matyska, Department of Geophysics, Faculty of Mathematics and Physics, Charles University, V Holešovičkách 2, 18000 Praha 8, Czech Republic. (e-mail: Ladislav.Hanyk@mff.cuni.cz)

D. A. Yuen, Minnesota Supercomputer Institute and Department of Geology and Geophysics, University of Minnesota, 1200 Washington Avenue South, Minneapolis, MN 55415-1227. (e-mail: davey@krissy.msi.umn.edu)

(Received September 23, 1998; revised November 25, 1998; accepted January 7, 1999.)