

Secure Bilinear Diffie-Hellman Bits

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Abstract. The Weil and Tate pairings are a popular new gadget in cryptography and have found many applications, including identity-based cryptography. In particular, the pairings have been used for key exchange protocols.

This paper studies the bit security of keys obtained using protocols based on pairings (that is, we show that obtaining certain bits of the common key is as hard as computing the entire key). These results give insight into how many “hard-core” bits can be obtained from key exchange using pairings.

The results are of practical importance. For instance, Scott and Barreto have recently used our results to justify the security of their compressed pairing technique.

1 Introduction

Let p be a prime and let \mathbb{F}_p be the field of p elements, which we identify with the set $\{0, 1, \dots, p-1\}$. Let l be a prime which is coprime to p and define m to be the smallest positive integer such that $p^m \equiv 1 \pmod{l}$.¹ In this paper we consider a non-degenerate bilinear pairing

$$e : \mathbb{G}_1 \times \mathbb{G}_2 \longrightarrow \mathcal{G} \subseteq \mathbb{F}_{p^m}^*$$

where \mathbb{G}_1 , \mathbb{G}_2 and \mathcal{G} are cyclic groups of order l . The Weil and Tate pairings on subgroups of elliptic curves give rise to such pairings. One implementation which has $\mathbb{G}_1 = \mathbb{G}_2$ (given by Verheul [27]) is to take a supersingular elliptic curve E over \mathbb{F}_p such that $l \nmid \#E(\mathbb{F}_p)$ and such that E has a suitable “distortion map” ψ (which is a non- \mathbb{F}_p -rational endomorphism on E). Let $\mathbb{G}_1 = \mathbb{G}_2$ be the unique subgroup of $E(\mathbb{F}_p)$ of order l . The pairing $e(P, Q)$ is defined to be the Weil (or Tate) pairing of P with $\psi(Q)$. For more details about pairings on elliptic curves and their applications see [3, 7, 8, 15, 16, 19–21, 27].

Pairings have found many applications in cryptography including the tripartite key exchange protocol of Joux [15] (also see the variations by Al-Riyami and Paterson [1] and Verheul [27]) and the identity-based key exchange protocol of Smart [26]. These protocols enable a set of users to agree a random element K in the subgroup \mathcal{G} of $\mathbb{F}_{p^m}^*$, and the “key” is then derived from K .

We recall the *tripartite Diffie-Hellman protocol* in the original formulation of Joux [15]: To set up the system, three communicating parties A , B and C choose

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¹ Typical values of these parameters are $m = 2$, p a 512-bit prime and l a prime dividing $p + 1$ of at least 160 bits; or $m = 6$, $p \approx 2^{180}$ and $l \approx 2^{160}$. More generally we can replace p by a prime power.

suitable groups \mathbb{G}_1 and \mathbb{G}_2 of order l and points $P \in \mathbb{G}_1$ and $Q \in \mathbb{G}_2$ with $e(P, Q) \neq 1 \in \mathbb{F}_{p^m}^*$.

To create a common secret key, A , B and C choose secret numbers $a, b, c \in [0, l - 1]$ and publish pairs

$$(aP, aQ), \quad (bP, bQ), \quad (cP, cQ).$$

Now each of them is able to compute the common key

$$K = e(P, Q)^{abc}.$$

For example, A can compute K as follows,

$$e(bP, cQ)^a = e(P, cQ)^{ab} = e(P, Q)^{abc} = K \in \mathbb{F}_{p^m}^*.$$

Note that K is an element of order l in $\mathbb{F}_{p^m}^*$.

Since p^m is very large (at least 1024 bits), and since K is an element of a smaller subgroup of order l , it makes sense to derive a key of about this size. One natural approach is to simply use a single component (or portion of it) of the representation of K in \mathbb{F}_{p^m} as a vector space over \mathbb{F}_p . As is well known (see [18] Theorem 2.24) selecting a component can be described in terms of the trace. Hence, in this paper we consider the representation-independent framework of considering the trace of K with respect to $\mathbb{F}_{p^m}/\mathbb{F}_p$ to obtain an element of \mathbb{F}_p . We represent elements of \mathbb{F}_p as integers in $[0, p - 1]$ and obtain corresponding bitstrings in the usual way. We show that the trace is a secure key derivation function.

Note that the identity-based encryption scheme of Boneh and Franklin [3] is a hybrid scheme (key transport followed by symmetric encryption) and so our results shed light on the security in this case.

The use of part of the representation of K has been proposed by Scott and Barreto [24]. Our results show that there is no loss of security from using their technique.

The results follow from several recently established results [17, 25] on the *hidden number problem with trace* in extension fields. Detailed surveys of bit security results and discussions of their meaning and importance are given in [9, 10]; several more recent results can be found in [4–6, 11–14, 17, 23, 25].

We obtain an almost complete analogue of the results of [6, 11] for $m = 2$ (for example, for the elliptic curves used by Joux [15] and Verheul [27]) and much weaker, but nontrivial, results for $m \geq 3$. For example, in the case that $m = 2$ and p is a 512 bit prime, our results imply that, if the bilinear-Diffie-Hellman problem is hard, then the 128 most significant bits of the trace of K can be used to derive a secure key.

Finite fields of characteristic 2 or 3 are sometimes used in pairing-based systems. In these cases one can obtain a key by taking some components of the representation of the finite field element as a vector space over \mathbb{F}_p ($p = 2, 3$). Using linear algebra it is trivial to obtain very strong bit security results in these cases. The details are left to the reader.

Note that we allow all our constants to depend on m while p and l are growing parameters. Throughout the paper $\log z$ denotes the binary logarithm of $z > 0$.

2 Hidden Number Problem with Trace

We denote by

$$\mathrm{Tr}(z) = \sum_{i=0}^{m-1} z^{p^i} \quad \text{and} \quad \mathrm{Nm}(z) = \prod_{i=0}^{m-1} z^{p^i}$$

the *trace* and *norm* of $z \in \mathbb{F}_{p^m}$ to \mathbb{F}_p , see Section 2.3 of [18].

For an integer x we define

$$\|x\|_p = \min_{a \in \mathbb{Z}} |x - ap|$$

and for a given $k > 0$, we denote by $\mathrm{MSB}_{k,p}(x)$ any integer u , $0 \leq u \leq p-1$, such that

$$\|x - u\|_p \leq p/2^{k+1}.$$

Roughly speaking, a value of $\mathrm{MSB}_{k,p}(x)$ gives the k most significant bits of the residue of x modulo p . Note that in the above definition k need not be an integer.

The *hidden number problem with trace* over a subgroup $\mathcal{G} \subseteq \mathbb{F}_{p^m}^*$ can be formulated as follows: Given r elements $t_1, \dots, t_r \in \mathcal{G} \subseteq \mathbb{F}_{p^m}^*$, chosen independently and uniformly at random, and the values $\mathrm{MSB}_{k,p}(\mathrm{Tr}(\alpha t_i))$ for $i = 1, \dots, r$ and some $k > 0$, recover the number $\alpha \in \mathbb{F}_{p^m}$.

The case of $m = 1$ and $\mathcal{G} = \mathbb{F}_p^*$ corresponds to the hidden number problem introduced in [6] (for the case $\mathcal{G} \subset \mathbb{F}_p^*$ see [11]). The case of $m \geq 2$ is more difficult. Nevertheless in some special cases results of a comparable strength have been obtained in [17]. In other cases, an alternative method from [25] can be used, leading to weaker results.

The following statement is a partial case of Theorem 2 of [17].

We denote by \mathcal{N} the set of $z \in \mathbb{F}_{p^m}$ with norm equal to 1, thus $|\mathcal{N}| = (p^m - 1)/(p - 1)$.

Lemma 1. *Let p be a sufficiently large prime number and let \mathcal{G} be a subgroup of \mathcal{N} of order l with $l \geq p^{(m-1)/2+\rho}$ for some fixed $\rho > 0$. Then for*

$$k = \left\lceil 2\sqrt{\log p} \right\rceil \quad \text{and} \quad r = \left\lceil 4(m+1)\sqrt{\log p} \right\rceil$$

there is a deterministic polynomial time algorithm \mathcal{A} as follows. For any $\alpha \in \mathbb{F}_{p^m}$, if t_1, \dots, t_r are chosen uniformly and independently at random from \mathcal{G} and if $u_i = \mathrm{MSB}_{k,p}(\mathrm{Tr}(\alpha t_i))$ for $i = 1, \dots, r$, the output of \mathcal{A} on the $2r$ values (t_i, u_i) satisfies

$$\Pr_{t_1, \dots, t_r \in \mathcal{G}} [\mathcal{A}(t_1, \dots, t_r; u_1, \dots, u_r) = \alpha] \geq 1 - p^{-1}.$$

For smaller groups a weaker result is given by Theorem 1 of [25].

Lemma 2. *Let p be a sufficiently large prime number and let \mathcal{G} be a subgroup of $\mathbb{F}_{p^m}^*$ of prime order l with $l \geq p^\rho$ for some fixed $\rho > 0$. Then for any $\varepsilon > 0$, let*

$$k = \lceil (1 - \rho/m + \varepsilon) \log p \rceil \quad \text{and} \quad r = \lceil 4m/\varepsilon \rceil$$

there is a deterministic polynomial time algorithm \mathcal{A} as follows. For any $\alpha \in \mathbb{F}_{p^m}$, if t_1, \dots, t_r are chosen uniformly and independently at random from \mathcal{G} and if $u_i = \text{MSB}_{k,p}(\text{Tr}(\alpha t_i))$ for $i = 1, \dots, r$, the output of \mathcal{A} on the $2r$ values (t_i, u_i) satisfies

$$\Pr_{t_1, \dots, t_r \in \mathcal{G}} [\mathcal{A}(t_1, \dots, t_r; u_1, \dots, u_r) = \alpha] \geq 1 - p^{-1}.$$

3 Bit Security of Tripartite Diffie-Hellman

We have already described the tripartite Diffie-Hellman system of Joux. In that case an adversary sees $(P, Q), (aP, aQ), (bP, bQ)$ and (cP, cQ) and the key is derived from $\text{Tr}(e(P, Q)^{abc}) \in \{0, 1, \dots, p-1\}$. Note that if distortion maps are used then $Q = \psi(P)$, see [27]. In this section we study the bit security of keys obtained in this way. Later in this section we discuss the bit security of keys obtained from the protocols of Al-Riyami and Paterson [1].

Let $\omega_1, \dots, \omega_m$ be a fixed basis of \mathbb{F}_{p^m} over \mathbb{F}_p and let $\vartheta_1, \dots, \vartheta_m$ be the dual basis, that is,

$$\text{Tr}(\vartheta_j \omega_i) = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j \end{cases}$$

see Section 2.3 of [18]. Then any element $\alpha \in \mathbb{F}_{p^m}$ can be represented in the basis $\omega_1, \dots, \omega_m$ as

$$\alpha = \sum_{i=1}^m \text{Tr}(\vartheta_i \alpha) \omega_i.$$

Hence, selecting a component of the representation of an element $\alpha \in \mathbb{F}_{p^m}$ with respect to some basis $\{\omega_i\}$ is equivalent to considering $\text{Tr}(\vartheta \alpha)$ for a suitable element $\vartheta \in \mathbb{F}_{p^m}$.

We now assume that there is an algorithm which can provide some information about one of the components $\text{Tr}(\vartheta_i e(P, Q)^{abc})$ of the above representation and show that it leads to an efficient algorithm to compute the whole value $e(P, Q)^{abc}$ and hence the key $\text{Tr}(e(P, Q)^{abc})$. It follows that the partial information about one of the components is as hard as the whole key.

To make this precise, for every $k > 0$ we denote by \mathcal{O}_k the oracle which, for some fixed $\vartheta \in \mathbb{F}_{p^m}^*$ and any $a, b, c \in [0, l-1]$, takes as input the pairs

$$(P, Q), \quad (aP, aQ), \quad (bP, bQ), \quad (cP, cQ),$$

and outputs $\text{MSB}_{k,p}(\text{Tr}(\vartheta e(P, Q)^{abc}))$.

We start with the case $m = 2$ for which we obtain a result of the same strength as those known for the classical two-party Diffie-Hellman scheme over \mathbb{F}_p , see [6, 11]. Moreover, one can prove that there are infinitely many parameter choices to which our construction applies. Indeed, we know from [2] that there are infinitely many primes p such that $p+1$ has a prime divisor $l \geq p^{0.677}$. These arguments can be easily adjusted to show that the same holds for primes in the arithmetic progression $p \equiv 2 \pmod{3}$. When $p \equiv 2 \pmod{3}$, the elliptic curve given by the Weierstrass equation $Y^2 = X^3 + 1$ has $\#E(\mathbb{F}_p) = p+1$. Another infinite series of examples of $\#E(\mathbb{F}_p) = p+1$ can be obtained with primes $p \equiv 3 \pmod{4}$ and the elliptic curve given by the Weierstrass equation $Y^2 = X^3 + X$, see [16].

Theorem 1. *Assume that p is an n -bit prime (for sufficiently large n) and l is the order of the groups \mathbb{G}_1 and \mathbb{G}_2 such that $l|(p+1)$, $\gcd(l, p-1) = 1$ and $l \geq p^{1/2+\rho}$ for some fixed $\rho > 0$. Then there exists a polynomial time algorithm which, given the pairs*

$$(P, Q), \quad (aP, aQ), \quad (bP, bQ), \quad (cP, cQ)$$

for some $a, b, c \in \{0, \dots, l-1\}$, makes $O(n^{1/2})$ calls of the oracle \mathcal{O}_k with $k = \lceil 2n^{1/2} \rceil$ and computes $e(P, Q)^{abc}$ correctly with probability at least $1 - p^{-1}$.

Proof. The case when $ab = 0$ is trivial. In the general case choose a random $d \in \{0, \dots, l-1\}$ and call the oracle \mathcal{O}_k on the pairs

$$(P, Q), \quad (aP, aQ), \quad (bP, bQ), \quad ((c+d)P, (c+d)Q)$$

(the points $(c+d)P$ and $(c+d)Q$ can be computed from the values of cP , cQ and d). Let $\alpha = \vartheta e(P, Q)^{abc}$ be the hidden number and let $t = e(P, Q)^{abd}$ which can be computed as $t = e(aP, bQ)^d$. The oracle returns

$$\text{MSB}_{k,p}(\text{Tr}(\vartheta e(P, Q)^{ab(c+d)})) = \text{MSB}_{k,p}(\text{Tr}(\alpha t)).$$

Since l is prime and $ab \not\equiv 0 \pmod{l}$ it follows that the “multipliers” t are uniformly and independently distributed in \mathcal{G} , when the shifts d are chosen uniformly and independently at random from $\{0, \dots, l-1\}$. Now from Lemma 1 with $m = 2$ we derive the result. \square

Similarly, from Lemma 2 we derive:

Theorem 2. *Assume that p is an n -bit prime (for sufficiently large n) and l is the order of the groups \mathbb{G}_1 and \mathbb{G}_2 such that $l|(p^m - 1)$, $\gcd(l, p^i - 1) = 1$ for all $1 \leq i < m$, and $l \geq p^\rho$ for some fixed $\rho > 0$. Then, for any $\varepsilon > 0$, there exists a polynomial time algorithm which, given the pairs*

$$(P, Q), \quad (aP, aQ), \quad (bP, bQ), \quad (cP, cQ)$$

for some $a, b, c \in \{0, \dots, l-1\}$, makes $O(\varepsilon^{-1})$ calls of the oracle \mathcal{O}_k with $k = \lceil (1 - \rho/m + \varepsilon)n \rceil$ and computes $e(P, Q)^{abc}$ correctly with probability at least $1 - p^{-1}$.

We now consider the authenticated three party key agreement protocols of Al-Riyami and Paterson [1] (which are presented in the case $\mathbb{G}_1 = \mathbb{G}_2$). In this setting, users A, B and C have public keys aP, bP and cP and transmit ephemeral keys xP, yP and zP . We give details for the protocol TAK-3, which computes a shared key of the form

$$e(P, P)^{xyc+xzb+yz a}.$$

If a bitstring is derived from this key using the trace then results analogous to Theorems 1 and 2 are obtained (with the same values of k).

Suppose \mathcal{O}_k is an oracle which, on input $(P, aP, bP, cP, xP, yP, zP)$, outputs

$$\text{MSB}_{k,p}(\text{Tr}(\vartheta e(P, P)^{xyc+xzb+yz a}))$$

and let $\alpha = \vartheta e(P, P)^{xyc+xzb+yz a}$. Repeatedly choosing random w and calling \mathcal{O}_k on $(P, aP, bP, cP, xP, yP, zP + wP)$ yields

$$\text{MSB}_{k,p}(\text{Tr}(\alpha t)) \quad \text{where} \quad t = e(xP, bP)^w e(yP, aP)^w.$$

It is straightforward to obtain analogues of of Theorems 1 and 2.

Al-Riyami and Paterson [1] also propose the protocol TAK-4, which is related to the MQV protocol. Our methods do not provide bit security results in this case, as the use of the hash function does not preserve algebraic relationships. It is an interesting open problem to give bit security results for this protocol.

4 Bit Security of Identity-based Key Exchange

The first identity-based key exchange protocol is due to Sakai, Ohgishi and Kasahara [22], but we consider the protocol of Smart [26] as it has better security properties.

The trusted authority defines two groups \mathbb{G}_1 and \mathbb{G}_2 , such that there is a suitable bilinear map as above. The authority chooses $P \in \mathbb{G}_2$ and a secret integer s , and publishes P and $P_{\text{pub}} = sP$. The identities of users A and B give rise to points $Q_A, Q_B \in \mathbb{G}_1$ (see Boneh and Franklin [3] for details about identity-based cryptography using pairings) and the trusted authority gives them sQ_A and sQ_B respectively.

The key agreement protocol is as follows. User A chooses a random integer a and transmits $T_A = aP$ to B. Similarly, user B transmits $T_B = bP$ to A. Both users can compute the common key

$$K = e(aQ_B + bQ_A, P_{\text{pub}})$$

for example user A computes $e(aQ_B, P_{\text{pub}})e(sQ_A, T_B)$. In practice, the key is derived from K using some key derivation function, which in this case we take to be the trace.

The bit security of this key-exchange protocol can be studied and results analogous to those above can be obtained. Suppose \mathcal{O}_k is an oracle such that, for any $a, b, c \in [0, l - 1]$, on input

$$(P, aP, bP, cP, Q_A, Q_B)$$

outputs $\text{MSB}_{k,p}(\text{Tr}(\vartheta e(aQ_B + bQ_A, cP)))$ for some fixed $\vartheta \in \mathbb{F}_{p^m}^*$. Let $\alpha = \vartheta e(aQ_B + bQ_A, P_{\text{pub}})$. Repeatedly choose random $d \in [0, l - 1]$ and call the oracle \mathcal{O}_k on

$$(P, T_A, T_B + dP, P_{\text{pub}}, Q_A, Q_B).$$

The oracle responses are of the form

$$\text{MSB}_{k,p}(\text{Tr}(\alpha t)) \quad \text{where} \quad t = e(Q_A, P_{\text{pub}})^d$$

and analogues of Theorems 1 and 2 are obtained.

5 Remarks

We remark that it would be valuable to extend our results (as well as the results of [4–6, 11, 12, 14, 17]) to case when the oracle works correctly only on a polynomially large fraction of all possible inputs. Unfortunately, at the moment it is not clear how to adjust the ideas of [6], underlying all further developments in this area, to work with such “unreliable” oracles.

It has been shown in [11] that for almost all primes p an analogue of Lemma 1 holds for subgroups $\mathcal{G} \in \mathbb{F}_p^*$ of cardinality $|\mathcal{G}| \geq p^\rho$, for any fixed $\rho > 0$. It is not immediately clear how to extend the underlying number theoretic techniques to extension fields, although this question definitely deserves further attention (see also the discussion in [25]).

Finally, we recall a different kind of bit security result (see [14]) concerning the value of the pairing $e(R, P)$ for an *unknown* point R , in case when $m = 1$ (although it is quite possible that the whole approach of [14] can be generalised to extension fields). In particular, if $l \geq p^{1/2+\rho}$ is a divisor of $p - 1$, where $\rho > 0$ is fixed, then an oracle producing about $(1 - \rho/5) \log p$ most significant bits of $e(R, P)$ for an *unknown* point $R \in \mathbb{G}_1$ and a *given* point $P \in \mathbb{G}_2$, can be used to construct a polynomial time algorithm to compute $e(R, P)$ exactly. It would be interesting to understand cryptographic implications of this result.

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