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Seeing an Exercise as a Single Mathematical Object:

Using Variation To Structure Sense-Making

In this theoretical paper we take an exercise to be a collection of procedural questions or tasks. It can be useful to treat such an exercise as a single object, with individual questions seen as elements in a mathematically and pedagogically structured set. We use the notions of 'dimensions of possible variation' and 'range of permissible change', derived from Ference Marton, to discuss affordances and constraints of some sample exercises. This gives insight into the potential pedagogical role of exercises, and shows how exercise analysis and design might contribute to hypotheses about learning trajectories. We argue that learners' response to an exercise has something in common with modeling that we might call 'micro-modeling', but we resort to a more inclusive description of mathematical thinking to describe learners' possible responses to a well-planned exercise. Finally we indicate how dimensions of possible variation inform the design and use of an exercise.

INTRODUCTION

When we are working with novice and experienced teachers on issues concerning lesson planning, we start from consideration of what learners might do, what they might see, hear and think, and how they might respond. Learners' perceptions of what is on offer in the mathematics classroom are the central starting point for considering learning. These include perceptions of social, cultural and environmental aspects of the classroom as well as mathematical content, and of course all of these are subject to interpretations arising from past experience and social, cultural, environmental and mathematical dispositions and practices.

Our focus for this paper is on the predictability of learners' mathematical responses to mathematical tasks. Our assumption about learning is that the starting point of making sense of any data is the discernment of variations within it (Marton & Booth, 1997). Because discernment of variation takes place in a complex world, with emotional and social as well as cognitive components, nothing is absolutely predictable. However, we are going to claim that tasks that carefully display constrained variation are generally likely to result in progress in ways that unstructured sets of tasks do not, as long as learners are working within mathematically supportive learning environments.

The notion of a *hypothetical learning trajectory* has been used as a technical term to describe part of the act of planning lessons (Simon & Tzur, 2004). Here *hypothetical* means conjectured rather than simply theoretical. At the micro or fine-grained level of task design and implementation we prefer to work with what can be observed in the space of a lesson, and so refer to a *hypothetical* or *conjectured (learner) response*. Learning cannot generally be predicted or identified in discrete chunks of time, say over one activity, or one lesson, or in a particular task sequence; this is why psychologists often use the research practice of delayed post-testing. Learning takes place over time as a result of repeated experiences that are connected through personal sense-making (Griffin, 1989). All we can know in one lesson is something about learners' expressed responses.

While the notion of *learning* covers a broad spectrum from factual acquisition through conceptual reorganization and schema development to alteration in predisposition and perspective, the learning of particular interest to us here is conceptual development. This means to us that the learner experiences a shift between attending to relationships within and between elements of current experience (e.g. the doing of individual questions), and perceiving relationships as properties that might be applicable in other situations (Mason, 2004). Thus, for us, a mathematical concept is constructed by a learner. For example, a learner might begin to construct a concept of 'nine-ness' by naming and expressing properties observed when nine is subtracted from various numbers. The teacher hopes that these constructions will eventually match a conventional canon of developed and refined concepts, as she understands them, but the process is essentially bottom-up rather than top-down (Barsalou, 1998). Several different experiences in which a learner may detect similarities, and may hence conceptualize about the similarity, are necessary for such matching (Zawojewski & Silver, 1998).

Learning mathematics involveslong-term conceptual development, advances in abstract understanding, and improved applicability. Learning does not take place solely through learners observing some patterns in their work, even if they have generalized them explicitly. Indeed, pattern-spotting, generalizing, and reproducing patterns are merely ways in which sensate beings make sense of any succession of experiences. Learners can do this by focusing on surface syntactic structures rather than deeper mathematical meaning – just following a process with different numbers rather than understanding how the sequence of actions produces an answer. Even in a highly structured situation in which most learners appear to arrive at the same end point using apparently identical data, different learners may have had different experiences along the way, as we shall illustrate later. But experiences have taken place through such engagement with pattern that *may* contribute to progression in understanding a particular concept as understood by the teacher.

The metaphor of learning trajectory (Clements & Sarama, 2004), that seems to assume predictable and sequential development in the conventional canon, is challenged by the variety of responses learners make to events in a lesson sequence. Even the hypothetical version offered by Simon and Tzur (2004, p.100) seems to assume that something sequential can be said about learning, but in our experience this over-simplifies the responses to experience that can occur in or out of a classroom. Nevertheless, when planning, thinking in terms of *hypothetical learning trajectories* (HLTs) may contribute to teacher-confidence, giving them at least a place to start. To a certain extent any lesson plan involves an implicit, if not explicit, sense of possible experiences that it is to be hoped each learner will transform into a personal learning trajectory. After all, teaching takes place in sequential time, even though learners take different trajectories. The approach to planning tasks that we are going to describe has some features in common with HLTs and with the similar design activity described by Gravemeijer (2004) but we use mathematical variation to hypothesise micro-response rather than cognitive theories to hypothesise learning.

HLTs provide teachers with tools to promote learning of particular conventional understandings of mathematics. Simon and Tzur (2004) focus on designing sequences of tasks that invite learners to reflect on the effect of their actions in the hope that they will recognize key relationships. Recognising that these conventional understandings, which are the goals of mathematics teaching, are all generalizations of relationships between variable objects, whether they are examples of algorithms, concepts and so on, leads us to suggest that the generalizations that support the intentions of lesson plans might be more usefully seen as the possible outcomes of micro-modeling. By micro-modeling we mean the processes of trying to see, structure and exploit regularities in experiential data, so that learners are thus exposed to mathematical structure affording them enhanced possibilities for making their own sense of a *collection* of questions, or an exercise. In other words, can exercises be constructed in such a way that desirable regularities might emerge from the learners' engagement with the task, like categories emerging from experimental data? Posing this challenge invites a further, more pragmatic question: What regularities are available to learners and which are most *likely* to be observed in any given task?

The modeling perspective described by Lesh and others draws on learners' natural desire to engage with and make sense of experiential data (Lesh & Doerr, 2003). It is also natural for learners to test their ideas as much as they need to for personal conviction, or to make continued exploration possible. 'Model eliciting activities' (Lesh & Yoon, 2004) harness this kind of response to promote learners' engagement with mathematical ideas through making sense of mathematically complex problematic situations. However, to make mathematical progress the results of the images, models and generalizations thus created have to become tools for more sophisticated mathematics. We see *generalization* as sensing the possible variation in a relationship, and *abstraction* as shifting from seeing relationships as specific to the situation, to seeing them as potential properties of similar situations. Any task, particularly problem-solving and modeling tasks, can focus learners' attention to the immediate 'doing' (calculations, re-presentation, etc.) but unless special steps are taken to promote further engagement, there is seldom motivation for abstraction, rigor or conceptualization beyond that necessary for the current problem, a point recognized by Burkhardt (1981). Doerr also recognized this possible limitation and offered sequences of tasks through which learners shift from creating models of carefully chosen situations, to seeing those models as embodying structures that may find future application, and also for exploration as structured mathematical objects in their own right (Doerr, 2000).

Teachers cannot even be sure that learners will use the most recently-met ideas for their modeling work, or engage with structure, particularly if the teacher is playing a non-intervention role for some reason (Baroody, Cibulskis, Meng-lung, & Xai, 2004). For example, the diagonal distance across a rectangular space might be measured or estimated even if Pythagoras' theorem has been recently 'learned'. Burkhardt (1981) suggested that learners are unlikely to use spontaneously a technique, method or perspective that they first encountered more recently than a year or two.

Apart from these features, a modeling approach offers a reasonably informative description of learners' sense-making in *any* mathematical task, in that they engage with a range of experiential data and attempt to construct meanings that are then tested out against expectations, perceived implications and eventually, future experience. At a mundane level, this means that attempts to answer closed questions are checked out against the teacher's answers; at a sophisticated level, concept images are consciously adapted and enriched as more examples and implications are encountered. Since experience depends on the learners' perceptions of the mathematical tasks offered by a teacher or other authority, it seems appropriate to consider whether these can invite abstraction as a natural sense-making response. We claim that if the teacher offers data that systematically expose mathematical structure, the empiricism of modeling can give way to the dance of exemplification, generalization and conceptualization that characterizes formal mathematics. For this claim we make the same assumptions as those required for modeling, namely that learners cannot resist looking for, or imposing pattern, and hence creating generalizations, even if these are not expressed or recognized. These generalizations are then the raw material for mathematical conceptualization.

ANALYSIS OF RESPONSES TO AN EXERCISE

To illustrate how paying attention to variation can illuminate our understanding of learners' responses to sequences of questions, we use an exercise written by Krause (1986) to teach taxi-cab geometry. At first glance it appears to be of a typical 'do a few examples' kind but there is rather more sophistication than is first apparent. In his textbook, Krause does not say in advance what the exercise is about. In order to understand what follows, we advise the reader to do the task.

$Dt(P, A)$ is the shortest distance from P to A on a two-dimensional coordinate grid, using horizontal and vertical movement only. We call it the taxicab distance.

For this exercise $A = (-2,-1)$. Mark A on a coordinate grid. For each point P in (a) to (h) below calculate $Dt(P, A)$ and mark P on the grid (in the original, they are in a single column so there is no temptation to work across rows instead of in order down the columns):

(a) $P = (1, -1)$	(e) $P = (\frac{1}{2}, -1\frac{1}{2})$
(b) $P = (-2, -4)$	(f) $P = (-1\frac{1}{2}, -3\frac{1}{2})$
(c) $P = (-1, -3)$	(g) $P = (0, 0)$
(d) $P = (0, -2)$	(h) $P= (-2, 2)$

We have used this task with many groups of inservice and preservice teachers, primary and secondary, and with two differently-aged classes of school students - about two hundred people in all. Afterwards we asked them to report on their experiences in group discussion, and we made notes of what was said. On two occasions we supplemented this with requests for written comments, and we have also had verbal reports from people who have then used the task with other teachers and students. In this way we have collected, over three years, a comprehensive qualitative picture of learners' reactions to this set of questions. In nearly every case the word 'learner' is appropriate because very few of the participants had worked with taxicab geometry before.

Our approach is consonant with the development of grounded theory through naturalistic inquiry (Lincoln & Guba, 1985) concerning response in the natural setting of groups of people learning new mathematics, from a textbook exercise, in an interactive environment. It can also be seen as a form of action research, since each use of a task is informed by our past and current experiences of participants' thinking. What we present here can be taken as a form of phenomenographic analysis of observations arising from semiformal action research. The result is a collation of as complete a story as possible of the varieties of ways in which people have responded. Our conjecture is that past responses provide us with a good prediction of how people in the future are likely to respond. This conjecture is strengthened by the proximity of a theoretical mathematical explanation of likely response to the actual response as reported.

There was variation in the order in which people carried out the exercise: Some chose to plot all the points first and then calculate all the distances; others chose to calculate all the distances and then plot all the points; others did each point separately, finding the distance and plotting the point as Krause suggests. Whichever way they did the exercise, those who had not met this material before reported remarkably similar experiences, or at least, reported their experiences in remarkably similar ways. They found themselves making generalizations early on in the exercise: such as, that all the distances will be 3 and/or that all the points appeared to be on a 'straight line'. Many were not even aware they had a generalized expectation until the 'straight line' broke down at the seventh point (g). Their evident surprise revealed the presence of implicit conjectures and expectations, typical of the role of disturbance in triggering sense-making and possibly learning (Mason & Johnston-Wilder, 2004, pp.118,149). The break in pattern caused many to begin to think about the mathematics behind what they were doing. They found themselves asking more probing questions such as 'where would I expect points to be that are all a distance of 3 away from A?' or 'what has this straight line got to do with a distance of 3?'

For completeness we also report that some people did not immediately know what to do with non-integer coordinates, but used what they found from the first four points to interpolate and reach an interpretation that 'seems to work'. Many people also reported that they became quicker with each successive point, that they became more eager because they wanted to know what was going on, and that where they made errors they were able to self-correct because the inherent patterns caused possible mistakes to become obvious. Thus fluency, motivation and accuracy appeared to result from the exercise.

The next question Krause poses in his book asks learners to make up more points P that have $Dt(P, A) = 3$, to graph them, and to describe the complete set. In our experience, many learners did this for themselves before they were asked to by the textbook. The next question, therefore, merely asked them to express what they had already deduced and reaffirmed an enquiry-based stance to mathematics as a constructive enterprise (Watson & Mason, 2005).

But the main purpose of the exercise is not to get better at calculating distances and plotting points, nor to learn about Dt(P,A) = 3. Almost unanimously, people reported that this exercise evoked their natural propensity to look for similarities and to make conjectures in order to 'teach' them something about taxicab geometry; that they started by just 'doing' each separate point but were jolted into thinking mathematically by being offered points that broke their current sense of pattern; and that they had not realized they were aware of pattern until they were offered these points. Thus the combination of several similar examples and further not-quite-similar examples shifted them to work on a higher level than simply plotting and calculating, leading to conceptual learning as well as fluency, accuracy, motivation and interest. Many found that their knowledge of Euclidean circles came to mind and this amplified their interest. This did not happen for everyone, possibly because not everyone had a well-developed definition of a circle, as the locus of a point equidistant from a fixed point, readily available.

A few people operated at a higher level throughout the exercise. As soon as they completed the second example they were already asking themselves 'why 3?' (seeing 3 as a dimension of possible variation that has been fixed for some reason, not simply as an accidental number). Some did not plot points at all, assuming this to be some visual extra that the 'teacher' required but that they did not themselves need; some of these proceeded to try to generalize algebraically. We have found very few other ways of responding to this exercise, and no examples of people who claimed to have merely performed the required tasks without voluntary conjecturing and predicting.

ANALYSIS OF VARIATION IN THE EXERCISE AS GIVEN

What intrigues us, given the very different backgrounds, mathematical knowledge, goals and social contexts of the groups with whom we have worked, is that there is an almost universal global response to the exercise, an articulation of taxi-cab circles. The structure of the local responses, contributing to the global response, varies according to the sequence in which the individual tasks are undertaken. It is the structure of the exercise as a whole, not the individual items, that promotes individual disturbance and common mathematical sense-making. Marton's identification of 'dimensions of variation', initially used as a way to characterize differential learning, offers a way to look at exercises in terms of what is available for the learner to notice (Marton & Booth, 1997; Marton, Runesson & Tsui, 2004). This particular strength of Marton's variation theory provides us with the means to justify in mathematical terms our empirical observations of learning. Applied to mathematics it provides ways of describing learning, for relating learning to mathematical structures as afforded to, and *perceived by*, learners whether intended by the teacher or not. Applied to mathematical text, this approach offers a structured and structural approach to exposing underlying mathematical form. Variation is also a tool to scaffold the construction of different tasks that are conceptually related (Zawijewski & Silver, 1998).

We therefore analysed the exercise to find out what variations were available to be discerned by the learner, and when. Our analysis would vary slightly according to the way someone chose to approach the exercise, so here we shall focus on the case in which a learner both calculates and plots each point in turn. We look at what aspects are fixed, what is varied (different aspects constitute different *dimensions of possible variation*), and how it is varied (which indicates a *range of permissible change*) throughout the exercise, and what is thus available for discernment by the learners, agreeing in part with Runesson and Mok who say: 'If the particular aspect is present as a dimension of variation, it is likely to be discerned ' (2004, p.218). We would modify this slightly and claim that such an aspect is more likely to be discerned if its variation is foregrounded against relative invariance of other features. If everything is varying, nothing may be discerned. More recently, Runesson says 'the enacted object of learning reveals the space of variation and invariance that is possible for the learner to experience' (2005, p.85)

Point A is fixed, hence the learner can focus on comparing relationships to A, rather than having to take pairs of points into account. With too many things varying, individual variation is obscured, and learners may form an impression of jumping about randomly if they are aware at all of a potential *example space* (Watson & Mason, 2005) from which the tasks are drawn. Later, A can be varied to generalize such relationships. The point A is not obviously special, so it might in some sense stand for any generic point, but it might also have some hidden particularities which would need to be distinguished. If (0,0) had been chosen – a somewhat special point – learners may not see anything that happens as a generality.

The first three points P fix $Dt(P, A)$ as 3. In this case, the fixing of 3 is essential to the mathematical plan; the dimension of possible variation has been constrained to points whose distance from A is a constant. Most report that the answer '3' becomes an expectation quite quickly and, eventually, their work shifts from 'calculating distance' to 'verifying that the distance is 3'. An elementary conjecture has emerged from the subconscious based on something varying and something relatively invariant. The range of change used so far is restricted to integer coordinates, positive and negative. The fourth point provides a confirmation that the points appear to be visually in a straight line, and so offers a self-checking opportunity. The fifth point alters the range of permissible change from integer coordinates to fractions, but within a context that provides ways to conjecture and test how these are interpreted. By this time, most people have decided that the answer they have to make is 3, and the points are going to make a straight line.

Then comes the point (0, 0). This is a familiar point, so not one that is expected to cause any difficulty in itself, but definitely one that is not on the straight line. It is not unusual for people to make little noises when they get to this point, indicating surprise, puzzlement, or a sense of error. Until now, the two possible generalizations have been that the points have been chosen so that the distance of 3 is fixed, and the position on a particular straight line. It has even been possible to conjecture a relationship. But Krause knew that the full range of permissible change for points that are a distance 3 away from *A* has not, at this point, been encountered or exposed, and hence people may be tempted to make unwise or incomplete generalizations. The inclusion of (0, 0) makes it possible to discern a difference between the shape of the locus and a straight line, yet the straight line conjecture allows learners to cope with fractions and set up an expectation that allows difference to be noticed.

The final point gives more information about what shape might emerge, and Krause stops there, leaving space for learners to think for themselves and to make up their minds about what else might happen. There is still the position of point *A* that can be varied, and the fixed distance, but the central generalization about the locus may have already been made through the development of what, at first sight, were 'practice' examples.

This analysis of what is possible at each stage of the exercise matches almost exactly the reported experience of the majority of people to whom we have offered it, although the order might vary according to the order of doing the task, or the way different people focused on different features. It matters whether they focus on dimensions of variation (variables) or ranges of change (values) at various stages in the exercise. This is reminiscent of Krutetskii's distinction (1976) between those students who worked on variables and those who worked with value, but here we could not claim that those who worked on variables were any more 'gifted' than those who stayed with value, since both approaches appeared in every group. People do not necessarily use the most sophisticated techniques available to them in novel, public, situations. Further differences are experienced by people who have leapt to a level of abstraction much earlier than the exercise expects, or those who have restricted themselves to a purely algebraic approach. Yet people displaying these exceptions ended up with a similar understanding of taxicab ‘circles’ as others, albeit by a different route or pathway, and each had different experiences along the way, drawing in and developing different skills, awareness and ancillary knowledge.

We do not underestimate the role of discussion, which was always a genuine attempt to learn about learners’ experiences but also provided a model and structure for reflective abstraction that otherwise may not have taken place for some people. A full analysis of the potential offered by the task has to include the discussion that followed. However, rather than asking ‘what did you notice about the maths?’ as if there was an intended answer in our minds we asked ‘what was your experience as a learner?’ No one can invent stories of variation, generalization and so on that they have not in some way imagined or experienced, even at a meta-level. Perception, conscious or subconscious, must precede articulations about perception (Barsalou,1998).

The close relationship between Krause’s choice of dimensions of variation and associated ranges of change, and the learning experiences of those doing the exercise, suggests that he had in mind that all learners would encounter taxicab circles, and that the route through which they met them could involve a personal mixture of spatial, numerical and symbolic experiences, practice of techniques, visualization, conjecture, generalization, expectation, surprise, conceptualization and affirmation. We do not know how long it took Krause to develop this exercise, but such artistry and precision in helping a learner learn does not come instantly. Constructing tasks that use variation and change optimally is a design project in which reflection about learner responses leads to further refinement and further precision of example choice and sequence, as Gravemeijer (2004) describes. This process cannot be done by textbook authors working alone under tight publication deadlines, but it can be done by teachers for themselves. Hewitt (2003), commenting on the effectiveness of a lesson that was videoed for national distribution, took care to point out that that particular lesson drew upon many years of experimentation, honing the tasks with a wide range of learners until he could be confident that it would ‘work’ with virtually any group of learners. As already mentioned, there are no guarantees, no matter how carefully an exercise is structured, because of the many other relevant factors influencing the situation. That said, greater commonality can be achieved through careful structuring than through apparently random collections of questions treated as individual tasks by learners. Hewitt's tasks typically involve far more repetition of linguistic and symbolic structures, with focused and controlled variation, than any textbook provides.

We are not claiming that there can be a deterministic, unproblematic, relationship between teaching and learning. What is actually discerned in any situation, out of all the varying features, is dependent on dispositions to engage with mathematics, prior experience of mathematical practices, current conceptualizations, social and affective aspects of the situation, and much more. What we are claiming is that, if the intention is that learners should understand, say, the invariance of sine for a fixed angle, it makes sense to keep the angle and hence the ratio constant while changing other features so many times that the invariance of the ratio becomes more and more insistent and obvious in students' perceptions. The generalization that 'opposite is related to hypotenuse in some way' (or equivalent expression in algebraic or diagrammatic form) then becomes a property of right-angled triangles to explore. The angle can now be varied, while other features are kept constant, so that the nature of the co-variation of 'opposite' and hypotenuse becomes more and more obvious – a property of angles that can eventually be named and discussed and compared to other properties. Thus the mathematical relationship between variables can be exploited as a pedagogical experience and tool.

EXERCISES SEEN AS MATHEMATICAL OBJECTS

We use the word 'object' to mean 'that which is the focus of attention', somewhat in the way in that Barnard and Tall (1997) use *cognitive unit*. In some senses this relates to Marton's use of the phrase 'object of learning' (Marton & Booth, 1997), but for us the 'object' is not necessarily an objective, either for the learner or the teacher, nor is it necessarily a mathematical object in the sense that it has been conventionally defined and/or has a meaning delineated by mathematical and linguistic processes. Rather, we use it to mean a thing on which a learner focuses and acts intelligently and mathematically by observing, analyzing, exploring, questioning, transforming, and so on. Thus an object could be a symbol, some text, a diagram, a theorem, a line of a theorem, a material object, an equation and so on. It is the ‘this’ for which a teacher might say ‘look at this’ or for which a learner might say ‘I am looking at this’ or even ‘I am thinking about this’. It will be immediately clear that the teacher cannot fully control what becomes an object for a learner. A teacher could think that a complete worked example is the object to which her students are attending, while a well-engaged learner might read one line of it over and over again and thinking ‘what on earth does that mean?’, or ‘where did that come from?’, thus having that line or symbol as the object. A teacher might think that the slope of a line relative to the x-axis is the object to which students are attending, while a well-engaged learner might be attending to the position of the intersection of the line and the axis. (The cursors in dynamic geometry software are better at asking ‘what am I supposed to be looking at?’ than the average student.)

The actions that the learner brings to this observation and exploration are at the very least the natural propensities to observe variation and similarity, and to seek pattern in the variation, either by identifying or imposing pattern on the experience of perception, and relationships between this variation and others in their experience. Perceptions of objects and their associated patterns are starting points for the more complex actions of meaning-making that involve experience, context, dialogue, enthusiasm and so on.

For example, in this sequence: 1, 2, 3,… the shapes of the symbols change wildly from one element to the next whereas in this sequence: /, //, /// … the changes of shape are systematic and easier to observe, to describe, and to imbue with meaning. But ten or twenty terms along, numerals start to be much easier to deal with than tally marks! We know that learning mathematics includes understanding that the meanings in the first sequence transcend concern about the shapes of the symbols, but agreement about variation *in what we can observe* in the latter case is more universal than in the former. Marton's focus on variation gives us a language for saying why this is, namely that the only variations in the second case are frequency and position, whereas in the first case variation is very hard to describe. Indeed, those children who claim that the symbol 1 has one 'pointy bit', 2 has two 'pointy bits' and 3 has three 'pointy bits' are following a natural desire to impose pattern in an attempt to construct meaning. Of course it matters where attention is focused, and we have used the word 'shape' to guide the reader towards a dimension of possible variation. Even so, one could have chosen to focus on the curves rather than the points or lines. If we had suggested 'value' as a focus the analysis would be more complex, because value is symbolically represented in the first sequence, and thus socially and discursively constructed, but iconically represented in the second. To perceive the sequences in terms of value requires interpretation and knowledge.

It does not matter for the purposes of our discussion whether these characteristics are seen in a Platonic sense as having an existence outside the learner, or are brought into being by the learner's perception, through the learner having a point of view (Dorfler, 2002), or as the duals of process (Gray & Tall, 1994; Tall, Thomas, Davis, Gray, & Simpson, 2000). All we need to know is that learners discern differences between and within objects through attending to variation. Teachers can therefore aim to constrain the number and nature of the differences they present to learners and thus increase the likelihood that attention will be focused on mathematically crucial variables. Further, we can treat a set of objects as an object in its own right (Dorfler, 2002, p.344) and see variations within it as its properties.

Here are three contrasting examples of exercises in which learners have to find the gradients of straight line segments between two given points on coordinate grids, and represent them on a diagram. Again, we have offered these exercises to several groups of teachers and teacher educators to learn more about the range of possible reactions:

Gradient exercise 1: Find the gradients between each of the following pairs of points

(4, 3) and (8, 12)	(-2, -1) and (-10, 1)
(7, 4) and (-4, 8)	(8, -7) and (11, -1)
(6, -4) and (6, 7)	(-5, 2) and (10, 6)
(-5, 2) and (-3, -9)	(-6, -9) and (-6, -8)
(8, 9) and (2, -9)	(7, -8) and (-7, 5)
(-9, -7) and (1, 4)	(-4, -3) and (4, -2)
(2, -5) and (-3, -7)	(1, 6) and (-1, -3)
(-1, 0) and (5, -1)	(-3, 5) and (-3, 2)

Gradient exercise 2: Find the gradient between each of the following pairs of points

(i) (4, 3) and (8, 12)	(ii) (-2, -3) and (4, 6)
(iii) (5, 6) and (10, 2)	(iv) (-3, 4) and (8, -6)
(v) (-5, 3) and (2, 3)	(vi) (2, 1) and (2, 9)
(vii) (p, q) and (r, s)	(viii) (0, a) and (a, 0)
(ix) (0, 0) and (a, b)	

(adapted from Backhouse and Houldsworth, 1957)

Gradient exercise 3: Find the gradient between each of the following pairs of points

(4, 3) and (8, 12)	(4, 3) and (4, 12)
(4, 3) and (7, 12)	(4, 3) and (3, 12)
(4, 3) and (6, 12)	(4, 3) and (2, 12)
(4, 3) and (5, 12)	(4, 3) and (1, 12)

In the first exercise the variation of signs is controlled so that all possible pairs occur, but numbers and relative values vary, and there is little apparent system even in the variation of signs. A diagram would add little sense to the variety that is produced and it is not clear that the concept of gradient, as a slope property, would be the focus of attention. It is more likely, because the variation is in the signs, that subtracting and dividing directed numbers will be the focus. In the second exercise, there is selective practice of various possibilities followed quickly by an invitation to write gradient as a formula, expressing a generality that the learner must already know in order to do the earlier questions. This is then followed by questions about special cases of the formula, so that we can see the author is intending *generalization* of gradient calculations to be the focus. In the last exercise, variation is tightly controlled by only varying the x-coordinate of the second point. Some technical practice is offered, but this breaks down when the x-coordinate is 4 because this means you have to divide by zero. An accompanying diagram would help to make sense of this. Some have found the third exercise too constrained for the development of technical fluency, even boring, while others have found that it is the most interesting of the three because it invites questions and further exploration, and includes a little shock that does not yield to an algorithmic approach. The focus is on varying gradient in a controlled way so that gradient changes from being a name for the answer to a calculation, to being a relationship, to being a property and also an arena for conjecture.

By asking the highly mathematical question 'what changes and what stays the same?', and by examining the nature of the changes offered, we can be precise about what an exercise together with an established way of working (collection of social practices) affords the learner, and with what constraints (Greeno, 1994; Watson, 2003 & 2004). Each of the exercises above affords a different kind of generalization, and learners' engagement with gradient as a concept will be influenced by what they can easily generalize from the exercise.

MICRO-MODELING AND MATHEMATICAL THINKING

Lesh and Yoon (2004) propose that model-eliciting activities provide situations for learners to make mathematical sense. Sense is made by identifying objects, relations, operations, transformations, patterns, regularities and quantifying, dimensionalising, coordinating, and systematizing them, using problem-solving strategies, often organized heuristically (p.210). This list of characteristics and activities sounds to us like a remarkably comprehensive description of mathematical thinking (Mason, Burton & Stacey, 1982; Watson & Mason, 1998). Furthermore, every part of it can be applied to situations that are entirely mathematical and not contextualized externally at all. For example, the concept of linearity can be developed through modeling data that is generated by situations known already by the expert to be linear, such as coordinate pairs, families of graphs, and equations that look similar in form, in addition to covariation of particular ‘real life’ variables.

Opportunities to practise skills, to select and represent variables, to express relationships and generalities, to gain mathematising tools with which to engage economically and critically with the world, are overwhelmingly present in modeling activities. What is often lacking is, firstly, any guarantee that new mathematical ideas will be encountered without the intervention of a mathematically-aware expert redirecting learners’ attention and, secondly, opportunity for purposeful entrance into the abstract world of mathematics. Learners may have experience of mathematical processes, may have consolidated and reorganized what they already know, and may be creative in its application, but may not have extended their knowledge of the conventional mathematics canon. Typically, their attention will have been on the range of possible change within the dimensions of variation offered by the modeled situation; the generalities that guide their exploration might be the generalities of shopping, or weather, or sewing, or whatever the modeling context is, rather than the generalities of mathematics. To engage fully with mathematical structure, it is the dimensions themselves and how they interrelate that have to become the focus of attention.

To make this shift requires a move away from the structures afforded by the modeled situation, but there may be no motivation within the modeling activity to engage with what has been produced in a more abstract way.

The term ‘modeling’ gives an invigorating picture of the messy, cyclic, human activity that is seldom made explicit in classrooms while pure mathematics is going on, but to extend its use to include everything else that is involved in learning mathematics seems to us to lose the power achieved by restricting its use to moving from material world phenomena to mathematical re-presentations. From a modeling perspective the term ‘micro-modeling’ may be helpful to describe learners’ response to exercises in which dimensions of variation have been carefully controlled, since the aim is to promote generalization of the dimensions being varied in the exercise, and thence to focus on mathematical relationships between dimensions. To move beyond that initial generalization to more abstract engagement with the concepts, such as moving from taxi-cab geometry to metric spaces, or moving from calculating straight line gradients to considering gradient functions, requires an approach to task design that goes beyond modeling, one that harnesses further processes of abstraction. Exercises that offer controlled variation have the potential to lay the groundwork for rapid engagement with structure, relationships and properties.

PUTTING VARIATION INTO PRACTICE

Teachers and textbook authors typically describe the use of repetitive exercises as providing ‘practice’ for learners, often without stating what such practice is supposed to achieve. Practice can mean the use of repetitive tasks to build up fluency, speed and accuracy in performing technical tasks. The importance of this in mathematics has long been recognized, from the use of Vedic sutras for arithmetic (Joseph, 1991), through Mary Boole’s urgings to use subconscious powers (Tahta, 1972), to Hewitt’s search for economy in learning repeated actions (Hewitt, 1996). However, many textbook exercises do not seem to offer practice of this type at all. Questions are more likely to be slightly different in a seemingly arbitrary way so that learners tend to proceed in a stop-start fashion.

Consider this selection of textbook questions on ratio:

Reduce to simplest terms:

(a) $\frac{4}{12}$	(b) $\frac{36}{12}$	(c) $\frac{240}{300}$	(d) 5 : 5
(e) $ab : ab$	(g) $2\frac{1}{4} : 1$	(h) $\frac{6ab}{3b}$	

(from one used by Lerman (2001))

The authors presumably hope that the learner will learn about a variety of forms of ratio, but the specific instruction is to express the ratios in simplest form. To get the answers the learner does not have to focus on notation, meaning or representation, merely on finding common terms to cancel. The exercise can be completed without any engagement with the ratio concept. It can also be completed without any increase in fluency because each question is different enough to need some fresh thinking. In classrooms, we observe that such exercises seem to result in a slowing down of pace and an increase in effort, rather than speeding up and reduced effort, unless teachers explicitly engage learners with the goal of getting faster and becoming ‘experts’. Without some pedagogic intervention little can be achieved except for counting the right answers, and the analysis of errors to inform future teaching.

The following is adapted from a larger exercise in Tuckey (1904):

Multiply each of the terms in the top row by each of the terms in the bottom row in pairs:

$x - 1$	$x + 1$	$x + 2$	$x + 3$
$x - 1$	$x + 1$	$x + 2$	$x + 3$

Apart from being a simple way to produce a long exercise, this appears to offer enough similarity to encourage fluency and some awareness of, and control over, change that may allow learners to get a sense of underlying structure while doing the examples. It is not sufficient just to 'do' all the products. Learners need to contemplate relationships, to consider effects of changes in one particular aspect (e.g. the sign). If the actual multiplication takes time, learners may not experience these variations and their effects, because there is too much time between one result and the next. In Tuckey's exercise there is sufficient similarity in the calculations and in the *appearance* of the algebra that regularities can easily be experienced and, through discussion, brought to articulation.

A well-wrought example of this approach is demonstrated by Yizhu Liu in his interpretation of Marton's theory, used to design a series of textbooks in China (2004). Readers are typically offered a range of similar but slightly different mathematical situations and asked an exploratory question from which a generalization is invited. The generalization is arrived at by comparing variations in their observations. To enable learners to relate the new idea to their previous experiences they are then asked to compare the new and old concepts in some way – for example, when absolute value is introduced they are asked to describe similarities and differences between, say -2.5 and |-2.5|. The exercise deliberately directs the learner towards classification, towards comparing the 'new' idea of modulus to the familiar idea of number. Doing this for a few examples provides territory for making an initial conjecture and verifying or refining it. In this kind of 'contrastive' question learners note variations and similarities between the new and old concepts. Learners have to step back from mere calculation to look for relationships and invariants, but the relationships are very easy to perceive.

A further example found in a typical elementary school textbook provides subtraction practice of this kind:

17 – 9 =   27 - 9 =   37 - 9 =   …

Clearly the aim here is more than repeated use of a subtraction strategy. The whole exercise is intended to be the object of study. Learners are likely to begin to develop a concept of 'nine-ness', particularly if the teacher supports this move. Without this support, learners who have not developed the habit of reflecting on work done and conjecturing about relationships may notice something special, but might not realize that this kind of noticing is the essence of mathematics.

FUTURE QUESTIONS

When we describe the potential power of certain questions, prompts and task-types, our aim is not to design lessons for teachers to use, or even to design lessons with teachers. This would be simply unsustainable in most teaching situations. However, it is important to find out how accessible and practical these ideas are for teachers so that they can design mathematical tasks without needing to access research for each separate topic. We have not researched this systematically, but have conducted informal trials with several novices and experienced teachers who then put 'exercise as mathematical object' into practice by designing and using question sets with deliberate attention to dimensions of variation, *as they understand it*. We report this very briefly here to indicate future research issues.

All the teachers had a training session in which the ideas were introduced and demonstrated. All were enthusiastic; one experienced teacher, (who had a psychology degree), said that the language of 'variation' helped her describe why she did not like textbook exercises that appeared to offer only random variation. All agreed to construct exercises in which dimensions of variation were carefully controlled. Some used deliberate strategies, such as selecting one attribute to be varied while holding others constant (Brown & Walter, 1983), or selecting one variable to be held constant while others vary systematically, or focusing on systematic generation of equivalent forms. Others acted more instinctively in topic-specific ways, such as focusing on the emergence of special cases as surprises within varying sets (Movshovits-Hadar, 1988).

All the teachers were able to design question sets that, on analysis, showed systematic attention to variation within the exercise as a whole so that correctly-performed techniques were only the starting point for mathematization. For example, one class was offered:

Simplify these:

6/10	18/20	6/8	14/16

Now simplify these:

15/25	45/50	15/20	35/40

Compare the answers

Another class, also studying fractions, was given a set of questions in which '7' was used as a generic placeholder, increasing in complexity:

1/7 x 7/2	1/7 x 7/3	1/7 x 7/4… etc.
3/7 x 7/8	3/7 x 14/15	3/7 x 21/22 … etc.
9/21 x 14/21		

Students' comments were collected in this case and showed that the questions they found most interesting were those where the role of '7' suddenly changed. Learners' attention moved from the details of the calculation to the multiplicative relationships between numerators and denominators. They could see how the questions had been constructed and saw more meaning in multiplying fractions than they had before. It had been possible, in the earlier questions, to treat '7' as merely a matching symbol that you had to 'cross out', but later they had to rethink what this 'crossing out' was all about.

The role of '7' drew our attention to the powerful role that perception plays in the discernment of variation. In all the examples we have given, direct perception has been a key feature in all responses. It is 'looking the same' or 'looking different' that seems to matter first;

without visual similarity comparing underlying meaning, or matching one representation to another, are less accessible activities that need deliberate prompting.

Whereas these teachers had seen potential power in controlling variation in exercises, and had been happy, even excited, about using this as a design principle, knowing more about impact on learning is going to take more experimentation and longer immersion.

Our conclusions after three years of work in a range of natural settings are that control of dimensions of variation and ranges of change is a powerful design strategy for producing exercises that encourage learners to engage with mathematical structure, to generalize and to conceptualize even when doing apparently mundane questions. This power is easily recognized by teachers, teacher educators and other professionals in mathematics education. Variation and change can also be used to analyse the affordances and constraints of exercises within particular settings and situations (Watson, 2003 & 2004).

We therefore offer the following processes for planning teaching sequences that start from the learners' perceptions of mathematical objects:

- Analysis of concepts in the conventional canon that one hopes learners will encounter.
- Identification of regularities in conventional examples of that concept (and its related techniques, images, language, contexts) that might help learners (re)construct generalities associated with the concept. Even an algorithm can be seen as a generality.
- Identification of variation(s) that would exemplify these generalities; decide which dimensions to vary and how to vary them;
- Construct exercises that offer micro-modeling opportunities, by presenting controlled variation, so that learners might observe regularities and differences, develop expectations, make comparisons, have surprises, test, adapt and confirm their conjectures within the exercise;

- Provide sequences of micro-modeling opportunities, based on sequences of hypothetical responses to variation, that nurture shifts between focusing on changes, relationships, properties, and relationships between properties.

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