

# Seiberg–Witten Curve for $E$ -String Theory Revisited

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## Abstract

We discuss various properties of the Seiberg–Witten curve for the  $E$ -string theory which we have obtained recently in hep-th/0203025. Seiberg–Witten curve for the  $E$ -string describes the low-energy dynamics of a six-dimensional  $(1, 0)$  SUSY theory when compactified on  $\mathbf{R}^4 \times T^2$ . It has a manifest affine  $E_8$  global symmetry with modulus  $\tau$  and  $E_8$  Wilson line parameters  $\{m_i\}, i = 1, 2, \dots, 8$  which are associated with the geometry of the rational elliptic surface. When the radii  $R_5, R_6$  of the torus  $T^2$  degenerate  $R_5, R_6 \rightarrow 0$ ,  $E$ -string curve is reduced to the known Seiberg–Witten curves of four- and five-dimensional gauge theories.

In this paper we first study the geometry of rational elliptic surface and identify the geometrical significance of the Wilson line parameters. By fine tuning these parameters we also study degenerations of our curve corresponding to various unbroken symmetry groups. We also find a new way of reduction to four-dimensional theories without taking a degenerate limit of  $T^2$  so that the  $SL(2, \mathbf{Z})$  symmetry is left intact. By setting some of the Wilson line parameters to special values we obtain the four-dimensional  $SU(2)$  Seiberg–Witten theory with 4 flavors and also a curve by Donagi and Witten describing the dynamics of a perturbed  $\mathcal{N} = 4$  theory.

## 1. Introduction

In our previous paper [1] we have constructed a Seiberg–Witten curve for the  $E$ -string (or  $E_8$  non-critical string) theory [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. Seiberg–Witten curve for  $E$ -strings describes the low-energy dynamics of a (1,0) SUSY theory in six dimensions partially compactified on  $\mathbf{R}^4 \times T^2$ .  $E$ -string is essentially one half of the  $E_8 \times E_8$  heterotic string and describes the degrees of freedom which appear on the M5-brane in the M-theory description of small  $E_8$  instanton singularities.

Our curve has a manifest affine  $E_8$  global symmetry with the modulus  $\tau$  of  $T^2$  and possesses  $E_8$  Wilson line parameters  $\{m_i\}, i = 1, 2, \dots, 8$ . When the radii of the torus  $T^2$  degenerate  $R_5, R_6 \rightarrow 0$ , the curve reduces to the Seiberg–Witten curves of four- and five-dimensional gauge theories [15, 16, 17, 18, 10] and  $\{m_i\}$  are identified as the mass parameters of the matter hypermultiplets.

It is known for some time that the  $E$ -string is associated with the geometry of the rational elliptic surface. In fact our curve has the form

$$\mathcal{C}: y^2 = 4x^3 - f(u; \tau, m_i)x - g(u; \tau, m_i) \quad (1.1)$$

where the function  $f, g$  are polynomials of degree 4, 6 in  $u$  which denotes the coordinate of  $\mathbf{P}^1$ . (1.1) describes the structure of an elliptic fibration over the base  $\mathbf{P}^1$ , i.e. the rational elliptic surface. The parameter  $\tau$  is identified as the modulus of the elliptic fiber  $E_\infty$  at  $u = \infty$ . Modulus  $\tilde{\tau}$  of the fiber at finite  $u$  depends on  $u$  and  $\tau, m_i$  and is identified as the inverse gauge coupling constant in the usual manner. See Figure 1.

Rational elliptic surface may be constructed by blowing up 9 points of  $\mathbf{P}^2$  and is also called as the (almost) del Pezzo surface  $\mathcal{B}_9$ . Its homology two-cycles form a lattice  $\Gamma^{9,1} = \Gamma_8 \oplus \Gamma^{1,1}$  where  $\Gamma_8$  denotes the  $E_8$  root lattice and  $\Gamma^{1,1}$  is a Lorentzian lattice. Thus  $b_2^+ = 1, b_2^- = 9$ . It is well-known that the rational elliptic surfaces appear in many ways like one half of the  $K_3$  surface, so that they are sometimes called as the  $\frac{1}{2}K_3$ . In fact when M5-brane of M-theory is wrapped over  $K_3$  surface, one obtains a heterotic string while when M5 is wrapped on  $\frac{1}{2}K_3$ , one obtains an  $E$ -string [12].

Using a chain of string dualities it has been pointed out that the BPS states of  $E$ -string are in one-to-one correspondence with the holomorphic curves in  $\frac{1}{2}K_3$  and also the instantons of  $\mathcal{N} = 4$  gauge theory on  $\frac{1}{2}K_3$  [12]. Let us denote the number of BPS states of  $E$ -string with winding number  $n$  and momentum  $k$  as  $N_{n,k}^{\text{BPS}}$  when it is compactified on a circle. Using the duality between F-theory and M-theory it is possible to show that  $N_{n,k}^{\text{BPS}}$

agrees with the number  $N_{n,k}^{\text{curve}}$  of holomorphic curves of  $\frac{1}{2}K_3$  in the class  $n[\Sigma]+k[E]$  where  $\Sigma$  denotes the base  $\mathbf{P}^1$  and  $E$  the elliptic fiber. Furthermore by considering the M5-brane wrapped around  $\frac{1}{2}K_3 \times T^2$  it was shown that  $N_{n,k}^{\text{BPS}}$  also agrees with the  $k$  instanton amplitude  $N_{n,k}^{\text{inst}}$  of the  $\mathcal{N} = 4$   $U(n)$  gauge theory on  $\frac{1}{2}K_3$ . Thus we have

$$N_{n,k}^{\text{BPS}} = N_{n,k}^{\text{curve}} = N_{n,k}^{\text{inst}} \equiv N_{n,k}. \tag{1.2}$$

Prepotential is then defined as usual [19] by

$$\mathcal{F}(\phi, \tau) = \mathcal{F}_{\text{classical}} - \frac{1}{(2\pi i)^3} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} N_{n,k} \text{Li}_3(e^{2\pi i n \phi + 2\pi i k \tau}) \tag{1.3}$$

where  $\text{Li}_3(x)$  is the tri-logarithm function  $\text{Li}_3(x) = \sum_{m=1}^{\infty} (x^m/m^3)$  and  $\phi$  denotes the size of the base  $\mathbf{P}^1$ . Due to the global  $E_8$  symmetry of the theory BPS states etc. of  $E$ -string fall into  $E_8$  Weyl orbits and thus  $N_{n,k}$  may be expanded as

$$N_{n,k} = \sum_{\mathcal{O}} \dim(\mathcal{O}) N_{n,k}^{\mathcal{O}} \tag{1.4}$$

where  $\dim(\mathcal{O})$  denotes the dimension of the Weyl orbit  $\mathcal{O}$ . When we introduce the  $E_8$  Wilson line parameters  $m_i, i = 1, 2, \dots, 8$ , the prepotential is modified as

$$\mathcal{F}(\phi, \tau, \vec{m}) = \mathcal{F}_{\text{classical}} - \frac{1}{(2\pi i)^3} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\mathcal{O}} N_{n,k}^{\mathcal{O}} \sum_{\vec{\nu} \in \mathcal{O}} \text{Li}_3(e^{2\pi i n \phi + 2\pi i k \tau + i \vec{\nu} \cdot \vec{m}}). \tag{1.5}$$

Here  $\vec{\nu}$  runs over the weights on the Weyl orbit  $\mathcal{O}$ . Partition functions  $Z_n$  of  $U(n)$  gauge theories on  $\frac{1}{2}K_3$  are then defined by

$$\mathcal{F}(\phi, \tau, \vec{m}) = \mathcal{F}_{\text{classical}} - \frac{1}{(2\pi i)^3} \sum_{n=1}^{\infty} q^{n/2} Z_n(\vec{m}; \tau) e^{2\pi i n \phi}, \quad q = e^{2\pi i \tau}. \tag{1.6}$$

(An extra factor of  $q^{n/2}$  has been introduced in the right-hand side so that  $Z_n$  has a simpler modular property). Prepotential (1.5), (1.6) is also interpreted as the generating function for the number of  $E$ -string BPS states or the holomorphic curves of  $\frac{1}{2}K_3$ . Variable  $u$  and  $\phi$  are related to each other by a mirror-type transformation of Seiberg–Witten theory.

It is known that the instanton amplitudes  $Z_n(\vec{m}; \tau)$  are Jacobi forms and possess good modular properties except for the ‘‘anomaly’’: i.e. they contain contributions of  $E_2$ , the Eisenstein series with weight 2 which possesses an

anomalous transformation law. Dependence of  $Z_n$  on  $E_2$  is determined by the holomorphic anomaly equation [11]

$$\frac{\partial Z_n}{\partial E_2} = \frac{1}{24} \sum_{m=1}^{n-1} m(n-m) Z_m Z_{n-m}. \quad (1.7)$$

Holomorphic anomaly represents the contributions of reducible connections of gauge theory on  $\frac{1}{2}K_3$  which possesses  $b_2^+ = 1$ .

It is known that the amplitude  $Z_n$  also obeys the “gap” condition,

$$N_{n,k} = 0 \quad \text{for } k < n. \quad (1.8)$$

Namely, holomorphic curves in the class  $n[\Sigma] + k[E]$  exist only for  $k \geq n$  in  $\frac{1}{2}K_3$ . Amplitudes  $Z_n$  have been determined for lower values of  $n$  perturbatively in [12] by making use of the holomorphic anomaly equation (1.7) and gap conditions (1.8).

It is well-known that given a Seiberg–Witten curve one can compute the amplitudes  $Z_n$  and generate the instanton expansion (1.5) by following the standard steps of the Seiberg–Witten theory. In our previous paper we have taken an inverse procedure: we first computed  $Z_n$  up to sufficiently high-orders in  $n$  using holomorphic anomaly and gap condition and then used these data to determine the Seiberg–Witten curve  $\mathcal{C}$ . It turned out that functions  $f(u; \tau, \vec{m})$  and  $g(u; \tau, \vec{m})$  are expressed in terms of the characters of affine  $E_8$  algebra up to level 4 and 6 [1]. We have checked the consistency of our curve in various ways: it possesses the correct modular properties and in particular reproduces the known five- and four-dimensional curves in the degenerate limit  $R_6, R_5 \rightarrow 0$ .

In this paper we first study holomorphic sections of the rational elliptic surface by making use of our curve  $\mathcal{C}$ . In the mathematical literature [20] it is known that meromorphic sections of an elliptic surface form a lattice under a suitable definition for their addition and their inner product. In the case of a rational elliptic surface [21, 22], this lattice (Mordell–Weil lattice) coincides with the root lattice of  $E_8$ . In particular there exist 240 holomorphic sections corresponding to non-zero roots of  $E_8$ .

We first construct a holomorphic section explicitly which corresponds to a given root of  $E_8$  by making use of affine  $E_7$  characters. Then the rest of 240 sections can be simply obtained by the application of  $E_8$  Weyl transformations to this section.

We shall show that the Wilson line parameters  $\{m_i\}$  are identified as the coordinates of the points on the elliptic fiber  $E_\infty$  where holomorphic sections

intersect. This is in fact the geometrical significance of the parameters  $\{m_i\}$  suggested in the literature [8, 13]. We also present the calculation of sections of five-dimensional curves for the sake of illustrations.

In the latter half of this paper we investigate various specializations of our Seiberg–Witten curve. First we study the relation between sets of special values of Wilson line parameters  $\{m_i\}$  and the corresponding degenerations of elliptic fibration: for instance, we find an unbroken symmetry  $E_7 \oplus A_1$  when all the parameters  $m_i$  are set equal to  $\pi/2$  while we have an unbroken  $E_8$  symmetry when they are all set to zero  $m_i = 0$ . In general when one adjusts the parameters  $\{m_i\}$  so that they preserve a symmetry under a subgroup of  $E_8$ , elliptic fibration exhibits corresponding degenerations. Degenerate fibers form *ADE*-type singularities in Kodaira’s classification and correspond precisely to the unbroken symmetry group.

Partial specification of Wilson line parameters in our curve also produces interesting results. As we mentioned already, our curve contains a series of known Seiberg–Witten curves for five- and four-dimensional gauge theories and the latter are obtained by taking the degenerate limit of  $T^2$  and sending some of the mass parameters to  $\infty$ . In this way of reduction, however, we necessarily lose  $SL(2, \mathbf{Z})$  symmetry of  $T^2$  and can not naturally recover the  $SL(2, \mathbf{Z})$  symmetry of  $N_f = 4$  theory in four dimensions.

In this paper we propose new type of reductions to four dimensions without taking the degenerate limit of  $T^2$  so that the  $SL(2, \mathbf{Z})$  symmetry is left intact. In fact, for instance, by setting four of the parameters  $\{m_i\}$  to half-periods  $(0, \pi, \pi + \pi\tau, \pi\tau)$  we obtain the curve for the  $SU(2)$   $N_f = 4$  gauge theory directly from our six-dimensional curve. As a remarkable by-product, the  $SO(8)$  triality of the  $N_f = 4$  theory is derived from the  $SL(2, \mathbf{Z})$  symmetry of the six-dimensional curve in a natural manner.

The organization of this paper is as follows: In section 2 we recall the basic symmetry properties of our six-dimensional curve (we call it as the  $\widehat{E}_8$  curve) and its five-dimensional counterparts ( $E_n$  curves). In section 3 we derive some holomorphic sections for  $E_n$  curves for the sake of illustrations. In section 4 we present holomorphic sections for the  $\widehat{E}_8$  curve and identify the geometrical significance of the Wilson line parameters. In section 5 we present examples which exhibit the correspondence between special values of Wilson line parameters and patterns of degenerations of elliptic fibration. In section 6 we discuss a new reduction to four dimensions and show how to obtain the Seiberg–Witten curve for the  $SU(2)$   $N_f = 4$  gauge theory. We also present a reduction to four-dimensional Donagi–Witten theory [23]. Section 7 is devoted to discussions. There are three Appendices in this paper:

in Appendix A we present the explicit form of the  $\widehat{E}_8$  curve and in Appendix B we list those of the  $E_n$  curves together with their holomorphic sections. In Appendix C we also present a list of special configurations of Wilson line parameters which lead to various unbroken subgroups of  $E_8$ .

## 2. Basics of the $\widehat{E}_8$ Curve and $E_n$ Curves

A rational elliptic surface possesses the structure of an elliptic fibration over a base  $\mathbf{P}^1$ . It can be described as a family of elliptic curves in the Weierstrass form

$$y^2 = 4x^3 - (a_0u^4 + a_1u^3 + a_2u^2 + a_3u + a_4)x - (b_0u^6 + b_1u^5 + b_2u^4 + b_3u^3 + b_4u^2 + b_5u + b_6) \quad (2.1)$$

where  $u$  denotes the coordinate for the base  $\mathbf{P}^1$ . Curve (2.1) possesses 12 moduli out of which 2 may be eliminated by the scaling and shift of the variable  $u$  in agreement with  $h^{1,1} = 10$  for the rational elliptic surface. We choose a convention  $a_1 = 0$  in this paper. Actually we can also fix the degree of freedom of the rescaling  $(u, x, y) \rightarrow (Lu, L^2x, L^3y)$  so that we are left with nine parameters  $\tau, \{m_i, i = 1, 2, \dots, 8\}$  to parametrize the coefficient functions  $\{a_j, b_j\}$ . Their explicit form is given in Appendix A.

The functions  $a_j(m_i; \tau), b_j(m_i; \tau)$  are expressed in terms of the characters of the  $\widehat{E}_8$  Weyl orbits at level  $j$ . Thus the curve possesses manifest  $\widehat{E}_8$  Weyl group symmetry and thereby we call it  $\widehat{E}_8$  curve. Strictly speaking the curve exhibits three kinds of automorphisms: the  $E_8$  Weyl group symmetry, the double periodicity in  $\vec{m}$ , and the modular property in  $\tau$ .

First the curve is invariant under the following Weyl reflections

$$\begin{aligned} &\bullet m_i \leftrightarrow m_j && (i \neq j), \\ &\bullet m_i \leftrightarrow -m_j && (i \neq j), \\ &\bullet m_i \rightarrow m_i - \frac{1}{4} \sum_{j=1}^8 m_j && (i = 1, \dots, 8). \end{aligned} \quad (2.2)$$

The entire  $E_8$  Weyl group is generated by the combination of these operations. On the other hand the double periodicity in  $\vec{m}$  takes the following form: let  $\vec{\alpha}$  be any vector of the root lattice  $\Gamma_8$ . Then the  $\widehat{E}_8$  curve is invariant under

$$\bullet \vec{m} \rightarrow \vec{m} + 2\pi\vec{\alpha}, \quad (2.3)$$

$$\bullet \vec{m} \rightarrow \vec{m} + 2\pi\tau\vec{\alpha} \quad \text{with} \quad (u, x, y) \rightarrow (Lu, L^2x, L^3y) \quad \text{where} \quad L = e^{-i\vec{m} \cdot \vec{\alpha}} q^{-\frac{1}{2}|\vec{\alpha}|^2}. \quad (2.4)$$

Curve also remains invariant under the action of the  $SL(2, \mathbf{Z})$  symmetry generated by

- $\tau \rightarrow \tau + 1,$  (2.5)

- $\tau \rightarrow -\frac{1}{\tau}, \vec{m} \rightarrow \frac{\vec{m}}{\tau}$  with  $(u, x, y) \rightarrow (\tau^{-6}Lu, \tau^{-10}L^2x, \tau^{-15}L^3y)$   
where  $L = e^{\frac{i}{4\pi\tau}|\vec{m}|^2}.$  (2.6)

Next let us review the reduction of our  $\widehat{E}_8$  curve down to the five-dimensional situation. By taking the limit  $\text{Im}\tau \rightarrow \infty$  ( $q \rightarrow 0$ ) we obtain a curve whose coefficients are written in terms of characters of the finite-dimensional  $E_8$  algebra. We call it the  $E_8$  curve. There exists another ‘ $E_8$  curve’ in four-dimensional theory [17] which can be obtained by taking a further degeneration [1]. Properties of the latter  $E_8$  curve have been studied in [17, 18], so we concentrate on the former.

Let us introduce some notations associated with the  $E_8$  algebra. Let  $\vec{\Lambda}$  be some dominant weight and  $\vec{\mu}_1, \dots, \vec{\mu}_8$  be the fundamental weights of  $E_8$ . We then introduce a notation

$$\vec{\Lambda} = \begin{bmatrix} & & n_2 & & & & & \\ n_1 & n_3 & n_4 & n_5 & n_6 & n_7 & n_8 & \end{bmatrix} = \sum_{i=1}^8 n_i \vec{\mu}_i, \quad n_i \in \mathbf{Z}_{\geq 0} \quad (2.7)$$

where  $\{n_i\}$  denote Dynkin indices of  $\vec{\Lambda}$ . We fix a labeling of the fundamental weights by placing eight indices at eight nodes of the Dynkin diagram. Next we define a character for an irreducible representation  $\mathcal{R}$  of highest weight  $\vec{\Lambda}$  by

$$\chi_{\mathcal{R}}^{E_8}(m_i) = \chi_{[n_1 n_3 n_4 n_5 n_6 n_7 n_8]}(m_i) \equiv \sum_{\vec{\nu} \in \mathcal{R}} e^{i\vec{m} \cdot \vec{\nu}} \quad (2.8)$$

where  $\vec{\nu}$  runs over all weights of representation  $\mathcal{R}$ . The variable  $\vec{m}$  takes its values on the  $\mathbf{C}$ -extended root space. Then characters of fundamental representations are written as

$$\begin{aligned} \chi_1^{E_8} &= \chi_{[1000000]}, & \chi_2^{E_8} &= \chi_{[0000000]}, & \chi_3^{E_8} &= \chi_{[0100000]}, \\ \chi_4^{E_8} &= \chi_{[0010000]}, & \chi_5^{E_8} &= \chi_{[0001000]}, & \chi_6^{E_8} &= \chi_{[0000100]}, \\ \chi_7^{E_8} &= \chi_{[0000010]}, & \chi_8^{E_8} &= \chi_{[0000001]}. \end{aligned} \quad (2.9)$$

With this preparation we can now express the  $E_8$  curve in terms of these characters. The explicit form is given in Appendix B. We have applied a shift in  $x$  for the compactness of the expression. Weierstrass form can be immediately recovered by shifting  $x$  to cancel the  $x^2$  term.

The  $E_7$  curve is obtained from the  $E_8$  curve by decomposing the  $E_8$  characters into those of  $E_7 \times U(1)$  and factoring out the  $U(1)$  parts. The operation is carried out by the following limit

$$\begin{aligned} & (\chi_1^{E_8}, \chi_2^{E_8}, \chi_3^{E_8}, \chi_4^{E_8}, \chi_5^{E_8}, \chi_6^{E_8}, \chi_7^{E_8}, \chi_8^{E_8}) \\ & \rightarrow (L^2 \chi_1^{E_7}, L^3 \chi_2^{E_7}, L^4 \chi_3^{E_7}, L^6 \chi_4^{E_7}, L^5 \chi_5^{E_7}, L^4 \chi_6^{E_7}, L^3 \chi_7^{E_7}, L^2), \end{aligned} \quad (2.10)$$

$$(u, x, y) \rightarrow (Lu, L^2 x, L^3 y) \quad \text{with } L \rightarrow \infty \quad (2.11)$$

where  $\chi_1^{E_7}, \dots, \chi_7^{E_7}$  denote characters of fundamental representations of  $E_7$

$$\begin{aligned} \chi_1^{E_7} &= \chi[100000], & \chi_2^{E_7} &= \chi[000000], & \chi_3^{E_7} &= \chi[010000], & \chi_4^{E_7} &= \chi[001000], \\ \chi_5^{E_7} &= \chi[000100], & \chi_6^{E_7} &= \chi[000010], & \chi_7^{E_7} &= \chi[000001]. \end{aligned} \quad (2.12)$$

The  $E_6$  curve is obtained from the  $E_7$  curve in a similar manner as

$$\begin{aligned} & (\chi_1^{E_7}, \chi_2^{E_7}, \chi_3^{E_7}, \chi_4^{E_7}, \chi_5^{E_7}, \chi_6^{E_7}, \chi_7^{E_7}) \\ & \rightarrow (L^2 \chi_1^{E_6}, L^3 \chi_2^{E_6}, L^4 \chi_3^{E_6}, L^6 \chi_4^{E_6}, L^5 \chi_5^{E_6}, L^4 \chi_6^{E_6}, L^3), \end{aligned} \quad (2.13)$$

$$(u, x, y) \rightarrow (Lu, L^2 x, L^3 y) \quad \text{with } L \rightarrow \infty \quad (2.14)$$

where  $\chi_1^{E_6}, \dots, \chi_6^{E_6}$  denote characters of fundamental representations of  $E_6$

$$\begin{aligned} \chi_1^{E_6} &= \chi[10000], & \chi_2^{E_6} &= \chi[00000], & \chi_3^{E_6} &= \chi[01000], & \chi_4^{E_6} &= \chi[00100], \\ \chi_5^{E_6} &= \chi[00010], & \chi_6^{E_6} &= \chi[00001]. \end{aligned} \quad (2.15)$$

Lower  $E_n$  curves are obtained in a similar fashion.

Five-dimensional  $E_n$  curves exhibit  $E_n$  Weyl group symmetry and a single periodicity on the mass parameters  $\{m_i\}$ . In fact the curves remain invariant under

$$\vec{m} \rightarrow \vec{m} + 2\pi\vec{\alpha} \quad (2.16)$$

where  $\vec{\alpha}$  is any vector of  $E_n$  root lattice.

### 3. Holomorphic Sections for $E_n$ Curves

Let us begin with the case of  $E_6$  curve in order to illustrate our method. The  $E_6$  curve has the form

$$\begin{aligned} y^2 &= 4x^3 + \left(-u^2 + 4\chi_1^{E_6}\right)x^2 + \left((2\chi_2^{E_6} - 12)u + (4\chi_3^{E_6} - 4\chi_6^{E_6})\right)x \\ &\quad + 4u^3 + 4\chi_6^{E_6}u^2 + (4\chi_5^{E_6} - 4\chi_1^{E_6})u + (4\chi_4^{E_6} - \chi_2^{E_6}\chi_2^{E_6}). \end{aligned} \quad (3.1)$$

A holomorphic section is given by a pair of polynomials  $(x = x(u), y = y(u))$  in  $u$  since  $x, y$  have to be holomorphic on the entire  $u$ -plane. To cancel the



$u^2$  term in the coefficient of  $x^2$  in (3.1),  $x(u)$  must be at least linear in  $u$ . So we assume the following form for a section

$$\sigma_{27} \equiv \begin{cases} x(u) = x_1 u + x_2, \\ y(u) = y_1 u^2 + y_2 u + y_3. \end{cases} \quad (3.2)$$

By substituting this ansatz into the  $E_6$  curve (3.1), we obtain a polynomial of degree 4 in  $u$  vanishing identically. Coefficients of  $u^k$   $k = 0, \dots, 4$  provide five equations for five unknowns  $x_1, x_2, y_1, y_2, y_3$ . By eliminating  $y_1, y_2, y_3, x_2$  we obtain two sets of 27th-degree equations in  $x_1$ . Solutions of these equations give holomorphic sections corresponding to **27** and  $\overline{\mathbf{27}}$  representations of  $E_6$ .

To obtain the explicit expression of sections, we first note that every weight of **27** preserves the  $E_5(= D_5)$  Weyl group symmetry. So let us consider the  $E_6 \rightarrow E_5 \times U(1)$  decomposition. Fundamental characters branch as:

$$\chi_1^{E_6} = \chi_1^{E_5} \Lambda^2 + \chi_2^{E_5} \Lambda^{-1} + \Lambda^{-4}, \quad (3.3)$$

$$\chi_2^{E_6} = \chi_2^{E_5} \Lambda^3 + (\chi_3^{E_5} + 1) + \chi_5^{E_5} \Lambda^{-3}, \quad (3.4)$$

$$\chi_3^{E_6} = \chi_3^{E_5} \Lambda^4 + \chi_1^{E_5} \chi_2^{E_5} \Lambda + (\chi_1^{E_5} + \chi_4^{E_5}) \Lambda^{-2} + \chi_2^{E_5} \Lambda^{-5}, \quad (3.5)$$

$$\chi_4^{E_6} = \chi_4^{E_5} \Lambda^6 + \chi_2^{E_5} \chi_3^{E_5} \Lambda^3 + (\chi_1^{E_5} \chi_4^{E_5} + \chi_3^{E_5}) + \chi_3^{E_5} \chi_5^{E_5} \Lambda^{-3} + \chi_4^{E_5} \Lambda^{-6}, \quad (3.6)$$

$$\chi_5^{E_6} = \chi_5^{E_5} \Lambda^5 + (\chi_1^{E_5} + \chi_4^{E_5}) \Lambda^2 + \chi_1^{E_5} \chi_5^{E_5} \Lambda^{-1} + \chi_3^{E_5} \Lambda^{-4}, \quad (3.7)$$

$$\chi_6^{E_6} = \Lambda^4 + \chi_5^{E_5} \Lambda + \chi_1^{E_5} \Lambda^{-2} \quad (3.8)$$

where  $\chi_1^{E_5}, \dots, \chi_5^{E_5}$  denote the  $E_5$  fundamental characters

$$\begin{aligned} \chi_1^{E_5} &= \chi[1000], & \chi_2^{E_5} &= \chi[0001], & \chi_3^{E_5} &= \chi[0100], \\ \chi_4^{E_5} &= \chi[0010], & \chi_5^{E_5} &= \chi[0001]. \end{aligned} \quad (3.9)$$

$\Lambda \equiv e^{i\lambda}$  represents the  $U(1)$  character with unit charge where  $\lambda$  is the projection of  $\vec{m}$  in the direction orthogonal to  $E_5$ . (We have suitably normalized  $\lambda$  so that all the weights have integer  $U(1)$  charges.) (3.3)–(3.8) show the standard branching of  $E_6$  representations; for example, (3.3) represents the decomposition  $\mathbf{27} = \mathbf{10}_2 + \mathbf{16}_{-1} + \mathbf{1}_{-4}$ . With these decompositions it is not difficult to find the following solution for the simultaneous algebraic equations

$$x(u) = -\Lambda^4 u - (\Lambda^8 + \chi_1^{E_5} \Lambda^2 + \Lambda^{-4}), \quad (3.10)$$

$$\begin{aligned} y(u) &= i\Lambda^4 u^2 + i(3\Lambda^8 + \chi_1^{E_5} \Lambda^2 - \Lambda^{-4})u \\ &\quad + i(2\Lambda^{12} + 2\chi_1^{E_5} \Lambda^6 - \chi_2^{E_5} \Lambda^3 + \chi_3^{E_5} + 1 - \chi_5^{E_5} \Lambda^{-3}). \end{aligned} \quad (3.11)$$

Explicit dependence of  $\Lambda$  and  $\chi_j^{E_5}$  on the parameters  $\{m_i\}$  is fixed in **27** ways corresponding to each weight of the representation **27**. Thus the above formula compactly represents 27 holomorphic sections for  $E_6$  curve.

Also we can immediately obtain the sections corresponding to  $\overline{27}$  by reflecting the overall sign of the solution  $y(u)$ , i.e. right-hand side of (3.11).

Next let us consider sections of the form

$$\sigma_{72} \equiv \begin{cases} x(u) = x_0 u^2 + x_1 u + x_2, \\ y(u) = y_0 u^3 + y_1 u^2 + y_2 u + y_3. \end{cases} \quad (3.12)$$

By eliminating variables  $y_0, y_1, y_2, y_3, x_2, x_1$  one obtains a 36th-degree equation in  $x_0$  which represents 36 pairs of roots of the adjoint representation of  $E_6$ . In this case,  $A_5$  is the surviving symmetry orthogonal to a root of  $E_6$ , so we consider the following decomposition:

$$\chi_1^{E_6} = 2\chi_1^{A_5} \cos \lambda + \chi_4^{A_5}, \quad (3.13)$$

$$\chi_2^{E_6} = 2 \cos 2\lambda + 2\chi_3^{A_5} \cos \lambda + \chi_1^{A_5} \chi_5^{A_5}, \quad (3.14)$$

$$\chi_3^{E_6} = 2\chi_2^{A_5} \cos 2\lambda + 2\chi_1^{A_5} \chi_4^{A_5} \cos \lambda + (\chi_1^{A_5} \chi_1^{A_5} + \chi_3^{A_5} \chi_5^{A_5} - \chi_2^{A_5}), \quad (3.15)$$

$$\begin{aligned} \chi_4^{E_6} = & 2\chi_3^{A_5} \cos 3\lambda + 2\chi_2^{A_5} \chi_4^{A_5} \cos 2\lambda + 2\chi_1^{A_5} \chi_3^{A_5} \chi_5^{A_5} \cos \lambda \\ & + (\chi_1^{A_5} \chi_1^{A_5} \chi_4^{A_5} + \chi_2^{A_5} \chi_5^{A_5} \chi_5^{A_5} - \chi_1^{A_5} \chi_5^{A_5} + \chi_3^{A_5} \chi_3^{A_5} - 2\chi_2^{A_5} \chi_4^{A_5} + 1), \end{aligned} \quad (3.16)$$

$$\chi_5^{E_6} = 2\chi_4^{A_5} \cos 2\lambda + 2\chi_2^{A_5} \chi_5^{A_5} \cos \lambda + (\chi_1^{A_5} \chi_3^{A_5} + \chi_5^{A_5} \chi_5^{A_5} - \chi_4^{A_5}), \quad (3.17)$$

$$\chi_6^{E_6} = 2\chi_5^{A_5} \cos \lambda + \chi_2^{A_5} \quad (3.18)$$

where  $\chi_1^{A_5}, \dots, \chi_5^{A_5}$  denote  $A_5$  fundamental characters

$$\begin{aligned} \chi_1^{A_5} &= \chi_{[10000]}, & \chi_2^{A_5} &= \chi_{[01000]}, & \chi_3^{A_5} &= \chi_{[00100]}, \\ \chi_4^{A_5} &= \chi_{[00010]}, & \chi_5^{A_5} &= \chi_{[00001]}. \end{aligned} \quad (3.19)$$

$\lambda$  is the projection of  $\vec{m}$  orthogonal to  $A_5$ . Then we find the following solution:

$$x(u) = \frac{1}{4 \sin^2 \lambda} \left[ u^2 + (2\chi_5^{A_5} \cos \lambda) u - 4\chi_4^{A_5} \sin^2 \lambda + \chi_5^{A_5} \chi_5^{A_5} \right], \quad (3.20)$$

$$\begin{aligned} y(u) = & \frac{1}{4 \sin^3 \lambda} \times \\ & \left[ (\cos \lambda) u^3 + \left( \chi_5^{A_5} \cos 2\lambda + 2\chi_5^{A_5} \right) u^2 \right. \\ & + \left( \chi_4^{A_5} \cos 3\lambda - 2\chi_1^{A_5} \cos 2\lambda + (3\chi_5^{A_5} \chi_5^{A_5} - \chi_4^{A_5}) \cos \lambda + 2\chi_1^{A_5} \right) u \\ & - \cos 5\lambda + \chi_3^{A_5} \cos 4\lambda + (-\chi_1^{A_5} \chi_5^{A_5} + 3) \cos 3\lambda \\ & + (2\chi_4^{A_5} \chi_5^{A_5} - 4\chi_3^{A_5}) \cos 2\lambda + (\chi_1^{A_5} \chi_5^{A_5} - 2) \cos \lambda \\ & \left. + (\chi_5^{A_5} \chi_5^{A_5} \chi_5^{A_5} - 2\chi_4^{A_5} \chi_5^{A_5} + 3\chi_3^{A_5}) \right]. \end{aligned} \quad (3.21)$$

This expression represents 72 sections corresponding to the roots of  $E_6$ .

In Appendix B we present holomorphic sections for the  $E_7$  and  $E_8$  curve.

### 4. Holomorphic Sections for the $\widehat{E}_8$ Curve

In the case of  $\widehat{E}_8$  curve the derivation of holomorphic sections become much more complex than  $E_n$  curves. Thus instead of solving algebraic equations in the generic case of 8 parameters we first consider the simpler case of only 2 non-vanishing Wilson line parameters. We then recover the general case by making use of the modular invariance and  $\widehat{E}_7$  symmetry.

Let us consider sections of the form

$$\sigma_{240} \equiv \begin{cases} x(u) = x_0u^2 + x_1u + x_2, \\ y(u) = y_0u^3 + y_1u^2 + y_2u + y_3. \end{cases} \tag{4.1}$$

In order to maintain the modular invariance, coefficients  $x_k, y_k$  must transform as

$$x_k(\vec{m}; \tau + 1) = x_k(\vec{m}; \tau), \tag{4.2}$$

$$y_k(\vec{m}; \tau + 1) = y_k(\vec{m}; \tau) \tag{4.3}$$

and

$$x_k\left(\frac{\vec{m}}{\tau}; -\frac{1}{\tau}\right) = \tau^{2-6k} e^{\frac{ik}{4\pi\tau}|\vec{m}|^2} x_k(\vec{m}; \tau), \tag{4.4}$$

$$y_k\left(\frac{\vec{m}}{\tau}; -\frac{1}{\tau}\right) = \tau^{3-6k} e^{\frac{ik}{4\pi\tau}|\vec{m}|^2} y_k(\vec{m}; \tau). \tag{4.5}$$

We assume that each section preserves an  $E_7$  Weyl symmetry and can be expressed in terms of  $\widehat{E}_7$  characters. In the following  $\widehat{E}_7$  characters are presented as explicit functions of  $m_i$  and  $\tau$ . We introduce an orthogonal coordinate basis  $\{\vec{e}_j\}$   $j = 1, \dots, 8$  for the  $E_8$  root lattice so that

$$\begin{cases} \pm\vec{e}_i \pm \vec{e}_j & i \neq j, \\ \frac{1}{2}(\pm\vec{e}_1 \pm \vec{e}_2 \pm \dots \pm \vec{e}_8) & \text{with even number of +’s} \end{cases} \tag{4.6}$$

form the 240 roots as usual. Corresponding to each one of these roots there exists a holomorphic section which is invariant under  $E_7$  Weyl group orthogonal to the root. As an illustration we consider a root

$$\vec{e}_7 + \vec{e}_8 \tag{4.7}$$

and the  $E_7$  root lattice orthogonal to it.  $\widehat{E}_7$  has two Weyl orbits at level one

whose characters are expressed as

$$w_{\mathfrak{b}}^{\widehat{E}_7}(m_i; \tau) = \frac{1}{2} \left[ \left( \prod_{j=1}^6 \vartheta_1(m_j|\tau) + \prod_{j=1}^6 \vartheta_2(m_j|\tau) \right) \vartheta_2(2m_-|2\tau) \right. \\ \left. + \left( \prod_{j=1}^6 \vartheta_3(m_j|\tau) + \prod_{j=1}^6 \vartheta_4(m_j|\tau) \right) \vartheta_3(2m_-|2\tau) \right], \quad (4.8)$$

$$w_{\mathfrak{f}}^{\widehat{E}_7}(m_i; \tau) = \frac{1}{2} \left[ \left( -\prod_{j=1}^6 \vartheta_1(m_j|\tau) + \prod_{j=1}^6 \vartheta_2(m_j|\tau) \right) \vartheta_3(2m_-|2\tau) \right. \\ \left. + \left( \prod_{j=1}^6 \vartheta_3(m_j|\tau) - \prod_{j=1}^6 \vartheta_4(m_j|\tau) \right) \vartheta_2(2m_-|2\tau) \right] \quad (4.9)$$

where  $m_- = (m_7 - m_8)/2$ . Note that the level-one  $\widehat{E}_8$  Weyl orbit character

$$w_{\mathfrak{b}}^{\widehat{E}_8}(m_i; \tau) = P(m_i; \tau) = \frac{1}{2} \sum_{\ell=1}^4 \prod_{j=1}^8 \vartheta_{\ell}(m_j|\tau) \quad (4.10)$$

decomposes as

$$w_{\mathfrak{b}}^{\widehat{E}_8}(m_i; \tau) = w_{\mathfrak{b}}^{\widehat{A}_1}(m_+; \tau) w_{\mathfrak{b}}^{\widehat{E}_7}(m_i; \tau) + w_{\mathfrak{f}}^{\widehat{A}_1}(m_+; \tau) w_{\mathfrak{f}}^{\widehat{E}_7}(m_i; \tau) \quad (4.11)$$

where the branching functions are the level-one characters of  $\widehat{A}_1$

$$w_{\mathfrak{b}}^{\widehat{A}_1}(m_+; \tau) = \vartheta_3(2m_+|2\tau), \quad w_{\mathfrak{f}}^{\widehat{A}_1}(m_+; \tau) = \vartheta_2(2m_+|2\tau) \quad (4.12)$$

and  $m_+ = (m_7 + m_8)/2$ .

With the help of the transformation rules (4.2)–(4.5) and the  $\widehat{E}_7$  symmetry, we need not solve the simultaneous algebraic equations with the most generic Wilson line parameters. It turns out that the case of two non-zero parameters is enough to determine general sections completely. On the other hand, it is known that sections with two non-zero parameters can be relatively easily obtained [10]. By rewriting these sections in terms of an  $\widehat{E}_8$  invariant coordinate and recovering all the Wilson line parameters, we obtain holomorphic sections for the  $\widehat{E}_8$  curve. Holomorphic section corresponding to the root  $\vec{\mathfrak{e}}_7 + \vec{\mathfrak{e}}_8$  is given explicitly as

$$x_0 = \wp(2m_+), \quad (4.13)$$

$$x_1 = C_{\mathfrak{b}}(m_+, \tau) w_{\mathfrak{b}}^{\widehat{A}_1}(m_+; \tau) w_{\mathfrak{b}}^{\widehat{E}_7}(m_i; \tau) + C_{\mathfrak{f}}(m_+, \tau) w_{\mathfrak{f}}^{\widehat{A}_1}(m_+; \tau) w_{\mathfrak{f}}^{\widehat{E}_7}(m_i; \tau) \quad (4.14)$$

where

$$C_b(m_+, \tau) = -\frac{\vartheta_2^8(\vartheta_3^4 + \vartheta_4^4)}{4\eta^{24}E_4}\wp(2m_+) + \frac{\vartheta_2^8}{48\eta^{24}} + \frac{\vartheta_2(2m_+|2\tau)^4}{\eta^{18}\vartheta_1(2m_+|\tau)^2}, \quad (4.15)$$

$$C_f(m_+, \tau) = -\frac{\vartheta_2^8(\vartheta_3^4 + \vartheta_4^4)}{4\eta^{24}E_4}\wp(2m_+) + \frac{\vartheta_2^8}{48\eta^{24}} + \frac{\vartheta_3(2m_+|2\tau)^4}{\eta^{18}\vartheta_1(2m_+|\tau)^2}, \quad (4.16)$$

$$\begin{aligned} x_2 = & \frac{1}{2}D(m_+, \tau) \left( w_b^{\hat{A}_1}(2m_+; 2\tau) + w_f^{\hat{A}_1}(2m_+; 2\tau) \right) \\ & \left( w_b^{\hat{E}_7}(2m_i; 2\tau) + w_f^{\hat{E}_7}(2m_i; 2\tau) \right) \\ & + \frac{1}{2}D(m_+, \tau + 1) \left( w_b^{\hat{A}_1}(2m_+; 2\tau) - w_f^{\hat{A}_1}(2m_+; 2\tau) \right) \\ & \left( w_b^{\hat{E}_7}(2m_i; 2\tau) - w_f^{\hat{E}_7}(2m_i; 2\tau) \right) \\ & + \frac{\tau^{14}}{2^4}D\left(\frac{m_+}{\tau}, -\frac{1}{\tau}\right)w_b^{\hat{A}_1}\left(m_+; \frac{\tau}{2}\right)w_b^{\hat{E}_7}\left(m_i; \frac{\tau}{2}\right) \\ & + \frac{\tau^{14}}{2^4}D\left(\frac{m_+}{\tau}, -\frac{1}{\tau} + 1\right)w_f^{\hat{A}_1}\left(m_+; \frac{\tau}{2}\right)w_f^{\hat{E}_7}\left(m_i; \frac{\tau}{2}\right) \\ & + \frac{(\tau + 1)^{14}}{2^4}D\left(\frac{m_+}{\tau + 1}, -\frac{1}{\tau + 1}\right)w_b^{\hat{A}_1}\left(m_+; \frac{\tau + 1}{2}\right)w_b^{\hat{E}_7}\left(m_i; \frac{\tau + 1}{2}\right) \\ & + \frac{(\tau + 1)^{14}}{2^4}D\left(\frac{m_+}{\tau + 1}, -\frac{1}{\tau + 1} + 1\right)w_f^{\hat{A}_1}\left(m_+; \frac{\tau + 1}{2}\right)w_f^{\hat{E}_7}\left(m_i; \frac{\tau + 1}{2}\right) \end{aligned} \quad (4.17)$$

where

$$D(m_+, \tau) = \frac{\vartheta_3^4\vartheta_4^4}{\eta^{36}E_4^2} \left[ (\vartheta_3^{12} - 28\eta^{12})\wp(2m_+) + \frac{1}{12}E_4(2\vartheta_2^8 + \vartheta_3^8) \right] \quad (4.18)$$

Coefficient functions  $y_k$  can be expressed in terms of the above  $\{x_k\}$  and coefficients  $\{a_k, b_k\}$  of the  $\hat{E}_8$  curve as

$$y_0 = \wp'(2m_+), \quad (4.19)$$

$$y_1 = \frac{1}{y_0} \left( 6x_1x_0^2 - \frac{1}{2}a_0x_1 - \frac{1}{2}b_1 \right), \quad (4.20)$$

$$y_2 = \frac{1}{y_0} \left( -\frac{1}{2}y_1^2 + 6x_2x_0^2 + 6x_1^2x_0 - \frac{1}{2}a_0x_2 - \frac{1}{2}a_2x_0 - \frac{1}{2}b_2 \right), \quad (4.21)$$

$$y_3 = \frac{1}{y_0} \left( -y_2y_1 + 12x_2x_1x_0 + 2x_1^3 - \frac{1}{2}a_2x_1 - \frac{1}{2}a_3x_0 - \frac{1}{2}b_3 \right). \quad (4.22)$$

Here  $\wp(z)$  is the Weierstrass  $\wp$ -function

$$\wp(z) = \frac{1}{z^2} + \sum_{m,n \in \mathbf{Z}^2 \setminus (0,0)} \left[ \frac{1}{(z - \Omega_{m,n})^2} - \frac{1}{\Omega_{m,n}^2} \right], \quad \Omega_{m,n} = 2\pi(m + n\tau) \quad (4.23)$$

and  $\wp'(z) = \frac{\partial}{\partial z} \wp(z)$ . We have verified the above formulas by substituting them into the  $\widehat{E}_8$  curve and checking the equation order by order in  $q$  for a sufficiently high degree.

All the other holomorphic sections corresponding to 240 roots of  $E_8$  are immediately obtained by the application of  $E_8$  Weyl transformations. Moreover, all the meromorphic sections may also be generated from the holomorphic ones if one uses the addition law of Mordell–Weil lattice. The lattice is constructed by the following addition rule and identification of the zero element: Given two sections  $S^{(i)} = (x^{(i)}(u), y^{(i)}(u))$   $i = 1, 2$  of an elliptic fibration  $y^2 = 4x^3 - f(u)x - g(u)$  we can construct a 3rd section  $S^{(3)} = (x^{(3)}(u), y^{(3)}(u))$  by

$$x^{(3)} = -x^{(1)} - x^{(2)} + \frac{1}{4} \left( \frac{y^{(2)} - y^{(1)}}{x^{(2)} - x^{(1)}} \right)^2, \quad (4.24)$$

$$y^{(3)} = \frac{(x^{(3)} - x^{(2)})y^{(1)} - (x^{(3)} - x^{(1)})y^{(2)}}{x^{(2)} - x^{(1)}}. \quad (4.25)$$

This defines an addition law  $S^{(3)} = S^{(1)} + S^{(2)}$  of the Mordell–Weil lattice. The zero element of the lattice is defined by the zero section  $(x(u) = \infty, y(u) = \infty)$  and the inverse of a section  $(x(u), y(u))$  is given by  $(x(u), -y(u))$ .

For example, let us denote the section (4.1) with coefficients (4.13)–(4.22) as  $S_{\vec{e}_7 + \vec{e}_8}$  and construct sections  $S_{\vec{e}_6 - \vec{e}_7}$ ,  $S_{\vec{e}_6 + \vec{e}_8}$  by Weyl reflections. Then one can verify that  $S_{\vec{e}_7 + \vec{e}_8}$  and  $S_{\vec{e}_6 - \vec{e}_7}$  actually add up to  $S_{\vec{e}_6 + \vec{e}_8}$  by the above addition law. Meromorphic sections correspond to weights with length-squared greater than 2 and can be generated by repeated use of the addition law.

The explicit form of holomorphic sections tells us the geometrical meaning of the Wilson line parameters. We first note that if we divide the  $\widehat{E}_8$  curve by  $u^6$  and take the limit  $u \rightarrow \infty$ , it is reduced to

$$\left( \frac{y}{u^3} \right)^2 = 4 \left( \frac{x}{u^2} \right)^3 - a_0 \left( \frac{x}{u^2} \right) - b_0 = 4 \left( \frac{x}{u^2} \right)^3 - \frac{E_4(\tau)}{12} \left( \frac{x}{u^2} \right) - \frac{E_6(\tau)}{216}. \quad (4.26)$$

This describes the fiber  $E_\infty$  at  $u = \infty$  with modulus  $\tau$ . Then the leading coefficients  $(x_0, y_0)$  of the section  $(x(u) = x_0 u^2 + \dots, y(u) = y_0 u^3 + \dots)$  are

identified as the coordinates on  $E_\infty$

$$y_0^2 = 4x_0^3 - \frac{E_4(\tau)}{12}x_0 - \frac{E_6(\tau)}{216}. \tag{4.27}$$

As we see in (4.13), (4.19),  $(x_0, y_0)$  are in fact expressed by the Weierstrass function and its derivative.

We note that the argument of  $\wp$ -functions in (4.13), (4.19) can be written as

$$2m_+ = m_7 + m_8 = (\vec{\mathbf{e}}_7 + \vec{\mathbf{e}}_8) \cdot \vec{\mathbf{m}}. \tag{4.28}$$

In general we find that a section corresponding to a root  $\vec{\alpha}$  intersects the fiber  $E_\infty$  at  $z = \vec{\alpha} \cdot \vec{\mathbf{m}}$

$$(x_0, y_0) = (\wp(\vec{\alpha} \cdot \vec{\mathbf{m}}), \wp'(\vec{\alpha} \cdot \vec{\mathbf{m}})). \tag{4.29}$$

Therefore the Wilson line parameters are identified as the coordinates on the fiber at  $u = \infty$ . See Figure 1. This is in fact the identification suggested in the literature [8, 13]. In our previous work [1] the precise dependence of our Seiberg–Witten curve on the parameters  $\{m_i\}$  has been determined so that they reproduce the correct mass dependence of the lower-dimensional curves after reduction. We have now confirmed that they also possess the correct geometrical significance in six dimensions.

### 5. Adjusting Parameters to Special Values

Let us now investigate various specializations of our Seiberg–Witten curve. We study the relation between sets of special values of Wilson line parameters  $\{m_i\}$  and the corresponding degenerations of elliptic fibration. In general when one adjusts the parameters  $\{m_i\}$  so that they preserve a symmetry under a subgroup of  $E_8$ , elliptic fibration exhibits corresponding degeneration. Degenerate fibers form *ADE*-type singularities in Kodaira’s classification and correspond precisely to the unbroken symmetry group. In this section we will demonstrate this correspondence in various cases.

As is well-known, we can identify the types of singularities by the behaviors of the functions  $f, g$  and the discriminant

$$y^2 = 4x^3 - f(u)x - g(u), \quad \Delta = f^3 - 27g^2 \tag{5.1}$$

near the singularity as shown in Table 1.

In subsequent discussions the patterns of degenerations of  $\widehat{E}_8$  curve are totally independent of the value of  $\tau$ . So we fix  $\tau$  to  $i\infty$  and consider the five-dimensional  $E_8$  curve.

Table 1: Singular elliptic fibers

order( $f$ )	order( $g$ )	order( $\Delta$ )	fiber-type	singularity-type
$\geq 0$	$\geq 0$	0	smooth	—
0	0	$n$	$I_n$	$A_{n-1}$
$\geq 1$	1	2	$II$	—
1	$\geq 2$	3	$III$	$A_1$
$\geq 2$	2	4	$IV$	$A_2$
2	$\geq 3$	$n+6$	$I_n^*$	$D_{n+4}$
$\geq 2$	3	$n+6$	$I_n^*$	$D_{n+4}$
$\geq 3$	4	8	$IV^*$	$E_6$
3	$\geq 5$	9	$III^*$	$E_7$
$\geq 4$	5	10	$II^*$	$E_8$

- $E_8$

When we set all  $\{m_i\}$  to 0, the whole  $E_8$  Weyl group symmetry is preserved. In this limit, each character becomes equal to the dimension of the corresponding representation

$$\begin{aligned} \chi_1 &= 3875, & \chi_2 &= 147250, & \chi_3 &= 6696000, & \chi_4 &= 6899079264, \\ \chi_5 &= 146325270, & \chi_6 &= 2450240, & \chi_7 &= 30380, & \chi_8 &= 248. \end{aligned} \quad (5.2)$$

By substituting these into the  $E_8$  curve, one obtains

$$y^2 = 4x^3 - u^2x^2 + 4u^5. \quad (5.3)$$

$f(u), g(u)$  of the Weierstrass form and its discriminant are given immediately as

$$f = \frac{1}{12}u^4, \quad (5.4)$$

$$g = \frac{1}{216}u^5(u - 864), \quad (5.5)$$

$$\Delta = u^{10}(u - 432). \quad (5.6)$$

According to Table 1, one in fact finds the degeneration of type  $II^*$  or  $E_8$  at  $u = 0$ .

- $D_8$

As is well-known,  $E_8$  representations contain two types of weights, those with integer components and half-integer components. They form independent  $D_8$  representations. If we set the parameters at  $\vec{m} = (0, 0, 0, 0, 0, 0, 0, 2\pi)$ , every integer weight contributes +1 while half-integer weight contributes  $-1$  to the character. Then the “twisted” characters take values

$$\begin{aligned} \chi_1 &= 35, & \chi_2 &= 50, & \chi_3 &= -960, & \chi_4 &= -41888, \\ \chi_5 &= 3094, & \chi_6 &= 832, & \chi_7 &= -84, & \chi_8 &= -8. \end{aligned} \quad (5.7)$$



Substituting these into the  $E_8$  curve, we obtain

$$f = \frac{1}{12}(u - 64)^2(u^2 + 128u - 8192), \tag{5.8}$$

$$g = \frac{1}{216}(u - 64)^3(u - 32)(u^2 - 640u + 28672), \tag{5.9}$$

$$\Delta = (u - 64)^{10}(u - 48). \tag{5.10}$$

One finds a  $I_4^*$  or  $D_8$  singularity at  $u = 64$ .

•  $E_7 \oplus A_1$

Let us next set all  $m_i$  to  $\pi/2$ . Then we obtain

$$f = \frac{1}{12}(u - 48)^3(u + 144), \tag{5.11}$$

$$g = \frac{1}{216}(u - 48)^5(u - 624), \tag{5.12}$$

$$\Delta = (u - 48)^9(u - 112)^2. \tag{5.13}$$

There exists a singularity of type  $III^*$  at  $u = 48$  and one of type  $I_2$  at  $u = 112$ . This corresponds to the unbroken symmetry  $E_7 \oplus A_1$ . In general when we set all  $\{m_i\}$  equal to a common value  $\mu$  and  $\mu$  is generic, one finds an  $E_7$  symmetry. We have  $E_7 \oplus A_1$  symmetry at  $\mu = (n + 1/2)\pi$  and  $E_8$  symmetry at  $\mu = n\pi$ .

$D_8$  and  $E_7 \oplus A_1$  are maximal regular subalgebras of  $E_8$ . Examples of other maximal subalgebras are listed in Appendix C. It is shown that when the Wilson line parameters are proportional to the fundamental weights  $\vec{\mu}_j$  of  $E_8$

$$\vec{m} \propto \vec{\mu}_j, \quad j = 1, 2, \dots, 8 \tag{5.14}$$

there appears an unbroken symmetry group which is obtained from the  $E_8$  Dynkin diagram by deleting its  $j$ -th vertex.

Let us now reinstate the  $\tau$  dependence of the  $\widehat{E}_8$  curve and discuss a few interesting cases.

•  $D_4 \oplus D_4$

We set the parameters as

$$m_1 = m_2 = 0, \quad m_3 = m_4 = \pi, \quad m_5 = m_6 = \pi + \pi\tau, \quad m_7 = m_8 = \pi\tau. \tag{5.15}$$

Then the  $\widehat{E}_8$  curve becomes

$$y^2 = 4x^3 - \frac{E_4}{12} \left(u^2 - \frac{4}{q\eta^{24}}\right)^2 x - \frac{E_6}{216} \left(u^2 - \frac{4}{q\eta^{24}}\right)^3. \tag{5.16}$$

Discriminant is given by

$$\Delta = \eta^{24} \left( u - \frac{2}{q^{1/2}\eta^{12}} \right)^6 \left( u + \frac{2}{q^{1/2}\eta^{12}} \right)^6. \quad (5.17)$$

We find two  $D_4$  singularities at  $u = \pm 2/(q^{1/2}\eta^{12})$ . The J-invariant  $J = 1728f^3/\Delta$  of the curve is  $u$ -independent and this describes a half of the orbifold  $T^4/\mathcal{I}_4$  discussed in [24].

We note the following interesting similarities among specializations of the  $\widehat{E}_8$  curve:

$$(a) \quad m_1 = \cdots = m_8 = 0$$

$$y^2 = 4x^3 - \frac{1}{12}E_4u^4x - \frac{1}{216}E_6u^6 + 4u^5, \quad (5.18)$$

$$(b) \quad m_1 = m_2 = m_3 = m_4 = m_5 = 0, \quad m_6 = \pi, \quad m_7 = \pi + \pi\tau, \quad m_8 = \pi\tau$$

$$y^2 = 4x^3 - \frac{1}{12}E_4u^4x - \frac{1}{216}E_6u^6 - \frac{4}{q^{1/2}\eta^{12}}u^4, \quad (5.19)$$

$$(c) \quad \{m_j\} = \left\{ \frac{2\pi}{3}(k + \ell\tau) \right\}, \quad k, \ell = -1, 0, 1 \text{ except } (k, \ell) = (0, 0)$$

$$y^2 = 4x^3 - \frac{1}{12}E_4u^4x - \frac{1}{216}E_6u^6 + \frac{4}{q\eta^{24}}u^3. \quad (5.20)$$

J-invariant  $J = 1728f^3/(f^3 - 27g^2)$  of these curves can be written commonly as

$$J = \frac{E_4^3v^2}{(\eta^{24}v^2 + E_6v - 432)} \quad (5.21)$$

with

$$(a) \quad v = u, \quad (b) \quad v = -q^{1/2}\eta^{12}u^2, \quad (c) \quad v = q\eta^{24}u^3. \quad (5.22)$$

## 6. Partial Fixing of Parameters

### 6.1. Extraction of the 4-dim $SU(2) N_f = 4$ Curve and $SO(8)$ Triality

Now let us discuss a new type of reduction of our six-dimensional curve to four-dimensional ones which maintains the manifest  $SL(2, \mathbf{Z})$  symmetry. We claim that when we set four of the Wilson line parameters at half-periods

$$m_5 = 0, \quad m_6 = \pi, \quad m_7 = \pi + \pi\tau, \quad m_8 = \pi\tau \quad (6.1)$$

we recover the four-dimensional  $SU(2)$  Seiberg–Witten curve with  $N_f = 4$  flavors.

Let us first recall the  $N_f = 4$  curve [15], whose explicit form is given by

$$\tilde{y}^2 = 4[W_1W_2W_3 + A(W_1T_1(e_2 - e_3) + W_2T_2(e_3 - e_1) + W_3T_3(e_1 - e_2)) - A^2N] \tag{6.2}$$

with

$$W_i = \tilde{x} - e_i\tilde{u} - e_i^2R, \tag{6.3}$$

$$A = (e_1 - e_2)(e_2 - e_3)(e_3 - e_1), \tag{6.4}$$

$$R = \frac{1}{2} \sum_i M_i^2, \tag{6.5}$$

$$T_1 = \frac{1}{12} \sum_{i>j} M_i^2M_j^2 - \frac{1}{24} \sum_i M_i^4, \tag{6.6}$$

$$T_2 = -\frac{1}{2} \prod_i M_i - \frac{1}{24} \sum_{i>j} M_i^2M_j^2 + \frac{1}{48} \sum_i M_i^4, \tag{6.7}$$

$$T_3 = \frac{1}{2} \prod_i M_i - \frac{1}{24} \sum_{i>j} M_i^2M_j^2 + \frac{1}{48} \sum_i M_i^4, \tag{6.8}$$

$$N = \frac{3}{16} \sum_{i>j>k} M_i^2M_j^2M_k^2 - \frac{1}{96} \sum_{i\neq j} M_i^2M_j^4 + \frac{1}{96} \sum_i M_i^6 \tag{6.9}$$

where  $M_1, \dots, M_4$  denote bare masses of matter hypermultiplets and

$$e_1 = \frac{1}{12}(\vartheta_3^4 + \vartheta_4^4), \quad e_2 = \frac{1}{12}(\vartheta_2^4 - \vartheta_4^4), \quad e_3 = \frac{1}{12}(-\vartheta_2^4 - \vartheta_3^4). \tag{6.10}$$

We note that when four of the Wilson line parameters  $\{m_i\}$  are set at zeros of the theta-function (6.1), all the odd coefficients  $a_3, b_1, b_3, b_5$  vanish identically in our curve. This is due to the structure of Hecke-type transformation in the coefficient functions  $\{a_i, b_i\}$ . See Appendix A.

Then if we redefine  $u^2$  as  $u$ , the  $\widehat{E}_8$  curve becomes

$$y^2 = 4x^3 - (a_0u^2 + a_2u + a_4)x - (b_0u^3 + b_2u^2 + b_4u + b_6). \tag{6.11}$$

By comparing (6.2) with (6.11) one finds that these curves are in fact the same if one makes the following identifications of four-dimensional masses

with Wilson line parameters

$$M_1 = \left( \prod_{j=1}^4 \vartheta_1(m_j|\tau) - \prod_{j=1}^4 \vartheta_2(m_j|\tau) \right) \Big/ \prod_{j=1}^4 \vartheta_1(m_j|\tau), \quad (6.12)$$

$$M_2 = \left( \prod_{j=1}^4 \vartheta_1(m_j|\tau) + \prod_{j=1}^4 \vartheta_2(m_j|\tau) \right) \Big/ \prod_{j=1}^4 \vartheta_1(m_j|\tau), \quad (6.13)$$

$$M_3 = \left( \prod_{j=1}^4 \vartheta_3(m_j|\tau) - \prod_{j=1}^4 \vartheta_4(m_j|\tau) \right) \Big/ \prod_{j=1}^4 \vartheta_1(m_j|\tau), \quad (6.14)$$

$$M_4 = \left( \prod_{j=1}^4 \vartheta_3(m_j|\tau) + \prod_{j=1}^4 \vartheta_4(m_j|\tau) \right) \Big/ \prod_{j=1}^4 \vartheta_1(m_j|\tau). \quad (6.15)$$

(Common denominator of  $\{M_i\}$  can be chosen arbitrarily. We have fixed it to  $\prod_j \vartheta_1(m_j|\tau)$  so that transformation laws of  $\{M_i\}$  fit with the convention of [15].)  $x, y, u$  of the curve (6.11) and  $\tilde{x}, \tilde{y}, \tilde{u}$  of the  $N_f = 4$  curve (6.2) are related as

$$u = L^2 \tilde{u} - \frac{1}{24\eta^{24}} \left( \frac{1}{12} E_4^2 a_2 - E_6 b_2 \right), \quad (6.16)$$

$$x = L^2 \left( \tilde{x} - \frac{1}{144} E_4 \sum_{i=1}^4 M_i^2 \right), \quad (6.17)$$

$$y = L^3 \tilde{y} \quad (6.18)$$

with

$$L = \frac{1}{q^{1/4} \eta^{18}} \prod_{j=1}^4 \vartheta_1(m_j|\tau). \quad (6.19)$$

The parameter  $\tau$  was interpreted in the *E*-string theory as the modulus of the torus  $T^2$  of fifth and sixth dimensions. Now we identify it with the bare coupling of four-dimensional gauge theory. We have derived the  $SL(2, \mathbf{Z})$  symmetry of four-dimensional  $N_f = 4$  theory from the geometry of six dimensions. This relationship has in fact been suggested before [8].

The structure of four-dimensional masses  $\{M_j\}$  (6.12)–(6.15) is quite interesting; they are invariant under the subgroup  $\Gamma(2)$  of the modular group which consists of matrices of the form  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{2}$  with  $ad - bc = 1$ . On the other hand they do transform under  $S$  and  $T$  transformations of the modular group. In fact the quotient  $SL(2, \mathbf{Z})/\Gamma(2)$  equals

the symmetric group  $\mathbf{S}_3 = \{I, S, T, ST, TST, T^{-1}S\}$ .  $\{M_j\}$  transform under the action of  $\mathbf{S}_3$  as

$\tau \rightarrow \tau + 1$  :

$$M_1 \rightarrow M_1, \tag{6.20}$$

$$M_2 \rightarrow M_2, \tag{6.21}$$

$$M_3 \rightarrow M_3, \tag{6.22}$$

$$M_4 \rightarrow -M_4, \tag{6.23}$$

$\tau \rightarrow -\frac{1}{\tau}$  :

$$M_1 \rightarrow \frac{1}{2}(M_1 + M_2 + M_3 - M_4), \tag{6.24}$$

$$M_2 \rightarrow \frac{1}{2}(M_1 + M_2 - M_3 + M_4), \tag{6.25}$$

$$M_3 \rightarrow \frac{1}{2}(M_1 - M_2 + M_3 + M_4), \tag{6.26}$$

$$M_4 \rightarrow \frac{1}{2}(-M_1 + M_2 + M_3 + M_4). \tag{6.27}$$

This is exactly the action of  $SO(8)$  triality transformation of the  $N_f = 4$  theory proposed in [15]. Triality has been postulated for the consistency of physical interpretation of four-dimensional gauge theory. Here it has been derived naturally from a six-dimensional setting.

Note that the four-dimensional masses  $M_j(m_i; \tau)$  are proportional to characters of  $\widehat{D}_4$  Weyl orbits at level one

$$w_{\mathbf{b}}^{\widehat{D}_4}(m_i; \tau) = \frac{1}{2} \left( \prod_{j=1}^4 \vartheta_3(m_j|\tau) + \prod_{j=1}^4 \vartheta_4(m_j|\tau) \right), \tag{6.28}$$

$$w_{\mathbf{v}}^{\widehat{D}_4}(m_i; \tau) = \frac{1}{2} \left( \prod_{j=1}^4 \vartheta_3(m_j|\tau) - \prod_{j=1}^4 \vartheta_4(m_j|\tau) \right), \tag{6.29}$$

$$w_{\mathbf{s}}^{\widehat{D}_4}(m_i; \tau) = \frac{1}{2} \left( \prod_{j=1}^4 \vartheta_2(m_j|\tau) + \prod_{j=1}^4 \vartheta_1(m_j|\tau) \right), \tag{6.30}$$

$$w_{\mathbf{c}}^{\widehat{D}_4}(m_i; \tau) = \frac{1}{2} \left( \prod_{j=1}^4 \vartheta_2(m_j|\tau) - \prod_{j=1}^4 \vartheta_1(m_j|\tau) \right) \tag{6.31}$$

where subscripts (b, v, s, c) refer to the (basic, vector, spinor, conjugate-spinor) representations of  $SO(8)$ , respectively.

## 6.2. Donagi–Witten Curve

Recall that if one sets the four masses  $M_i$  of the  $N_f = 4$  curve at

$$M_1 = M_2 = \frac{M}{2}, \quad M_3 = M_4 = 0, \quad (6.32)$$

one obtains the Seiberg–Witten curve for the  $SU(2)$  theory with an adjoint matter [15]

$$\tilde{y}^2 = 4 \prod_{k=1}^3 \left( \tilde{x} - e_k \tilde{u} - \frac{1}{4} e_k^2 M^2 \right). \quad (6.33)$$

This is in fact the curve discussed by Donagi and Witten [23]. To obtain this curve directly from the  $\widehat{E}_8$  curve, one may set

$$m_2 = 0, \quad m_3 = m_4 = \pi, \quad m_5 = m_6 = \pi + \pi\tau, \quad m_7 = m_8 = \pi\tau \quad (6.34)$$

and transform the variables as

$$u^2 = L^2 \left( \tilde{u} - \wp(m_1) \right), \quad (6.35)$$

$$x = L^2 \left( \tilde{x} - \frac{1}{72} E_4 \right), \quad (6.36)$$

$$y = L^3 \tilde{y} \quad (6.37)$$

with

$$L = \frac{2i}{q^{1/2} \eta^{15}} \vartheta_1(m_j | \tau). \quad (6.38)$$

Then one obtains a curve

$$\tilde{y}^2 = 4 \prod_{k=1}^3 \left( \tilde{x} - e_k \tilde{u} - e_k^2 \right), \quad (6.39)$$

which agrees with (6.33) after a rescaling.

A characteristic feature of the curve (6.33) is the factored form of the cubic in  $x$ . This form ensures that each singularity of the  $u$ -plane is at least doubled since the discriminant of a curve of the form

$$y^2 = 4(x - \alpha)(x - \beta)(x - \gamma) \quad (6.40)$$

is given by

$$\Delta = 16(\alpha - \beta)^2(\beta - \gamma)^2(\gamma - \alpha)^2. \quad (6.41)$$

A more general factorized curve may be obtained from the  $\widehat{E}_8$  curve. Let us leave  $m_1, m_2$  free and fix the others at

$$m_3 = m_4 = \pi, \quad m_5 = m_6 = \pi + \pi\tau, \quad m_7 = m_8 = \pi\tau. \quad (6.42)$$

Then one obtains a curve

$$y^2 = 4 \prod_{k=1}^3 \left( x - \left( x_0^{(k)} u^2 + x_1^{(k)} u + x_2^{(k)} \right) \right) \tag{6.43}$$

where

$$x_0^{(k)} = e_k, \tag{6.44}$$

$$x_1^{(1)} = \frac{2\vartheta_2^4 \vartheta_1(m_1|\tau) \vartheta_1(m_2|\tau)}{q^{1/2} \eta^6 E_4}, \tag{6.45}$$

$$x_1^{(2)} = \frac{2(-\vartheta_3^4) \vartheta_1(m_1|\tau) \vartheta_1(m_2|\tau)}{q^{1/2} \eta^6 E_4}, \tag{6.46}$$

$$x_1^{(3)} = \frac{2\vartheta_4^4 \vartheta_1(m_1|\tau) \vartheta_1(m_2|\tau)}{q^{1/2} \eta^6 E_4}, \tag{6.47}$$

$$x_2^{(k)} = \frac{\vartheta_1(m_1|\tau)^2 \vartheta_1(m_2|\tau)^2}{q \eta^{36}} \left[ -4e_k \wp(m_1) \wp(m_2) + \left( \frac{1}{18} E_4 - 4e_k^2 \right) \left( \wp(m_1) + \wp(m_2) \right) + E_6 \left( \frac{1}{108} - \frac{7}{6} \frac{e_k^2}{E_4} + 24 \frac{e_k^4}{E_4^2} \right) \right]. \tag{6.48}$$

This curve describes six  $I_2$  fibers and thus corresponds to  $A_1^{\oplus 6}$  symmetry.

### 7. Discussions

In this paper we have discussed various properties of the Seiberg–Witten curve for the  $E$ -string theory which we obtained previously. First we have studied its holomorphic sections and identified the geometrical meaning of the Wilson line parameters  $\{m_i\}$ . By fine tuning these parameters we have then studied various degenerations of the curve which correspond to several unbroken symmetry groups.

We have also presented a new way of reduction to four dimensions without taking the degenerate limit of  $T^2$  so that the  $SL(2, \mathbf{Z})$  symmetry is left intact. By setting some of the parameters  $\{m_i\}$  to special values we have obtained the four-dimensional Seiberg–Witten theory with  $N_f = 4$  flavors and also the curve by Donagi and Witten describing a perturbed  $\mathcal{N} = 4$  theory. In this reduction four-dimensional masses are expressed in terms of theta functions and possess exactly the proposed triality properties. Thus our

curve seems to serve successfully as a geometrical way of deriving  $SL(2, \mathbf{Z})$  symmetry from higher dimensions.

We should point out, however, we have redefined  $u^2$  as  $u$  in deriving four-dimensional curves in the new reduction. The geometrical significance of this change of variable is yet to be understood. We hope that we will be able to discuss this issue in a future publication.

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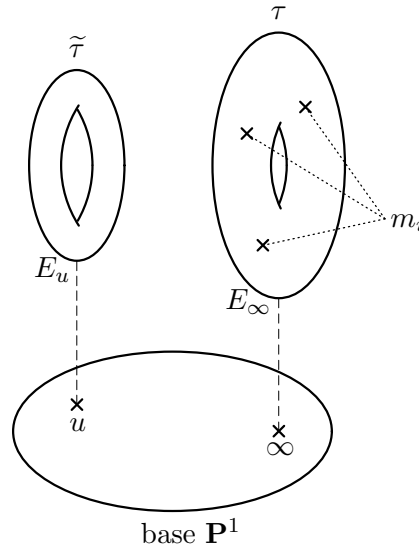


Figure 1: Elliptic fibration of rational elliptic surface

### Appendix A. $\widehat{E}_8$ Curve

Curve:

$$y^2 = 4x^3 - (a_0u^4 + a_1u^3 + a_2u^2 + a_3u + a_4)x - (b_0u^6 + b_1u^5 + b_2u^4 + b_3u^3 + b_4u^2 + b_5u + b_6). \quad (\text{A.1})$$



Coefficients:

$$a_0 = \frac{1}{12}E_4, \quad b_0 = \frac{1}{216}E_6, \quad a_1 = 0, \quad b_1 = -4\frac{P(m_i; \tau)}{E_4}, \quad (\text{A.2})$$

$$a_2 = \frac{1}{E_4\Delta} \left[ f_{a2,0}(\tau)P(2m_i; 2\tau) + f_{a2,1}(\tau)P\left(m_i; \frac{\tau}{2}\right) + f_{a2,1}(\tau + 1)P\left(m_i; \frac{\tau + 1}{2}\right) \right] \quad (\text{A.3})$$

where

$$f_{a2,0}(\tau) = \frac{2}{3}(E_4(\tau) - 9\vartheta_2(\tau)^8), \quad (\text{A.4})$$

$$f_{a2,1}(\tau) = \frac{1}{24}(E_4(\tau) - 9\vartheta_4(\tau)^8), \quad (\text{A.5})$$

$$b_2 = \frac{1}{E_4^2\Delta} \left[ f_{b2,0}(\tau)P(2m_i; 2\tau) + f_{b2,1}(\tau)P\left(m_i; \frac{\tau}{2}\right) + f_{b2,1}(\tau + 1)P\left(m_i; \frac{\tau + 1}{2}\right) \right] \quad (\text{A.6})$$

where

$$f_{b2,0}(\tau) = \frac{1}{36}(\vartheta_3(\tau)^4 + \vartheta_4(\tau)^4)(E_4(\tau)^2 + 60E_4(\tau)\vartheta_2(\tau)^8 - 45\vartheta_2(\tau)^{16}), \quad (\text{A.7})$$

$$f_{b2,1}(\tau) = -\frac{1}{576}(\vartheta_3(\tau)^4 + \vartheta_2(\tau)^4)(E_4(\tau)^2 + 60E_4(\tau)\vartheta_4(\tau)^8 - 45\vartheta_4(\tau)^{16}), \quad (\text{A.8})$$

$$a_3 = \frac{1}{E_4^2\Delta^2} \left[ f_{a3,0}(\tau)P(3m_i; 3\tau) + f_{a3,1}(\tau)P\left(m_i; \frac{\tau}{3}\right) + f_{a3,1}(\tau + 1)P\left(m_i; \frac{\tau + 1}{3}\right) + f_{a3,1}(\tau + 2)P\left(m_i; \frac{\tau + 2}{3}\right) + \frac{2}{3}E_6(\tau)P(m_i; \tau)^3 \right] \quad (\text{A.9})$$

where

$$f_{a3,0}(\tau) = \frac{1}{3}E_4(\tau)h_2(\tau)^2(7E_4(\tau) - 9h_0(\tau)^4), \quad (\text{A.10})$$

$$f_{a3,1}(\tau) = -\frac{1}{3^8}E_4(\tau)h_3\left(\frac{\tau}{3}\right)^2\left(7E_4(\tau) - h_0\left(\frac{\tau}{3}\right)^4\right), \quad (\text{A.11})$$

$$\begin{aligned}
b_3 = & \frac{1}{E_4^3 \Delta^2} \left[ f_{b3,0}(\tau) P(3m_i; 3\tau) + f_{b3,1}(\tau) P\left(m_i; \frac{\tau}{3}\right) \right. \\
& + f_{b3,1}(\tau+1) P\left(m_i; \frac{\tau+1}{3}\right) + f_{b3,1}(\tau+2) P\left(m_i; \frac{\tau+2}{3}\right) \\
& \left. + \frac{1}{54} (8E_4^3 - 5E_6^2) P(m_i; \tau)^3 \right] \tag{A.12}
\end{aligned}$$

where

$$f_{b3,0}(\tau) = \frac{1}{18} E_4(\tau)^2 h_2(\tau)^2 (32h_2(\tau)^2 + 48h_2(\tau)h_0(\tau)^3 - 81h_0(\tau)^6), \tag{A.13}$$

$$\begin{aligned}
f_{b3,1}(\tau) = & \frac{1}{2 \cdot 3^{12}} E_4(\tau)^2 h_3\left(\frac{\tau}{3}\right)^2 \left( 32h_3\left(\frac{\tau}{3}\right)^2 + 48h_3\left(\frac{\tau}{3}\right)h_0\left(\frac{\tau}{3}\right)^3 \right. \\
& \left. - 81h_0\left(\frac{\tau}{3}\right)^6 \right), \tag{A.14}
\end{aligned}$$

$$\begin{aligned}
a_4 = & \frac{1}{E_4 \Delta^2} \left[ 16\vartheta_4(2\tau)^8 P(4m_i; 4\tau) + \frac{1}{256} \vartheta_2\left(\frac{\tau}{2}\right)^8 P\left(m_i; \frac{\tau}{4}\right) \right. \\
& + \frac{1}{256} \vartheta_2\left(\frac{\tau+1}{2}\right)^8 P\left(m_i; \frac{\tau+1}{4}\right) + \frac{1}{256} \vartheta_2\left(\frac{\tau+2}{2}\right)^8 P\left(m_i; \frac{\tau+2}{4}\right) \\
& + \frac{1}{256} \vartheta_2\left(\frac{\tau+3}{2}\right)^8 P\left(m_i; \frac{\tau+3}{4}\right) + \vartheta_4(2\tau)^8 P\left(2m_i; \tau + \frac{1}{2}\right) \\
& \left. + E_4 P(2m_i; \tau) - \frac{3}{2} \Delta^2 a_2^2 + \frac{3}{8} E_4 \Delta a_2 b_1^2 - \frac{9}{128} E_4^2 b_1^4 \right], \tag{A.15}
\end{aligned}$$

$$\begin{aligned}
b_4 = & \frac{1}{E_4 \Delta^2} \left[ f_{b4,0}(\tau) P(4m_i; 4\tau) + f_{b4,1}(\tau) P\left(m_i; \frac{\tau}{4}\right) \right. \\
& + f_{b4,1}(\tau+1) P\left(m_i; \frac{\tau+1}{4}\right) + f_{b4,1}(\tau+2) P\left(m_i; \frac{\tau+2}{4}\right) \\
& + f_{b4,1}(\tau+3) P\left(m_i; \frac{\tau+3}{4}\right) + f_{b4,2}(\tau) P\left(2m_i; \tau + \frac{1}{2}\right) \\
& \left. + \frac{5}{48} E_6 P(2m_i; \tau) - \frac{1}{96} E_4 E_6 \Delta b_3 b_1 - 3E_4 \Delta b_2 b_1^2 - \frac{55}{384} E_4 E_6 b_1^4 \right] \tag{A.16}
\end{aligned}$$

where

$$f_{b4,0}(\tau) = -\frac{8}{9} \left( 32\vartheta_3(2\tau)^{12} - 75\vartheta_3(2\tau)^4 \vartheta_4(2\tau)^8 + 2\vartheta_4(2\tau)^{12} \right), \tag{A.17}$$

$$f_{b4,1}(\tau) = \frac{1}{2^{11} \cdot 9} \left( 32\vartheta_3\left(\frac{\tau}{2}\right)^{12} - 75\vartheta_3\left(\frac{\tau}{2}\right)^4 \vartheta_2\left(\frac{\tau}{2}\right)^8 + 2\vartheta_2\left(\frac{\tau}{2}\right)^{12} \right) \tag{A.18}$$

$$f_{b4,2}(\tau) = -\frac{1}{18} \left( 32\vartheta_2(2\tau)^{12} - 75\vartheta_2(2\tau)^4 \vartheta_4(2\tau)^8 - 2\vartheta_4(2\tau)^{12} \right), \tag{A.19}$$

$$\begin{aligned}
 b_5 = & \frac{1}{E_4^2 \Delta^3} \left[ f_{b5,0}(\tau)P(5m_i; 5\tau) + f_{b5,1}(\tau)P\left(m_i; \frac{\tau}{5}\right) + f_{b5,1}(\tau+1)P\left(m_i; \frac{\tau+1}{5}\right) \right. \\
 & + f_{b5,1}(\tau+2)P\left(m_i; \frac{\tau+2}{5}\right) + f_{b5,1}(\tau+3)P\left(m_i; \frac{\tau+3}{5}\right) \\
 & + f_{b5,1}(\tau+4)P\left(m_i; \frac{\tau+4}{5}\right) \\
 & + \left( g_{b5,0}(\tau)P(4m_i; 4\tau) + g_{b5,1}(\tau)P\left(m_i; \frac{\tau}{4}\right) \right. \\
 & + g_{b5,1}(\tau+1)P\left(m_i; \frac{\tau+1}{4}\right) + g_{b5,1}(\tau+2)P\left(m_i; \frac{\tau+2}{4}\right) \\
 & \left. + g_{b5,1}(\tau+3)P\left(m_i; \frac{\tau+3}{4}\right) + g_{b5,2}(\tau)P\left(2m_i; \tau + \frac{1}{2}\right) \right) P(m_i; \tau) \left. \right] \\
 & + \frac{1}{\Delta^3} \left[ \frac{3}{2} \Delta^2 b_3 b_1^2 + \frac{3}{16} E_6 \Delta b_2 b_1^3 + \frac{1}{9216} (53E_4^3 + 67E_6^2) b_1^5 \right] \tag{A.20}
 \end{aligned}$$

where

$$\begin{aligned}
 f_{b5,0}(\tau) = & \frac{2}{3} \eta(\tau)^{16} \eta(5\tau)^{16} \tag{A.21} \\
 & \times \left( 5^{11} + 4 \cdot 5^9 \frac{\eta(\tau)^6}{\eta(5\tau)^6} + 22 \cdot 5^6 \frac{\eta(\tau)^{12}}{\eta(5\tau)^{12}} + 4 \cdot 5^4 \frac{\eta(\tau)^{18}}{\eta(5\tau)^{18}} - 31 \frac{\eta(\tau)^{24}}{\eta(5\tau)^{24}} \right),
 \end{aligned}$$

$$\begin{aligned}
 f_{b5,1}(\tau) = & \frac{2}{3} \eta(\tau)^{16} \eta\left(\frac{\tau}{5}\right)^{16} \\
 & \times \left( \frac{1}{5} + 4 \frac{\eta(\tau)^6}{\eta\left(\frac{\tau}{5}\right)^6} + 22 \frac{\eta(\tau)^{12}}{\eta\left(\frac{\tau}{5}\right)^{12}} + 20 \frac{\eta(\tau)^{18}}{\eta\left(\frac{\tau}{5}\right)^{18}} - 31 \frac{\eta(\tau)^{24}}{\eta\left(\frac{\tau}{5}\right)^{24}} \right), \tag{A.22}
 \end{aligned}$$

$$\begin{aligned}
 g_{b5,0}(\tau) = & \frac{2}{3} \vartheta_3(2\tau)^4 \vartheta_4(2\tau)^8 \left( -1024 \vartheta_3(2\tau)^{12} + 1664 \vartheta_3(2\tau)^8 \vartheta_4(2\tau)^4 \right. \\
 & \left. - 644 \vartheta_3(2\tau)^4 \vartheta_4(2\tau)^8 + 55 \vartheta_4(2\tau)^{12} \right), \tag{A.23}
 \end{aligned}$$

$$\begin{aligned}
 g_{b5,1}(\tau) = & \frac{1}{2^{19} \cdot 3} \vartheta_3\left(\frac{\tau}{2}\right)^4 \vartheta_2\left(\frac{\tau}{2}\right)^8 \left( -1024 \vartheta_3\left(\frac{\tau}{2}\right)^{12} + 1664 \vartheta_3\left(\frac{\tau}{2}\right)^8 \vartheta_2\left(\frac{\tau}{2}\right)^4 \right. \\
 & \left. - 644 \vartheta_3\left(\frac{\tau}{2}\right)^4 \vartheta_2\left(\frac{\tau}{2}\right)^8 + 55 \vartheta_2\left(\frac{\tau}{2}\right)^{12} \right), \tag{A.24}
 \end{aligned}$$

$$\begin{aligned}
 g_{b5,2}(\tau) = & -\frac{1}{24} \vartheta_2(2\tau)^4 \vartheta_4(2\tau)^8 \left( 1024 \vartheta_2(2\tau)^{12} + 1664 \vartheta_2(2\tau)^8 \vartheta_4(2\tau)^4 \right. \\
 & \left. + 644 \vartheta_2(2\tau)^4 \vartheta_4(2\tau)^8 + 55 \vartheta_4(2\tau)^{12} \right), \tag{A.25}
 \end{aligned}$$

$$\begin{aligned}
b_6 = & \frac{1}{E_4\Delta^3} \left[ f_{b6,0}(\tau)P(6m_i; 6\tau) + f_{b6,1}(\tau)P\left(m_i; \frac{\tau}{6}\right) \right. \\
& + f_{b6,1}(\tau+1)P\left(m_i; \frac{\tau+1}{6}\right) + f_{b6,1}(\tau+2)P\left(m_i; \frac{\tau+2}{6}\right) \\
& + f_{b6,1}(\tau+3)P\left(m_i; \frac{\tau+3}{6}\right) + f_{b6,1}(\tau+4)P\left(m_i; \frac{\tau+4}{6}\right) \\
& + f_{b6,1}(\tau+5)P\left(m_i; \frac{\tau+5}{6}\right) + f_{b6,2}(\tau)P\left(3m_i; \frac{3\tau}{2}\right) \\
& + f_{b6,2}(\tau+1)P\left(3m_i; \frac{3\tau+1}{2}\right) + f_{b6,3}(\tau)P\left(2m_i; \frac{2\tau}{3}\right) \\
& \left. + f_{b6,3}(\tau+1)P\left(2m_i; \frac{2\tau+2}{3}\right) + f_{b6,3}(\tau+2)P\left(2m_i; \frac{2\tau+1}{3}\right) \right] \\
& + \frac{a_2}{E_4\Delta^2} \left[ g_{b6,0}(\tau)P(4m_i; 4\tau) + g_{b6,1}(\tau)P\left(m_i; \frac{\tau}{4}\right) \right. \\
& + g_{b6,1}(\tau+1)P\left(m_i; \frac{\tau+1}{4}\right) + g_{b6,1}(\tau+2)P\left(m_i; \frac{\tau+2}{4}\right) \\
& \left. + g_{b6,1}(\tau+3)P\left(m_i; \frac{\tau+3}{4}\right) + g_{b6,2}(\tau)P\left(2m_i; \tau + \frac{1}{2}\right) \right] \\
& + \frac{1}{\Delta^3} \left[ -\frac{83}{7344}E_6\Delta^2b_5b_1 + \frac{83}{408}\Delta^2b_4b_1^2 + \frac{1}{34}\Delta^2b_3b_2b_1 \right. \\
& + \frac{29}{6528}E_6\Delta b_3b_1^3 + \frac{669}{1088}\Delta b_2b_1^4 - \frac{5}{7344}E_4\Delta^2a_3a_2b_1 \\
& - \frac{419}{235008}E_4^2\Delta a_3b_1^3 - \frac{1}{36864}E_4E_6\Delta a_2^2b_1^2 + \frac{1}{1536}E_4^2\Delta a_2b_2b_1^2 \\
& \left. - \frac{1}{256}E_6\Delta b_2^2b_1^2 + \frac{1215}{69632}E_6b_1^6 \right] \tag{A.26}
\end{aligned}$$

where

$$f_{b6,0}(\tau) = -\frac{4}{17}\left(h_0(\tau) + h_0(2\tau)\right)\left(h_0(\tau) - 2h_0(2\tau)\right)^2 \tag{A.27}$$

$$\times \left(27h_0(\tau)^3 + 84h_0(\tau)^2h_0(2\tau) + 72h_0(\tau)h_0(2\tau)^2 - 32h_0(2\tau)^3\right),$$

$$f_{b6,1}(\tau) = \frac{1}{297432}\left(2h_0\left(\frac{\tau}{3}\right) + h_0\left(\frac{\tau}{6}\right)\right)\left(h_0\left(\frac{\tau}{3}\right) - h_0\left(\frac{\tau}{6}\right)\right)^2 \tag{A.28}$$

$$\times \left(27h_0\left(\frac{\tau}{3}\right)^3 + 42h_0\left(\frac{\tau}{3}\right)^2h_0\left(\frac{\tau}{6}\right) + 18h_0\left(\frac{\tau}{3}\right)h_0\left(\frac{\tau}{6}\right)^2 - 4h_0\left(\frac{\tau}{6}\right)^3\right),$$

$$f_{b6,2}(\tau) = \frac{1}{136}\left(-2h_0(\tau) + h_0\left(\frac{\tau}{2}\right)\right)\left(h_0(\tau) + h_0\left(\frac{\tau}{2}\right)\right)^2 \tag{A.29}$$

$$\times \left(27h_0(\tau)^3 - 42h_0(\tau)^2h_0\left(\frac{\tau}{2}\right) + 18h_0(\tau)h_0\left(\frac{\tau}{2}\right)^2 + 4h_0\left(\frac{\tau}{2}\right)^3\right)$$

$$f_{b6,3}(\tau) = \frac{4}{37179} \left( h_0\left(\frac{\tau}{3}\right) - h_0\left(\frac{2\tau}{3}\right) \right) \left( h_0\left(\frac{\tau}{3}\right) + 2h_0\left(\frac{2\tau}{3}\right) \right)^2 \tag{A.30}$$

$$\times \left( 27h_0\left(\frac{\tau}{3}\right)^3 - 84h_0\left(\frac{\tau}{3}\right)^2 h_0\left(\frac{2\tau}{3}\right) + 72h_0\left(\frac{\tau}{3}\right) h_0\left(\frac{2\tau}{3}\right)^2 + 32h_0\left(\frac{2\tau}{3}\right)^3 \right),$$

$$g_{b6,0}(\tau) = -\frac{640}{51}\vartheta_3(2\tau)^4 + \frac{32}{9}\vartheta_4(2\tau)^4, \tag{A.31}$$

$$g_{b6,1}(\tau) = \frac{5}{408}\vartheta_3\left(\frac{\tau}{2}\right)^4 - \frac{1}{288}\vartheta_2\left(\frac{\tau}{2}\right)^4, \tag{A.32}$$

$$g_{b6,2}(\tau) = -\frac{40}{51}\vartheta_2(2\tau)^4 - \frac{2}{9}\vartheta_4(2\tau)^4. \tag{A.33}$$

Here

$$P(m_i; \tau) = \frac{1}{2} \sum_{\ell=1}^4 \prod_{j=1}^8 \vartheta_\ell(m_j|\tau) \tag{A.34}$$

is the  $\widehat{E}_8$  Weyl orbit character at level one. Theta functions are defined as

$$\vartheta_1(z|\tau) = i \sum_{n \in \mathbf{Z}} (-1)^n y^{n-1/2} q^{(n-1/2)^2/2}, \tag{A.35}$$

$$\vartheta_2(z|\tau) = \sum_{n \in \mathbf{Z}} y^{n-1/2} q^{(n-1/2)^2/2}, \tag{A.36}$$

$$\vartheta_3(z|\tau) = \sum_{n \in \mathbf{Z}} y^n q^{n^2/2}, \tag{A.37}$$

$$\vartheta_4(z|\tau) = \sum_{n \in \mathbf{Z}} (-1)^n y^n q^{n^2/2} \tag{A.38}$$

where  $y = e^{iz}$ ,  $q = e^{2\pi i\tau}$  and we abbreviate  $\vartheta_\ell(0|\tau)$  to  $\vartheta_\ell(\tau)$  or just  $\vartheta_\ell$ .  $E_{2n}$  is the Eisenstein series with weight  $2n$ ,

$$E_{2n}(\tau) = 1 + \frac{(2\pi i)^{2n}}{(2n-1)! \zeta(2n)} \sum_{m=1}^{\infty} \frac{m^{2n-1} q^m}{1-q^m}, \tag{A.39}$$

and  $\Delta(\tau) \equiv \eta(\tau)^{24} = \frac{1}{1728} (E_4(\tau)^3 - E_6(\tau)^2)$ . Functions  $\{h_i\}$  are defined by

$$h_0(\tau) = \sum_{n_1, n_2 = -\infty}^{\infty} q^{n_1^2 + n_2^2 - n_1 n_2} = \vartheta_3(2\tau)\vartheta_3(6\tau) + \vartheta_2(2\tau)\vartheta_2(6\tau), \tag{A.40}$$

$$h_2(\tau) = \frac{\eta(\tau)^9}{\eta(3\tau)^3}, \quad h_3(\tau) = 27 \frac{\eta(3\tau)^9}{\eta(\tau)^3}. \tag{A.41}$$

## Appendix B. $E_n$ Curves and Holomorphic Sections

$E_8$  curve:

$$\begin{aligned}
y^2 = & 4x^3 \\
& + \left( -u^2 + 4\chi_1 - 100\chi_8 + 9300 \right) x^2 \\
& + \left( (2\chi_2 - 12\chi_7 - 70\chi_1 + 1840\chi_8 - 115010)u \right. \\
& \quad + (4\chi_3 - 4\chi_6 - 64\chi_1\chi_8 + 824\chi_8\chi_8 - 112\chi_2 \\
& \quad \quad \left. + 680\chi_7 + 8024\chi_1 - 205744\chi_8 + 9606776) \right) x \\
& + \left( 4u^5 \right. \\
& \quad + (4\chi_8 - 992)u^4 \\
& \quad + (4\chi_7 - 12\chi_1 - 680\chi_8 + 93620)u^3 \\
& \quad + (4\chi_6 - 8\chi_1\chi_8 + 92\chi_8\chi_8 - 28\chi_2 - 540\chi_7 + 2320\chi_1 + 30608\chi_8 \\
& \quad \quad - 3823912)u^2 + (4\chi_5 - 4\chi_1\chi_7 - 20\chi_2\chi_8 + 116\chi_7\chi_8 + 8\chi_1\chi_1 \\
& \quad \quad - 52\chi_3 - 416\chi_6 + 1436\chi_1\chi_8 - 17776\chi_8\chi_8 + 4180\chi_2 + 16580\chi_7 \\
& \quad \quad \left. - 182832\chi_1 + 1103956\chi_8 + 18130536)u \right. \\
& \quad + (4\chi_4 - \chi_2\chi_2 + 4\chi_1\chi_1\chi_8 - 40\chi_3\chi_8 + 36\chi_6\chi_8 + 248\chi_1\chi_8\chi_8 \\
& \quad \quad - 2232\chi_8\chi_8\chi_8 + 2\chi_1\chi_2 - 232\chi_5 + 224\chi_1\chi_7 + 1124\chi_2\chi_8 \\
& \quad \quad - 6580\chi_7\chi_8 - 457\chi_1\chi_1 + 4980\chi_3 + 8708\chi_6 - 88136\chi_1\chi_8 \\
& \quad \quad + 1129964\chi_8\chi_8 - 146282\chi_2 + 66612\chi_7 + 6123126\chi_1 \\
& \quad \quad \left. - 104097420\chi_8 + 2630318907) \right) \tag{B.1}
\end{aligned}$$

where  $\chi_1, \dots, \chi_8$  denote characters of  $E_8$  fundamental representations and are defined as

$$\begin{aligned}
\chi_1 &= \chi[1000000], & \chi_2 &= \chi[0001000], & \chi_3 &= \chi[0100000], \\
\chi_4 &= \chi[0010000], & \chi_5 &= \chi[0000100], & \chi_6 &= \chi[0000100], \\
\chi_7 &= \chi[0000010], & \chi_8 &= \chi[0000001].
\end{aligned} \tag{B.2}$$

240 sections for  $E_8$  curve:

$$\begin{aligned}
 x = & \frac{1}{4 \sin^2 \lambda} \times \\
 & \left[ u^2 + \left( 2 \cos 4\lambda - 24 \cos 2\lambda + 2\chi_7^{E_7} \cos \lambda - 90 \right) u \right. \\
 & + \left( 2 \cos 6\lambda + (2\chi_1^{E_7} - 134) \cos 4\lambda \right. \\
 & \quad - 16\chi_7^{E_7} \cos 3\lambda + (-20\chi_1^{E_7} + 2\chi_6^{E_7} + 2396) \cos 2\lambda \\
 & \quad \left. \left. - 96\chi_7^{E_7} \cos \lambda + \chi_7^{E_7} \chi_7^{E_7} + 18\chi_1^{E_7} - 2\chi_6^{E_7} + 872 \right) \right], \tag{B.3}
 \end{aligned}$$

$$\begin{aligned}
 y = & \frac{1}{4 \sin^3 \lambda} \times \\
 & \left[ (\cos \lambda) u^3 + \left( -\cos 5\lambda + 5 \cos 3\lambda + \chi_7^{E_7} \cos 2\lambda - 172 \cos \lambda + 2\chi_7^{E_7} \right) u^2 \right. \\
 & + \left( -3 \cos 7\lambda + (-\chi_1^{E_7} + 124) \cos 5\lambda + (5\chi_1^{E_7} + \chi_6^{E_7} - 575) \cos 3\lambda \right. \\
 & \quad + (-2\chi_2^{E_7} - 108\chi_7^{E_7}) \cos 2\lambda + (3\chi_7^{E_7} \chi_7^{E_7} - 4\chi_1^{E_7} - \chi_6^{E_7} + 9862) \cos \lambda \\
 & \quad \left. \left. + 2\chi_2^{E_7} - 228\chi_7^{E_7} \right) u \right. \\
 & + \left( -2 \cos 9\lambda + (-2\chi_1^{E_7} + 176) \cos 7\lambda + \chi_2^{E_7} \cos 6\lambda \right. \\
 & \quad + (62\chi_1^{E_7} - \chi_3^{E_7} - \chi_6^{E_7} - 3822) \cos 5\lambda \\
 & \quad + (\chi_1^{E_7} \chi_7^{E_7} - 3\chi_2^{E_7} + \chi_5^{E_7} + 3\chi_7^{E_7}) \cos 4\lambda \\
 & \quad + (-\chi_2^{E_7} \chi_7^{E_7} - 286\chi_1^{E_7} + 3\chi_3^{E_7} - 53\chi_6^{E_7} + 16534) \cos 3\lambda \\
 & \quad + (2\chi_6^{E_7} \chi_7^{E_7} + 115\chi_2^{E_7} - 4\chi_5^{E_7} + 2906\chi_7^{E_7}) \cos 2\lambda \\
 & \quad + (\chi_2^{E_7} \chi_7^{E_7} - 168\chi_7^{E_7} \chi_7^{E_7} + 226\chi_1^{E_7} - 2\chi_3^{E_7} + 54\chi_6^{E_7} - 188502) \cos \lambda \\
 & \quad \left. \left. + \chi_7^{E_7} \chi_7^{E_7} \chi_7^{E_7} - \chi_1^{E_7} \chi_7^{E_7} - 2\chi_6^{E_7} \chi_7^{E_7} - 113\chi_2^{E_7} + 3\chi_5^{E_7} + 6499\chi_7^{E_7} \right) \right]. \tag{B.4}
 \end{aligned}$$

where  $\chi_1^{E_7}, \dots, \chi_7^{E_7}$  denote  $E_7$  fundamental representation characters and are defined as

$$\begin{aligned}
 \chi_1^{E_7} &= \chi[100000], & \chi_2^{E_7} &= \chi[000000], & \chi_3^{E_7} &= \chi[010000], & \chi_4^{E_7} &= \chi[001000], \\
 \chi_5^{E_7} &= \chi[000100], & \chi_6^{E_7} &= \chi[000010], & \chi_7^{E_7} &= \chi[000001]. \tag{B.5}
 \end{aligned}$$

$E_7$  curve:

$$\begin{aligned}
y^2 = & 4x^3 + \left(-u^2 + 4\chi_1^{E_7} - 100\right)x^2 \\
& + \left((2\chi_2^{E_7} - 12\chi_7^{E_7})u + (4\chi_3^{E_7} - 4\chi_6^{E_7} - 64\chi_1^{E_7} + 824)\right)x \\
& + \left(4u^4 + 4\chi_7^{E_7}u^3 + (4\chi_6^{E_7} - 8\chi_1^{E_7} + 92)u^2 \right. \\
& \quad + (4\chi_5^{E_7} - 4\chi_1^{E_7}\chi_7^{E_7} - 20\chi_2^{E_7} + 116\chi_7^{E_7})u \\
& \quad \left. + (4\chi_4^{E_7} - \chi_2^{E_7}\chi_2^{E_7} + 4\chi_1^{E_7}\chi_1^{E_7} - 40\chi_3^{E_7} + 36\chi_6^{E_7} + 248\chi_1^{E_7} - 2232)\right). \tag{B.6}
\end{aligned}$$

56 sections for  $E_7$  curve:

$$x = -\left(\Lambda^3 + \Lambda^{-3}\right)u - \left(\Lambda^6 + \chi_1^{E_6}\Lambda^2 - 8 + \chi_6^{E_6}\Lambda^{-2} + \Lambda^{-6}\right), \tag{B.7}$$

$$\begin{aligned}
y = & i\left(\Lambda^3 - \Lambda^{-3}\right)u^2 + i\left(3\Lambda^6 + \chi_1^{E_6}\Lambda^2 - \chi_6^{E_6}\Lambda^{-2} - 3\Lambda^{-6}\right)u \\
& + i\left(2\Lambda^9 + 2\chi_1^{E_6}\Lambda^5 - \chi_2^{E_6}\Lambda^3 + (\chi_3^{E_6} + \chi_6^{E_6})\Lambda \right. \\
& \quad \left. + (-\chi_1^{E_6} - \chi_5^{E_6})\Lambda^{-1} + \chi_2^{E_6}\Lambda^{-3} - 2\chi_6^{E_6}\Lambda^{-5} - 2\Lambda^{-9}\right). \tag{B.8}
\end{aligned}$$

where  $\chi_1^{E_6}, \dots, \chi_6^{E_6}$  denote  $E_6$  fundamental representation characters and are defined as

$$\begin{aligned}
\chi_1^{E_6} &= \chi[10000], & \chi_2^{E_6} &= \chi[00000], & \chi_3^{E_6} &= \chi[01000], & \chi_4^{E_6} &= \chi[00100], \\
\chi_5^{E_6} &= \chi[00010], & \chi_6^{E_6} &= \chi[00001]. \tag{B.9}
\end{aligned}$$

126 sections for  $E_7$  curve:

$$\begin{aligned}
x = & \frac{1}{4\sin^2\lambda} \times \\
& \left[ u^2 + \left(2\chi_6^{D_6} \cos\lambda\right)u + \left(2\cos 4\lambda + (2\chi_5^{D_6} - 20)\cos 2\lambda \right. \right. \\
& \quad \left. \left. + \chi_6^{D_6}\chi_6^{D_6} - 2\chi_5^{D_6} + 18\right) \right], \tag{B.10}
\end{aligned}$$



$$\begin{aligned}
y = & \frac{1}{4 \sin^3 \lambda} \times \\
& \left[ (\cos \lambda) u^3 + \left( \chi_6^{D_6} \cos 2\lambda + 2\chi_6^{D_6} \right) u^2 \right. \\
& + \left( -\cos 5\lambda + (\chi_5^{D_6} + 5) \cos 3\lambda - 2\chi_2^{D_6} \cos 2\lambda \right. \\
& \quad \left. + (3\chi_6^{D_6} \chi_6^{D_6} - \chi_5^{D_6} - 4) \cos \lambda + 2\chi_2^{D_6} \right) u \\
& + \left( -\chi_1^{D_6} \cos 5\lambda + (\chi_4^{D_6} + \chi_6^{D_6}) \cos 4\lambda + (-\chi_2^{D_6} \chi_6^{D_6} + 3\chi_1^{D_6}) \cos 3\lambda \right. \\
& \quad + (2\chi_5^{D_6} \chi_6^{D_6} - 4\chi_4^{D_6}) \cos 2\lambda + (\chi_2^{D_6} \chi_6^{D_6} - 2\chi_1^{D_6}) \cos \lambda \\
& \quad \left. \left. + \chi_6^{D_6} \chi_6^{D_6} \chi_6^{D_6} - 2\chi_5^{D_6} \chi_6^{D_6} + 3\chi_4^{D_6} - \chi_6^{D_6} \right) \right] \quad (\text{B.11})
\end{aligned}$$

where  $\chi_1^{D_6}, \dots, \chi_6^{D_6}$  denote  $D_6$  fundamental representation characters and are defined as

$$\begin{aligned}
\chi_1^{D_6} &= \chi[00000], & \chi_2^{D_6} &= \chi[10000], & \chi_3^{D_6} &= \chi[01000], & \chi_4^{D_6} &= \chi[00100], \\
\chi_5^{D_6} &= \chi[00010], & \chi_6^{D_6} &= \chi[00001].
\end{aligned} \quad (\text{B.12})$$

### Appendix C. Special $E_8$ Curves with Singular Fibers

- $\vec{m} = \frac{2\pi}{2} \vec{\mu}_1 = (0, 0, 0, 0, 0, 0, 0, 2\pi)$  (or  $\vec{m} = (0, 0, 0, 0, \pi, \pi, \pi, \pi)$ )

$$f = \frac{1}{12}(u - 64)^2(u^2 + 128u - 8192), \quad (\text{C.1})$$

$$g = \frac{1}{216}(u - 64)^3(u - 32)(u^2 - 640u + 28672), \quad (\text{C.2})$$

$$\Delta = (u - 64)^{10}(u - 48). \quad (\text{C.3})$$

$\Rightarrow$  degenerate fiber:  $I_4^*$       symmetry:  $D_8$

- $\vec{m} = \frac{2\pi}{3} \vec{\mu}_2 = \left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}\right)$  (or  $\vec{m} = \left(\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, -\frac{2\pi}{3}\right)$ )

$$f = \frac{1}{12}(u - 54)(u^3 + 54u^2 - 16524u + 583200), \quad (\text{C.4})$$

$$g = \frac{1}{216}u^6 - 4u^5 + 873u^4 - \frac{172179}{2}u^3 + 4376916u^2 - 112140612u + 1147184289, \quad (\text{C.5})$$

$$\Delta = (u - 63)^9(u^2 - 117u + 3429). \quad (\text{C.6})$$

$\Rightarrow$  degenerate fiber:  $I_9$       symmetry:  $A_8$

- $\vec{m} = \frac{2\pi}{4}\vec{\mu}_3 = (-\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{7\pi}{4})$  (or  $\vec{m} = (\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, -\frac{7\pi}{4})$ )

$$f = \frac{1}{12}u^4 - \frac{4864}{3}u^2 + 122880u - \frac{7847936}{3}, \quad (\text{C.7})$$

$$g = \frac{1}{216}(u^2 - 96u + 2176)(u^4 - 768u^3 + 112640u^2 - 6094848u + 113115136), \quad (\text{C.8})$$

$$\Delta = (u - 64)^8(u - 56)^2(u - 60). \quad (\text{C.9})$$

$\Rightarrow$  degenerate fiber:  $I_8, I_2$       symmetry:  $A_7 \oplus A_1$

- $\vec{m} = \frac{2\pi}{6}\vec{\mu}_4 = (0, 0, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3})$

$$f = \frac{1}{12}(u - 54)(u^3 + 54u^2 - 16308u + 569160), \quad (\text{C.10})$$

$$g = \frac{1}{216}(u^2 - 96u + 2196)(u^4 - 768u^3 + 112104u^2 - 6035904u + 111493584), \quad (\text{C.11})$$

$$\Delta = (u - 66)^6(u - 57)^3(u - 58)^2. \quad (\text{C.12})$$

$\Rightarrow$  degenerate fiber:  $I_6, I_3, I_2$       symmetry:  $A_5 \oplus A_2 \oplus A_1$

- $\vec{m} = \frac{2\pi}{5}\vec{\mu}_5 = (0, 0, 0, \frac{2\pi}{5}, \frac{2\pi}{5}, \frac{2\pi}{5}, \frac{2\pi}{5}, \frac{8\pi}{5})$

$$f = \frac{1}{12}u^4 - \frac{4750}{3}u^2 + 118750u - \frac{7512500}{3}, \quad (\text{C.13})$$

$$g = \frac{1}{216}(u^2 - 114u + 3250)(u^4 - 750u^3 + 98750u^2 - 4687500u + 74687500), \quad (\text{C.14})$$

$$\Delta = (u^2 - 125u + 3875)^5(u - 57). \quad (\text{C.15})$$

$\Rightarrow$  degenerate fiber:  $I_5, I_5$       symmetry:  $A_4 \oplus A_4$

- $\vec{m} = \frac{2\pi}{4}\vec{\mu}_6 = (0, 0, 0, 0, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2})$  (or  $\vec{m} = (0, 0, 0, 0, 0, \pi, \pi, \pi)$ )

$$f = \frac{1}{12}(u - 56)^2(u^2 + 112u - 9152), \quad (\text{C.16})$$

$$g = \frac{1}{216}(u - 56)^3(u - 40)(u^2 - 656u + 33856), \quad (\text{C.17})$$

$$\Delta = (u - 56)^7(u - 72)^4. \quad (\text{C.18})$$

$\Rightarrow$  degenerate fiber:  $I_1^*, I_4$       symmetry:  $D_5 \oplus A_3$

- $\vec{m} = \frac{2\pi}{3}\vec{\mu}_7 = (0, 0, 0, 0, 0, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3})$  (or  $\vec{m} = (0, 0, 0, 0, 0, \frac{4\pi}{3}, \frac{4\pi}{3}, \frac{4\pi}{3})$ )

$$f = \frac{1}{12}(u - 54)^3(u + 162), \tag{C.19}$$

$$g = \frac{1}{216}(u - 54)^4(u^2 - 648u + 26244), \tag{C.20}$$

$$\Delta = (u - 54)^8(u - 81)^3. \tag{C.21}$$

$\Rightarrow$  degenerate fiber:  $IV^*$ ,  $I_3$       symmetry:  $E_6 \oplus A_2$

- $\vec{m} = \frac{2\pi}{2}\vec{\mu}_8 = (0, 0, 0, 0, 0, 0, \pi, \pi)$  (or  $\vec{m} = (\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$ )

$$f = \frac{1}{12}(u - 48)^3(u + 144), \tag{C.22}$$

$$g = \frac{1}{216}(u - 48)^5(u - 624), \tag{C.23}$$

$$\Delta = (u - 48)^9(u - 112)^2. \tag{C.24}$$

$\Rightarrow$  degenerate fiber:  $III^*$ ,  $I_2$       symmetry:  $E_7 \oplus A_1$   $\vec{\mu}_1, \dots, \vec{\mu}_8$  denotes the fundamental weights of  $E_8$ :

$$\vec{\mu}_1 = 2\vec{e}_8, \tag{C.25}$$

$$\vec{\mu}_2 = \frac{1}{2}\vec{e}_1 + \frac{1}{2}\vec{e}_2 + \frac{1}{2}\vec{e}_3 + \frac{1}{2}\vec{e}_4 + \frac{1}{2}\vec{e}_5 + \frac{1}{2}\vec{e}_6 + \frac{1}{2}\vec{e}_7 + \frac{5}{2}\vec{e}_8, \tag{C.26}$$

$$\vec{\mu}_3 = -\frac{1}{2}\vec{e}_1 + \frac{1}{2}\vec{e}_2 + \frac{1}{2}\vec{e}_3 + \frac{1}{2}\vec{e}_4 + \frac{1}{2}\vec{e}_5 + \frac{1}{2}\vec{e}_6 + \frac{1}{2}\vec{e}_7 + \frac{7}{2}\vec{e}_8, \tag{C.27}$$

$$\vec{\mu}_4 = \vec{e}_3 + \vec{e}_4 + \vec{e}_5 + \vec{e}_6 + \vec{e}_7 + 5\vec{e}_8, \tag{C.28}$$

$$\vec{\mu}_5 = \vec{e}_4 + \vec{e}_5 + \vec{e}_6 + \vec{e}_7 + 4\vec{e}_8, \tag{C.29}$$

$$\vec{\mu}_6 = \vec{e}_5 + \vec{e}_6 + \vec{e}_7 + 3\vec{e}_8, \tag{C.30}$$

$$\vec{\mu}_7 = \vec{e}_6 + \vec{e}_7 + 2\vec{e}_8, \tag{C.31}$$

$$\vec{\mu}_8 = \vec{e}_7 + \vec{e}_8. \tag{C.32}$$

We show the labeling of the fundamental weights of  $E_8$ :

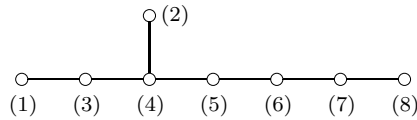


Figure 2: Dynkin diagram for  $E_8$

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