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SEIBERG–WITTEN–LIKE EQUATIONS ON THE STRICTLY–PSEUDOCONVEX CR 7–MANIFOLDS

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Abstract. In this paper, Seiberg−Witten−like equations are constructed on 7−manifolds endowed with G_2-structure, lifted by $SU(3)-$structure. Then a global solution is obtained on the strictly−Pseudoconvex CR 7−manifolds for a given negative and constant scalar curvature.

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1. INTRODUCTION

The exceptional Lie group G_2 is the automorphisms group of the octonion algebra $\mathbb{O}$ which is a subgroup of $SO(7)$. A manifold whose structure group is G_2 is called a G_2-manifold. G_2-manifolds have been studied in terms of the covariant derivation of the fundamental 3-form and the parallelism of this form with respect to the Levi−Civita connection [5, 12, 14]. In addition to this, compact G_2 manifolds are currently being studied [4, 11, 17–19].

A 7−manifold M equipped with G_2-structure is a Riemannian manifold whose structure group is a reduction of the tangent bundle from $Gl(7,\mathbb{R})$ to the subgroup G_2, which is also a subgroup of $SO(7)$. This implies that the 7−dimensional manifold equipped with G_2-structure is an orientable Riemannian manifold. Also G_2-structure on the 7−manifolds determines a non−degenerate global three form Φ on M and G_2-structure is the stabiliser of Φ. The action of G_2 on the tangent bundle induces an action of G_2 on $\Lambda^2(M)$ and gives the following orthogonal decomposition of $\Lambda^2(M)$:

$$\Lambda^2(M) = \Lambda^2_7(M) \oplus \Lambda^2_{14}(M)$$

where

$$\Lambda^2_7(M) = \{\beta \in \Lambda^2(M) | *(\beta \wedge \Phi) = -2\beta\},$$
$$\Lambda^2_{14}(M) = \{\beta \in \Lambda^2(M) | *(\beta \wedge \Phi) = \beta\}$$

and $*$ is the Hodge star operator [3]. These two decompositions are used to define self−duality and anti self−duality concept on G_2-manifolds [9].

On a 6−manifold N equipped with $SU(3)$−structure, $SU(3)$−structure acts on the tangent bundle and thus it induces an action of $SU(3)$−structure on the space of two-forms $\Lambda^2(N)$. According to this $SU(3)$−structure is the stabiliser in $SO(6)$ of a non−degenerate 2− form ω and a normalized 3−form Ψ. Then, by using ω one can obtain the following decomposition:

$$\Lambda^2(N) = \Lambda^2_1(N) \oplus \Lambda^2_6(N) \oplus \Lambda^2_8(N)$$

where

$$\Lambda^2_1(N) = \{\beta \in \Lambda^2(N) | * (\beta \wedge \omega) = 2\beta\},$$
$$\Lambda^2_6(N) = \{\beta \in \Lambda^2(N) | * (\beta \wedge \omega) = \beta\},$$
$$\Lambda^2_8(N) = \{\beta \in \Lambda^2(N) | * (\beta \wedge \omega) = -\beta\}.$$

Let N be a subset of M endowed with $SU(3)$−structure. The relation between $SU(3)$ and G_2−structure is given by the inclusion $SU(3) \subset G_2$−structure. This inclusion is characterized by the orthogonal decomposition

$$\mathbb{R}^7 = \mathbb{R}^6 \oplus \alpha\mathbb{R} \tag{1.1}$$

where $\alpha = e^7$ annihilates $\mathbb{R}^6$ at each point.

Then, a non−degenerate 3−form Φ, determined by G_2−structure on a 7−manifold M, is decribed by

$$\Phi = \omega \wedge \alpha + \Psi_+ \tag{1.2}$$

where Ψ_+ is the real part of a normalized 3−form Ψ [6, 15, 22]. This implies that $\omega \wedge \alpha$ determines $SU(3)$− structure on G_2−manifolds [6]. In the following, Seiberg−Witten equations are briefly reminded.

Seiberg−Witten equations were defined firstly by Witten on any smooth 4− manifold [23]. The solutions of these equations play an important role in the topology of 4−manifolds. Later on, Seiberg−Witten equations have been investigated in higher dimensional manifolds by several authors [7,9,16]. In 7−dimension, Seiberg−Witten equations are defined on the manifolds equipped with G_2−structure by Degirmenci and Ozdemir[9]. In their study they gave a local non−trivial solution to these equations on $\mathbb{R}^7$. In this paper we extend this solution to a global one on the strictly pseudoconvex CR 7−manifolds for a given negative and constant scalar curvature. Since G_2−structure is lifted by $SU(3)$−structure, it has a non degenerate 2−form which is the stabilizer of $SU(3)$−structure. According to this, if wedge product of α is taken by the stabilizer of $SU(3)$−structure, one gets 3−form which is also stabilizer of $SU(3)$. By using this 3−form, one can decompose the space of 2−form. According to this decomposition, self−duality concept can be defined.

This paper is organized as follows. At first, some basic facts concerning $SU(3)$-structures contained in G_2−structure is introduced. In section 2, the space of two-forms $\Omega^2(M)$ is decomposed by considering induced $SU(3)$−structure. Then the space of self−dual two-forms is defined. In section 3, Seiberg−Witten−like equations is defined on the 7−manifold endowed with G_2−structure lifted by an $SU(3)$−structure. Finally, we give a global solution to these equations on the strictly-Pseudoconvex CR 7−manifolds for a given negative and constant scalar curvature.

2. $SU(3)$−STRUCTURE ON 7−DIMENSIONAL MANIFOLDS

Let us consider $\mathbb{R}^7$ with a basis $\{e_1,...,e_7\}$ and its metric dual $\{e^1,...,e^7\}$. An inclusion of $SU(3)$−structure into G_2−structure is defined and characterised by the orthogonal decomposition $\mathbb{R}^7 = \mathbb{R}^6 \oplus \alpha\mathbb{R}$ where α annihilates $\mathbb{R}^6$ at each point.

Definition 1. On the 7−manifold M, an $SU(3)$−structure is a triple $(\alpha,\omega,\Psi) \in \Omega^1(M)\times\Omega^2(M)\times\Omega^3(M,\mathbb{C})$ with model tensor

$$\left(\alpha,\omega,\Psi\right) := \left(e^7, e^{12}+e^{34}+e^{56}, e^1_{\mathbb{C}}\wedge e^2_{\mathbb{C}}\wedge e^3_{\mathbb{C}}\right) \in (\mathbb{R}^7)^* \times \Lambda^2(\mathbb{R}^7)^* \times \Lambda^3(\mathbb{R}^7)^*$$

where $e^j_{\mathbb{C}} := e^{2j-1} - i e^{2j}$ for $j = 1,...,3$ and $e^{ij} = e^i \wedge e^j$.

By setting $\Psi_+ := Re(\Psi)$ and $\Psi_- := Im(\Psi)$, the complex−valued $(3,0)$ form Ψ can be written as $\Psi := \Psi_+ + i\Psi_-$ [6].

On the 6−dimensional manifold N, an $SU(3)$−structure (α,ω,Ψ) can be lifted to G_2−structure, which is the holonomy group of the 7−dimensional manifold M, as follows [15]:

$$\Phi = \omega\wedge\alpha + \Psi_+.$$

According to this, there is a natural 6−dimensional distribution $H := TN = Ker_\alpha$ and complementary 1−dimensional distribution Ker_ω. Moreover, the Reeb vector field ξ of (α,ω,Ψ) $SU(3)$−structure is the section of the vector bundle $H \subset TM$ with $\alpha(\xi) = 1$.

Then we have an almost Hermitian structure (g, J_H) on H with respect to $SU(3)$-structure. Since $J_H^2 = -I_d$, the following eigenspaces decomposition can be given by:

$$\Lambda^1_H(M) = H\otimes_{\mathbb{R}}\mathbb{C} = \Lambda^{1,0}_H(M)\oplus\Lambda^{0,1}_H(M)$$

where

$$\Lambda^{1,0}_H(M) = \{Z \in H\otimes_{\mathbb{R}}\mathbb{C} | J_H Z = iZ\},$$
$$\Lambda^{0,1}_H(M) = \{Z \in H\otimes_{\mathbb{R}}\mathbb{C} | J_H Z = -iZ\}.$$

The complexification of $\Lambda^s_H(M)$ is decomposed as follows

$$\Lambda^s_H(M) = \sum_{q+r=s}\Lambda^{q,r}_H(M),$$

where $\Lambda^{q,r}(M)_H = span\{u\wedge v | u \in \Lambda^q\left(\Lambda^{1,0}_H(M)\right), v \in \Lambda^r\left(\Lambda^{0,1}_H(M)\right)\}$.

The endomorphism map J_H on M induces an endomorphism on $\Lambda^s_H(M)$ and satisfies the identity $J_H^2 = (-1)^r I_d$. The natural action of J_H on 2−form θ is given by

$$J_H\theta(V,W) = \theta(J_H V, J_H W).$$

Then, the following is obtained

$$\Lambda^{1,1}_H(M) = \{\theta \in \Lambda^2_H(M) | J_H\theta = \theta\},$$
$$\Lambda^{2,0}_H(M) \oplus \Lambda^{0,2}_H(M) = \{\theta \in \Lambda^2_H(M) | J_H\theta = -\theta\}.$$

Since $(H, d\alpha\big|_H)$ is a symplectic vector bundle equipped with an almost complex structure J_H on M, an almost contact structure can be defined by extending J_H to an endomorphism J of the tangent bundle TM by setting $J\xi = 0$. In that case, an almost contact structure on TM is given as

$$J^2 = -Id + \alpha \otimes \xi.$$

Moreover, A contact manifold (M,α) endowed with an almost contact structure can be endowed by the Riemannian metric g_α on TM such that

$$g_\alpha(V,W) = d\alpha(V, JW) + \alpha(V)\alpha(W)$$

for any $V,W \in \Gamma(TM)$.

After that we denoted contact metric manifold by $(M, g_\alpha, \alpha, J, \xi)$. On the contact metric manifold $(M, g_\alpha, \alpha, J, \xi)$, the generalized Webster-Tanaka connection is given by :

$$\nabla^{TW}_V W = \nabla_V W - \big(\nabla_V \alpha\big)(W)\xi - \alpha(V)\nabla_W \xi - \alpha(V)\alpha(W),$$

where ∇ is the Levi−Cita connection and $V,W \in \chi(M)$ [21]. Webster−Tanaka connection satisfies the condition $\nabla^{TW}\alpha = 0$ and $\nabla^{TW} g_\alpha = 0$. Also, if $\nabla^{TW} J = 0$, then $(M, g_\alpha, \alpha, J, \xi)$ is called strictly pseudoconvex CR manifold [20].

3. SELF−DUAL 2−FORMS ON THE CONTACT METRIC MANIFOLDS OF DIMENSION 7

Let $(M, g_\alpha, \alpha, J, \xi)$ be a 7−dimensional contact metric manifold endowed with G_2−structure which is lifted by $SU(3)$−structure. Then any 2−form $\eta \in \Omega^2(M)$ splits into $\eta = \eta_H + \eta_\xi$, where $\eta_H = \eta \circ \pi$, $\pi : TM \to H$ is the canonical projection and $\eta_\xi = \eta \wedge \iota(\xi)\eta$ where ι is the contraction operator. In addition, if $\iota(\xi)\eta = 0$, then η is called a horizontal 2−form. Also, $\Omega^2(M)$ can be decomposed with respect to the bundles of horizontal forms $\Omega^2_H(M)$ and $\Omega^1_H(M)$, as [10]

$$\Omega^2(M) = \Omega^2_H(M) \oplus \alpha \wedge \Omega^1_H(M).$$

Let $\Phi = \omega \wedge \alpha + \Psi_+$ be a fundamental 3−form induced by $SU(3)$−structure whose stabilizer is G_2. In an orthonormal basis $\{e_i\} i = 1, ..., 7$, the fundamental 3−form Φ is described as

$$\Phi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}$$

where $e^{ijk} = e^i \wedge e^j \wedge e^k$.

Φ defines T_Φ duality operator on G_2-manifold as follows

$$\begin{aligned} T_\Phi : \Omega^2(TM) &\longrightarrow \Omega^2(TM) \\ \beta &\longmapsto T_\Phi(\beta) := *(\Phi \wedge \beta) \end{aligned}$$

with 7 and 14 dimensional eigenspaces corresponding to eigenvalues 2 and -1, respectively [3].

Let consider the 3$-$form $\Phi_\omega^{'} = \omega \wedge \alpha$ which is the stabilizer of the $SU(3)-$structure contained in G_2-structure. By considering induced $SU(3)$structure corresponding with 3$-$form $\Phi_\omega^{'} = \omega \wedge \alpha$ one can obtain another decomposition of $\Omega^2(M)$ [6]. In an orthonormal basis $\{e_i\}$, $i = 1, ..., 7$, the fundamental 3$-$form $\Phi_\omega^{'}$ can be written as:

$$\Phi_\omega^{'} = e^{127} + e^{347} + e^{567}.$$

Also $\Phi_\omega^{'}$ defines $T_{\Phi'}$ duality operator on 2$-$forms as

$$\begin{aligned} T_{\Phi'} : \Omega^2(TM) &\longrightarrow \Omega^2(TM) \\ \beta &\longmapsto T_{\Phi'}(\beta) := *(\Phi^{'} \wedge \beta) \end{aligned}$$

with $1, 6, 6$ and 8 dimensional eigenspaces corresponding to eigenvalues $2, 1, 0$ and -1, respectively. A basis consisting of the corresponding eigenvalues is given below:

$$\Omega^2(M) = \Omega_H^2(M) \oplus \alpha \wedge \Omega_H^1(M).$$

Eigenvector associated with the eigenvalue 2:

$$\omega = e^1 \wedge e^3 + e^3 \wedge e^4 + e^5 \wedge e^6. \tag{3.1}$$

Eigenvectors associated with the eigenvalue 1:

$$\begin{aligned} a_1 &= -e^1 \wedge e^3 + e^2 \wedge e^4 & a_2 &= e^1 \wedge e^4 + e^2 \wedge e^3 \\ a_3 &= -e^1 \wedge e^5 + e^2 \wedge e^6 & a_4 &= e^1 \wedge e^6 + e^2 \wedge e^5 \\ a_5 &= -e^3 \wedge e^5 + e^4 \wedge e^6 & a_6 &= e^3 \wedge e^6 + e^4 \wedge e^5. \end{aligned}$$

Eigenvectors associated with the eigenvalue -1:

$$\begin{aligned} b_1 &= -e^1 \wedge e^2 + e^3 \wedge e^4 & b_2 &= -e^1 \wedge e^2 + e^5 \wedge e^6 \\ b_3 &= e^1 \wedge e^3 + e^2 \wedge e^4 & b_4 &= -e^1 \wedge e^4 + e^2 \wedge e^3 \\ b_5 &= e^1 \wedge e^5 + e^2 \wedge e^6 & b_6 &= -e^1 \wedge e^6 + e^2 \wedge e^5 \\ b_7 &= e^3 \wedge e^5 + e^4 \wedge e^6 & b_8 &= -e^3 \wedge e^6 + e^4 \wedge e^5 \end{aligned}$$

Eigenvectors associated with the eigenvalue 0:

$$c_1 = e^1 \wedge e^7 \qquad c_2 = e^2 \wedge e^7$$
$$c_3 = e^3 \wedge e^7 \qquad c_4 = e^4 \wedge e^7$$
$$c_5 = e^5 \wedge e^7 \qquad c_6 = e^6 \wedge e^7$$

Considering the natural action of $SU(3)-$structure on the space of $two-$forms $\Omega^2_H(M)$, the following orthogonal eigenspace decomposition is obtained [2].

$$\Omega^2_H(M) = \Omega^{2,1}_H(M) \oplus \Omega^{2,6}_H(M) \oplus \Omega^{2,8}_H(M)$$

where

$$\Omega^{2,1}_H(M) = \{k\omega : k \in \mathbb{R}\},$$
$$\Omega^{2,6}_H(M) = \{\theta \in \Omega^2_H(M) : J\theta = -\theta\},$$
$$\Omega^{2,8}_H(M) = \{\theta \in \Omega^2_H(M) : J\theta = \theta \text{ and } \theta \wedge \omega \wedge \omega = 0\}.$$

By complexifying the space of $two-$forms $\Omega^2_H(M)$, we get the following:

$$\Omega^2_H(M) \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}\omega \oplus \left(\Omega^{2,8}_H(M) \otimes_{\mathbb{R}} \mathbb{C}\right) \oplus \Omega^{2,6}_H(M) \otimes_{\mathbb{R}} \mathbb{C}.$$

The space $\Omega^2_H(M)^+ = \mathbb{C}\omega \oplus \left(\Omega^{2,6}_H(M) \otimes_{\mathbb{R}} \mathbb{C}\right)$ is called as the space of self−dual $two-$forms. Similarly, the space $\Omega^2_H(M)^-$ is called the space of anti self−dual $two-$ forms[24]. Locally, we can express the space of self−dual 2−forms relative to Φ'_ω by $\{\omega, a_1, a_2, a_3, a_4, a_5, a_6\}$.

4. DIRAC OPERATOR ON THE CONTACT METRIC MANIFOLDS

In this section, we talk about the canonical $Spin^c-$structure of a contact metric manifold and its spinor bundle with associated connection.

Contact metric manifold is defined by a contact distribution and with its complementary. Since contact distribution has an almost−hermitian structure, the results of it can be extended to a contact metric manifold. Since the structure group of any contact metric manifold of dimension $2n+1$ is $U(n)$, it admits a canonical $Spin^c-$structure given by:

$$P_{Spin^c(n)} = P_{U(n)} \times_F Spin^c(n)$$

where $F : U(n) \longrightarrow Spin^c(2n)$ is the lifting map [13, 20]. The associated canonical spinor bundle then has the form:

$$\mathbb{S}^{\mathbb{C}} \cong \Omega^{0,*}(M).$$

where $\Omega^{0,*}(M)$ is the direct sum of $\Omega(M)^{0,1} \oplus \Omega(M)^{0,2} \oplus \cdots \oplus \Omega(M)^{0,i}$, $i \in \mathbb{N}$. Also, on this spinor bundle, the Clifford multiplication is given by:

$$V \cdot \psi = \sqrt{2}\left((V^{0,1}_H)^* \wedge \psi - \iota(V^{0,1}_H)\psi\right) + i(-1)^{deg\,\psi+1}\eta(V)\psi. \tag{4.1}$$

where V_H denotes the horizontal part of V. According to these multiplication one can easily obtain $\xi\psi = i(-1)^{deg\psi+1}\psi$.

As in the almost−Hermitian case, given a metric−connection called Levi−Civita ∇ on TM, there are two ways to include a connection on $\mathbb{S}$:

The first of these is obtained by the extension of the connection to forms and the latter is obtained via $Spin^c$−structure. In this work, we mainly focused on the canonical $Spin^c$−structure with the following isomorphism:

$$\mathbb{S}^{\mathbb{C}} \cong \Omega_H^{0,*}(M).$$

On this bundle, we described Dirac operator defined on $\mathbb{S}$ and we give the relation with the Dirac−type operator defined on $\Omega_H^{0,*}(M)$.

In the case of contact metric manifold endowed with a canonical $Spin^c$structure, there is a spinorial connection ∇^A on the associated spinor bundle $\mathbb{S}^{\mathbb{C}}$ induced by an unitary connection 1−form A on the determinant line bundle L together with the generalized Webster−Tanaka connection ∇^{TW}. Also, on the associated spinor bundle one can describe Dirac operator as follows:

Let $\{e_i\}$ $i = 1, \dots, 2n$ be a local orthonormal frame on H. Then the Kohn−Dirac operator D_H^A is given by:

$$D_H^A = \sum_{i=1}^{2n} e_i \cdot \nabla_{e_i}^A. \tag{4.2}$$

Hence, Dirac operator on the $2n + 1$ dimensional contact metric manfold is [20]:

$$D^A = D_H^A + \xi \cdot \nabla_\xi^A. \tag{4.3}$$

Moreover, by considering strictly Pseudoconvex CR manifolds with $\Omega_H^{0,*}(M)$ associated spinor bundle the Dirac type operator is defined as follows

Let

$$\overline{\partial}_H : \Omega_H^{0,r}(M) \longrightarrow \Omega_H^{0,r+1}(M),\ \overline{\partial}_H^* : \Omega_H^{0,r}(M) \longrightarrow \Omega_H^{0,r-1} \tag{4.4}$$

respectively given by:

$$\overline{\partial}_H = \sum_{i=1}^{n} \overline{Z}_i^* \wedge \nabla_{\overline{Z_i}}^{TW},\ \overline{\partial}_H^* = -\sum_{i=1}^{n} \iota(\overline{Z_i})^* \wedge \nabla_{\overline{Z_i}}^{TW}$$

where ∇^{TW} is the extension of the generalized Webster−Tanaka connection to $\Omega_H^{0,*}(M)$ and ι is the contraction operator.

It follows from (4.1) that we have on $\Omega_M^{0,*}(\overline{M})$

$$\mathcal{H} = \sqrt{2}\sum_{r=0}^{n}\left(\overline{\partial}_H + \overline{\partial}_H^*\right) + \sum_{r=0}^{n}(-1)^{r+1}\sqrt{-1}\cdot\nabla_\xi^{TW}. \tag{4.5}$$

Since $\mathbb{S}^{\mathbb{C}} \cong \Omega_H^{0,*}(M)$, (4.3) coincides with (4.5). In this paper we consider the following spinor representation $\kappa : \mathbb{R}^7 \longrightarrow \mathbb{C}(8)$:

$$\kappa(e_1) = m_4 \otimes m_1 \otimes m_3, \qquad \kappa(e_3) = -m_1 \otimes m_3 \otimes m_3,$$
$$\kappa(e_5) = -m_3 \otimes m_3 \otimes m_3, \qquad \kappa(e_2) = I \otimes I \otimes m_2,$$
$$\kappa(e_4) = m_4 \otimes m_2 \otimes m_3, \qquad \kappa(e_6) = -m_2 \otimes m_3 \otimes m_3,$$
$$\kappa(e_7) = I \otimes I \otimes m_1,$$

where

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ m_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},\ m_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix},\ m_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ m_4 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}.$$

In 7−dimension, Seiberg−Witten equations are described by the Dirac equation and Curvature equation. Although, Dirac equation has a common definition on any smooth manifold endowed with $Spin^c$−structure, the definition of the curvature equation shows some difference with respect to the chosen self−duality concept. In this paper we use the definition given in [9].

An imaginary valued 2−form $\sigma(\psi)$ given by

$$\sigma(\psi)(V,W) = \langle V \cdot W \cdot \psi, \psi \rangle + \langle V, W \rangle |\psi|^2$$

where $V, W \in \Gamma(TM)$ and $\langle , \rangle$ is the Hermitian inner product on the spinor bundle $\mathbb{S}^c$. The restriction of $\sigma(\psi)$ to H is denoted by $\sigma_H(\psi) := \sigma(\psi)\big|_H$.

Definition 2. Let M be the 7−manifold endowed with G_2−structure, lifted by $SU(3)$−structure. For any unitary connection 1− form A and spinor field $\psi \in \Gamma(S)$, the Seiberg−Witten equations are defined by:

$$\begin{aligned} D_A\psi &= 0, \\ F_A^+ &= \frac{1}{4}\sigma(\psi)^+ \end{aligned} \tag{4.6}$$

where F_A^+ is the self−dual part of the curvature F_A and $\sigma(\psi)^+$ the self−dual part of the 2−form $\sigma(\psi)$ corresponding with the spinor field $\psi \in \Gamma(S)$.

In the following, the method applied by Ş. Bulut in order to give a global solution is used [8].

5. Global solution to the Seiberg−Witten−like equations on the strictly−Pseudoconvex CR 7−manifolds

Let $(M, g_\alpha, \alpha, J, \xi)$ be a strictly−Pseudoconvex CR 7−manifold endowed with a canonical $Spin^c$−structure and $\{e_1, e_2 = J(e_1), e_3, e_4 = J(e_3), e_5, e_6 = J(e_5), \xi\}$ be a local frame with dual basis $\{e^1, e^2, e^3, e^4, e^5, e^6, \alpha\}$. The spinor bundle $\mathbb{S}^{\mathbb{C}}$ decomposes into eigensubbundles under the action $d\alpha = e^1 \wedge e^2 + e^3 \wedge e^4 + e^5 \wedge e^6$,

$$\mathbb{S}^{\mathbb{C}} \cong \Lambda_H^{0,0}(M) \oplus \Lambda_H^{0,1}(M) \oplus \Lambda_H^{0,2}(M) \oplus \Lambda_H^{0,3}(M).$$

Each $\Lambda_H^{0,0}(M)$, $\Lambda_H^{0,1}(M)$, $\Lambda_H^{0,2}(M)$, $\Lambda_H^{0,3}(M)$ is associated with the eigenvalue $-3i,-i,i,3i$ with dimension $1,3,3,1$. Also $\mathbb{S}^{\mathbb{C}}$ can be described as [8]

$$\mathbb{S}^{\mathbb{C}} = \mathbb{S}^{\mathbb{C},+} \oplus \mathbb{S}^{\mathbb{C},-}$$

where

$$\mathbb{S}^{\mathbb{C},+} \cong \Lambda_H^{0,0}(M) \oplus \Lambda_H^{0,2}(M),$$
$$\mathbb{S}^{\mathbb{C},-} \cong \Lambda_H^{0,1}(M) \oplus \Lambda_H^{0,3}(M).$$

This gives the following isomorphisms

$$\Lambda_H^{0,0}(M) \oplus \Lambda_H^{0,2}(M) \cong S_i^{\mathbb{C}} \oplus S_{-3i}^{\mathbb{C}},$$
$$\Lambda_H^{0,1}(M) \oplus \Lambda_H^{0,3}(M) \cong S_{-i}^{\mathbb{C}} \oplus S_{3i}^{\mathbb{C}}.$$

where $S_i^{\mathbb{C}} = \{\psi \in \Gamma(S), \omega \cdot \psi = i\psi\}$. Let ψ_0 be the spinor in $\mathbb{S}_{-3i}^{\mathbb{C}} \cong \Lambda_H^{0,0}(M)$ corresponding to constant function 1, in the chosen coordinates

$$\psi_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

As a result we get $\sigma_H(\psi_0) = -i d\alpha$.

Theorem 1. *Let $(M, g_\alpha, \alpha, J, \xi)$ be a 7−dimensional strictly Pseudoconvex CR manifold. Then, for a given negative and constant scalar curvature s_H, $\big(A, \psi = \sqrt{-\frac{2}{3}s_H}\,\psi_0\big)$ is a solution of the Seiberg−Witten−like equations.*

Proof. Since $\psi = \sqrt{-\frac{2}{3}s_H}\,\psi_0 \in \Lambda^{0,0}(M)$ and the spinor ψ_0 is a spinor corresponding to the constant function 1, we get $\mathcal{H} + \sum_{q=0}^{n} (-1)^{q+1}\sqrt{-1} \cdot \nabla \xi^{TW} = 0$. This means $D_A\psi = 0$. Then, only satisfying the second equation is left. The relation between a curvature of the connection 1−form A and a Ricci form ρ_{ric}^H is given as:

$$F_A = Ric = i\rho_{ric}^H$$

where the unitary connection 1−form A induced by means of ∇^{TW} in the line bundle $K=\Omega_H^{0,n}(M)$ [1]. Then, by using the definition of the Ricci form ρ^H_{ric} given by

$$\rho^H_{ric}(V,W)=Ric(V,J_HW)=g(V,J_H\circ RicW)$$

for any $V,W\in\Gamma(TM)$, one gets

$$\begin{aligned}\rho^H_{ric}&=-R_{11}e_1\wedge e_2+R_{14}(e_1\wedge e_3+e_2\wedge e_4)+R_{13}(e_2\wedge e_3-e_1\wedge e_4)\\&\quad-R_{33}e_3\wedge e_4-R_{26}(e_1\wedge e_5+e_2\wedge e_6)+R_{15}(-e_1\wedge e_6+e_2\wedge e_5)\\&\quad+R_{36}(e_3\wedge e_5+e_4\wedge e_6)+R_{35}(-e_3\wedge e_6+e_4\wedge e_5)-R_{55}e_5\wedge e_6.\end{aligned}\tag{5.1}$$

Eliminating anti-self dual 2−form in (5.1), one has self−dual part of ρ^H_{ric} as follows

$$\begin{aligned}\rho^{H,+}_{ric}&=\frac{-R_{11}-R_{33}-R_{55}}{3}d\alpha=-\left(\frac{R_{11}+R_{22}+R_{33}+R_{44}+R_{55}+R_{66}}{3}\right)d\alpha\\&=-\frac{s}{6}d\alpha.\end{aligned}$$

The following is obtained

$$F_A^+=Ric^+=i\rho^{H,+}_{ric}=-i\frac{s_H}{6}d\alpha=\frac{1}{4}\sigma_H(\Psi)=\frac{1}{4}\sigma_H^+(\Psi)=\frac{1}{4}\sigma^+(\psi).$$

As a consequence the pair $\left(A,\psi=\sqrt{-\frac{2}{3}s_H}\,\psi_0\right)$ is a solution of the Seiberg−Witten like equations in (4.6). □

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