Letters in Mathematical Physics **39:** 213–228, 1997.

Seiberg–Witten Monopoles in Three Dimensions

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(Received: 17 October 1995, revised version: 25 September 1996)

Abstract. Dimensional reduction of the Seiberg–Witten equations leads to the equations of motion of a U(1) Chern–Simons theory coupled to a massless spinorial field. A topological quantum field theory is constructed for the moduli space of gauge equivalence classes of solutions of these equations. The Euler characteristic of the moduli space is obtained as the partition function which yields an analogue of Casson's invariant. A mathematically rigorous definition of the invariant is developed for homology spheres using the theory of spectral flow of self-adjoint Fredholm operators.

Mathematics Subject Classifications (1991): 81T13, 58D29, 81T70, 58D27.

Key words: Seiberg–Witten monopoles, Casson invariants, topological Lagrangians.

1. Introduction

The field of low-dimensional geometry and topology has undergone a dramatic period of progress in recent years, prompted, to a large extent, by new ideas and discoveries in mathematical physics. Indeed, the study of conformal field theory, quantum Chern–Simons theory and more importantly, quantum groups, reshaped knot theory and the theory of 3-manifolds; investigations of the classical Yang–Mills theory led to the creation of the Donaldson theory of four-manifolds [6]; and very recently, a set of powerful invariants of 4-manifolds, the Seiberg–Witten invariants [20], were discovered in the study of supersymmetric gauge theories.

Since its inception, the Seiberg–Witten theory has been intensively analysed by both physicists and mathematicians. Many new results have been obtained which have had a profound impact on the theory of 4-manifolds (for a concise review and references, see [11]).

The purpose of the present Letter is to explore possible applications of the Seiberg–Witten theory to 3-manifolds. It is well known that Floer cohomology [7] and Casson's invariant [1] for homological spheres have a deep connection with Donaldson theory. Therefore, it is natural for us to expect that by reducing Seiberg–Witten theory to three dimensions, analogues of Floer and Casson invariants should also arise. This is indeed the case as we shall show here.

Dimensionally reducing the Seiberg–Witten equations to a 3-manifold, we obtain the equations of motion of a U(1) Chern–Simons theory coupled to a massless spinorial field. In Section 2, we use topological field theoretical techniques to study the moduli space of the gauge equivalence classes of solutions of these equations. The Euler characteristic of the moduli space, which is obtained as the partition function of the topological field theory, can be regarded as the Seiberg–Witten version of Casson's invariant. In Section 3, we outline a mathematical approach similar to Taubes' construction of Casson's invariant using gauge theory [18]. This section contains a rigorous definition of the invariant in terms of the (mod 2) spectral flow of a family of self-adjoint Fredholm operators. The proof that this spectral flow invariant does not depend on the choice of a 'good' metric relies on contemporaneous work of Wang [19] in which the invariant of this Letter is shown to be the Euler characteristic of a Seiberg–Witten–Floer homology.

This Letter is a modified version of [5].

2. Topological Field Theory

2.1. SEIBERG–WITTEN INVARIANTS

Let X be a compact Riemannian 4-manifold with Spin_c bundles $W^{\pm}$. Denote the determinant line bundle of W^+ by L. Let A be a U(1)-connection on L and let M be a smooth section of W^+. The Seiberg–Witten monopole equations are classical field theoretical equations for A and M, which read

$$F^+ = \tfrac{1}{4}\bar{M}e_i.e_j.Me^i \wedge e^j, \qquad D_A M = 0, \tag{1}$$

where D_A is the twisted Dirac operator, $\{e_i\}_{i=1}^4$ is the orthonormal frame for TX, $\{e^i\}_{i=1}^4$ is its dual, e_i acts on a spinor by Clifford multiplication, $e_ie_j + e_je_i = -2\delta_{ij}$, and F^+ represents the self-dual part of the curvature of L with connection A.

Denote by $\mathcal{M}$ the moduli space of solutions of the Seiberg–Witten monopole equations up to gauge transformations. In general (we assume $b_2^+(X) \geqslant 2$) [8, 11, 20], after perturbing the Seiberg–Witten equations by adding a generic self-dual two-form to the curvature equation, the parametrized moduli space is a smooth compact manifold with the dimension given by

$$\dim \mathcal{M} = -\frac{2\chi(X) + 3\sigma(X)}{4} + \frac{c_1(L)^2}{4}, \tag{2}$$

where $\chi(X)$ is the Euler character of X, $\sigma(X)$ its signature index and $c_1(L)^2$ is the square of the first Chern class of L evaluated on X in the standard way. Also this moduli space is orientable in the sense that the top exterior power of the tangent bundle of the parametrized moduli space is orientable. This orientation is determined by a choice of orientations of $H^0(X, \mathbb{R}), H^1(X, \mathbb{R})$ and $H^2_+(X, \mathbb{R})$.

When $\dim \mathcal{M}$ equals zero, i.e. for $c_1(L)^2 = 2\chi(X) + 3\sigma(X)$, the moduli space consists of a finite number of points. Then the orientation of $\mathcal{M}$ assigns in a systematic way a sign ε_p to each point $p \in \mathcal{M}$ [20] and the Seiberg–Witten invariant of the 4-manifold X with respect to the given Spin_c structure is defined by

$$\sum_{p\in\mathcal{M}} \varepsilon_p, \tag{3}$$

which will depend on the differential structures of X, but not on the metric and the choice of perturbation (provided that $b_2^+ \geqslant 2$).

The Seiberg–Witten invariants can be reproduced by a topological field theory [4, 21]. Introduce a Lie superalgebra with an odd generator Q and two even generators U and δ obeying the following (anti)commutation relations

$$[U,Q] = Q, \qquad [Q,Q] = 2\delta, \qquad [Q,\delta] = 0. \tag{4}$$

We will call U the ghost number operator, and Q the BRST operator. Define the action of the superalgebra on the fields A and M by requiring that δ coincide with a gauge transformation with a gauge parameter $\phi \in \Omega^0(X,\mathbb{R})$. The field multiplets associated with A and M furnishing representations of the superalgebra are, respectively, (A,ψ,ϕ), and (M,N), where $\psi \in \Omega^1(X,i\mathbb{R})$, $\phi \in \Omega^0(X,\mathbb{R})$, and N is a section of $S^+ \otimes \mathcal{L}$. They transform under the action of the superalgebra according to

$$\begin{aligned} &[Q,A_i] = \psi_i, \qquad [Q,M] = N,\\ &[Q,\psi_i] = -i\partial_i\phi, \qquad [Q,N] = i\phi M, \qquad [Q,\phi] = 0. \end{aligned} \tag{5}$$

We assume that both A and M have ghost number 0, and thus will be regarded as bosonic fields when we study their quantum theory. The ghost numbers of other fields can be read off the above transformation rules.

In order to construct a quantum field theory which will reproduce the Seiberg–Witten invariants as correlation functions, anti-ghosts and Lagrangian multipliers are also required. We introduce the anti-ghost multiplet $(\lambda,\eta) \in \Omega^0(X,\mathbb{R})$, such that

$$[U,\lambda] = -2\lambda, \qquad [Q,\lambda] = \eta, \qquad [Q,\eta] = 0, \tag{6}$$

and the Lagrangian multipliers $(\chi,H) \in \Omega^{2,+}(X,i\mathbb{R})$, and $(\mu,\nu) \in S^- \otimes \mathcal{L}$ such that

$$\begin{aligned} &[U,\chi] = -\chi, \qquad [Q,\chi] = H, \qquad [Q,H] = 0;\\ &[U,\mu] = -\mu, \qquad [Q,\mu] = \nu, \qquad [Q,\nu] = i\phi\mu. \end{aligned} \tag{7}$$

With the given fields, we construct the following functional which has ghost number -1

$$V = \int_X \{[-\nabla_k \psi^k + (\bar{N}M - \bar{M}N)]i\lambda + \chi^{kl}(H_{kl} - F^+_{kl} - \bar{M}\Gamma_{kl}M) - \\ - \bar{\mu}(\nu - D_A M) - \overline{(\nu - D_A M)}\mu\}, \tag{8}$$

where $\Gamma_{kl} = \frac{1}{2}[e_k, e_l]$ and the indices of the tensorial fields are raised and lowered by a given Riemannian metric on X. Following the standard procedure, we take the classical action of our topological field theory to be

$$S = [Q, V], \tag{9}$$

which has ghost number 0. One can easily show that S is invariant under the BRST superalgebra (4), in particular, $[Q, S] = 0$.

The bosonic Lagrangian multiplier fields H and ν do not have any dynamics and thus can be eliminated from the action by using their equations of motion, leading to

$$S = \int_X \{[-\Delta\phi + \bar{M}M\phi - i\bar{N}N]\lambda - [-\nabla_k \psi^k + (\bar{N}M - \bar{M}N)]i\eta + \\ + 2i\phi\bar{\mu}\mu\overline{(D_A N - \gamma.\psi M)}\mu - \bar{\mu}(D_A N - \gamma.\psi M) - \\ - \chi^{kl}[(\nabla_k \psi^l - \nabla_l \psi^k)^+ + (\bar{M}\Gamma_{kl}N + \bar{N}\Gamma_{kl}M)]\} + \\ + S_0, \tag{10}$$

where S_0 is given by

$$S_0 = \int_X \{\tfrac{1}{4}|F^+ + \bar{M}\Gamma M|^2 + \tfrac{1}{2}|D_A M|^2\}. \tag{11}$$

The partition function of the quantum field theory defined by the classical action S reads

$$Z = \int \exp\left(-\frac{1}{e^2}S\right),$$

where $e \in \mathbb{R}$ is the coupling constant. It depends on neither the coupling constant nor the metric of X. In the case when the moduli space $\mathcal{M}$ is zero-dimensional, Z yields the Seiberg–Witten invariants.

2.2. DIMENSIONAL REDUCTION AND CASSON'S INVARIANT

Our aim is to study 3-manifolds using the Seiberg–Witten theory. To do that, we observe that the topological field theory constructed above is also well defined for a compact 4-manifold X *with* boundary. We take X to be of the form $Y \times [0, 1]$ with Y being a compact 3-manifold without boundary and the metric on X to be

$$(\mathrm{d}s)^2 = (\mathrm{d}t)^2 + \sum_{i,j}^{3} g_{ij}(x)\, \mathrm{d}x^i\, \mathrm{d}x^j,$$

where the t-independent functions $g_{ij}(x)$ give the Riemannian metric on Y. We assume that Y admits a spin structure which is compatible with the Spin$_c$ structure of X, i.e., if we think of Y as embedded in X, then this embedding induces maps from the Spin$_c$ bundles $S^{\pm} \otimes \mathcal{L}$ of X to $\tilde{S} \otimes \mathcal{L}$, where $\tilde{S}$ is a spin bundle and $\mathcal{L}$ is a line bundle over Y. We also impose the condition that all fields are t independent. The action now reduces to

$$\begin{aligned} S \;=\; \int \sqrt{g}\, \mathrm{d}^3 y \{ & [-\Delta\phi + \bar{M}M\phi - i\bar{N}N]\lambda - \\ & -[-\nabla_k \psi^k + (\bar{N}M - \bar{M}N)]i\eta + 2i\phi\bar{\mu}\mu + \\ & +\overline{[(D_A + ib)N - (\sigma.\psi - i\tau)M]}\mu - \bar{\mu}[(D_A + ib)N - \\ & -(\sigma.\psi - i\tau)M] - 2\chi^k[-i\partial_k\tau - *(\nabla\psi)_k - \bar{M}\sigma_k N - \bar{N}\sigma_k M] + \\ & +\tfrac{1}{4}| * F - i\partial b - \bar{M}\sigma M|^2 + \tfrac{1}{2}|(D_A + ib)M|^2 \}, \end{aligned} \qquad (12)$$

where $\sigma = \sum_{k=1}^{3} \sigma_k e^k$, and $\{\sigma_k\}_{k=1}^{3}$ are the Pauli matrices which are the representation of the Clifford multiplication for the orthonormal frame $\{e_i\}_{i=1}^{3}$ of TY

$$[\sigma_i, \sigma_j] = 2\varepsilon_{ijk}\sigma_k, \qquad \sigma_i\sigma_j + \sigma_j\sigma_i = -2\delta_{ij}.$$

The fields $b, \tau \in \Omega^0(Y, \mathbb{R})$, respectively, arose from A_0 and ψ_0 of the four-dimensional theory, while the meanings of the other fields are clear. The BRST symmetry in four-dimensions carries over to the three-dimensional theory. The BRST transformations rules for (A_i, ψ_i, ϕ), $i = 1, 2, 3$, (M, N), and (λ, η) are the same as before, but for the other fields, we have

$$\begin{aligned} &[Q, b] = \tau, \qquad [Q, \tau] = 0, \\ &[Q, \chi_k] = \tfrac{1}{2}(*F_k - i\partial_k b - \bar{M}\sigma_k M), \qquad [Q, \mu] = \tfrac{1}{2}(D_A + ib)M. \end{aligned} \qquad (13)$$

The action S is cohomological in the sense that $S = [Q, V_3]$, with V_3 being the dimensionally reduced version of V defined by (8), and $[Q, S] = 0$. Thus, it gives rise to a topological field theory upon quantization. The partition function of the theory

$$Z = \int \exp\left(-\frac{1}{e^2} S\right),$$

does not depend on either the coupling constant or the metric of Y. It can be computed exactly in the limit $e^2 \to 0$, yielding

$$Z = \sum_p \exp\left(-\frac{1}{e^2} S_{cl}^{(p)}\right) \int \exp(-S_q^{(p)}),$$

where $S_q^{(p)}$ is the quadratic part of S expanded around a classical configuration with the classical parts for the fields A, M, b being A^o, M^o, b^o, while those for all the other fields being zero. The classical action $S_{\text{cl}}^{(p)}$ is given by

$$S_{\text{cl}}^{(p)} = \int_Y \{\tfrac{1}{4}| *F^o - i\,\mathrm{d}b^o - \bar{M}^o \sigma M^o|^2 + \tfrac{1}{2}|(D_{A^o} + b^o)M^o|^2\},$$

which can be rewritten as

$$S_{\text{cl}}^{(p)} = \int_Y \{\tfrac{1}{4}| *F^o - \bar{M}^o \sigma M^o|^2 + \tfrac{1}{2}|D_{A^o} M^o|^2 + \tfrac{1}{2}|\,\mathrm{d}b^o|^2 + \tfrac{1}{2}|b^o M^o|^2\}.$$

In order for the classical configuration to have nonvanishing contributions to the partition function, all the terms in $S_{\text{cl}}^{(p)}$ should vanish separately. Therefore,

$$*F^o - \bar{M}^o \sigma M^o = 0, \qquad D_{A^o} M^o = 0, \quad \text{and} \quad b^o = 0, \tag{14}$$

where the last condition requires some explanation. When we have a trivial solution of Equations (14), i.e., $M^o = 0$, then the requirement for b^o can be replaced by the less stringent condition $\mathrm{d}b^o = 0$.

Let us define an operator

$$\begin{aligned} &\hat{T}\colon \Omega^0(Y, \mathbb{R}) \oplus \Omega^1(Y, i\mathbb{R}) \oplus (\hat{S} \otimes L) \to \Omega^0(Y, \mathbb{R}) \oplus \Omega^1(Y, i\mathbb{R}) \oplus (\hat{S} \otimes L), \\ &(\tau, \psi, N) \mapsto (\mathrm{d}^*\psi + (\bar{N}M - \bar{M}N), \qquad *(\mathrm{d}\psi) - i\,\mathrm{d}\tau - \bar{N}\sigma M - \bar{M}\sigma N, \\ &D_A N - (\sigma.\psi - i\tau)M), \end{aligned} \tag{15}$$

where the complex bundle $\hat{S} \otimes L$ should be regarded as a real one with twice the rank. This operator can be shown to be self-adjoint Fredholm. In terms of $\hat{T}$, the equations of motion of the fields χ^i and μ can be expressed as

$$\hat{T}^{(p)}(\tau, \psi, N) = 0, \tag{16}$$

where $\hat{T}^{(p)}$ is the operator $\hat{T}$ with the background fields (A^o, M^o) belonging to the gauge class p of classical configurations.

When the kernel of $\hat{T}$ is zero, the moduli space $\mathcal{M}$ of the gauge equivalence classes of solutions of Equation (14) is generically of dimension 0. A result of [8] asserts that $\mathcal{M}$ is compact, thus consisting of a finite number of isolated points. In this case, the partition function Z does not vanish identically. An easy computation leads to

$$Z = \sum_{p \in \mathcal{M}} \varepsilon^{(p)}, \tag{17}$$

where $\varepsilon^{(p)}$ is the sign of the determinant of $\hat{T}^{(p)}$.

For a 3-manifold Y with nontrivial Spin^C structure, we can regard the partition function (17) as the Seiberg–Witten analogue of Casson's invariant, since it is the Euler characteristic of the Seiberg–Witten–Floer group [11]. However, for a homology sphere, we find that this group and the analogue of Casson's invariant depend on the metric and perturbation. In this case, we need a wall-crossing formula to see how the Casson invariant depends on the chamber structure in the space of metrics and perturbations. We refer the reader to [12] for an analysis of this wall-crossing formula.

Needless to say, our approach to this invariant is heuristic; a mathematically rigorous definition for it is required [11, 12, 19]. We will sketch such a definition in the next section, but first we examine the geometric meaning of (17).

2.3. GEOMETRICAL INTERPRETATION

To elucidate the geometric meaning of the three-dimensional theory, we now cast it into the framework of Atiyah and Jeffrey [2]. Let us briefly recall the geometric set-up of the Mathai–Quillen formula as reformulated in reference [2]. Let P be a Riemannian manifold of dimension $2m + \dim G$, and G be a compact Lie group acting on P by isometries. Then $P \to P/G$ is a principle bundle. Let V be a $2m$-dimensional real vector space which furnishes a representation $G \to \mathrm{SO}(2m)$. Form the associated vector bundle $P \times_G V$. Now the Thom form of $P \times_G V$ can be expressed as

$$\begin{aligned} U \;=\; & \frac{\exp(-x^2)}{(2\pi)^{\dim G}\pi^m} \times \\ & \times \int \exp\left\{ \frac{i\chi\phi\chi}{4} + i\chi\,\mathrm{d}x - i\langle \delta\nu, \lambda\rangle - \langle \phi, R\lambda\rangle + \langle \nu, \eta\rangle \right\} \times \\ & \times \mathcal{D}\eta\mathcal{D}\chi\mathcal{D}\phi\mathcal{D}\lambda, \end{aligned} \tag{18}$$

where $x = (x^1, ..., x^{2m})$ is the coordinates of V, ϕ and λ are bosonic variables in the Lie algebra g of G, and η and χ are Grassmannian variables valued in the Lie algebra and the tangent space of the fiber, respectively. In the above equation, C maps any $\eta \in g$ to the element of the vertical part of TP generated by η; ν is the g-valued one form on P defined by $\langle \nu(\alpha), \eta\rangle = \langle \alpha, C(\eta)\rangle$, for all vector fields α; and $R = C^*C$. Also, δ is the exterior derivative on P.

Now we choose a G invariant map $s\colon P \to V$, and pull back the Thom form U. Then the top form on P in s^*U is the Euler class. If $\{\delta p\}$ forms a basis of the cotangent space of P (note that ν and δs are one forms on P), we replace it by a set of Grassmannian variables $\{\psi\}$ in s^*U, then integrate them away. We arrive at

$$\frac{1}{(2\pi)^{\dim G}\pi^m}\times$$
$$\times \int \exp\left\{-|s|^2 + \frac{i\chi\phi\chi}{4} + i\chi\delta s - i\langle\delta\nu,\lambda\rangle - \langle\phi, R\lambda\rangle + \langle\psi, C\eta\rangle\right\}\times$$
$$\times\, \mathcal{D}\eta\mathcal{D}\chi\mathcal{D}\phi\mathcal{D}\lambda\mathcal{D}\psi, \tag{19}$$

the precise relationship of which, with the Euler character of $P\times_G V$, is

$$\int_P (19) = \mathrm{Vol}(G)\chi(P\times_G V).$$

We wish to show that the partition function of the three-dimensional theory yields the Euler number of $\mathcal{W} = (\mathcal{A}\times(\tilde{S}\otimes\mathcal{L}))/\mathcal{G}$. Let P be a principal bundle over P/G, V, V' be two orthogonal represention of G. Suppose there is an embedding from $P\times_G V'$ to $P\times_G V$ via a G-map $\gamma(p)\colon V'\to V$ for $p\in P$. Denote the resulting quotient bundle as E. In order to derive the Thom class for E, we need to choose a section of E, or equivalently, a G-map $s\colon P\to V$ such that $s(p)\in(\mathrm{Im}\,\gamma(p))^{\perp}$. Then the Euler class of E can be expressed as $\pi_*\rho^*U$, where U is the Thom class of $P\times_G V$, ρ is a G-map: $P\times V'\to P\times V$ defined by

$$\rho(p,\tau) = (p, \gamma(p)\tau + s(p)),$$

and π_* is the integration along the fiber for the projection $\pi\colon P\times V'\to P/G$. Explicitly,

$$\pi_*\rho^*(U)$$
$$= \int \exp\{-|\gamma(p)\tau + s(p)|^2 + i\chi\phi\chi + i\chi\delta(\gamma(p)\tau + s(p)) -$$
$$- i\langle\delta\nu,\lambda\rangle - \langle\phi, R\lambda\rangle + \langle\nu, C\eta\rangle\}\mathcal{D}\chi\mathcal{D}\phi\mathcal{D}\tau\mathcal{D}\eta\mathcal{D}\lambda. \tag{20}$$

Consider the exact sequence

$$0 \longrightarrow (\mathcal{A}\times\Gamma(W))\times_{\mathcal{G}}\Omega^0(Y,\mathbb{R})$$
$$\xrightarrow{j} (\mathcal{A}\times\Gamma(W))\times_{\mathcal{G}}(\Omega^1(Y,i\mathbb{R})\times\Gamma(W))$$

where $j_{(A,M)}\colon b\mapsto(-i\,\mathrm{d}b, ibM)$. (We assume that $M\neq 0$.) Then the tangent bundle of $\mathcal{A}\times_{\mathcal{G}}\Gamma(W)$ can be regarded as the quotient bundle

$$(\mathcal{A}\times\Gamma(W))\times_{\mathcal{G}}(\Omega^1(Y,i\mathbb{R})\times\Gamma(W))/\mathrm{Im}(j).$$

We define a vector field on $\mathcal{A} \times_{\mathcal{G}} \Gamma(W)$ by

$$s(A, M) = (*F_A - \bar{M}\sigma M, D_A M),$$

which lies in $(\mathrm{Im}\, j)^{\perp}$

$$-\int_Y (*F_A - \bar{M}\sigma M) \wedge *(-i\,\mathrm{d}b) + \\ + \int_Y \sqrt{g}\,\mathrm{d}^3 y(\langle D_A M, ibM\rangle + \langle ibM, D_A M\rangle) = 0, \qquad (21)$$

where we have used the notation $\langle M_1, M_2\rangle = \bar{M}_1 M_2$.

Formally applying formula (20) to the present infinite-dimensional situation, we obtain the Euler class $\pi_* \rho^*(U)$ for the tangent bundle $T(\mathcal{A} \times_{\mathcal{G}} \Gamma(W))$, where ρ is the $\mathcal{G}$-invariant map ρ is defined by

$$\rho\colon \Omega^0(Y, \mathbb{R}) \to \Omega^1(Y, i\mathbb{R}) \times \Gamma(W),$$

$$\rho(b) = (-i\,\mathrm{d}b + *F_A - \bar{M}\sigma M, (D_A + ib)M),$$

π is the projection

$$(\mathcal{A} \times \Gamma(W)) \times_{\mathcal{G}} \Omega^0(Y, \mathbb{R}) \to \mathcal{A} \times_{\mathcal{G}} \Gamma(W),$$

and π_* signifies the integration along the fiber. Also, U is the Thom form of the bundle

$$(\mathcal{A} \times \Gamma(W)) \times_{\mathcal{G}} (\Omega^1(Y, i\mathbb{R}) \times \Gamma(W)) \to \mathcal{A} \times_{\mathcal{G}} \Gamma(W).$$

To understand the Thom form more concretely, we need to explain the geometry of this bundle. The metric on Y and the Hermitian metric $\langle .\,,\,.\rangle$ on $\Gamma(W)$ naturally defines a connection. The Maurer–Cartan connection on $\mathcal{A} \to \mathcal{A}/\mathcal{G}$ is flat, while the Hermitian connection has the curvature $i\phi\mu \wedge \bar{\mu}$. This gives the expression for the term $i(\chi, \mu)\phi(\chi, \mu)$ in (19) in our case.

In our infinite-dimensional setting, the map C is given by

$$C\colon \Omega^0(Y, \mathbb{R}) \to T_{(A,M)}(\mathcal{A} \times \Gamma(W)),$$

$$C(\eta) = (-i\,\mathrm{d}\eta, i\eta M)$$

and its dual is given by

$$C^*\colon \Omega^1(Y, i\mathbb{R}) \times \Gamma(W) \to \Omega^0(Y, \mathbb{R}),$$

$$C^*(\psi, N) = i(\mathrm{d}^*\psi - \bar{M}N + \bar{N}M)i.$$

The one form $\langle \nu, \eta \rangle$ on $\mathcal{A} \times \Gamma(W)$ takes the value

$$\langle (\psi, N), C\eta \rangle = \int_Y i\eta(\mathrm{d}^*\psi - \bar{M}N + \bar{N}M)\sqrt{g}\,\mathrm{d}^3y$$

on the vector field (ψ, N). We also easily obtain $R(\lambda) = -\Delta\lambda + \bar{M}M\lambda$, where $\Delta = \mathrm{d}^*\,\mathrm{d}$. Then $\langle \delta\nu, \lambda \rangle$ is a 2-form on $\mathcal{A} \times \Gamma(W)$ whose value on $(\psi_1, N_1), (\psi_2, N_2)$ is $-(\bar{N}_1 N_2 + \bar{N}_2 N_1)\lambda$.

Combining all the information together, we arrive at the formula

$$\begin{aligned} \pi_*\rho^*(U) \;=\; \int \exp\{&-\tfrac{1}{2}|\rho|^2 + i(\chi, \mu)\delta\rho + 2i\phi\mu\bar{\mu} + \\ &+ \langle \Delta\phi, \lambda \rangle - \phi\lambda\langle M, M \rangle + i\langle N, N \rangle\lambda + \langle \nu, \eta \rangle\} \times \\ &\times \mathcal{D}\chi\mathcal{D}\phi\mathcal{D}\lambda\mathcal{D}\eta\mathcal{D}b. \end{aligned} \tag{22}$$

Note that the 1-form $i(\chi, \mu)\delta\rho$ on $\mathcal{A} \times \Gamma(W) \times \Omega^0(Y, \mathbb{R})$ contracted with the vector field (ϕ, N, b), leads to

$$\begin{aligned} &2\chi^k\,[-i\partial_k\tau - *(\nabla\psi)_k - \bar{M}\sigma_k N - \bar{N}\sigma_k M] + \\ &\quad +2\,\mathrm{Re}\langle \mu, [(D_A + ib)N - (\sigma.\psi - i\tau)M] \rangle \end{aligned} \tag{23}$$

and relation (21) gives

$$|\rho|^2 = |*F - \bar{M}\sigma M|^2 + |\mathrm{d}b|^2 + |D_A M|^2 + b^2|M|^2.$$

Finally, we obtain the Eular character

$$\pi_*\rho^*(U) = \int \exp(-S)\mathcal{D}\chi\mathcal{D}\phi\mathcal{D}\lambda\mathcal{D}\eta\mathcal{D}b, \tag{24}$$

where S is the action (12) of the three-dimensional theory defined on the manifold Y. Integrating (24) over $\mathcal{A} \times_{\mathcal{G}} \Gamma(W)$ leads to the generalised Euler number.

3. A Spectral Flow Definition of the Three-Manifold Invariant

In this section, we will sketch the main idea on how to use Seiberg–Witten monopoles to define an invariant analogous to that of Casson for homology spheres. A key proof depends on the contemporaneous work of [19] on a Seiberg–Witten–Floer complex.

Let Y be a closed, oriented, compact three-dimensional Riemannian manifold with the homology of a 3-sphere. We choose the L^2_1-metric on the configuration space $\mathcal{C}(Y) = \mathcal{A} \times \Gamma(W)$, where $\mathcal{A}$ is an L^2_1-connection space on $\det(W)$ and

$\Gamma(W)$ is the L^2_1-sections of the spinor bundle W. The L^2-norm for the tangent space of $\mathcal{A}$ is given by

$$\langle\psi_1,\psi_2\rangle_{L^2} = -\int_Y \psi_1 \wedge *\psi_2,$$

where $\psi_1,\psi_2 \in \Omega^1(Y,i\mathbb{R})$ and $*$ is the Hodge star operator on Y. The L^2-metric for the spinors is given by the integration of twice the real part of the Hermitian metric $\langle N_1,N_2\rangle = \bar{N}_1 N_2$ on W as follows:

$$\langle N_1,N_2\rangle_{L^2} = \int_Y (\bar{N}_1 N_2 + \bar{N}_2 N_1)\sqrt{g}\,\mathrm{d}^3y.$$

Within this setting, we can discuss the Morse theory on $\mathcal{C}(Y)/\mathcal{G} = \mathcal{B}$ with the quotient topology where $\mathcal{G}$ is the L^2_2-gauge transformation group. Since the gauge transformation acts nonfreely on the reducible points, we denote by $\mathcal{R}$ the set of reducible points (where the spinor part is zero), and let $\mathcal{B}^* = (\mathcal{C}(Y)\backslash\mathcal{R})/\mathcal{G}$. Then we can think of $\mathcal{B}^*$ as an infinite-dimensional manifold with the tangent space at $[A,M]$ given by the orthogonal complement of $T_{(A,M)}(\mathcal{G}.(A,M))$ in $T_{(A,M)}(\mathcal{C}(Y))$, i.e.

$$T_{[A,M]}(\mathcal{B}^*) = \{(\psi,N) \in \Omega^1(Y,i\mathbb{R}) \oplus \Gamma(W) | \mathrm{d}^*\psi = \bar{M}N - \bar{N}M\}.$$

Note that Equation (14) is just the equation of motion of a U(1) Chern–Simons theory coupled to a massless spinor, arising from the critical points of the following functional

$$C(A,M) = \int_Y (A-A_0)\wedge F_A + \int_Y \sqrt{g}\,\mathrm{d}^3y\bar{M}D_AM, \tag{25}$$

where A_0 is a fixed connection.

For a homology sphere Y, the functional (25) descends to $\mathcal{B}$ as a function. We can define an L^2-gradient vector field ∇C on $\mathcal{B}$ which is formally defined on $\mathcal{B}$ as the vector

$$\nabla C_{[A,M]} = (*F_A - \bar{M}\sigma M, D_AM). \tag{26}$$

It is easy to check that $(*F_A - \bar{M}\sigma M, D_AM)$ is orthogonal to the tangent space to the gauge orbit through (A,M) due to the fact that

$$\mathrm{d}^*(\bar{M}\sigma M) = \langle D_AM, M\rangle - \langle M, D_AM\rangle.$$

Therefore, $\nabla C_{[A,M]}$ is an L^2-section of the L^2-tangent bundle on $\mathcal{B}$, its zeros on $\mathcal{B}$ are the gauge orbits of the solutions for the following equations:

$$*F_A = \bar{M}\sigma M, \qquad D_AM = 0. \tag{27}$$

These equations are the reduced Seiberg–Witten equations on a three-dimensional manifold Y.

Fix an open interval $I = [0,1]$, write the configuration $(A(t), M(t))$ in a temporal gauge for the Spin$_c$ manifold $Y \times [0,1]$ and think of $(A(t), M(t))$ as a path in $\mathcal{C}(Y)$, then the downward gradient flow equation of C

$$\frac{\partial}{\partial t}(A(t), M(t)) = \nabla C_{[A(t),M(t)]} \tag{28}$$

is the Seiberg–Witten equation for $(A(t), M(t))$ on $Y \times [0,1]$ in the temporal gauge ($A_0 = 0$),

$$\frac{\mathrm{d}A}{\mathrm{d}t} = *F_A - \bar{M}\sigma M, \qquad \frac{\mathrm{d}M}{\mathrm{d}t} = D_A M. \tag{29}$$

In order to make the gradient flow run between the critical points in infinite time, we put the following finite energy condition on the solution of (29) (the Seiberg–Witten equations on $Y \times \mathbb{R}$)

$$\int_{-\infty}^{+\infty} \left(\left\| \frac{\mathrm{d}A}{\mathrm{d}t} \right\|^2_{L^2(Y)} + \left\| \frac{\mathrm{d}M}{\mathrm{d}t} \right\|^2_{L^2(Y)} \right) \mathrm{d}t < \infty.$$

We rely on a result of [19] that the solution $[A(t), M(t)]$ of (29) decays exponentially in the L^2_1-topology to the solutions of (27) as $|t| \to \infty$.

The Hessian operator for C at any $[A, M]$ is the covariant derivative of ∇C at $[A, M]$

$$\begin{aligned} &T\colon T_{[A,M]}(\mathcal{B}) \to T_{[A,M]}(\mathcal{B}), \\ &(\psi, N) \mapsto (*\,\mathrm{d}\psi - \bar{N}\sigma M - \bar{M}\sigma N - i\,\mathrm{d}\tau, D_A N - (\sigma.\psi - i\tau)M), \end{aligned} \tag{30}$$

where the $\tau \in \Omega^0(Y, \mathbb{R})$ is the unique solution for the equation

$$\mathrm{d}^*\,\mathrm{d}\tau + 2\tau\bar{M}M = i(\bar{N}D_A M - \overline{D_A M}N).$$

Note that at any solution $[A, M]$ for (27) on $\mathcal{B}^*$, $\tau = 0$. Therefore, the Hessian operator at the critical point of C is precisely the linearisation of the Seiberg–Witten equations on Y. From the Weitzenbock formula, Sobolev inequalities and the Rellich lemma, we know that T is a closed, self-adjoint Fredholm operator from the L^2_1-completion of $T_{[A,M]}(\mathcal{B})$ to the L^2-completion of $T_{[A,M]}(\mathcal{B})$, and has a discrete spectrum without accumulation points.

The previous discussion suggests that there should be a Seiberg–Witten invariant for homology spheres whose development parallels that of Floer instanton theory. In order to use the Morse theory for the infinite-dimensional manifold $\mathcal{B}$, we need to perturb the gradient vector field (26) such that the critical points for the perturbed functional are nondegenerate.

There is a standard perturbation of the curvature equation

$$*F_A = \bar{M}\sigma M + *\,\mathrm{d}\beta, \qquad D_A M = 0, \tag{31}$$

where β is an imaginary-valued 1-form on Y.

It is easy to see that for a homology sphere the reducible solution for (31) is the orbit through $(\beta, 0)$. The perturbed functional is

$$C'(A, M) = \int_Y (A - A_0) \wedge (F_A - 2\beta) + \int_Y \sqrt{g}\,\mathrm{d}^3 y(\bar{M}(D_A M). \tag{32}$$

There is an interesting phenomenon for Seiberg–Witten theory in the homology sphere case. To make the invariant and the Floer cohomology group well-defined, we need a condition on the perturbing form β such that Dirac operator D_β has no kernel. The set of D_β having nontrivial kernel is a subset of codimension greater than or equal to one in the space of perturbing forms. Moreover, the condition: $\ker D_\beta = 0$ depends on the metric. Let us call the metric 'good' if $\ker D_\beta = 0$.

LEMMA. *Suppose D_β has no kernel, then the reducible solution for* (31) *is isolated in $\mathcal{B}$, and nondegenerate as a critical point.*

Proof. This is a straightforward result by the Kuranishi model for the singular point. However, we can also prove this lemma by a direct calculation. Suppose a solution, near $(\beta, 0)$, is $(\beta + \omega, M)$, so that

$$\mathrm{d}^*\omega = 0, \qquad *\,\mathrm{d}\omega = \bar{M}\sigma M, \qquad D\beta(M) + \omega.M = 0,$$

where

$$\omega = \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots, \qquad M = \varepsilon M_1 + \varepsilon^2 M_2 + \cdots,$$

for a very small ε. Then we have $D_\beta M_1 = 0$, which implies $M_1 = 0$ since $\ker(D_\beta) = 0$. By the curvature equation in (31) and $\mathrm{d}^*\omega = 0$, we get $\omega_1 = \omega_2 = \omega_3 = 0$, these enforce that $M_2 = M_3 = M_4 = 0$, the procedure continues to show that (ω, M) must be a trivial solution. Therefore, the reducible solution is isolated. The nondegenerate property follows from the fact that Y is a homology sphere and $\ker(D_\beta) = 0$.

Applying the Sard–Smale theory, we can find β (sufficiently small) such that all the critical points for (32) are nondegenerate and regular in $\mathcal{B}^*$, with only the one reducible critical point which is nondegenerate (and isolated). We denote $\mathcal{M}^* = \mathcal{M}\backslash[\beta, 0]$.

That $\mathcal{M}^*$ is regular means $\mathcal{M}^*$ is a smooth 0-dimensional manifold, i.e., the linearisation is surjective at that solution in $\mathcal{B}^*$.

We claim that $\mathcal{M}$ is sequentially compact and, hence, so too is $\mathcal{M}^*$, which implies that $\mathcal{M}^*$ is a finite set of points in $\mathcal{B}^*$.

To illustrate the compactness of the moduli space for the generic perturbed Seiberg–Witten equations, we show that there exist a-priori bounds for the solutions. Start with the Weitzenbock formula for the Dirac operator

$$D_A^* D_A M = \nabla_A^* \nabla_A M + \tfrac{1}{4} s M + \tfrac{1}{2} * F_A . M, \tag{33}$$

where s is the scalar curvature of Y for the given Riemannian metric g. Pairing (33) with M pointwisely, we obtain a differential inequality

$$\tfrac{1}{2}\,\mathrm{d}^*\,\mathrm{d}|M|^2 + |\nabla_A M|^2 + \tfrac{1}{4}|M|^4 < C_0 |M|^2 \tag{34}$$

for some constant C_0 (depending on the metric g). The maximum principle gives the pointwise bound for M and L^2-bound for $\nabla_A M$,

$$|M(x)|^2 < C_0, \qquad ||\nabla_A M||_{L^2}^2 < C_0^2\,\mathrm{Vol}(Y).$$

These bounds, combined with the standard bootstrapping arguments, show that the moduli space to the perturbed Equations (31) is compact. This is discussed in more detail in [19].

Up to an overall sign, we can define the Seiberg–Witten analogue of Casson's invariant [1] for a homology 3-sphere Y by

$$|\lambda(Y,g)| = \left| \sum_{[A,M]\in\mathcal{M}^*} (-1)^{\varepsilon^0_{[A,M]}} \right|, \tag{35}$$

where $\varepsilon^0_{[A,M]}$ is the mod 2 spectral flow of a family self-adjoint operators T (which is the linearisation of (31)) connecting the fixed zero point $[A_0, M_0]$ to $[A, M]$ in $\mathcal{M}^*$.

We now follow Taubes [18] to define this mod 2 spectral flow. Choose a path $\gamma\colon [0,1] \to \mathcal{W} = \mathcal{A} \times_{\mathcal{G}} \Gamma(W)^{\times}$ connecting $[A_0, M_0]$ and $[A, M]$, where (A_0, M_0) and (A_1, M_1) are two nondegenerate critical points of C'. Then T_γ defines a family of bounded self-adjoint, index zero Fredholm operators. The path γ is chosen to be generic, i.e., T_γ intersects transversely with the subspace of operators which have nonempty kernel. Note that this subspace is of codimension at least 1 in the space of the bounded self-adjoint operators [9], and the operators $T_t|_{t=0,1}$ have trivial kernels. We can always perturb the path so that T_γ intersects that subspace at a finite number of points. Define

$$\begin{aligned} \varepsilon^0_{[A,M]} &= (\mathrm{mod}\,2) \quad \text{spectral flow of } T_\gamma \\ &= \#\{t | \mathrm{Ker}\, T_t \neq 0\}\ (\mathrm{mod}\,2). \end{aligned} \tag{36}$$

Note that $\varepsilon^0_{[A,M]}$ is independent of the chosen path for the homology sphere Y, since T_γ pulls back the first Stiefel–Whitney class ω_1 in $H^1(\mathcal{W}, \mathbb{Z}_2)$, which is zero.

In order to determine the sign of the invariant $\lambda(Y, g)$, we extend the operator T (30) to the whole of the configuration space $\mathcal{A} \times_{\mathcal{G}} \Gamma(W)$. Choose a sufficiently small $\varepsilon > 0$ such that $-\varepsilon$ is not in the spectrum of T at $\mathcal{M}^*$ and let $[A^0, 0]$ denote a reducible point where A^0 is the trivial connection. Then substitute for T, the operator $T + \varepsilon$ in the above definition of the spectral flow. Denote this modified spectral flow by sf_ε. The standard topological argument [3] shows that sf_ε is independent of the chosen ε. Denote by $sf_\varepsilon([A^0, 0], [A, M])$, this spectral flow from the trivial solution to the irreducible solution $[A, M] \in \mathcal{M}^*$. Now we define $\lambda(Y, g)$ as

$$\lambda(Y, g) = \sum_{[A,M]\in\mathcal{M}^*} (-1)^{sf_\varepsilon([A^0,0],[A,M])}. \tag{37}$$

Now we come to the question of whether $\lambda(Y, g)$ is dependent on metric and perturbation parameter β. If Y has first Betti number $\geqslant 1$, then by suitably perturbating we find that the Chern–Simons type functional (25) has no reducible critical points even for the trivial Spin^c structure. Thus, $\lambda(Y, g)$ is independent of metric g and the perturbation β. In the homology sphere case, the reducible critical points are always present. We need the reducible solution to be nondegenerate and from Lemma 3 we know that this means D_β must have trivial kernel. We call (g, β) a 'good' pair if the corresponding Dirac operator D_β has no kernel. The space of 'good' metrics and perturbations consists of components separated by the 'wall' (where (g, β) is such that $\mathrm{Ker}(D_\beta)$ is nontrivial). A standard cobordism argument shows that $\lambda(Y, g)$ does not depend on the choice of good pairs (metric and perturbation) in the same chamber, but clearly we cannot assert that it has the same value on different connected components of the space of good metrics. To relate our Casson invariant on each side of the wall, we need to prove a 'wall-crossing formula'. This problem is solved in [12] by using an equivariant Seiberg–Witten–Floer homology theory. However, a full explanation would be too long for us to include here. For this Casson invariant $\lambda(Y, g)$, Kronheimer proposed a conjecture which relates it to the usual Casson invariant in instanton theory: let X be a four-dimensional spin manifold bounded by a homology sphere Y, then the instanton Casson invariant is expected to be

$$\lambda(Y, g) - (\mathrm{Ind}_{\mathbb{C}} D_X + 1/8\sigma(X)),$$

where D_X is the Dirac operator on X, $\sigma(X)$ is the signature of X. It would be interesting to prove this conjecture, in particular, to understand it using the Mathai–Quillen formalism in quantum topological field theory as in Section 2.3.

Acknowledgements

We thank the referee for detecting an oversight in an earlier version of this Letter (we did not notice that the reducible point could not be shown to be isolated by standard arguments when the metric was not ‘good’). This research was supported by the Australian Research Council.

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