

SEIBERG-WITTEN PREPOTENTIAL FROM INSTANTON COUNTING

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Abstract

Direct evaluation of the Seiberg-Witten prepotential is accomplished following the localization programme suggested in [1]. Our results agree with all low-instanton calculations available in the literature. We present a two-parameter generalization of the Seiberg-Witten prepotential, which is rather natural from the M-theory/five dimensional perspective, and conjecture its relation to the tau-functions of KP/Toda hierarchy.

To Arkady Vainshtein on his 60th anniversary

M.Jourdain: *Par ma foi!
il y a plus de quarante ans
que je dis de la prose
sans que j'en susses rien ;
et je vous suis le plus obligé du monde
de m'avoir appris cela....*

*Le Bourgeois gentilhomme.
J. -B. Molière*

1 Introduction

The dynamics of gauge theories is a long and fascinating subject. The dynamics of supersymmetric gauge theories is a subject with shorter history. However, more facts are known about susy theories, and with better precision [2] yet with rich enough applications both in physics and mathematics. In particular, the solution of Seiberg and Witten [3] of $\mathcal{N} = 2$ gauge theory using the constraints of special geometry of the moduli space of vacua led to numerous achievements in understanding of the strong coupling dynamics of gauge theory and as well as string theory backgrounds of which the gauge theories in question arise as low energy limits. The low energy effective Wilsonian action for the massless vector multiplets (a_i) is governed by the prepotential $\mathcal{F}(a; \Lambda)$, which receives one-loop perturbative and instanton non-perturbative corrections (here Λ is the dynamically generated scale):

$$\mathcal{F}(a; \Lambda) = \mathcal{F}^{pert}(a; \Lambda) + \mathcal{F}^{inst}(a; \Lambda) \quad (1.1)$$

In spite of the fact that these instanton corrections were calculated in many indirect ways, their gauge theory calculation is lacking beyond two instantons[4][5]. The problem is that the instanton measure seems to get very complicated with the growth of the instanton charge, and the integrals are hard to evaluate.

The present paper attempts to solve this problem via the localization technique, proposed long time ago in [1][6][7]. Although we tried to make the paper readable to both mathematicians and physicists we don't expect it to be quite understandable without some background material, which we suggest to look up in [3][8][9].

Notations. Let G be a semi-simple Lie group, T is maximal torus, $\mathfrak{g} = Lie(G)$ its Lie algebra, $\mathfrak{t} = Lie(T)$ its Cartan subalgebra, $W = N(T)/T$ denote its Weyl group, $\mathcal{U} = (\mathfrak{t} \otimes \mathbb{C})/W$ denotes the complexified space of

conjugacy classes in \mathfrak{g} . We consider the moduli space $M_k(G)$ of framed G -instantons: the anti-self-dual gauge fields A , $F_A^+ = 0$, in the principal G -bundle \mathcal{P} over the 4-sphere $\mathbf{S}^4 = \mathbb{R}^4 \cup \infty$ with

$$k = -\frac{1}{8h\pi^2} \int_{\mathbb{R}^4} \text{Tr } F_A \wedge F_A \tag{1.2}$$

considered up to the gauge transformations $g : A \mapsto g^{-1}Ag + g^{-1}dg$, s.t. $g(\infty) = 1$. We also consider several compactifications of the space $M_k(G)$: the Uhlenbeck compactification $\tilde{M}_k(G)$ and the Gieseker compactification $\widetilde{\mathcal{M}}_k$ for $G = U(N)$ or $SU(N)$. In the formula (1.2) we use the trace in the adjoint representation, and h stands for the dual Coxeter number of G .

Field theory description. We calculate vacuum expectation value of certain gauge theory observables. These observables are annihilated by a combination of the supercharges, and their expectation value is not sensitive to various parameters, the energy scale in particular. Hence, one can do the calculation in the ultraviolet, where the theory is weakly coupled and the instantons dominate. Or, one can do the calculation in the infrared, and relate the answer to the prepotential of the effective low-energy theory. By equating these two calculations we obtain the desired formula.

Mathematical description. We study $G \times \mathbf{T}^2$ equivariant cohomology of the moduli space $\widetilde{\mathcal{M}}_k$, where G acts by rotating the gauge orientation of the instantons at infinity, and \mathbf{T}^2 is the maximal torus of $SO(4)$ – the group of rotations of \mathbb{R}^4 which also acts naturally on the moduli space². Let $p : \widetilde{\mathcal{M}}_k \rightarrow pt$ be the map collapsing the moduli space to a point. We consider the following generating function:

$$Z^{inst}(a, \epsilon_1, \epsilon_2; q) = \sum_{k=0}^{\infty} q^k \oint_{\widetilde{\mathcal{M}}_k} 1 \tag{1.3}$$

where $\oint 1$ denotes the *localization* of the pushforward p_*1 of $1 \in H_{G \times \mathbf{T}^2}^*(\widetilde{\mathcal{M}}_k)$ in $H_{G \times \mathbf{T}^2}^*(pt) = \mathbb{C}[\mathcal{U}, \epsilon_1, \epsilon_2]$. We denote the coordinates on \mathfrak{t} by a and the coordinates on the Lie algebra of \mathbf{T}^2 by ϵ_1, ϵ_2 . In explicit calculations³ we represent 1 by a cohomologically equal form which allows to replace $\oint 1$ by an ordinary integral:

²Throughout the paper we mostly consider the $SU(N)$ instantons (or $U(N)$ noncommutative instantons). We use the notation $\widetilde{\mathcal{M}}_{k,N}$ when we want to emphasize that the gauge group is $U(N)$.

³For $G = SU(N)$ we actually use $a = (a_1, \dots, a_N)$ s.t. $\sum_l a_l = 0$

$$\oint_{\widetilde{\mathcal{M}}_k} 1 = \int_{\widetilde{\mathcal{M}}_k} \exp [\omega + \mu_G(a) + \mu_{\mathbf{T}^2}(\epsilon_1, \epsilon_2)] \tag{1.4}$$

where ω is a symplectic form on $\widetilde{\mathcal{M}}_k$, invariant under the $G \times \mathbf{T}^2$ action, and $\mu_G, \mu_{\mathbf{T}^2}$ are the corresponding moment maps.

Our first claim is

$$Z^{inst}(a, \epsilon_1, \epsilon_2; q) = \exp \left(\frac{\mathcal{F}^{inst}(a, \epsilon_1, \epsilon_2; q)}{\epsilon_1 \epsilon_2} \right)$$

(1.5)

where the function \mathcal{F}^{inst} is analytic in ϵ_1, ϵ_2 near $\epsilon_1 = \epsilon_2 = 0$.

We also have the following explicit expression for Z in the case⁴ $\epsilon_1 = -\epsilon_2 = \hbar$ for⁵ $G = SU(N)$:

$$Z^{inst}(a, \hbar, -\hbar; q) = \sum_{\vec{\mathbf{k}}} q^{|\mathbf{k}|} \prod_{(l,i) \neq (n,j)} \frac{a_{ln} + \hbar(k_{l,i} - k_{n,j} + j - i)}{a_{ln} + \hbar(j - i)}$$

(1.6)

Here $a_{ln} = a_l - a_n$, the sum is over all colored partitions: $\vec{\mathbf{k}} = (\mathbf{k}_1, \dots, \mathbf{k}_N)$, $\mathbf{k}_l = \{k_{l,1} \geq k_{l,2} \geq \dots k_{l,n_l} \geq k_{l,n_l+1} = k_{l,n_l+2} = \dots = 0\}$,

$$|\vec{\mathbf{k}}| = \sum_{l,i} k_{l,i} ,$$

and the product is over $1 \leq l, n \leq N$, and $i, j \geq 1$.

Already (1.6) can be used to make rather powerful checks of the Seiberg-Witten solution. But the checks are more impressive when one considers the theory with fundamental matter. To get there one studies the bundle V over $\widetilde{\mathcal{M}}_k$ of the solutions of the Dirac equation in the instanton background. Let us consider the theory with N_f flavors. It can be shown that the gauge theory instanton measure calculates in this case (cf. [11]):

$$Z(a, m, \epsilon_1, \epsilon_2; q) = \sum_k q^k \oint_{\widetilde{\mathcal{M}}_k} \text{Eu}_{G \times \mathbf{T}^2 \times U(N_f)}(V \otimes M) \tag{1.7}$$

⁴in the general case we also have a formula, but it looks less transparent
⁵a simple generalization to SO and Sp cases will be presented in [10]

where $M = \mathbb{C}^{N_f}$ is the flavor space, where acts the flavor group $U(N_f)$, $m = (m_1, \dots, m_{N_f})$ are the masses = the coordinates on the Cartan subalgebra of the flavor group Lie algebra, and finally $\text{Eu}_{G \times \mathbb{T}^2 \times U(N_f)}$ denotes the equivariant Euler class.

The formula (1.6) generalizes in this case to:

$$Z^{inst}(a, m, \epsilon_1, \epsilon_2; q) = \sum_{\vec{k}} (q\hbar^{N_f})^{|\vec{k}|} \prod_{(l,i)} \prod_{f=1}^{N_f} \frac{\Gamma(\frac{a_l+m_f}{\hbar} + 1 + k_{l,i} - i)}{\Gamma(\frac{a_l+m_f}{\hbar} + 1 - i)} \times \prod_{(l,i) \neq (n,j)} \frac{a_{ln} + \hbar(k_{l,i} - k_{n,j} + j - i)}{a_{ln} + \hbar(j - i)} \tag{1.8}$$

Again, we claim that

$$\mathcal{F}^{inst}(a, m, \epsilon_1, \epsilon_2; q) = \epsilon_1 \epsilon_2 \log Z^{inst}(a, m, \epsilon_1, \epsilon_2; q) \tag{1.9}$$

is analytic in $\epsilon_{1,2}$.

The formulae (1.6) were checked against the Seiberg-Witten solution [12]. Namely, we claim that

$\mathcal{F}^{inst}(a, m, \epsilon_1, \epsilon_2)|_{\epsilon_1=\epsilon_2=0}$ = the instanton part of the prepotential of the low-energy effective theory of the $\mathcal{N} = 2$ gauge theory with the gauge group G and N_f fundamental matter hypermultiplets.

Mathematical formulation. The latter statement means that \mathcal{F}^{inst} is related to periods of a family of curves. More precisely, consider the following family of curves⁶ Σ_u (here we formulate things for $G = SU(N)$ but the generalization to general G is well-known [12]):

$$w + \frac{\Lambda^{2N-N_f} Q(\lambda)}{w} = \mathbf{P}(\lambda) = \prod_{l=1}^N (\lambda - \alpha_l) \tag{1.10}$$

where $Q(\lambda) = \prod_{f=1}^{N_f} (\lambda + m_f)$. The base of the family (1.10) is the space $\mathcal{U} = \mathbb{C}^{N-1} \ni u$ of the polynomials \mathbf{P} (we set $\sum_l \alpha_l = 0$). Consider the

⁶ Λ, m_f are fixed for the family

region $\mathcal{U}^{pert} \subset \mathcal{U}$ where $|\alpha_l|, |\alpha_l - \alpha_n| \gg |\Lambda|, |m_f|$. In \mathcal{U}^{pert} we can pass from the local coordinates (α_l) to the local coordinates (a_l) given by:

$$a_l = \frac{1}{2\pi i} \oint_{A_l} \lambda \frac{dw}{w} \tag{1.11}$$

where the cycle A_l can be described as encircling the cut on the λ plane connecting the points $\alpha_l^\pm = \alpha_l + o(\Lambda)$ which solve the equations:

$$\pm 2\Lambda^{N-\frac{N_f}{2}} Q^{\frac{1}{2}}(\alpha_l^\pm) = \mathbf{P}(\alpha_l^\pm) \tag{1.12}$$

The sum $\sum_l A_l$ vanishes in the homology of Σ_u , therefore we get $N - 1$ independent coordinates, as we should have. Now, define the dual coordinates

$$a_l^D = \frac{1}{2\pi i} \oint_{B_l} \lambda \frac{dw}{w} \tag{1.13}$$

where B_l encircles the cut connecting α_l^+ and $\alpha_{(l+1) \bmod N}^-$. Then, one can show [3][12] that

$$\sum_l da_l \wedge da_l^D = 0$$

on \mathcal{U} , and as a consequence, there exists a (locally defined) function, called prepotential, $\mathcal{F}(a; m, \Lambda)$ such that

$$\sum_l a_l^D da_l = d\mathcal{F}(a) \tag{1.14}$$

In the region \mathcal{U}^{pert} the prepotential has the expansion:

$$\begin{aligned} \mathcal{F}(a; m, \Lambda) &= \mathcal{F}^{pert}(a) + \mathcal{F}^{inst}(a) \\ \mathcal{F}^{pert}(a) &= \frac{1}{2} \sum_{l \neq n} (a_l - a_n)^2 \log \left(\frac{a_l - a_n}{\Lambda} \right) - \sum_{l, f} (a_l + m_f)^2 \log \left(\frac{a_l + m_f}{\Lambda} \right) \end{aligned} \tag{1.15}$$

where \mathcal{F}^{inst} is a power series in Λ . Our claim is that

\mathcal{F}^{inst} defined by the formula (1.15) coincides with $\mathcal{F}^{inst}(a, m, \epsilon_1, \epsilon_2)|_{\epsilon_1 = \epsilon_2 = 0}$.

We have checked this claim by an explicit calculation for up to five instantons, against the formulae in [13].

There is also a generalization of (1.6) to the case of adjoint matter. It is presented in the main body of the paper.

* * *

This paper is a short version of a longer manuscript [10], which will contain various details. In this paper we mostly state the results.

The paper is organized as follows. In the next section we describe the physical idea of our calculation. We define the observable of interest, and sketch two calculations of its expectation value – in the weak coupling regime in the ultraviolet, and the infrared low-energy effective theory calculation. The section 3 provides more details on the instanton calculation and generalizes the pure gauge theory calculation to the case of the theories with matter. We also discuss explicit low instanton charge calculations. In the section 4 we discuss our results from the M-theory viewpoint, consider some generalizations, present our conjectures and describe future directions.

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2 Field theory expectations

In this section we explain our approach in the field theory language. We exploit the fact that the supersymmetric gauge theory on flat space has

a large collection of observables whose correlation functions are saturated by instanton contribution in the limit of weak coupling. In addition, in the presence of the adjoint scalar vev these instantons tend to shrink to zero size. Moreover, the observables we choose have the property that the instantons which contribute to their expectation values are localized in space. This solves the problem of the runaway of point-like instantons, pointed out in [1].

2.1 Supersymmetries and twisted supersymmetries

The $\mathcal{N} = 2$ theory has eight conserved supercharges, $Q_\alpha^i, Q_{\dot{\alpha}}^i$, which transform under the global symmetry group $SU(2)_L \times SU(2)_R \times SU(2)_I$ of which the first two factors belong to the group of spatial rotations and the last one is the R -symmetry group. The indices $\alpha, \dot{\alpha}, i$ are the doublets of these respective $SU(2)$ factors. The basic multiplet of the gauge theory is the vector multiplet. Here is the spin content of its members:

Field	$SU(2)_L$	$SU(2)_R$	$SU(2)_I$
A_μ	$\frac{1}{2}$	$\frac{1}{2}$	0
ψ_α^i	$\frac{1}{2}$	0	$\frac{1}{2}$
$\psi_{\dot{\alpha}}^i$	0	$\frac{1}{2}$	$\frac{1}{2}$
$\phi, \bar{\phi}$	0	0	0

It is useful to work in the notations which make only $SU(2)_L \times SU(2)_d$ part of the global symmetry group manifest. Here $SU(2)_d$ is the diagonal subgroup of $SU(2)_R \times SU(2)_I$. If we call this subgroup a “Lorentz group”, then the supercharges, superspace, and the fermionic fields of the theory split as follows:

Fermions: $\psi_\mu, \chi_{\mu\nu}^+, \eta;$

Superspace: $\theta^\mu, \bar{\theta}_{\mu\nu}^+, \bar{\theta};$

Superfield: $\Phi = \phi + \theta^\mu \psi_\mu + \frac{1}{2} \theta^\mu \theta^\nu F_{\mu\nu} + \dots;$

Supercharges: $Q, Q_{\mu\nu}^+, G_\mu.$

The supercharge Q is a scalar with respect to the “Lorentz group” and is usually considered as a BRST charge in the topological quantum field theory version of the susy gauge theory. It is conserved on any four-manifold.

In [14] E. Witten has employed a self-dual two-form supercharge $Q_{\mu\nu}^+$ which is conserved on Kähler manifolds.

Our idea is to use other supercharges G_μ as well. Their conservation is tied up with the isometries of the four-manifold on which one studies the gauge theory. Of course, the idea to regularize the supersymmetric theory by subjecting it to the twisted boundary conditions is very common both in physics [15], and in mathematics [16][17][18][16].

At this point we should mention that the idea to apply localization techniques to the instanton integrals has been recently applied in [20] in the one- and two-instanton cases. Without \mathbf{T}^2 -localization this is still rather complicated, yet simpler, calculation than the direct evaluation [5]. We refer the interested reader to the beautiful review [9] for more details.

2.2 Good observables: UV

In the applications of the susy gauge theory to Donaldson theory, where one works with the standard topological supercharge Q , the observables one is usually interested in are the gauge invariant polynomials $\mathcal{O}_{P,x}^{(0)} = P(\phi(x))$ in the adjoint scalar ϕ , evaluated at space-time point x , and its descendants: $\mathcal{O}_{P,\Sigma}^{(k)} = \int_\Sigma P(\Phi)$, where Σ is a k -cycle. Unfortunately for $k > 0$ all such cycles are homologically trivial on \mathbb{R}^4 and no non-trivial observables are constructed in such a way. One constructs an equivalent set of dual observables by integration over \mathbb{R}^4 of a product of a closed k -form $\omega = \frac{1}{(4-k)!} \omega_{\mu_1 \dots \mu_k} \theta^{\mu_1} \dots \theta^{\mu_k}$ and the $4 - k$ -form part of $P(\Phi)$:

$$\mathcal{O}_P^\omega = \int d^4x d^4\theta \omega(x, \theta) P(\Phi) \tag{2.1}$$

Again, most of these observables are Q -exact, as any closed k -form on \mathbb{R}^4 is exact for $k > 0$.

However, if we employ the rotational symmetries of \mathbb{R}^4 and work equivariantly, we find new observables.

Namely, consider the fermionic charge

$$\tilde{Q} = Q + E_a \Omega_{\mu\nu}^a x^\nu G_\mu \tag{2.2}$$

Here $\Omega^a = \Omega_{\mu\nu}^a x^\nu \partial_\mu$ for $a = 1 \dots 6$ are the vector fields generating $SO(4)$ rotations, and $E \in Lie(SO(4))$ is a formal parameter.

With respect to the charge \tilde{Q} the observables $\mathcal{O}_{P,\Sigma}^{(k)}$ are no longer invariant⁷.

⁷except for $\mathcal{O}_{P,0}^{(0)}$ where $0 \in \mathbb{R}^4$ is the origin, left fixed by the rotations

However, the observables (2.1) can be generalized to the new setup, producing a priori nontrivial \tilde{Q} -cohomology classes. Namely, let us take any $SO(4)$ -equivariant form on \mathbb{R}^4 . That is, take an inhomogeneous differential form $\Omega(E)$ on \mathbb{R}^4 which depends also on an auxiliary variable $E \in Lie(SO(4))$ which has the property that for any $g \in SO(4)$:

$$g^*\Omega(E) = \Omega(g^{-1}Eg) \quad (2.3)$$

where we take pullback defined with the help of the action of $SO(4)$ on \mathbb{R}^4 by rotations. Such E -dependent forms are called equivariant forms. On the space of equivariant forms acts the so-called equivariant differential,

$$D = d + \iota_{V(E)} \quad (2.4)$$

where $V(E)$ is the vector field on \mathbb{R}^4 representing the infinitesimal rotation generated by E . For equivariantly closed (i.e. D -closed) form $\Omega(E)$ the observable:

$$\mathcal{O}_P^{\Omega(E)} = \int_{\mathbb{R}^4} \Omega(E) \wedge P(\Phi) \quad (2.5)$$

is \tilde{Q} -closed.

Any $SO(4)$ invariant polynomial in E is of course an example of the D -closed equivariant form. Such a polynomial is characterized by its restriction onto the Cartan subalgebra of $SO(4)$, where it must be Weyl-invariant. The Cartan subalgebra of $SO(4)$ is two-dimensional. Let us denote the basis in this subalgebra corresponding to the decomposition $\mathbb{R}^4 = \mathbb{R}^2 \oplus \mathbb{R}^2$ into a orthogonal direct sum of two dimensional planes, by (ϵ_1, ϵ_2) . Under the identification $Lie(SO(4)) \approx Lie(SU(2)) \oplus Lie(SU(2))$ these map to $(\epsilon_1 + \epsilon_2, \epsilon_1 - \epsilon_2)$. The Weyl ($= \mathbb{Z}_2 \times \mathbb{Z}_2$) invariant polynomials are polynomials in $\sigma = \epsilon_1^2 + \epsilon_2^2$ and $\chi = \epsilon_1 \epsilon_2$. As $SO(4)$ does not preserve any forms except for constants we should relax the $SO(4)$ symmetry to get interesting observables.

Thus, let us fix in addition a translationally invariant symplectic form ω on \mathbb{R}^4 . Its choice breaks $SO(4)$ down to $U(2)$ – the holonomy group of a Kähler manifold. Let us fix this $U(2)$ subgroup. Then we have a moment map:

$$\mu : \mathbb{R}^4 \longrightarrow Lie(U(2))^*, \quad d\mu(E) = \iota_{V(E)}\omega, \quad E \in Lie(U(2)) \quad (2.6)$$

And therefore, the form $\omega - \mu(E)$ is D -closed. One can find such euclidean coordinates x^ν , $\nu = 1, 2, 3, 4$ that the form ω reads as follows:

$$\omega = dx^1 \wedge dx^2 + dx^3 \wedge dx^4 \quad (2.7)$$

The Lie algebra of $U(2)$ splits as a direct sum of one-dimensional abelian Lie algebra of $U(1)$ and the Lie algebra of $SU(2)$. Accordingly, the moment map μ splits as (h, μ^1, μ^2, μ^3) . In the x^μ coordinates

$$h = \sum_{\mu} (x^\mu)^2, \quad \mu^a = \frac{1}{2} \eta_{\mu\nu}^a x^\mu x^\nu, \quad (2.8)$$

where $\eta_{\mu\nu}^a$ is the anti-self-dual 't Hooft symbol.

Finally, the choice of ω also defines a complex structure on \mathbb{R}^4 , thus identifying it with \mathbb{C}^2 with complex coordinates z_1, z_2 given by: $z_1 = x^1 + ix^2$, $z_2 = x^3 + ix^4$. For E in the Cartan subalgebra $H = \mu(E)$ is given by the simple formula:

$$H = \epsilon_1 |z_1|^2 + \epsilon_2 |z_2|^2 \quad (2.9)$$

After all these preparations we can define the correlation function of our interest:

$$Z(a, \epsilon) = \left\langle \exp \frac{1}{(2\pi i)^2} \int_{\mathbb{R}^4} (\omega \wedge \text{Tr} (\phi F + \frac{1}{2} \psi \psi) - H \text{Tr} (F \wedge F)) \right\rangle_a \quad (2.10)$$

where we have indicated that the vacuum expectation value is calculated in the vacuum with the expectation value of the scalar ϕ in the vector multiplet given by $a \in \mathfrak{t}$. More precisely, a will be the central charge of $\mathcal{N} = 2$ algebra corresponding to the W -boson states (cf.[3]) .

Remarks.

- 1.) Note that the observable in (2.10) makes the widely separated instantons suppressed. More precisely, if the instantons form clusters around points $\vec{r}_1, \dots, \vec{r}_l$ then they contribute $\sim \exp - \sum_m H(\vec{r}_m)$ to the correlation function.
- 2.) One can expand (2.10) as a sum over different instanton sectors:

$$Z(a, \epsilon) = \sum_{k=0}^{\infty} q^k Z_k(a, \epsilon)$$

where $q \sim \Lambda^{2N}$ is the dynamically generated scale – for us – simply the generating parameter.

3.) The observable (2.10) is formally cohomologous to the identity, as

$$\int (\omega + H) \operatorname{Tr} \Phi^2 = \tilde{Q} \int d^4 x d^4 \theta \mathbf{A}(x, \theta) \operatorname{Tr} \Phi^2 ,$$

where $\mathbf{A}(x, \theta) = \omega_{\mu\nu} x^\mu \theta^\nu$. We cannot eliminate it without having to perform the full path integral, however, as it serves as a supersymmetric regulator. On the other hand, in the presence of this observable the path integral can be drastically simplified, the fact we shall exploit below. The analogous manipulation in the context of two dimensional supersymmetric Landau-Ginzburg models was done in the first reference in [15].

The supersymmetry guarantees that (2.10) is saturated by instantons. Moreover, the superspace of instanton zero modes is acted on by a finite dimensional version of the supercharge \tilde{Q} which becomes an equivariant differential on the moduli space of framed instantons. Localization with respect to this supercharge reduces the computation to the counting of the isolated fixed points and the weights of the action of the symmetry groups (a copy of gauge group and $U(2)$ of rotations) on the tangent spaces. This localization can be understood as a particular case of the Duistermaat-Heckman formula [19], as (2.10) calculates essentially the integral of the exponent of the Hamiltonian of a torus action (Cartan of G times \mathbf{T}^2) against the symplectic measure.

The counting of fixed points can be nicely summarized by a contour integral (see below). This contour integral also can be obtained by transforming the integral over the ADHM moduli space of the observable (2.10) evaluated on the instanton configuration, by adding \tilde{Q} -exact terms, as in [7][6]. It also can be derived from Bott's formula [21].

2.3 Good observables: IR

The nice feature of the correlator (2.10) is its simple relation to the prepotential of the low-energy effective theory. In order to derive it let us think of the observable (2.10) as of a slowly varying changing of the microscopic coupling constant. If we could completely neglect the fact that H is not constant, then its addition would simply renormalize the effective low-energy scale $\Lambda \rightarrow \Lambda e^{-H}$.

However, we should remember that H is not constant, and regard this renormalization as valid up to terms in the effective action containing derivatives of H . Moreover, H is really a bosonic part of the function $\mathcal{H}(x, \theta)$ on the

(chiral part of) superspace (in [1] such superspace-dependent deformations of the theory on curved four-manifolds were considered):

$$\mathcal{H}(x, \theta) = H(x) + \frac{1}{2} \omega_{\mu\nu} \theta^\mu \theta^\nu$$

Together these terms add up to the making the standard Seiberg-Witten effective action determined by the prepotential $\mathcal{F}(a; \Lambda)$ to the one with the superspace-dependent prepotential

$$\mathcal{F}(a; \Lambda e^{-\mathcal{H}(x, \theta)}) = \mathcal{F}(a; \Lambda e^{-H}) + \omega \Lambda \partial_\Lambda \mathcal{F}(a; \Lambda e^{-H}) + \frac{1}{2} \omega^2 (\Lambda \partial_\Lambda)^2 \mathcal{F}(a; \Lambda e^{-H}) \quad (2.11)$$

This prepotential is then integrated over the superspace (together with the conjugate terms) to produce the effective action.

Now, let us go to the extreme infrared, that is let us scale the metric on \mathbb{R}^4 by a very large factor t (keeping ω intact). On flat \mathbb{R}^4 the only term which may contribute to the correlation function in question in the limit $t \rightarrow \infty$ is the last term in (2.10) as the rest will (after integration over superspace) necessarily contain couplings to the gauge fields which will require some loop diagrams to get non-trivial contractions, which all will be suppressed by inverse powers of t , except for the one-loop contribution, which requires more care.

The last term in (2.11) gives:

$$Z^{inst}(a; \epsilon) = \exp - \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \omega \wedge \omega \frac{\partial^2 \mathcal{F}(a; \Lambda e^{-H})}{(\partial \log \Lambda)^2} + O(\epsilon) \quad (2.12)$$

where we used the fact that the derivatives of H are proportional to $\epsilon_{1,2}$. Recalling (2.7) the integral in (2.12) reduces to:

$$Z^{inst}(a; \epsilon_1, \epsilon_2) = \exp \frac{\mathcal{F}^{inst}(a; \Lambda) + O(\epsilon)}{\epsilon_1 \epsilon_2} \quad (2.13)$$

where

$$\mathcal{F}^{inst}(a; \Lambda) = \int_0^\infty \partial_H^2 \mathcal{F}(a; \Lambda e^{-H}) H dH$$

thereby explaining our claim about the analytic properties of Z and \mathcal{F} . Incidentally, the careful evaluation of the one-loop contribution gives the perturbative part of the prepotential, thus making the full partition function equal to:

$$Z(a; \epsilon_1, \epsilon_2) = \exp \frac{1}{\epsilon_1 \epsilon_2} (\mathcal{F}(a; \Lambda) + O(\epsilon)) \quad (2.14)$$

We shall return to this issue below.

3 Instanton measure and its localization

3.1 ADHM data

The moduli space $\mathcal{M}_{k,N}$ of instantons with fixed framing at infinity has dimension $4kN$. It has the following convenient description. Take two complex vector spaces V and W of the complex dimensions k and N respectively. These spaces should be viewed as Chan-Paton spaces for $D(p-4)$ and Dp branes in the brane realization of the gauge theory with instantons.

Let us also denote by L the two dimensional complex vector space, which we shall identify with the Euclidean space $\mathbb{R}^4 \approx \mathbb{C}^2$ where our gauge theory lives.

Then the ADHM [22] data consists of the following maps between the vector spaces:

$$V \xrightarrow{\tau} V \otimes L \oplus W \xrightarrow{\sigma} V \otimes \Lambda^2 L \quad (3.1)$$

where

$$\tau = \begin{pmatrix} B_2 \\ -B_1 \\ J \end{pmatrix}, \quad \sigma = (B_1 \quad B_2 \quad I) \quad (3.2)$$

$$B_{1,2} \in \text{End}(V), \quad I \in \text{Hom}(W, V), \quad J \in \text{Hom}(V, W)$$

The ADHM construction represents the moduli space of $U(N)$ instantons on \mathbb{R}^4 of charge k as a hyperkähler quotient [23] of the space of operators (B_1, B_2, I, J) by the action of the group $U(k)$ for which V is a fundamental representation, B_1, B_2 transform in the adjoint, I in the fundamental, and J in the anti-fundamental representations.

More precisely, the moduli space of proper instantons is obtained by taking the quadruples $(B_{1,2}, I, J)$ obeying the so-called ADHM equations:

$$\mu_c = 0, \quad \mu_r = 0, \quad (3.3)$$

where:

$$\begin{aligned}\mu_c &= [B_1, B_2] + IJ \\ \mu_r &= [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J\end{aligned}\tag{3.4}$$

and with the additional requirement that the stabilizer of the quadruple in $U(k)$ is trivial. This produces a non-compact hyperkähler manifold $M_{k,N}$ of instantons with fixed framing at infinity.

The framing is really the choice of the basis in W . The group $U(W) = U(N)$ acts on these choices, and acts on $M_{k,N}$, by transforming I and J in the anti-fundamental and the fundamental representations respectively.

This action also preserves the hyperkähler structure of $M_{k,N}$ and is generated by the hyperkähler moment maps:

$$\mathbf{m}_r = I^\dagger I - J J^\dagger, \quad \mathbf{m}_c = J I\tag{3.5}$$

Actually, $\text{Tr}_W \mathbf{m}_{r,c} = \text{Tr}_V \mu_{r,c}$, thus the central $U(1)$ subgroup of $U(N)$ acts trivially on $M_{k,N}$. Therefore it is the group $G = SU(N)/\mathbf{Z}_N$ which acts non-trivially on the moduli space of instantons.

3.2 Instanton measure

The supersymmetric gauge theory measure can be regarded as an infinite-dimensional version of the equivariant Matthai-Quillen representative of the Thom class of the bundle $\Gamma(\Omega^{2,+} \otimes \mathfrak{g}_P)$ over the infinite-dimensional space of all gauge fields \mathcal{A}_P in the principal G -bundle P (summed over the topological types of P). In physical terms, in the weak coupling limit we are calculating the supersymmetric delta-function supported on the instanton gauge field configurations. In the background of the adjoint Higgs vev, this supersymmetric delta-function is actually an equivariant differential form on the moduli space $M_{k,N}$ of instantons. It can be also represented using the finite-dimensional hyperkähler quotient ADHM construction of $M_{k,N}$ (as opposed to the infinite-dimensional quotient of the space of all gauge fields by the action of the group of gauge transformations, trivial at infinity) [7]:

$$\int \mathcal{D}\phi \mathcal{D}\bar{\phi} \mathcal{D}\bar{H} \mathcal{D}\bar{\chi} \mathcal{D}\eta \mathcal{D}\Psi \mathcal{D}B \mathcal{D}I \mathcal{D}J e^{\tilde{Q}(\bar{\chi} \cdot \bar{\mu}(B,I,J) + \Psi \cdot V(\bar{\phi}) + \eta[\phi, \bar{\phi}])}\tag{3.6}$$

where, say:

$$\begin{aligned}
\tilde{Q}B_{1,2} &= \Psi_{B_{1,2}}, & \tilde{Q}\Psi_{B_{1,2}} &= [\phi, B_{1,2}] + \epsilon_{1,2}B_{1,2} \\
\tilde{Q}I &= \Psi_I, & \tilde{Q}\Psi_I &= \phi I - Ia \\
\tilde{Q}J &= \Psi_J, & \tilde{Q}\Psi_J &= -J\phi + Ja - (\epsilon_1 + \epsilon_2)J \\
\tilde{Q}\chi_r &= H_r, & \tilde{Q}H_r &= [\phi, \chi_r], & \tilde{Q}\chi_c &= H_c, & \tilde{Q}H_c &= [\phi, \chi_c] + (\epsilon_1 + \epsilon_2)\chi_c \\
\tilde{\Psi} \cdot V(\bar{\phi}) &= \text{Tr} \left(\Psi_{B_1}[\bar{\phi}, B_1^\dagger] + \Psi_{B_2}[\bar{\phi}, B_2^\dagger] + \Psi_I[\bar{\phi}, I^\dagger] - \Psi_J[\bar{\phi}, J^\dagger] + c.c. \right)
\end{aligned} \tag{3.7}$$

(we refer to [7] for more detailed explanations). If the moduli space $M_{k,N}$ was compact and smooth one could interpret (3.6) as a certain topological quantity and apply the powerful equivariant localization techniques [8] to calculate it.

The non-compactness of the moduli space of instantons is of both ultraviolet and of infrared nature. The UV non-compactness has to do with the instanton size, which can be made arbitrarily small. The IR non-compactness has to do with the non-compactness of \mathbb{R}^4 which permits the instantons to run away to infinity.

3.3 Curing non-compactness

The UV problem can be solved by relaxing the condition on the stabilizer, thus adding the so-called point-like instantons. A point of the hyperkähler space $\tilde{M}_{k,N}$ with orbifold singularities which one obtains in this way (Uhlenbeck compactification) is an instanton of charge $p \leq k$ and a set of $k - p$ points on \mathbb{R}^4 :

$$\tilde{M}_{k,N} = M_{k,N} \cup M_{k-1,N} \times \mathbb{R}^4 \cup M_{k-2,N} \times \text{Sym}^2(\mathbb{R}^4) \cup \dots \cup \text{Sym}^k(\mathbb{R}^4) \tag{3.8}$$

The resulting space $\tilde{M}_{k,N}$ is a geodesically complete hyperkähler orbifold. Its drawback is the non-existence of the universal bundle with the universal instanton connection over $\tilde{M}_{k,N} \times \mathbb{R}^4$. We actually think that in principle one can still work with this space. Fortunately, in the case of $U(N)$ gauge group there exists a nicer space $\tilde{\mathcal{M}}_{k,N}$ which is obtained from $\tilde{M}_{k,N}$ by a sequence of blowups (resolution of singularities) which is smooth, and after some modification of the gauge theory (noncommutative[24][25][26] deformation) becomes a moduli space with the universal instanton. Technically this space is obtained [27] by the same ADHM construction except that now one performs the hyperkähler quotient at the non-zero level of the moment

map:

$$\mu_r = \zeta_r \mathbf{1}_V, \quad \mu_c = 0 \tag{3.9}$$

(one can also make $\mu_c \neq 0$ but this does not give anything new). The space of quadruples (B_1, B_2, I, J) obeying (3.9) is freely acted on by $U(k)$. The cohomology theory of $\widetilde{\mathcal{M}}_{k,N}$ is richer than that of $\widetilde{M}_{k,N}$ because of the exceptional divisors. However, our goal is to study the original gauge theory. Therefore we are going to consider the (equivariant) cohomology classes of $\widetilde{\mathcal{M}}_{k,N}$ lifted from $\widetilde{M}_{k,N}$.

As we stated in the introduction, we are going to utilize the equivariant symplectic volumes of $\widetilde{\mathcal{M}}_{k,N}$. This is not quite precise. We are going to consider the symplectic volumes, calculated using the closed two-form lifted from $\widetilde{M}_{k,N}$. This form vanishes when restricted onto the exceptional variety. This property ensures that we don't pick up anything not borne in the original gauge theory (don't pick up freckle contribution in the terminology of [28]).

The ADHM construction from the previous section gives rise to the instantons with fixed gauge orientation at infinity (fixed framing). The group $G = SU(N)/\mathbf{Z}_N$ acts on their moduli space $\mathcal{M}_{N,k}$ by rotating the gauge orientation. Also, the group of Euclidean rotations of \mathbb{R}^4 acts on $\mathcal{M}_{N,k}$. We are going to apply localization techniques with respect to both of these groups.

In fact, it is easier to localize first with respect to the groups $U(k) \times G \times \mathbf{T}^2$ acting on the vector space of ADHM matrices, and then integrate out the $U(k)$ part of the localization multiplet, to incorporate the quotient.

The action of \mathbf{T}^2 is free at "infinities" of $\widetilde{\mathcal{M}}_k$, thus allowing to apply localization techniques without worrying about the IR non-compactness. Physically, the integral (2.10) is Gaussian-like and convergent in the IR region.

3.4 Reduction to contour integrals

After the manipulations as in [7][6] we end up with the following integral[28]:

$$\begin{aligned}
 Z_k(a, \epsilon_1, \epsilon_2) = & \frac{1}{k!} \frac{\epsilon^k}{(2\pi i \epsilon_1 \epsilon_2)^k} \oint \prod_{I=1}^k \frac{d\phi_I Q(\phi_I)}{P(\phi_I)P(\phi_I + \epsilon)} \\
 & \times \prod_{1 \leq I < J \leq k} \frac{\phi_{IJ}^2 (\phi_{IJ}^2 - \epsilon^2)}{(\phi_{IJ}^2 - \epsilon_1^2)(\phi_{IJ}^2 - \epsilon_2^2)}
 \end{aligned} \tag{3.10}$$

where:

$$\begin{aligned}
 Q(x) &= \prod_{f=1}^{N_f} (x + m_f) \\
 P(x) &= \prod_{l=1}^N (x - a_l),
 \end{aligned}
 \tag{3.11}$$

ϕ_{IJ} denotes $\phi_I - \phi_J$, and $\epsilon = \epsilon_1 + \epsilon_2$.

We went slightly ahead of time and presented the formula which covers the case of the gauge theory with N_f fundamental multiplets. In fact, its derivation is rather simple if one keeps in mind the relation to the Euler class of the Dirac zeromodes bundle over the moduli space of instantons, stated in the introduction.

The integrals (3.10) should be viewed as contour integrals. As explained in [6] the poles at $\phi_{IJ} = \epsilon_1, \epsilon_2$ should be avoided by shifting $\epsilon_{1,2} \rightarrow \epsilon_{1,2} + i0$, those at $\phi_I = a_l$ similarly by $a_l \rightarrow a_l + i0$ (this case was not considered in [6] but actually was considered (implicitly) in [7]). The interested reader should consult [29] for more mathematically sound explanations of the contour deformations arising in the similar context in the study of symplectic quotients.

Perhaps a more illuminating way of understanding the contour integral (3.10) proceeds via the Duistermaat - Heckman formula [19]:

$$\int_X \frac{\omega^n}{n!} e^{-\mu[\xi]} = \sum_{f: V_\xi(f)=0} \frac{e^{-\mu[\xi](f)}}{\prod_{i=1}^n w_i[\xi](f)}
 \tag{3.12}$$

Here (X^{2n}, ω) is a symplectic manifold (in the original DH setup it should be compact, but the formula hold in more general situation which extends to our case) with a Hamiltonian action of a torus \mathbf{T}^r , $\mu : X \rightarrow \mathfrak{t}^*$ is the moment map, $\xi \in \mathfrak{t} = Lie(\mathbf{T})$ is the generator, $V_\xi \in Vect(X)$ is the vector field on X representing the \mathbf{T} action, generated by ξ , $f \in X$ runs through the set of fixed points, and $w_i[\xi](f)$ are the weights of the \mathbf{T} action on the tangent space to the fixed point.

In our case $X = \widetilde{\mathcal{M}}_k$ is the moduli space of instantons of charge k , \mathbf{T} is the product of the Cartan torus of G and the torus $\mathbf{T}^2 \subset SO(4)$, $\xi = (a, \epsilon_1, \epsilon_2)$. The fixed points f will be described in the next section. However, already without performing the detailed analysis of fixed points one can understand the meaning of (3.10)

Suppose f is a fixed point. It corresponds to some quadruple (B_1, B_2, I, J) such that the \mathbf{T} -transformed quadruple belongs to the same $U(k)$ -orbit. Working infinitesimally we derive that there must exist $\phi \in \text{Lie}(U(k))$, such that:

$$\begin{aligned} [B_1, \phi] &= \epsilon_1 B_1, & [B_2, \phi] &= \epsilon_2 B_2 \\ -\phi I + Ia &= 0, & -aJ + J\phi &= -(\epsilon_1 + \epsilon_2)J \end{aligned} \quad (3.13)$$

We can go to the bases in the spaces V, W where ϕ and a are diagonal. Then the equations (3.13) will read as follows:

$$\begin{aligned} (\phi_I - \phi_J + \epsilon_1) B_{1,IJ} &= 0 \\ (\phi_I - \phi_J + \epsilon_2) B_{2,IJ} &= 0 \\ (\phi_I - a_l) I_{I,l} &= 0 \\ (\phi_I + \epsilon_1 + \epsilon_2 - a_l) J_{l,I} &= 0 \end{aligned} \quad (3.14)$$

For (3.14) to have a non-trivial solution some of the combinations

$$\phi_{IJ} + \epsilon_{1,2}, \phi_I - a_l, \phi_I + \epsilon_1 + \epsilon_2 - a_l \quad (3.15)$$

must vanish. This is where the poles of the integrand (3.10) are located. It follows from the remark below (3.9) that the equations (3.14) specify ϕ uniquely, given $a, \epsilon_1, \epsilon_2$ and f . Thus, exactly k out of $2k^2 + 2kN$ combinations should vanish. Now let us look at the (holomorphic) tangent space $T_f \widetilde{\mathcal{M}}_k$. It is spanned by the quadruples $(\delta B_1, \delta B_2, \delta I, \delta J)$ obeying the linearized ADHM equations, and considered up to the linearized $U(k)$ transformations, that is, it can be viewed as first cohomology group of the following ‘‘complex’’:

$$\begin{aligned} C^0 = \text{End}(V) &\longrightarrow C^1 = \text{End}(V) \otimes L \oplus \text{Hom}(V, W) \oplus \text{Hom}(W, V) \\ &\longrightarrow C^2 = \text{End}(V) \otimes \Lambda^2 L \end{aligned} \quad (3.16)$$

where the first arrow is given by the infinitesimal gauge transformation, while the second is the linearized ADHM equation: $d\mu_c$. To calculate $\prod_i w_i[a, \epsilon_1, \epsilon_2](f)$ it is convenient to compute first the ‘‘Chern’’ character:

$$\text{Ch}(T_f \widetilde{\mathcal{M}}_k) = \sum_i e^{w_i[a, \epsilon_1, \epsilon_2](f)} \quad (3.17)$$

which is given by the alternating sum of the Chern characters of the terms in (3.16)

$$\begin{aligned} & Ch(C^1) - Ch(C^0) - Ch(C^2) \\ &= \sum_{I,J} e^{\phi_{I,J}} (e^{\epsilon_1} - 1) (1 - e^{\epsilon_2}) + \sum_{I,l} \left(e^{\phi_I - a_l} + e^{a_l - \phi_I - \epsilon} \right) \end{aligned} \quad (3.18)$$

Upon the standard conversion

$$\sum_{\alpha} n_{\alpha} e^{x_{\alpha}} \mapsto \prod_{\alpha} x_{\alpha}^{n_{\alpha}}$$

we arrive at (3.10) It remains to explain why not every possible k -tuple out of the combinations (3.15) contribute, only those which are picked by the $+i0$ prescription. This will be done in [10].

3.5 Classification of the residues

The poles which with non-vanishing contributions to the integral must have $\phi_{IJ} \neq 0$, for $I \neq J$, otherwise the numerator vanishes. This observation simplifies the classification of the poles. They are labelled as follows. Let $k = k_1 + k_2 + \dots + k_N$ be a partition of the instanton charge in N summands which have to be non-negative (but may vanish), $k_l \geq 0$. In turn, for all l such that $k_l > 0$ let Y_l denote a partition of k_l :

$$k_l = k_{l,1} + \dots + k_{l,\nu^{l,1}}, \quad k_{l,1} \geq k_{l,2} \geq \dots \geq k_{l,\nu^{l,1}} > 0$$

Let $\nu^{l,1} \geq \nu^{l,2} \geq \dots \nu^{l,k_{l,1}} > 0$ denote the dual partition. Pictorially one represents these partitions by the Young diagram with $k_{l,1}$ rows of the lengths $\nu^{l,1}, \dots, \nu^{l,k_{l,1}}$. This diagram has $\nu^{l,1}$ columns of the lengths $k_{l,1}, \dots, k_{l,\nu^{l,1}}$. Sometimes we also use the notation $n_l = \nu^{l,1}$, and we find it useful to extend the sequence $k_{l,i}$ all the way to infinity by zeroes: $\mathbf{k}_l = \{k_{l,1} \geq k_{l,2} \geq \dots k_{l,n_l+1} = k_{l,n_l+2} = \dots = 0\}$.

In total we have k boxes distributed among N Young tableaux (some of which could be empty, i.e. contain zero boxes). Let us label these boxes somehow (the ordering is not important as it is cancelled in the end by the factor $k!$ in (3.10)). Let us denote the collection of N Young diagrams by $\vec{Y} = (Y_1, \dots, Y_N)$. We denote by $|Y_l| = k_l$ the number of boxes in the l 'th diagram, and by $|\vec{Y}| = \sum_l |Y_l| = |\mathbf{k}| = k$.

Then the pole of the integral (3.10) corresponding to \vec{Y} is at ϕ_I with I corresponding to the box (α, β) in the l 'th Young tableau (so that $0 \leq \alpha \leq$

$\nu^{l,\beta}$, $0 \leq \beta \leq k_{l,\alpha}$ equal to:

$$\vec{Y} \longrightarrow \phi_I = a_l + \epsilon_1(\alpha - 1) + \epsilon_2(\beta - 1) \tag{3.19}$$

3.6 Residues and fixed points

The poles in the integral (3.10) correspond to the fixed points of the action of the groups $G \times \mathbf{T}^2$ on the resolved moduli space $\tilde{\mathcal{M}}_{k,N}$. Physically they correspond to the $U(N)$ (noncommutative) instantons which split as a sum of $U(1)$ noncommutative instantons corresponding to N commuting $U(1)$ subgroups of $U(N)$. The instanton charge k_l is the charge of the $U(1)$ instanton in the l 'th subgroup. Moreover, these abelian instantons are of special nature – they are fixed by the group of space rotations. If they were commutative (and therefore point-like) they had to sit on top of each other, and the space of such point-like configurations would have been rather singular. Fortunately, upon the noncommutative deformation the singularities are resolved. The instantons cannot sit quite on top of each other. Instead, they try to get as close to each other as the uncertainty principle lets them. The resulting abelian configurations were classified (in the language of torsion free sheaves) by H. Nakajima [30].

Now let us fix a configuration \vec{Y} and consider the corresponding contribution to the integral over instanton moduli. It is given by the residue of the integral (3.10) corresponding to (3.19)

$$R_{\vec{Y}} = \prod_l \prod_{\alpha=1}^{\nu^{l,1}} \prod_{\beta=1}^{k_{l,\alpha}} \frac{\mathcal{S}_l(\epsilon_1(\alpha - 1) + \epsilon_2(\beta - 1))}{(\epsilon(\ell(s) + 1) - \epsilon_2 h(s))(\epsilon_2 h(s) - \epsilon \ell(s))} \times \prod_{l < m} \prod_{\alpha=1}^{\nu^{l,1}} \prod_{\beta=1}^{k_{m,1}} \left(\frac{(a_{lm} + \epsilon_1(\alpha - \nu^{m,\beta}) + \epsilon_2(1 - \beta)) (a_{lm} + \epsilon_1 \alpha + \epsilon_2(k_{l,\alpha} + 1 - \beta))}{(a_{lm} + \epsilon_1 \alpha + \epsilon_2(1 - \beta)) (a_{lm} + \epsilon_1(\alpha - \nu^{m,\beta}) + \epsilon_2(k_{l,\alpha} + 1 - \beta))} \right)^2 \tag{3.20}$$

where we have used the following notations: $a_{lm} = a_l - a_m$,

$$\mathcal{S}_l(x) = \frac{Q(a_l + x)}{\prod_{m \neq l} (x + a_{lm})(x + \epsilon + a_{lm})}, \quad S_l(x) = \frac{Q(a_l + x)}{\prod_{m \neq l} (x + a_{lm})^2}, \tag{3.21}$$

and

$$\ell(s) = k_{l,\alpha} - \beta, \quad h(s) = k_{l,\alpha} + \nu^{l,\beta} - \alpha - \beta + 1 \tag{3.22}$$

Now, if we set $\epsilon_1 = \hbar = -\epsilon_2$ the formula (3.20) can be further simplified. After some reshuffling of the factors it becomes exactly the formula for the term in the sum (1.8) corresponding to the partition $\{k_{l,i}\}$.

Remark. The expressions (3.20) are the typical localization formulae for the instanton integrals. They commonly appear in the two dimensional sigma model instanton calculations, on the so-called A side. The Seiberg-Witten prepotential [3] is the typical type B expression. It is not easy to recognize in the type A expression the mirror manifold, and its periods. To illustrate this point, we suggest to look at the generating function of the number of holomorphic curves of genus zero in the Calabi-Yau quintic, computed using localization [17]. It requires some extra work to map it to the mirror calculation, yet it can be done [18]. In this paper we shall not complete the story in the sense that we shall not prove directly that our “type A” expression can be computed on the “B side” involving Seiberg-Witten curves. We shall, however, present a conjecture, which connects our calculation to its mirror counterpart (that is, we shall define the mirror computation).

We shall also make some explicit checks for low instanton numbers (up to five) to make sure we have computed the right thing.

3.7 The first three nonabelian instantons

We shall now give the formulae for the first three instanton contributions to the prepotential for the general $SU(N)$ case, with $N_f < 2N$.

We shall work with $\epsilon_1 = \hbar = -\epsilon_2$. It will be sufficient to derive the gauge theory prepotential.

Directly applying the rules (3.10) we arrive at the following expressions

for the moduli integrals:

$$\begin{aligned}
Z_1 &= \frac{1}{\epsilon_1 \epsilon_2} \sum_l S_l \\
Z_2 &= \frac{1}{(\epsilon_1 \epsilon_2)^2} \left(\frac{1}{4} \sum_l S_l (S_l(+\hbar) + S_l(-\hbar)) + \frac{1}{2} \sum_{l \neq m} \frac{S_l S_m}{\left(1 - \frac{\hbar^2}{a_{lm}^2}\right)^2} \right) \\
Z_3 &= \frac{1}{(\epsilon_1 \epsilon_2)^3} \left(\sum_l \frac{S_l (S_l(+\hbar)S_l(+2\hbar) + S_l(-\hbar)S_l(-2\hbar) + 4S_l(+\hbar)S_l(-\hbar))}{36} + \right. \\
&\quad \sum_{l \neq m} \frac{S_l S_m}{4 \left(1 - \frac{4\hbar^2}{a_{lm}^2}\right)} \left(S_l(+\hbar) \left(1 - \frac{2(\hbar/a_{lm})^2}{(1 - (\hbar/a_{lm}))}\right)^2 + \right. \\
&\quad \left. \left. S_l(-\hbar) \left(1 - \frac{2(\hbar/a_{lm})^2}{(1 + (\hbar/a_{lm}))}\right)^2 \right) + \right. \\
&\quad \left. \sum_{l \neq m \neq n} \frac{S_l S_m S_n}{6 \left(\left(1 - \frac{\hbar^2}{a_{lm}^2}\right) \left(1 - \frac{\hbar^2}{a_{ln}^2}\right) \left(1 - \frac{\hbar^2}{a_{mn}^2}\right) \right)^2} \right)
\end{aligned} \tag{3.23}$$

where

$$S_l = S_l(0), \quad S_l^{(n)} = \partial_x^n S_l(x)|_{x=0}$$

which yield:

$$\begin{aligned}
\mathcal{F}_1 &= \sum_l S_l \\
\mathcal{F}_2 &= \sum_l \frac{1}{4} S_l S_l^{(2)} + \sum_{l \neq m} \frac{S_l S_m}{a_{lm}^2} + O(\hbar^2) \\
\mathcal{F}_3 &= \sum_l \frac{S_l}{36} \left(S_l S_l^{(4)} + 2S_l^{(1)} S_l^{(3)} + 3S_l^{(2)} S_l^{(2)} \right) + \\
&\quad \sum_{l \neq m} \frac{S_l S_m}{a_{lm}^4} \left(5S_l - 2a_{lm} S_l^{(1)} + a_{lm}^2 S_l^{(2)} \right) + \\
&\quad \sum_{l \neq m \neq n} \frac{2S_l S_m S_n}{3(a_{lm} a_{ln} a_{mn})^2} (a_{ln}^2 + a_{lm}^2 + a_{mn}^2) + O(\hbar^2)
\end{aligned} \tag{3.24}$$

3.8 Four and five instantons

To collect more “experimental data-points” we have considered in more details the cases of the gauge groups $SU(2)$ and $SU(3)$ with fundamental matter. We have computed explicitly the prepotential for four and five instantons and found a perfect agreement (yet a few typos) with the results of [13]. We should stress that this is a non-trivial check. Just as an example, we quote here the expression for \mathcal{F}_5 for $SU(2)$ gauge theory with $N_f = 3$:

$$\mathcal{F}_5(a, m) = \frac{\mu_3}{8a^{18}} (35a^{12} - 210a^{10}\mu_2 + a^8 (207\mu_2^2 + 846\mu_4)$$

$$-1210a^6\mu_2\mu_4 + a^4 (1131\mu_4^2 + 3698\mu_3^2\mu_2) - 5250a^2\mu_3^2\mu_4 + 4471\mu_3^4),$$

where $2a = a_1 - a_2$, $\mu_2 = m_1^2 + m_2^2 + m_3^2$, $\mu_3 = m_1m_2m_3$, $\mu_4 = (m_1m_2)^2 + (m_2m_3)^2 + (m_1m_3)^2$.

3.9 Adjoint matter and other matters

So far we were discussing $\mathcal{N} = 2$ gauge theories with matter in the fundamental representations. Now we shall pass to other representations. It is simpler to start with the adjoint representation. The ϵ -integrals (3.10) reflect both the topology of the moduli space of instantons and also of the matter bundle.

The latter is the bundle of the Dirac zero modes in the representation of interest. For the adjoint representation, and on \mathbb{R}^4 , this bundle can be identified with the tangent bundle to the moduli space of instantons. Turning on a mass term for the adjoint hypermultiplet corresponds to working equivariantly with respect to a certain $U(1)$ subgroup of the extended R-symmetry group. The $U(1) \times G \times \mathbf{T}^2$ equivariant Euler class of the tangent bundle (= the $G \times \mathbf{T}^2$ equivariant Chern polynomial) is the instanton measure in the case of massive adjoint matter. This reasoning leads to the following ϵ -integral:

$$\begin{aligned} Z_k = & \frac{1}{k!} \left(\frac{(\epsilon_1 + \epsilon_2)(\epsilon_1 + m)(\epsilon_2 + m)}{2\pi i \epsilon_1 \epsilon_2 m (\epsilon - m)} \right)^k \oint \prod_{I=1}^k \frac{d\phi_I P(\phi_I + m)P(\phi_I + \epsilon - m)}{P(\phi_I)P(\phi_I + \epsilon)} \times \\ & \times \prod_{I < J} \frac{\phi_{IJ}^2(\phi_{IJ}^2 - \epsilon^2)(\phi_{IJ}^2 - (\epsilon_1 - m)^2)(\phi_{IJ}^2 - (\epsilon_2 - m)^2)}{(\phi_{IJ}^2 - \epsilon_1^2)(\phi_{IJ}^2 - \epsilon_2^2)(\phi_{IJ}^2 - m^2)(\phi_{IJ}^2 - (\epsilon - m)^2)} \end{aligned} \quad (3.25)$$

Note the similarity of this expression to the contour integrals appearing[6] in the calculations of the bulk contribution to the index of the supersymmetric quantum mechanics with 16 supercharges (similarly, (3.10) is related to the one with 8 supercharges). This is not an accident, of course.

Proceeding analogously to the pure gauge theory case we arrive at the following expressions for the first two instanton contributions to the prepotential (which agrees with [13]):

$$\begin{aligned}\mathcal{F}_1 &= m^2 \sum_l T_l \\ \mathcal{F}_2 &= \sum_l \left(-\frac{3m^2}{2} T_l^2 + \frac{m^4}{4} T_l T_l^{(2)} \right) \\ &\quad + m^4 \sum_{l \neq n} T_l T_n \left(\frac{1}{a_{ln}^2} - \frac{1}{2(a_{ln} + m)^2} - \frac{1}{2(a_{ln} - m)^2} \right)\end{aligned}\tag{3.26}$$

where $T_l(x) = \prod_{n \neq l} \left(1 - \frac{m^2}{(x+a_{ln})^2} \right)$, $T_l = T_l(0)$, $T_l^{(n)} = \partial_x^n T_l(x)|_{x=0}$ (cf. [20]).

3.10 Perturbative part

So far we were calculating the nonperturbative part of the prepotential. It would be nice to see the perturbative part somewhere in our setup, so as to combine the whole expression into something nice.

One way is to calculate carefully the equivariant Chern character of the tangent bundle to $\widetilde{\mathcal{M}}_k$ along the lines sketched in the end of the previous section[10]. The faster way in the $\epsilon_1 + \epsilon_2 = 0$ case is to note that the expression (1.6) is a sum over partition with the universal denominator, which is not well-defined without the non-universal numerator. Nevertheless, let us try to pull it out of the sum.

We get the infinite product (up to an irrelevant constant):

$$\begin{aligned}\prod_{i,j=1}^{\infty} \prod_{l \neq n} \frac{1}{a_{ln} + \hbar(i-j)} &\sim \\ \exp - \sum_{l \neq n} \int_0^{\infty} \frac{ds}{s} \frac{e^{-sa_{ln}}}{(e^{\hbar s} - 1)(e^{-\hbar s} - 1)} &\end{aligned}\tag{3.27}$$

If we regularize this by cutting the integral at $s \sim \varepsilon \rightarrow 0$, we get a finite expression, which actually has the form

$$\exp \frac{\mathcal{F}^{pert}(a, \epsilon_1, \epsilon_2)}{\epsilon_1 \epsilon_2},$$

with \mathcal{F}^{pert} being analytic in ϵ_1, ϵ_2 at zero. In fact

$$\mathcal{F}^{pert}(a, 0, 0) = \sum_{l \neq n} \frac{1}{2} a_{ln}^2 \log a_{ln} + \text{ambiguous quadratic polynomial in } a_{ln}.$$

The formula (3.27) is a familiar expression for the Schwinger amplitude of a mass a_{ln} particle in the electromagnetic field

$$F \propto \epsilon_1 dx^1 \wedge dx^2 + \epsilon_2 dx^3 \wedge dx^4 . \tag{3.28}$$

Its appearance will be explained in the next section.

Let us now combine \mathcal{F}^{inst} and \mathcal{F}^{pert} into a single ϵ -deformed prepotential

$$\mathcal{F}(a, \epsilon_1, \epsilon_2) = \mathcal{F}^{pert}(a, \epsilon_1, \epsilon_2) + \mathcal{F}^{inst}(a, \epsilon_1, \epsilon_2)$$

where for general ϵ_1, ϵ_2 we define:

$$\mathcal{F}^{pert}(a, \epsilon_1, \epsilon_2) = \sum_{l \neq n} \int_{\varepsilon}^{\infty} \frac{ds}{s} \frac{e^{-sa_{ln}}}{\sinh\left(\frac{s\epsilon_1}{2}\right) \sinh\left(\frac{s\epsilon_2}{2}\right)} \tag{3.29}$$

with the singular in ε part dropped. We define:

$$\mathcal{Z}(a, \epsilon_1, \epsilon_2; q) = \exp \frac{\mathcal{F}(a, \epsilon_1, \epsilon_2; q)}{\epsilon_1 \epsilon_2} \tag{3.30}$$

4 M- and K-theory inspired conjectures

In this section we suggest a physical interpretation to the ϵ -deformed prepotential $\mathcal{F}(a, \epsilon_1, \epsilon_2; q)$. We also conjecture a relation of $\mathcal{Z}(a, \hbar, -\hbar; q)$ to a tau-function of Toda hierarchy.

Philosophy. So far we studied the equivariant cohomology of the instanton moduli space, pushforwards, and localization. The equivariant cohomology is a quasiclassical limit of the equivariant K-theory (in the same sense in which the DH formula is the quasiclassical limit of the (super)character formula

for $\text{Tr}(-)^F e^{-\hat{\mu}[\xi]}$. Conversely, by studying the \mathbf{S}^1 -equivariant cohomology of the loop space of the original space one can get the index theorems natural in K-theory [31].

This is of course an old story. However, this old story might get a new meaning with the entering of M-theory on the scene.

4.1 Five dimensional viewpoint

It was understood long time ago that the instanton corrections to the prepotential of $\mathcal{N} = 2$ gauge theory can be interpreted as one-loop corrections in the five dimensional theory compactified on a circle, in the limit of vanishing circle radius [32].

Consider five dimensional $\mathcal{N} = 2$ gauge theory with the gauge group G , and possibly some matter. Consider a path integral in this theory on the space-time manifold which is a product $\mathbf{S}^1 \times \mathbb{R}^4$. We shall impose periodic boundary conditions on the fermions in the theory (up to a twist described momentarily). We shall also consider the vacuum in which the adjoint scalar in the vector multiples has vacuum expectation value $\varphi \in \mathfrak{t}$. We should also specify the holonomy $g \in T$ of the gauge field around the circle at infinity of \mathbb{R}^4 (which must commute with φ). Together they define an element $\mathbf{a} = g \exp \beta \varphi \sim \exp \beta a$ of the complexified Cartan *subgroup* $\mathbf{T}_{\mathbb{C}}$ of the gauge group [32].

Let us define the following generalized index:

$$\begin{aligned} \mathcal{Z}(\mathbf{a}, \epsilon_1, \epsilon_2, \epsilon_3; \beta) = \\ \text{Tr}_{\mathcal{H}_{\mathbf{a}}} (-)^{2(j_L + j_R)} \exp - [(\epsilon_1 - \epsilon_2) J_L^3 + (\epsilon_1 + \epsilon_2) J_R^3 + \epsilon_3 J_I^3 + \beta H] \end{aligned} \quad (4.1)$$

We now choose $\epsilon_3 = \epsilon_1 + \epsilon_2$. This is the counterpart of choosing the subgroup $SU(2)_d$ as we did in the section 2. Here we have used the little group $SO(4)$ spins j_L, j_R and the generator J_I^3 of the R-symmetry group $SU(2)_I$. With this choice of ϵ 's some of the supercharges of the five dimensional gauge theory will commute with the twists $e^{\epsilon \cdot J}$ and (4.1) will define a generalized index.

The five dimensional theory has two kinds of particles. The perturbative spectrum consists of the gauge bosons and their superpartners (we shall now consider the case of minimal susy theory for simplicity). In addition, the theory has solitons, coming from instanton solutions of four dimensional

gauge theory. To find the spectrum of these solitons and their bound states one can adopt the standard collective coordinate quantization scheme.

In the limit $\beta \rightarrow 0$ the infinite-dimensional version of the heat kernel expansion will reduce the supertrace in (4.1) to a path integral in the four dimensional gauge theory. Moreover, the arrangement of the twists is such that the theory will possess some supersymmetry, which we identify with \tilde{Q} . One can play with the gauge coupling to further reduce the path integral to a finite-dimensional integral over the instanton moduli space, of the kind we considered in this paper.

On the other hand, geometrically, the twists in (4.1) can be realized by replacing the flat five dimensional space-time by a twisted \mathbb{R}^4 bundle over the circle \mathbf{S}^1 of the radius $\beta/2\pi$ such that by going around this circle one twists the⁸ \mathbb{R}^4 according to (4.1) Let us denote the resulting (locally flat) space by X_ϵ .

Now imagine engineering [33] the five dimensional gauge theory by “compactifying” M-theory on a non-compact Calabi-Yau given by the appropriate fibration of the ALE singularity over the base \mathbb{P}^1 . By further compactifying on X_ϵ we get a background, which can be now analyzed string-theoretically.

If we view the base circle of X_ϵ as the M -theory circle, then we end up with the IIA Mellin-like background, where one has a vev of the RR 1-form field strength, which is actually

$$F = \epsilon_1 dx^1 \wedge dx^2 + \epsilon_2 dx^3 \wedge dx^4 \quad (4.2)$$

near $x = 0$. In fact, by going to the weak string coupling limit ($\beta \rightarrow 0$) one can make (4.2) to hold arbitrarily far away from the origin.

The five dimensional particles, going around the circle \mathbf{S}^1 appear in the four remaining dimensions like particles carrying some charge with respect to F . In calculating their contribution to the supertrace (4.1) we would perform the standard Schwinger calculation, as in [34].

This identification suggests the interpretation of \hbar . Let us expand $\mathcal{F}(a, \hbar, -\hbar)$ as a power series in \hbar :

$$-\log \mathcal{Z}(a, \hbar, -\hbar) = \frac{1}{\hbar^2} \mathcal{F}(a, \hbar, -\hbar) = \sum_{g=0}^{\infty} \hbar^{2g-2} \mathcal{F}_g(a) \quad (4.3)$$

⁸If $\epsilon_1 + \epsilon_2 \neq 0$ one should also twist the transverse six dimensional space

(the fact that only the even powers of \hbar appear follows from the obvious symmetry $\epsilon_1 \leftrightarrow \epsilon_2$). The expansion (4.3) suggests that \hbar has to play a role of the string coupling constant. To be more precise, in the setup in which the gauge theory is realized as IIA compactification on the ALE singularity fibered over \mathbb{P}^1 the prepotential is essentially calculated by the worldsheet instantons of genus zero. The higher genus corrections give rise to the $R^2 F^{2g-2}$ couplings [35], where R is the curvature of the four dimensional metric, and F is the graviphoton field strength. Collecting all the evidence above we conjecture that the ϵ -deformed prepotential captures the graviphoton couplings (even in the case $\epsilon_1 + \epsilon_2 \neq 0$, where the graviphoton field strength is not self-dual – in this case the theory should be properly twisted).

4.2 K-theory viewpoint

If we don't take the limit $\beta \rightarrow 0$ we can still give a finite dimensional expression for (1.6) The collective coordinate quantization leads [32] to the minimal supersymmetric quantum mechanics on $\widetilde{\mathcal{M}}_k$, whose ground states correspond to the harmonic spinors on $\widetilde{\mathcal{M}}_k$. The index (4.1) calculates the equivariant index of Dirac operator on $\widetilde{\mathcal{M}}_k$. The latter has the following Atiyah-Singer expression:

$$\mathcal{Z}(\mathbf{a}, \epsilon_1, \epsilon_2, \beta; q) = \sum_{k=0}^{\infty} q^k \oint_{\widetilde{\mathcal{M}}_k} \widehat{A}_\beta(\widetilde{\mathcal{M}}_k) \tag{4.4}$$

where $\widehat{A}(M)$ is the A-roof genus of the manifold M :

$$\text{ch}(TM) = \sum_i e^{x_i} \implies \widehat{A}_\beta(M) = \prod_i \frac{\beta x_i}{e^{\frac{\beta x_i}{2}} - e^{-\frac{\beta x_i}{2}}} \tag{4.5}$$

The localization technique (this time in equivariant K-theory [36]) leads to the following expression for (4.4)

$$\mathcal{Z}(\mathbf{a}, \hbar, -\hbar, 2\beta; q) = \sum_{\vec{k}} q^{|\vec{k}|} \prod_{(l,i) \neq (n,j)} \frac{\sinh \beta (a_{ln} + \hbar (k_{l,i} - k_{n,j} + j - i))}{\sinh \beta (a_{ln} + \hbar (j - i))} \tag{4.6}$$

It would be nice to analyze (4.6) further, relate it to the relativistic Toda chain spectral curves [32], and to the four dimensional analogue of Verlinde formula [37]. It also seems reasonable to expect applications of (4.6) to the DLCQ quantization of the M5-brane [38].

4.3 Chiral fermions and M5 brane

Another conjecture relates the expansion (1.6) to the dynamics of the Seiberg-Witten curve⁹. Denote, as before $q = \Lambda^{2N}$.

Consider the theory of a free complex chiral fermion ψ, ψ^* ,

$$\mathcal{S} = \int_{\Sigma} \psi^* \bar{\partial} \psi \quad (4.7)$$

living on the curve Σ :

$$w + \frac{\Lambda^{2N}}{w} = \mathbf{P}(\lambda), \quad \mathbf{P}(\lambda) = \prod_{l=1}^N (\lambda - \alpha_l) \quad (4.8)$$

embedded into the space $\mathbb{C} \times \mathbb{C}^*$ with the coordinates (λ, w) . This curve has two distinguished points $w = 0$ and $w = \infty$ which play a prominent role in the Toda integrable hierarchy [39]. Let us cut out small disks D_0 and D_{∞} around these two points.

The path integral on the surface Σ with two disks deleted will give a state in the tensor product $\mathcal{H}_0 \otimes \mathcal{H}_{\infty}^*$ of the Hilbert spaces $\mathcal{H}_0, \mathcal{H}_{\infty}$ associated to ∂D_0 and ∂D_{∞} respectively. It can also be viewed as an operator $G_{\Sigma} : \mathcal{H}_0 \rightarrow \mathcal{H}_{\infty}$.

Choose a vacuum state $|0\rangle \in \mathcal{H}_0$ and its dual $\langle 0| \in \mathcal{H}_{\infty}^*$ (we use the global coordinate w to identify \mathcal{H}_0 and \mathcal{H}_{∞}). Consider

$$\tau_{\Sigma} = \left\langle 0 \left| \exp \left(\frac{1}{\hbar} \oint_{\partial D_{\infty}} S J \right) G_{\Sigma} \exp \left(-\frac{1}{\hbar} \oint_{\partial D_0} S J \right) \right| 0 \right\rangle \quad (4.9)$$

where:

$$\begin{aligned} J &=: \psi^* \psi : \\ dS &= \frac{1}{2\pi i} \lambda \frac{dw}{w} \end{aligned} \quad (4.10)$$

and we choose the branch of S near $w = 0, \infty$ such that (cf. [40]) :

$$S = \frac{N}{2\pi i} w^{\mp \frac{1}{N}} + O(\lambda^{-1})$$

⁹For simplicity we consider the case $N_f = 0$. The conjecture for $N_f > 0$ case is easy to guess.

Let us represent Σ as a two-fold covering of the λ -plane. It has branch points at $\lambda = \alpha_l^\pm$ where

$$\mathbf{P}(\alpha_l^\pm) = \pm 2\Lambda^N$$

Let us choose the cycles A_l to encircle the cuts between α_l^- and α_l^+ . Of course, in $H_1(\Sigma, \mathbf{Z})$, $\sum_l A_l = 0$. Then, we define:

$$a_l = \oint_{A_l} dS .$$

Our final conjecture states:

$\mathcal{Z}(a, \hbar, -\hbar; q) = \tau_\Sigma$

(4.11)

Note that from this conjecture the fact that $\mathcal{F}_0(a, 0, 0)$ coincides with the Seiberg-Witten expression follows as a consequence of the Krichever universal formula [41]. The remaining paragraph is devoted to the explanation of the motivation behind (4.11)

Let us assume that we are in the domain where $\alpha_l - \alpha_m \gg \Lambda$. Then the surface Σ can be decomposed into two halves Σ_\pm by N smooth circles C_l which are the lifts to Σ of the cuts connecting α_l^- and α_l^+ . The path integral calculating the matrix element (4.9) can be evaluated by the cutting and sewing along the C_l 's. The path integral on Σ_\pm gives a state in

$$\otimes_{l=1}^N \mathcal{H}_{C_l}$$

(its dual). If we first pull the $\oint S J$ as close to C_l as possible, we shall get the Hilbert space obtained by quantization of the fermions which have $a_l + \frac{1}{2} \text{mod } \mathbf{Z}$ moding:

$$\psi(w) \sim \sum_{i \in \mathbf{Z}} \psi_{l,i} w^{a_l+i} \left(\frac{dw}{w} \right)^{\frac{1}{2}} \tag{4.12}$$

near $C_l \subset \Sigma$. In addition, the states in \mathcal{H}_{C_l} of fixed total $U(1)$ charge are labelled by the partitions $k_{l,i}$. We conjecture, that

$$\left\langle 0 \left| e^{\oint S J} \prod_{l,i} \psi_{l,k_{l,i}-i} \psi_{l,-i}^* \right| 0 \right\rangle_l \sim \prod_{(l,i) < (m,j)} (a_{lm} + \hbar(k_{l,i} - k_{m,j} + j - i)) \tag{4.13}$$

It is clear that (4.13) implies (4.11) For $N = 1$ (4.13) is of course a well-known fact (with the coefficient given by $\prod_{i < j} \frac{1}{j-i}$), which leads to the following

formula for (1.6) which can also be derived using Schur identities [42]:

$$Z_{N=1}(\hbar, -\hbar; q) = e^{-\frac{q}{\hbar^2}}$$

confirming the fact, that even though we worked with the resolved moduli space $\cup_k \widetilde{\mathcal{M}}_{k,1} = \cup_k (\mathbb{C}^2)^{[k]}$ the “symplectic” volume we got is that of $\cup_k \widetilde{\mathcal{M}}_{k,1} = \cup_k \text{Sym}^k(\mathbb{R}^4)$.

The conjecture (4.11) should in some sense trivially follow from the M5/NS5 realization of the four dimensional supersymmetric gauge theory [43][44]. The ϵ -twisting of the four dimensional part of the fivebrane worldvolume reduces the dynamical degrees of freedom down to those of the chiral boson on the curve, which after fermionization should lead to (4.11)

Finally, if (4.11) is true, it is natural to conjecture that the analogous equivariant generating functions for instantons on ALE spaces of the ADE type will be related to the ADE WZW theories on the SW curves.

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