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Seismic measurement of the depth of the solar convection zone

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ABSTRACT

The observed frequencies of solar oscillations have been used to determine the depth of the convection zone. The effect of diffusion of helium and heavy elements on this measurement is studied and it is found that the discontinuity in the composition gradient at the base of the CZ due to diffusion gives rise to some systematic errors in this measurement. Taking into account these systematic errors the base of the CZ is estimated to be at a radial distance of $(0.713 \pm 0.001)\,\mathrm{R}_\odot$. Further, the estimated opacity at the base of the CZ is found to be consistent with that calculated from the OPAL opacity tables using the current value of Z/X. Assuming that the OPAL tables correctly represent the opacity of solar material the surface Z/X is estimated to be 0.0245 ± 0.0008.

Key words: convection – Sun: interior – Sun: oscillations.

1 INTRODUCTION

The transition of the temperature gradient from adiabatic to radiative values at the base of the solar convection zone (CZ) leaves its signature on the sound speed. Thus helioseismic measurement of the sound speed enables a determination of the position of the base of the CZ (cf. Kosovichev & Fedorova 1991; Christensen-Dalsgaard, Gough & Thompson 1991, hereafter CDGT; Guzik & Cox 1993). The frequencies of solar oscillations are very sensitive to the depth of the CZ and hence it is possible to determine the depth very accurately from the observed frequencies. Conversely, if the depth of the CZ in a solar model is somehow adjusted to match the estimated value, the model in general turns out to be fairly close to the Sun, in terms of both the frequency differences and the sound speed, even when the input physics in the solar model is not quite correct.

Apart from temperature, the sound speed is also affected by the composition of solar material. Hence, a discontinuity in the composition gradient at the CZ base in models that incorporate diffusion of helium and heavy elements would also introduce differences in the frequencies of solar oscillations that are similar to those introduced by change in temperature gradient. Because the formulation of diffusion in the Sun is uncertain (cf. Gough et al. 1996), so is the composition profile. This will introduce some systematic error in the measurement of the depth of the CZ (Basu & Antia 1994a), which is investigated in this work. These errors are independent of those introduced because of observational errors in the solar oscillation frequencies. We also study the systematic errors introduced by other sources, such as those due to uncertainties in opacities and surface abundances, or those due to the presence of an overshoot layer below the CZ.

Since the base of the CZ is defined to be the point at which the radiative temperature gradient equals the adiabatic value $[\nabla_{\mathrm{ad}} = (\partial \ln T/\partial \ln P)_{\mathrm{s}}]$, it is possible to estimate the opacity at the base of the CZ. However, this estimate is found to be rather sensitive to the chemical composition of the solar envelope as well as to the composition gradients below the CZ.

Solar oscillation frequencies have been used to estimate the extent of overshoot below the solar CZ (Gough 1990; Monteiro, Christensen-Dalsgaard & Thompson 1994; Basu, Antia & Narasimha 1994). While these measurements provide only an upper limit on the extent of overshoot, they have effectively ruled out solar models with steep composition gradients below the base of the CZ (Basu & Antia 1994b; Basu 1997). These results are useful in the present work to constrain the composition gradient at the base of the CZ, as that is the main source of systematic errors in the measurement of the depth of the CZ.

In this work, we use results from the asymptotic sound speed inversion to estimate the depth of the CZ. Using a number of test models we identify various sources of systematic errors which may arise in helioseismic measurement of the depth of the CZ. Section 2 describes various models that are used for calibration and testing the technique, while the basic technique used to determine the depth of the CZ as well as the sources of systematic errors in this measurement are described in Section 3. Section 4.1 describes the results obtained for test models as well as those using the observed frequencies and Section 4.2 gives the estimate of the opacity at the base of the CZ. Finally, in Section 5 we discuss the results from this study.

2 MODELS USED

Since the uncertainties in abundance profiles introduce systematic errors in the measurement of the depth of the CZ, we have used three sets of calibration models with different abundance profiles to estimate the systematic errors. We use envelope models for this

Table 1. Solar models used.

Model	Type	Opacity	EOS	X-profile	Z-profile	Y_{sur}	Z_{sur}	$r_b/R_\odot$	Over-shoot
ND712	Env.	OPAL	OPAL	ND	ND	0.2460	0.01803	0.71200	0
NDF	Full	OPAL	OPAL	ND	ND	0.2772	0.02003	0.72329	0
DIF712	Env.	OPAL	OPAL	DIF	DIF	0.2460	0.01803	0.71200	0
DIFX	Env.	OPAL	OPAL	DIF	ND	0.2460	0.01803	0.71300	0
DIFF	Full	OPAL	OPAL	DIF	DIF	0.2615	0.02046	0.71331	0
INVF	Full	OPAL	OPAL	INV	INV	0.2456	0.01804	0.71340	0
TD	Full	OPAL	OPAL	TD	ND	0.2472	0.01800	0.71463	0
INVP	Full	OPAL	OPAL	INV	DIF	0.2510	0.01791	0.71461	0
ND713ov	Env.	OPAL	OPAL	ND	ND	0.2460	0.01803	0.71300	2.8 Mm
ND717ov	Env.	OPAL	OPAL	ND	ND	0.2460	0.01803	0.71702	2.8 Mm
NDY	Env.	OPAL	OPAL	ND	ND	0.2562	0.01779	0.71300	0
NDZ	Env.	OPAL	OPAL	ND	ND	0.2445	0.01950	0.71300	0
MHD	Env.	OPAL	MHD	ND	ND	0.2460	0.01803	0.71300	0
CT	Env.	CT	OPAL	ND	ND	0.2460	0.01803	0.71300	0

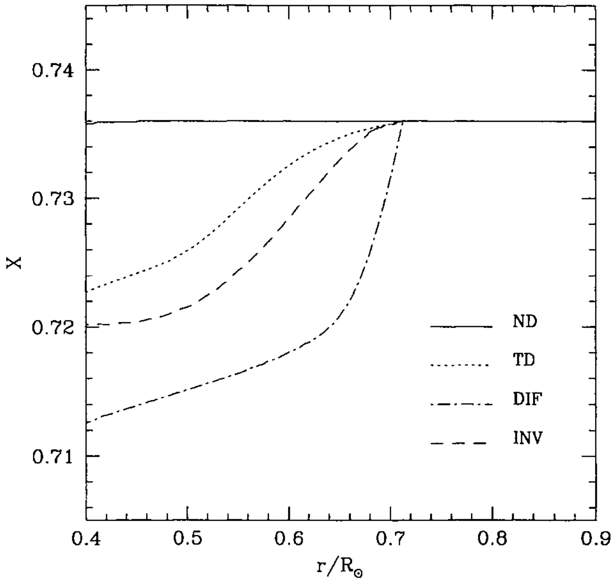

Figure 1. The hydrogen abundance profiles in different models. ND is the profile when gravitational settling of helium is ignored (Bahcall & Pinsonneault 1992). The profile DIF is from a model including diffusion of helium (Bahcall & Pinsonneault 1992). TD is the profile including the turbulent diffusion (Christensen-Dalsgaard et al. 1993), while INV is that obtained by helioseismic inversion (Antia & Chitre 1997).

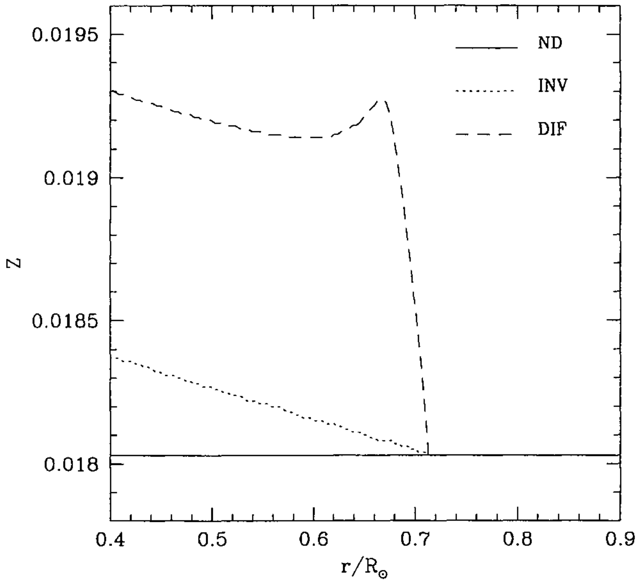

Figure 2. The different heavy element abundance profiles used in the models. ND is the homogeneous non-diffusion profile; DIF, from Proffitt (1994), is a result of non-turbulent diffusion. The profile INV is a smoothed version of DIF.

purpose since this enables the construction of models with a specified depth of the CZ and specified chemical abundances. Each set of calibration models consists of five models with $r_b = 0.709$, 0.711, 0.713, 0.715, and $0.717\,R_\odot$, where r_b is the radial distance of the base of the CZ. All models have been constructed using the OPAL equation of state (Rogers, Swenson & Iglesias 1996) and OPAL opacities (Rogers & Iglesias 1992; Iglesias & Rogers 1996). All models employ the formulation of Canuto & Mazzitelli (1991) for calculating the convective flux. These models have a helium abundance of 0.246 (cf. Basu & Antia 1995), and a surface Z/X of 0.0245 (cf. Grevesse & Noels 1993). Since these are envelope models, it is possible to scale the X and Z abundance profiles obtained from evolutionary solar models separately to obtain the desired surface abundances for all calibration models. Further, the Z value in the envelope is chosen to yield the required value of Z/X.

The three abundance profiles used are as follows.

ND: the hydrogen and heavy element abundance profiles from the no-diffusion model of Bahcall & Pinsonneault (1992).

DIF: the hydrogen abundance profile from the solar model of Bahcall & Pinsonneault (1992), incorporating the diffusion of helium and the Z profile from Proffitt (1994).

INV: the hydrogen abundance profile obtained from helioseismic inversions (Antia & Chitre 1997) and a smoothed version of the Z profile from Proffitt (1994).

In order to test our procedure, we have used a series of test models whose properties are summarized in Table 1. Most of the models have one of the three abundance profiles listed above. One model (TD) has the hydrogen abundance profile resulting from turbulent diffusion, TD2, from Christensen-Dalsgaard, Proffitt & Thompson (1993). In order to separate out the effect of X profiles from that of Z profiles we also have one model DIFX which incorporates only helium diffusion but not heavy element diffusion. Similarly, we have also constructed model INVP, with the unsmoothed Z profile from Proffitt (1994), but with the same X profile as INV. The X profiles in these models have been scaled (along with the mixing length) to match the central boundary conditions, while the Z profiles have been scaled to obtain the required value of Z/X in the envelope. The various X and Z profiles used in these models are shown in Figs 1 and 2 respectively.

In addition to models constructed with the OPAL equation of state, we have also computed one model using the MHD equation of state (cf. Hummer & Mihalas 1988; Mihalas, Däppen & Hummer 1988; Däppen et al. 1988) to enable us to study any systematic effect that may be introduced by uncertainties in the equation of state. We have also constructed models to estimate the influence of the helium abundance (model NDY), the heavy element abundance (model NDZ) and the presence of an overshoot layer (models ND713ov and ND717ov) below the CZ. We have also used a model (CT) that uses the Los Alamos opacities (Cox & Tabor 1976) to study the effect of errors in opacity.

To determine the depth of the CZ, we need the frequencies of acoustic modes with lower turning points around the base of the CZ. In this work we have used solar oscillation frequencies observed from the Big Bear Solar Observatory (BBSO) (Libbrecht, Woodard & Kaufman 1990). We have only used modes with frequencies less than 3.5 mHz as these are thought to be more reliable than the higher frequency modes. Similarly, only the intermediate-degree modes ($5 \le \ell \le 140$) are used, as these are determined by fitting individual modes, and hence are likely to be more reliable than the higher degree modes determined by ridge-fitting techniques. In fact, there are systematic differences between the frequencies of these intermediate-degree modes and those of higher degree (Antia 1996). Such systematic differences can introduce errors in inversion results and hence we do not use the high-degree modes in this work. These high-degree modes which are trapped in the outer layers of the Sun are not essential for the present work. Further, we only use modes with a lower turning point above $0.4\,\mathrm{R}_\odot$, i.e., $\nu/(\ell+.5) < 0.2$ mHz. The lower limit to the turning point was dictated by the fact that we use envelope models for calibration.

3 THE PROCEDURE

3.1 The basic technique

The temperature gradient in the lower part of the solar CZ is close to (although marginally larger than) the adiabatic value: it decreases and switches to the radiative value at the base of the CZ. Thus if there are two otherwise similar solar models with different depths of CZ, the model with a deeper CZ will have a larger sound speed than the other, just below the base of the CZ. In Fig. 3, we have shown the relative sound-speed differences between some solar envelope models, which differ only in their depth of the CZ. The observable difference of sound speed can be calibrated to find the CZ depth of a test model or of the Sun.

It is not necessary to perform a complete sound-speed inversion in order to determine the CZ depth of the Sun. Asymptotically, the frequency differences between a solar model and the Sun, or between two solar models, can be written as (Christensen-Dalsgaard, Gough & Thompson 1989)

$$S(w)\frac{\delta\omega}{\omega} = H_1(w) + H_2(\omega), \tag{1}$$

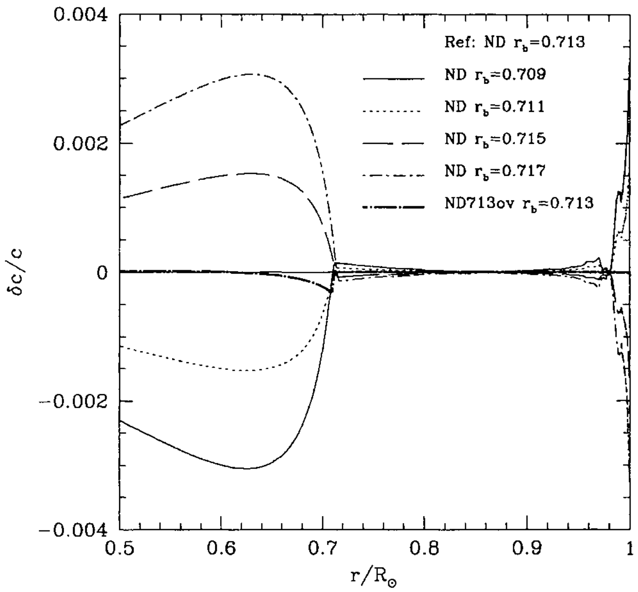

Figure 3. The relative sound-speed difference between models that differ in the depth of their CZ and extent of overshoot below the CZ. All differences are with respect to the model ND713, which has $r_b = 0.713\,\mathrm{R}_\odot$. The differences are in the sense $(c_{\mathrm{ND713}} - c_{\text{other model}})/c_{\mathrm{ND713}}$, where c is the sound speed.

where

$$S(w) = \int_{r_t}^{\mathrm{R}_\odot} \left(1 - \frac{c^2}{w^2 r^2}\right)^{-1/2} \frac{\mathrm{d}r}{c}. \tag{2}$$

Here ω is the frequency of a mode $w = \omega/(\ell + 1/2)$ and r_t is the lower turning point such that $c/r_t = w$. The functions $H_1(w)$ and $H_2(\omega)$ can be found by a least-squares fit to the known frequency differences. $H_1(w)$ can be expressed in terms of the sound-speed difference between the two models:

$$H_1(w) = \int_{r_t}^{\mathrm{R}_\odot} \left(1 - \frac{c^2}{r^2 w^2}\right)^{-1/2} \frac{\delta c}{c}\frac{1}{c}\,\mathrm{d}r, \tag{3}$$

and can be inverted to obtain the sound-speed difference, $\delta c/c$, between the reference model and the Sun. However, that is not required as $H_1(w)$ itself can also be used to determine the CZ depth. In Fig. 4, we have plotted $H_1(w)$ for the same model pairs that are shown in Fig. 3. We can see that the differences in sound speed are converted to differences in $H_1(w)$, hence $H_1(w)$ can be calibrated instead of sound speed.

There is a distinct advantage in using $H_1(w)$ instead of the inverted sound-speed difference. The use of $H_1(w)$ allows us to use the more reliable intermediate-degree modes only. Inversion of $H_1(w)$ to determine the sound-speed difference requires the shallowly penetrating (hence high-degree) modes, since the integration is carried out from the surface (cf. Christensen-Dalsgaard, Gough & Thompson 1989). In the absence of high-degree modes $H_1(w)$ will need to be extrapolated to a lower value of w that corresponds to the solar surface, and this introduces errors in the sound speed thus determined.

If $\phi(w)$ is the $H_1(w)$ between two envelope models that differ only in their depth of CZ, then $H_1(w)$ for any other pair of models can be written as

$$H_1(w) = \beta\phi(w) + H_s(w), \tag{4}$$

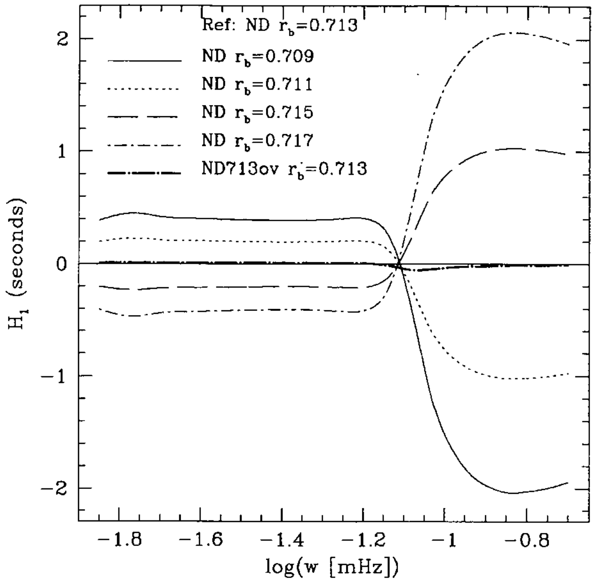

Figure 4. The function $H_1(w)$ for the models whose sound-speed difference is shown in Fig. 3.

where $H_s(w)$ is a smooth component of $H_1(w)$ which results from sound-speed differences that arise from differences in the equation of state, abundances, surface physics, etc., and the first term is the contribution to $H_1(w)$ due to the sound-speed difference caused by the difference in r_b, the position of the base of the CZ. Such a decomposition is possible since the integrand in equation (3) is linear in $\delta c/c$. Thus if β is determined by a least-squares fit, the unknown r_b of the Sun can be obtained. Although $H_1(w)$ can always be expressed in terms of the two components with a suitable choice for the smooth component $H_s(w)$, in practice the separation through fitting would be possible only if $H_s(w)$ were sufficiently smooth. Between two models that differ only in the depth of their CZ $H_s(w)$ would be essentially zero, but between two models that differ in hydrogen abundance profile below the CZ $H_s(w)$ could be significant and cause some ambiguity in separating out the two components. Since the calibration curve $\phi(w)$ has significant features only around $r = r_b$, it would be easier to separate the two components in equation (4) if β were determined by a fit over a reasonably narrow range of w around the base of the CZ. In this work we have generally determined β by a least-squares fit over the interval $-1.3 \le \log[w(\mathrm{mHz})] \le -1.1$. Although from Fig. 4 it can be seen that the variation in the calibration curves $\phi(w)$ continues up to $\log[w(\mathrm{mHz})] = -1.0$, the deeper part is not very useful in realistic circumstances as the model differences in those regions tend to be dominated by differences in abundance gradients. The actual limits of $\log w$ within which the fit to find β is performed were determined by looking at the variation of the obtained result for the test models. We find that the upper limit of the fitting range is more crucial to the results since it lies below the CZ base. A higher upper limit corresponds to fitting deeper into the radiative interior and hence effects due to the difference in abundance gradients far from the CZ base become difficult to remove. The selected upper limit of $\log[w(\mathrm{mHz})] = -1.1$ corresponds to a turning point at approximately $0.68\,\mathrm{R}_\odot$, and the results are quite stable around this range. We find that the shift in the computed r_b is less than $0.0002\,\mathrm{R}_\odot$ when the upper limit of the fitting interval is changed from $\log[w(\mathrm{mHz})] = -1.12$ to -1.08 In most cases the shift is much smaller. The results of the fit, on the other hand, are not very sensitive to the lower limit of the fitting range since that lies within the CZ where the temperature gradient is almost adiabatic.

Although β determined from one model is enough to estimate r_b, in practice we determine β for a series of calibration models with different r_b, and interpolate to find the position for which $\beta = 0$. This reduces some of the errors that may be introduced by an imperfect fit to equation (4).

In order to determine the uncertainty in the results due to errors in the observed frequencies, we have performed Monte Carlo simulations for a series of models with 25 realizations of errors in each. The variance in the results would give an estimate of errors. Apart from this there is also the additional error due to variation in the fitting range as explained earlier. It turns out that the random errors due to uncertainties in the frequencies are comparable to those arising from variation in the fitting range and both of these are very small compared with other systematic errors.

The main difference between the technique adopted in this work and that used by CDGT is that while they tried to estimate the location where there is a discontinuity in the sound-speed derivatives using the sound-speed inversion, in this work we do not directly attempt to determine the location of discontinuity but instead look for resulting sound-speed differences in the radiative interior. There will undoubtedly be systematic errors since the sound-speed difference below the CZ also depends on opacity gradients as well as composition gradients. We believe, however, that those are compensated by the simplicity of the approach and by the fact that because of the limited resolution that is available from a finite set of observed frequencies it is not possible to determine the location of the discontinuity in the derivatives of the sound speed to sufficient accuracy. Besides, with recent improvements in the input physics the difference in sound speed between a standard solar model and that in the Sun has decreased significantly (Christensen-Dalsgaard et al. 1996; Gough et al. 1996) and systematic errors introduced due to these uncertainties should be fairly small. The estimated position of the discontinuity will also have systematic errors from other sources. For example, there is no difficulty in detecting differences caused by even a shift of $0.0002\,\mathrm{R}_\odot$ in r_b by looking at the sound speed [or $H_1(w)$] below the CZ, but it may not be possible to detect such shifts in r_b by looking for the position of the discontinuity in derivatives of the sound speed because of the finite resolution of the inversions.

3.2 Possible systematic errors

Unfortunately, the change in temperature gradient is not the only factor that leaves its imprint on the sound speed near the base of the CZ, and any factor that changes the sound-speed profile below the CZ base introduces an error in the estimate of r_b. For example, the change in abundance gradients at the CZ base will also affect the sound-speed profile. It has been known for some time that helium and heavy elements diffuse below the CZ due to gravitational settling (cf. Cox, Guzik & Kidman 1989; Bahcall & Pinsonneault 1992; Christensen-Dalsgaard et al. 1993). Convective mixing in the CZ necessarily means that the abundance profile will be flat there, but there is a change at the base of the CZ. The excess helium causes an increase in the mean molecular weight below the base of the CZ, and this reduces the sound speed. Thus a model with helium diffusion will appear to have a shallower CZ, i.e., in regions just below the base of the CZ it will have a sound speed similar to a no-diffusion model with a shallower CZ. Thus two models with the same depth of CZ but different abundance profiles will also show a sound-speed difference quite similar to that in Fig. 3.

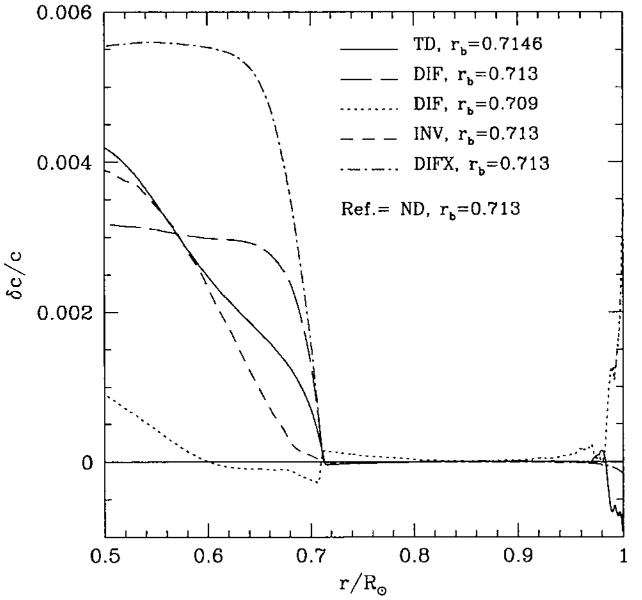

Figure 5. The relative sound-speed difference between the reference model of Fig. 3 and models that have different diffusion profiles.

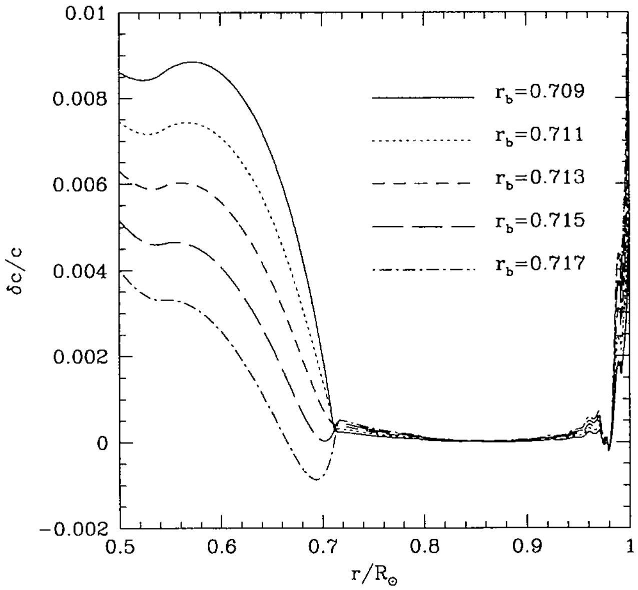

Figure 6. The relative sound-speed difference between the ND calibration models and the model CT.

The effect of different abundance profiles is illustrated in Fig. 5, where we have plotted the sound-speed difference between a model with no settling of helium and models that incorporate helium diffusion. The figure clearly shows that if we try to determine r_b for a model with an abundance profile that is different from that of the models used for calibration, then we will obtain results that are systematically different from the true result. From Fig. 5, we find that the presence of a steeper hydrogen abundance gradient in a test model compared with a reference model causes the sound speed in the test model to be lower than that in the reference model. From Fig. 3, we see than this means that the 'apparent' r_b of the test model is larger than the actual value, i.e., the CZ is shallower. The result is an overestimation of r_b. From Fig. 5 it is clear that the sound speed in a diffusion model with $r_b = 0.709\,R_\odot$ is close to that in a no diffusion model with $r_b = 0.713\,R_\odot$. Thus inclusion of helium and heavy element diffusion is expected to introduce a systematic error of about $0.004\,R_\odot$ in the measurement of the depth of the CZ (Basu & Antia 1994a). If the gradients in hydrogen abundance profiles do not differ too much at the base of the CZ (e.g., models TD and INV) then it may be possible to find r_b with better accuracy. This is confirmed by our results.

It may be noted that diffusion of heavy elements below the CZ also affects the sound-speed profile and hence introduces systematic errors in the measurement of the CZ depth. An increase in heavy element abundance results in an increase in opacity and hence the temperature (or sound-speed) gradient. Thus models with heavy element diffusion tend to have larger sound speeds compared with a model that does not include diffusion of heavy elements. Therefore, a model with steeper profile of Z below the CZ will appear to have a deeper CZ as compared with a model with same CZ depth and same surface abundances but with a less steep Z profile.

Apart from abundance profiles, the presence of an overshoot layer below the CZ could also affect the sound-speed profile. Since the overshoot layer is also expected to be adiabatically stratified, we may expect the present technique to yield the base of the overshoot layer rather than the base of the CZ. However, from the sound-speed profiles for models with overshoot in Fig. 3 it is clear that the presence of an overshoot layer has practically no effect on the sound-speed profile except in the narrow overshoot layer itself, which will not be resolved by inversions (see Fig. 4). It may be noted that because of limits on the extent of overshoot (cf. Basu & Antia 1994b) an overshoot layer with thickness large enough to be resolved by inversions is ruled out. Hence, the presence of an overshoot layer is not expected to affect the determination of the depth of the CZ in any significant manner. This implies that the value of r_b as measured from the frequencies will give the radius at which the radiative temperature gradient equals the adiabatic value, i.e., the top of the overshoot region, rather than its base.

The sound-speed below the CZ also depends on the opacities. Thus if we take an extreme case of a model using the Los Alamos opacities, then even for the same depth of CZ the sound-speed difference (cf. Fig. 6) shows a steep trend similar to that between models with different CZ depths. This will clearly introduce some systematic errors in the measurement of r_b. Since the Los Alamos opacities are lower than the OPAL opacities, the sound speed will be lower and the CZ will appear to be shallower. Looking at the sound-speed difference between the models CT and ND715, it may appear that if we look at a sufficiently narrow region below the CZ base it will be possible to estimate the depth correctly, but as can be seen from Fig. 7, which shows the corresponding $H_1(w)$ obtained from the frequency differences between these models, the small hump in the sound speed at the base of the CZ is completely missed because of limited resolution of inversions. Thus clearly errors in opacity which are of the order of the difference between CT and OPAL opacities can be expected to introduce systematic errors of about $0.002\,R_\odot$ in r_b. That is not, however, a very serious issue, since from the inverted sound-speed differences it is clear that Los Alamos opacities are not consistent with solar data. It can be seen from Fig. 6 that the sound-speed difference between the CT and OPAL models with the same CZ depth is as large as 0.6 per cent in the radiative interior. Such differences would be easily detectable from sound-speed inversions.

Since the abundance profile of hydrogen or heavy elements inside the Sun is not known, it becomes necessary to use calibration

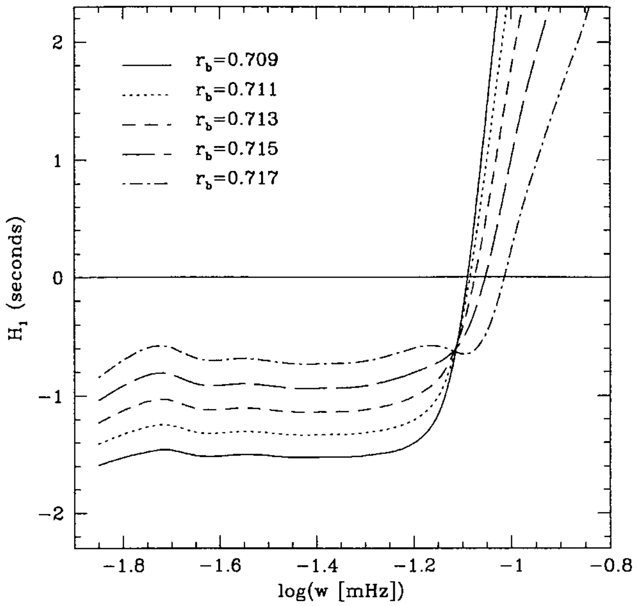

Figure 7. The function $H_1(w)$ between the ND calibration models and the model CT.

models with different abundance profiles to determine r_b for the Sun in order to estimate the systematic errors introduced by using an incorrect abundance profile. Hence, we have used three such sets of models. In addition, we have first tested out the method on a series of test models to check for other systematic errors. The models and the abundance profiles used in constructing these test and calibration models have been described in the previous section.

4 RESULTS

4.1 Depth of the solar CZ

Employing the technique described in Section 3, we have estimated

Table 2. Seismically inferred position of the CZ base.

Model	Exact r_b ($R_\odot$)	Calibration models ND ($R_\odot$)	DIF ($R_\odot$)	INV ($R_\odot$)
ND712	0.71200	0.71200	0.70846	0.71125
NDF	0.72329	0.72367	0.72016	0.72277
DIF712	0.71200	0.71555	0.71200	0.71476
DIFX	0.71300	0.71844	0.71489	0.71762
DIFF	0.71331	0.71698	0.71343	0.71617
INVF	0.71340	0.71416	0.71061	0.71339
TD	0.71463	0.71510	0.71155	0.71432
INVP	0.71461	0.71364	0.71009	0.71288
ND713ov	0.71300	0.71279	0.70925	0.71204
ND717ov	0.71702	0.71686	0.71330	0.71605
NDY	0.71300	0.71331	0.70975	0.71255
NDZ	0.71300	0.71292	0.70938	0.71217
MHD	0.71300	0.71330	0.70975	0.71254
CT	0.71300	0.71527	0.71193	0.71470
Sun				
BBSO data[1]	–	**0.71407**	**0.71051**	**0.71330**

[1]Monte Carlo simulations show a 1σ error of $0.0002\,R_\odot$.

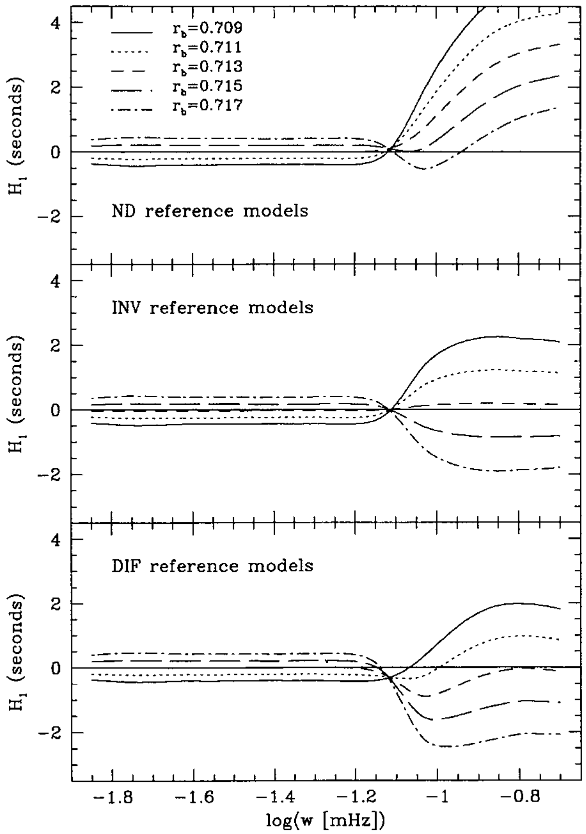

Figure 8. The function $H_1(w)$ for the model INVF using the three sets of calibration models as reference models.

the CZ depth in the test models using all three sets of calibration models. The results are shown in Table 2. Monte Carlo simulations show a 1σ error of about $0.0002\,R_\odot$ for the BBSO data.

Table 2 clearly shows that the position of the base of the CZ can be determined accurately if the calibration and test models have the same abundance profiles below the base of the CZ and use similar opacity tables, or, in other words, when the signal due to the change in the temperature gradient at the CZ base is not contaminated by the signal due to changes in abundances or opacities. From the results of models DIFX and INVP it is also evident that uncertainties in the Z profile too have a significant effect on the determination of r_b. However, the effect of the gradient in the Z profile is opposite to that in the Y profile. Further, the error in r_b due to uncertainty in the abundance profiles is much larger than those due to observational errors. For example, conventional treatment of helium diffusion introduces a systematic error of $0.0054\,R_\odot$, as can be seen from the results for model DIFX, while heavy element diffusion introduces a systematic error of $0.0019\,R_\odot$ in the opposite direction. The combined effect of helium and heavy element diffusion is $0.0036\,R_\odot$, which is more than an order of magnitude larger than the errors due to uncertainties in frequencies. Fig. 8 shows $H_1(w)$ for the model INVF with calibration models that have different abundance profiles. The effect of different composition profiles is clear from this figure.

Apart from composition profile, another major source of systematic error is the opacity. From the results for model CT, it is clear that error in opacity can introduce systematic errors of about $0.002\,R_\odot$ for the extreme-difference case of Los Alamos and OPAL opacities. The sound-speed difference between the CT and

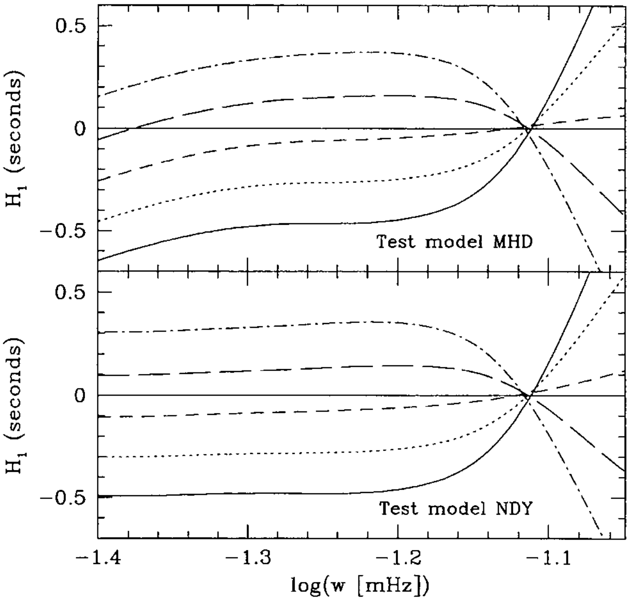

Figure 9. The function $H_1(w)$ for the models NDY and MHD using the ND set of calibration models as reference models. The various line types show the results for reference models with different r_b as marked in Fig. 8.

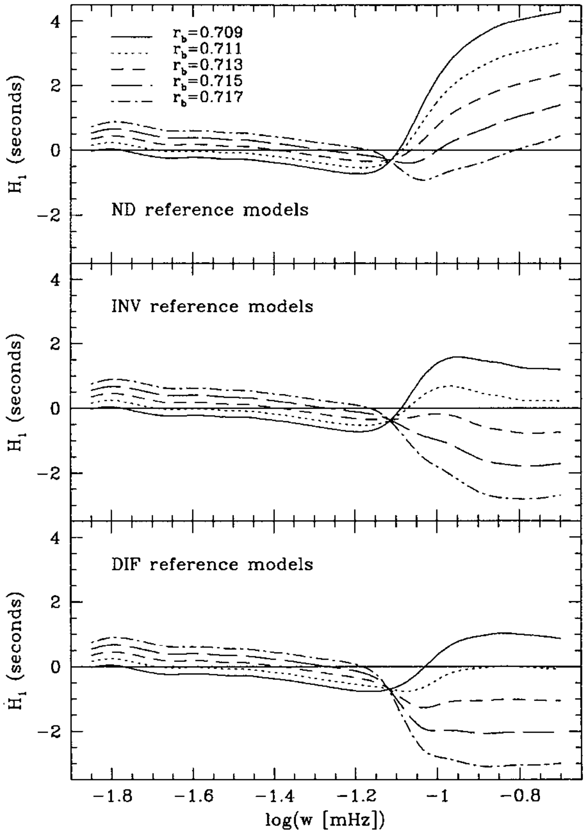

Figure 10. The functions $H_1(w)$ for the BBSO data using the three sets of calibration models as reference models.

ND models with the same CZ depth is about 0.6 per cent in regions below the CZ, while from inversions we know that the difference in sound speed between the Sun and current standard solar models is only of the order of 0.06 per cent (cf. Gough et al. 1996; Antia 1996; Basu et al. 1996) around $r = 0.6\,\mathrm{R}_\odot$. Thus for realistic errors in the latest OPAL opacity tables the systematic errors introduced may be an order of magnitude less. A more reasonable estimate of the systematic error due to opacities is probably provided by model NDZ, which has a different heavy element abundance and hence a different opacity. For this model the systematic error is only $0.0001\,\mathrm{R}_\odot$, which is less than the random errors introduced due to uncertainties in the observed frequencies. Thus, it appears that systematic errors due to realistic uncertainties in the current opacity tables are comparable to random errors in r_b.

The results for models ND713ov and ND717ov show that the presence of an overshoot layer below the CZ does not affect the determination of r_b significantly. It appears that the increased temperature gradient in the overshoot layer is almost compensated by a larger discontinuity at the base of the overshoot layer.

From the results for model NDY it appears that a difference in the helium abundance can cause some systematic errors in the estimated r_b, as is also clear from results for model NDF. The main point to note, however, is that the error due to uncertainties in helium abundance is much smaller than those due to the uncertainties in the abundance profiles of the models. While the uncertainty in r_b due to those in the profiles is as high as $0.0036\,\mathrm{R}_\odot$ in some cases, the uncertainties due to the helium abundance difference of 1 per cent (which is probably larger than the expected uncertainty in Y) are approximately $0.0003\,\mathrm{R}_\odot$, an order of magnitude lower.

Results for model MHD indicate that the equation of state also plays a role in the determination of r_b. This is a bit surprising as the equation of state is expected to affect only the smooth part in equation (4). Fig. 9 shows the function $H_1(w)$ between the model MHD and ND calibration models. It is clear that even in the lower part of the CZ where the material is more or less totally ionized, the difference in the equation of state is detectable. A close inspection of the OPAL and MHD models just above the base of the CZ indeed indicates significant differences in the adiabatic indices. For example, the MHD model has $\nabla_{\mathrm{ad}} = 0.396\,14$, while the OPAL model has $\nabla_{\mathrm{ad}} = 0.396\,29$. Because of this difference the temperature gradient and hence the sound speed and $H_1(w)$ are also different. Also shown in Fig. 9 is $H_1(w)$ for the model NDY. The two sets of $H_1(w)$ are almost identical in the range over which we perform the fits to find r_b. Thus the difference between the MHD and OPAL equations of state could be interpreted as a difference in helium abundance by about 1 per cent. This effect could give rise to some systematic errors in determining helium abundance from frequencies of solar oscillations. Since a significant fraction of the work on determining the helium abundance has been carried out using mode sets extending to $\ell = 140$–150, it is not possible to resolve the second helium ionization zone properly, and as a result the resulting value of Y may be determined to some extent by the conditions in the lower part of the CZ. The direct inversion techniques (Däppen et al. 1991; Dziembowski, Pamyatnykh & Sienkiewicz 1991; Kosovichev et al. 1992) for determining Y may thus be more sensitive to changes in the lower part of the CZ, possibly giving misleading results, unless high-degree modes capable of resolving the He II ionization zone are also used.

Having tested our technique for determining the depth of the CZ on the test models and with the possible sources of systematic errors identified, we now apply this technique to estimate the depth of the solar CZ, using the observed frequencies. These results are also listed in Table 2. The $H_1(w)$ curves for the BBSO data with various sets of calibration models are shown in Fig. 10. From the figure, it appears that ND models will give a high value of r_b, while DIF will give a low one. The ND models give $r_b = (0.7141 \pm 0.0002)\,R_\odot$ for BBSO data, which is consistent with that obtained by CDGT. However, the error estimate in the present work is more than an order of magnitude lower than those in CDGT. This difference is mostly because of improvements in the observed frequencies. For most modes, the parametrized error estimates in the observed frequencies used by CDGT are an order of magnitude larger than the actual errors in the BBSO data.

There is evidence that gravitational settling of helium and heavy elements occurs in the Sun. Hence, we would expect the ND models to actually give an upper limit to r_b. From the results of overshoot estimation (Basu & Antia 1994b), it appears that the abundance gradient at the CZ base in DIF models is greater than that in the Sun. In that sense, the value of r_b of $0.7105\,R_\odot$ is a lower limit to the position of the base of the CZ. The shift in the value of r_b obtained is consistent with that inferred by Basu & Antia (1994a) by looking at the sound-speed differences between different models and the Sun, as is also clear from Fig. 3. Using recent data from the GONG project (Hill et al. 1996) it turns out that models with DIF profiles are ruled out at approximately the 5σ level (Basu 1997). Thus realistic uncertainties because of uncertainties in the composition profile can be expected to be about a fifth of the difference between the values obtained using the calibration models ND and DIF.

Using the INV calibration models we obtain $r_b = (0.7133 \pm 0.0002)\,R_\odot$. This is probably the most reliable determination of r_b for the Sun. However, one has to remember that there is some remaining uncertainty due to the uncertainty in the Z profiles of the models, which has not been inferred by inversions. Further, it is not possible to estimate the gradient of X at the base of the CZ reliably by inversions. Considering all the uncertainties it is possibly reasonable to claim that $r_b = (0.713 \pm 0.001)\,R_\odot$, where error bars include a reasonable estimate of the systematic errors from all of the sources considered above.

4.2 Opacity at the base of the CZ

From the test models, it is found that although the sound-speed profile inside the CZ is not particularly sensitive to surface abundances and the depth of the CZ, the density profile is extremely sensitive to these changes. Hence, by comparing the density profiles in the solar models with that inferred from inversion of the observed frequencies it is possible to find those models that are close to the Sun. Comparing the density profile in the calibration models INV with that inferred from inversion of the BBSO data, along with the frequencies of low-degree modes from the BiSON network (Elsworth et al. 1994), we find that a model with its CZ base at $(0.7130 \pm 0.0005)\,R_\odot$ fits the best (cf. Fig. 11). The error estimates here correspond to an error estimate of 0.005 in the relative density difference. If the surface abundances used in the model are correct then it appears that opacity around the base of the CZ is essentially consistent with that calculated from OPAL tables. If the OPAL tables were overestimating the opacity, then we would have found an envelope model with a deeper CZ in agreement with the inverted density profile. On the other hand, it should also be possible to adjust the gradients of the X and Z profiles below the CZ such that the estimated position of the base of the CZ turns out to be below $0.712\,R_\odot$, requiring an increase in opacity. Thus the estimate of opacity at the base of the CZ is also affected by the uncertainties in the abundance profiles below the CZ. From the results on overshoot estimation (Basu & Antia 1994b; Basu 1997) it appears that composition profiles with a steep gradient at the base of the CZ are not consistent with observations, but it is not clear if small

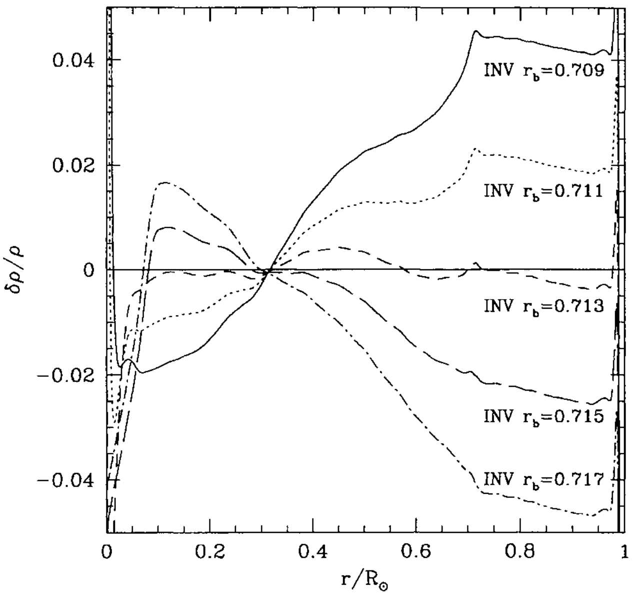

Figure 11. The relative difference in density between the set of INV calibration models and the Sun.

gradients can be ruled out by current data. Thus, as argued in the previous subsection, a shift by $0.001\,R_\odot$ in the base of the CZ may still be permissible.

Unfortunately, the density profile in the CZ is also sensitive to the surface abundances. If we use models with $Y = 0.249$ then the inverted density profile matches that of an envelope model with $r_b = 0.7140\,R_\odot$. Hence, it appears that expected uncertainties in the helium abundance of about 0.003 (cf. Basu & Antia 1995) can shift the required r_b by about $0.001\,R_\odot$. Considering the variation in the radiative gradient near the base of the CZ it appears that a change in opacity by about 2 per cent will cause the base of the CZ to shift by $0.001\,R_\odot$. Thus uncertainties in the depth of the CZ (by $0.001\,R_\odot$) and those in X (by 0.003) will each account for an uncertainty of about 2 per cent in estimated opacity at the base of the CZ. In addition an uncertainty of 1 per cent arises from those in density inversion, thus giving a total uncertainty of approximately 3 per cent. It may be noted that uncertainty in the abundances of heavy elements will also introduce errors in opacity. Thus changing Z/X by the quoted uncertainty of of 10 per cent could change the opacities by almost 10 per cent, which is much more than the uncertainties due to all other sources. Hence, we can conclude that with the current value of Z/X (Grevesse & Noels 1993) the latest OPAL opacity tables are consistent with the estimated opacity at the base of the CZ.

Considering the good agreement between the estimated opacity and that calculated from the OPAL tables using the currently accepted Z/X value, it appears that if the opacity tables are indeed correct, then the uncertainty in Z/X is much smaller than what has been quoted by Grevesse & Noels (1993). Hence the opacity estimates can be used to estimate the heavy element abundance in the solar envelope to much better accuracy than what is possible spectroscopically. Assuming that the OPAL opacity tables correctly represent the opacity of solar material we can estimate $Z/X = 0.0245 \pm 0.0008$.

5 DISCUSSION

In this work we have attempted to estimate the depth of the solar CZ

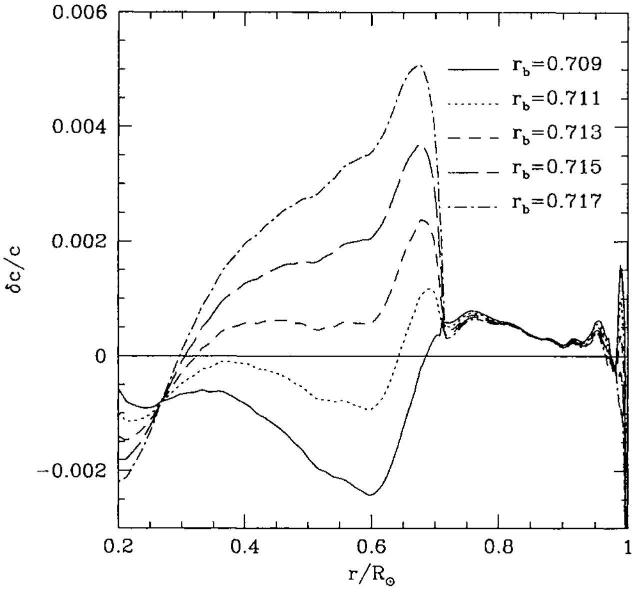

Figure 12. The relative difference in sound speed between the Sun and the set of DIF calibration models.

using the observed frequencies of solar oscillations. We find that there are systematic errors in the determination of the depth of the CZ due to uncertainties in the solar abundance gradients in the radiative interior. There are much smaller errors due to uncertainties in the actual value of the helium abundance and possible errors in the computed opacities.

Using calibration models without any abundance gradient at the CZ base, the position of the base of the solar CZ is found to be $(0.7141 \pm 0.0002)\,R_\odot$, which should be considered as the upper limit on r_b. However, calibration models with the conventional treatment of diffusion of helium and heavy elements give a value of $(0.7105 \pm 0.0002)\,R_\odot$, while models with hydrogen abundance profiles obtained from helioseismic inversions yield a result of $(0.7133 \pm 0.0002)\,R_\odot$. These results are consistent with the earlier estimate of $(0.713 \pm 0.003)\,R_\odot$ by CDGT, which was obtained using no-diffusion models. Similarly, the estimate of $r_b = (0.712 \pm 0.001)\,R_\odot$ obtained by Guzik & Cox (1993) using models with diffusion is also close to the value we obtain with similar models. Considering the fact that models with a steep composition profile at the base of the CZ are ruled out from other considerations (Basu & Antia 1994b; Basu 1997), a reasonable value including an estimate of the systematic errors is $r_b = (0.713 \pm 0.001)\,R_\odot$.

Apart from all the systematic errors considered in this work, the presence of the magnetic field near the base of the CZ could also introduce some error. However, from the study of frequency splittings of solar p modes it is possible to put limits on the strength of the possible magnetic field. Using current helioseismic data it is found that $v_A^2/c^2 \lesssim 10^{-4}$ (Gough et al. 1996), where v_A is the Alfvén speed. Thus a change in effective wave speed due to a possible magnetic field would be of this order. From Fig. 3 it is clear that changes of this magnitude could arise from a shift of $0.0001\,R_\odot$ in r_b. Hence, the systematic error arising due to a possible magnetic field would be very small as compared with other systematic errors and comparable to the random errors due to uncertainties in the observed frequencies.

It should also be realized that the presence of composition gradients below the CZ could cause misleading results to be obtained using comparison of sound-speed profiles in the model with that inferred in the Sun. To illustrate this point Fig. 12 shows the relative difference in sound speed between the calibration models DIF and the Sun as inferred by inversions. It can be seen that the models with $r_b \geq 0.713\,R_\odot$ show a prominent hump just below the CZ which has been interpreted as being due to differences in composition profiles below the CZ (Antia 1996; Christensen-Dalsgaard et al. 1996; Gough et al. 1996). However, the model with $r_b = 0.711\,R_\odot$ shows a smaller hump and after some averaging during inversions this may be ascribed to uncertainties in the opacities. All these models have the same composition profile below the CZ. Thus it is clear that by increasing the depth of the CZ we can at least partially suppress the hump below the CZ, even without any change in the composition profile. We believe this is responsible for the smaller hump found using models with mass loss (Gough et al. 1996). It may be noted that Guzik & Cox (1995) have adjusted the opacity to obtain a deeper CZ in their models with mass loss. This deeper CZ reduces the hump in the sound-speed difference below the CZ, even though the composition profiles in these models are not very different from those in the usual diffusion models. In order to actually test the model we should study the oscillatory contribution to the frequency arising from the discontinuity in the derivatives of the sound speed at the base of the CZ (Basu & Antia 1994b; Basu 1997). From such studies it appears that composition profiles of the type obtained by Guzik & Cox (1995) are not consistent with observed frequencies. Of course, if the models with mass loss also incorporate turbulent diffusion below the CZ, then the resulting composition profiles would be consistent with observations. However, in the presence of turbulent diffusion below the CZ, mass loss may not be required to explain lithium depletion.

The estimate of the depth of the CZ along with results from density inversion enable us to estimate the opacity at the base of the CZ. The estimated opacity is found to be close to those opacities calculated from the latest OPAL tables and the latest Z/X ratio, within the expected errors of about 3 per cent. Alternatively, assuming the OPAL opacity tables to be correct we estimate the heavy element abundance in the solar envelope to be $Z/X = 0.0245 \pm 0.0008$.

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REFERENCES

Antia H. M., 1996, A&A, 307, 609

Antia H. M., Chitre S. M., 1997, ApJ, submitted

Bahcall J. N., Pinsonneault M. H., 1992, Rev. Mod. Phys., 64, 885

Basu S., 1997, MNRAS, in press

Basu S., Antia H. M., 1994a, JA&A, 15, 143

Basu S., Antia H. M., 1994b, MNRAS, 269, 1137

Basu S., Antia H. M., 1995, MNRAS, 276, 1402

Basu S., Antia H. M., Narasimha D., 1994, MNRAS, 267, 209

Basu S., Christensen-Dalsgaard J., Schou J., Thompson M. J., Tomczyk S., 1996, ApJ, 460, 1064

Canuto V. M., Mazzitelli I., 1991, ApJ, 370, 295

Christensen-Dalsgaard J., Gough D. O., Thompson M. J., 1989, MNRAS, 238, 481

Christensen-Dalsgaard J., Gough D. O., Thompson M. J., 1991, ApJ, 378, 413 (CDGT)
Christensen-Dalsgaard J., Proffitt C. R., Thompson M. J., 1993, ApJ, 408, L75
Christensen-Dalsgaard J. et al., 1996, Sci, 272, 1286
Cox A. N., Tabor, J. E., 1976, ApJS, 31, 271
Cox A. N., Guzik J. A., Kidman R. B., 1989, ApJ, 342, 1187
Däppen W., Mihalas D., Hummer D. G., Mihalas B. W., 1988, ApJ, 332, 261
Däppen W., Gough D. O., Kosovichev A. G., Thompson M. J., 1991, in Gough D. O., Toomre J., eds, Challenges to theories of the structure of moderate-mass stars, Lecture Notes in Physics, Vol. 388. Springer, Heidelberg, p. 111
Dziembowski W. A., Pamyatnykh A. A., Sienkiewicz R., 1991, MNRAS, 249, 602
Elsworth Y., Howe R., Isaak G. R., McLeod C. P., Miller B. A., New R., Speake C. C., Wheeler S. J., 1994, ApJ, 434, 801
Gough D. O., 1990, in Osaki Y., Shibahashi H., eds, Lecture Notes in Physics 367. Springer, Berlin, p. 283
Gough D. O. et al, 1996, Sci, 272, 1296
Grevesse N., Noels A., 1993, in Prantzos N., Vangioni-Flam E., Cassé M., eds, Origin and evolution of the Elements. Cambridge Univ. Press, Cambridge, p. 15
Guzik J. A., Cox A. N., 1993, ApJ, 411, 394
Guzik J. A., Cox A. N., 1995, ApJ, 448, 905
Hill F. et al., 1996, Sci, 272, 1292
Hummer D. G., Mihalas D., 1988, ApJ, 331, 794
Iglesias C. A., Rogers F. J., 1996, ApJ, 464, 943
Kosovichev A. G., Fedorova A. V., 1991, SvA, 35, 507
Kosovichev A. G., Christensen-Dalsgaard J., Däppen W., Dziembowski W. A., Gough D. O., Thompson M. J., 1992, MNRAS, 259, 536
Libbrecht K. G., Woodard M. F., Kaufman J. M., 1990, ApJS, 74, 1129
Mihalas D., Däppen W., Hummer D. G., 1988, ApJ, 331, 815
Monteiro M. J. P. F. G., Christensen-Dalsgaard J., Thompson M. J., 1994, A&A, 283, 247
Proffitt C. R., 1994, ApJ, 425, 849
Rogers F. J., Iglesias C. A., 1992, ApJS, 79, 507
Rogers F. J., Swenson F. J., Iglesias C. A., 1996, ApJ, 456, 902

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