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Seismic-wave traveltimes in random media

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SUMMARY

A ray-theoretical relation is established between the autocorrelation function of the slowness fluctuations of a random medium and the autocorrelation function of the traveltime fluctuations on a profile perpendicular to the general propagation direction of an originally plane wave. Although this relation can be inverted exactly, it is preferable for applications to use the results of a forward calculation for a modified exponential autocorrelation function which represents slowness fluctuations with zero mean. The essential parameters of this autocorrelation function, standard deviation ε and correlation distance a, follow by simple relations from the maximum and the zero crossing of the corresponding autocorrelation function of the traveltime fluctuations. The traveltime analysis of 2-D finite-difference seismograms shows that ε and a can be reconstructed successfully, if the wavelength-to-correlation-distance ratio is 0.5 or less. Otherwise, ε is underestimated and a overestimated; however, both effects can be compensated for.

The average traveltime, as determined from the finite-difference seismograms, is slightly, but systematically shorter than the traveltime according to the average slowness, i.e. the wave prefers fast paths through the medium. This is in agreement with results of Wielandt (1987) for a spherical low-velocity inclusion in a full-space and with results of Soviet authors, summarized by Petersen (1990). The velocity shift is proportional to ε^2, it has dispersion similar to the dispersion related to anelasticity, and it increases with the pathlength of the wave.

Key words: average wave velocity, random media, scattering, traveltime fluctuations, velocity shift.

1 INTRODUCTION

Velocity or slowness fluctuations in the Earth with scalelengths less than a few kilometres in the crust and a few tens of kilometres in the mantle cannot usually be resolved with current seismic networks in a deterministic way. The main reason is that subsurface coverage with intersecting seismic rays is not sufficient. Such small-scale fluctuations often can be described by only a few statistical measures like a mean value, a standard deviation and a correlation distance, and knowing just these quantities is enough for characterizing the medium. These few unknowns may be determined with the aid of *simple* ray systems, corresponding to observations on one profile and to excitation by one point source or plane-wave source. Hence, whereas deterministic tomography is not usually possible in such media, statistical tomography may well be. The purpose of this paper is to contribute to statistical tomography by studying theoretically and numerically (1) the traveltimes of waves in random media and (2) the inferences that can be drawn from the traveltimes and their fluctuations on the slownesses and their fluctuations.

A basic work on traveltime or phase fluctuations in random media is the book by Chernov (1960). The background of his theory is the Born approximation for acoustic media. He mainly treated the case of monochromatic plane waves in a medium with a Gaussian autocorrelation function of the refractive-index (or slowness) fluctuations. The variance of the traveltime fluctuations turns out to be proportional to the variance of the relative slowness fluctuations, to the traveltime through the medium for the mean velocity, to the traveltime corresponding to the correlation distance, and to a factor which depends on frequency and increases with it. Chernov also derived, in the high-frequency or ray-theoretical limit, the autocorrelation function of the traveltime fluctuations on a profile perpendicular to the propagation direction and found agreement with the Gaussian autocorrelation function of the slowness fluctuations. Frankel & Clayton (1986) investigated, by finite-difference calculations, waves in media with different statistics (Gaussian, exponential, self-similar) and determined, among other things, the autocorrelation function of the traveltime fluctuations of an originally plane wave. They identified the half-width of this autocorrelation function with the correlation distance of the velocity fluctuations. Gudmundsson *et al.* (1990) treated traveltime fluctuations in the framework of ray theory and Fermat's principle and gave a general relation between the autocorrelation functions of slowness and traveltime fluctuations.

Other work related to wave propagation in random media has focused on the interpretation of traveltime and amplitude fluctuations, observed across large seismic arrays such as NORSAR. Flatté & Wu (1988) give an overview of papers dating back as far as 1973. Their own theory (see also Wu & Flatté 1990; Wu 1991) is based on the approximate description of forward scattering by a parabolic wave equation which to some extent includes frequency-dependent effects. They give expressions for various correlation functions (autocorrelation function of phase fluctuations and log-amplitude fluctuations, respectively, cross-correlation function of both fluctuations) and discriminate also between different receiver geometries with respect to the incident wave. In the terminology of these authors we are dealing in the present paper with the transverse autocorrelation function of traveltime fluctuations.

The studies mentioned so far dealt with the fluctuations of traveltime and slowness around average or reference values. An interesting question is also whether or not the average traveltime along a profile is in agreement with the average slowness or velocity of the medium. Wielandt (1987) concluded, on the basis of synthetic seismograms for a spherical inclusion in an otherwise homogeneous medium, that in deterministic tomography low-velocity inclusions would be more difficult to outline than high-velocity inclusions because of first arriving waves, diffracted *around* the slow inclusions in faster material. Therefore, the average velocity inferred from traveltime fluctuations would be biased towards higher values. Nolet (1987) has suggested calling this bias 'Wielandt effect'. However, Soviet authors, summarized by Petersen (1990), have obtained similar conclusions to Wielandt in the framework of ray theory for random media at least since 1983. Hence, a neutral abbreviation for this important effect is desirable. We will call it 'velocity shift' in the following.

The investigations of this paper are for 2-D acoustic media. In Section 2 we establish, in a way similar to that of Gudmundsson *et al.* (1990), the relationship between the autocorrelation functions of slowness and traveltime fluctuations. This integral relation can be inverted exactly, i.e. the autocorrelation function of the slowness fluctuations follows in a unique way from the autocorrelation function of the traveltime fluctuations. For practical applications, the *forward* relation is elaborated in Section 3 for a modified exponential autocorrelation function of the slowness fluctuations. This autocorrelation function corresponds to slowness fluctuations with mean value zero, and its free parameters are the variance and the correlation distance, as usual. Formulae are given by which these two quantities can be determined from the maximum and the zero crossing of the autocorrelation function of the traveltime fluctuations. In Section 4, media with the statistics just mentioned are constructed, and plane-wave propagation through them is calculated by a finite-difference method. Grid dispersion is compensated by special filtering of the synthetic seismograms, such that accurate traveltime determinations are possible. From the autocorrelation function of the traveltime fluctuations along a profile the variance and the correlation distance of the slowness fluctuations are estimated and compared with the pre-specified values. The success of the reconstruction depends on the ratio of the dominant wavelength to the correlation distance. In Section 5, the velocity shift is quantified, i.e. the velocity corresponding to the average traveltime on the profile is compared with the velocity corresponding to the average slowness of the structure. The velocity increase turns out to be proportional to the variance of the slowness fluctuations and to depend both on frequency and pathlength. Finally, Section 6 presents some conclusions.

2 SUMMARY OF THEORY

We consider a 2-D medium with slowness $u(x, z) = u_0 + \delta u(x, z)$, where u_0 is the average slowness and $\delta u(x, z)$ a weak, randomly distributed slowness fluctuation whose mean value vanishes. Rays, propagating generally in x-direction between $x = 0$ and $x = L$, have z-dependent traveltime fluctuations with respect to the traveltime $T_0 = Lu_0$. Their well-known linearized form is

$$\delta T(z) = \int_0^L \delta u(x, z)\, dx. \tag{1}$$

Because of the validity of Fermat's principle, integration in equation (1) can be performed to first order along the straight rays in the unperturbed structure.

The autocorrelation function of the traveltime fluctuations is the expectation (in the sense of the theory of stochastic processes with many realizations)

$$\begin{aligned}\phi(\zeta) &= E[\delta T(z)\, \delta T(z + \zeta)]\\ &= \int_0^L \int_0^L E[\delta u(x, z)\, \delta u(x', z + \zeta)]\, dx\, dx',\end{aligned}$$

where ζ is the lag corresponding to the z-coordinate. The integrand is the autocorrelation function of the slowness fluctuations. We assume that this function, called $F(r)$, depends only on the distance $r = [(x' - x)^2 + \zeta^2]^{1/2}$ of two points in the medium, i.e. there is no preferred orientation of the fluctuations. Then, the double integral

$$\phi(\zeta) = \int_0^L \int_0^L F(r)\, dx\, dx' \tag{2}$$

can be reduced to a single integral by introducing as new integration variables the sum $x' + x$ and the difference $x' - x$. Straightforward transformation yields

$$\phi(\zeta) = 2L \int_\zeta^N F(r) \frac{r}{(r^2 - \zeta^2)^{1/2}}\, dr - 2 \int_\zeta^N F(r) r\, dr, \tag{3}$$

where $N = (L^2 + \zeta^2)^{1/2}$. This result can be simplified by noting that L has to be much larger than the correlation distance a of the slowness fluctuations. Hence, at $r = N$ the integrands are vanishingly small, and N can be replaced by infinity. Moreover, for $\zeta = 0$ where $\phi(\zeta)$ is maximum, the second integral in (3) is of the order of a/L times the first integral; one way to show this is to use an exponential $F(r)$. Hence, the second integral in (3) can be neglected both for $\zeta = 0$ and $\zeta > 0$, and we obtain

$$\phi(\zeta) = 2L\int_\zeta^\infty F(r)\frac{r}{(r^2-\zeta^2)^{1/2}}\,dr, \tag{4}$$

in agreement with equation (15) of Gudmundsson *et al.* (1990).

In Section 3, equation (4) will be used to perform a forward calculation of $\phi(\zeta)$ for a suitably chosen $F(r)$ and to relate the variance and the correlation distance of the slowness fluctuations to prominent features of $\phi(\zeta)$. These formulae allow simple interpretations of traveltime fluctuations. The special structure of equation (4), however, makes possible an *exact* inversion of $F(r)$ from $\phi(\zeta)$ which is briefly explained in the following.

Equation (4) can be transformed, by the variable changes $x = 1/\zeta$ and $\xi = 1/r$, into the variant

$$f(x) = \int_0^x \frac{y(\xi)}{(x^2-\xi^2)^{1/2}}\,d\xi$$

of Abel's integral equation whose solution is (e.g. Frank & von Mises 1961)

$$y(\xi) = \frac{2}{\pi}\left\{f(0) + \xi\int_0^\xi \frac{f'(x)}{(\xi^2-x^2)^{1/2}}\,dx\right\}. \tag{5}$$

Here, the functions $f(x)$ and $y(\xi)$ are:

$$f(x) = \frac{1}{2Lx}\,\phi\left(\frac{1}{x}\right), \qquad y(\xi) = \frac{1}{\xi^2}F\left(\frac{1}{\xi}\right).$$

Using $f(0) = 0$, one obtains from (5)

$$F(r) = -\frac{1}{\pi L r^2}\int_r^\infty \frac{\zeta}{(\zeta^2-r^2)^{1/2}}\{\phi(\zeta) + \zeta\phi'(\zeta)\}\,d\zeta. \tag{6}$$

The variance of the slowness fluctuations, $F(0)$, can be found by the variable change $u = (\zeta^2 - r^2)^{1/2}$ in (6),

$$F(r) = -\frac{1}{\pi L r^2}\int_r^\infty \{\phi[(u^2+r^2)^{1/2}] + (u^2+r^2)^{1/2}\phi'[(u^2+r^2)^{1/2}]\}\,du, \tag{7}$$

and by Taylor expansion of the integrand in (7) for small r^2. The constant term gives a vanishing contribution to the integral, and from the r^2 term one finds

$$F(0) = -\frac{1}{\pi L}\int_0^\infty \frac{1}{\zeta}\phi'(\zeta)\,d\zeta. \tag{8}$$

The integrand of (8) has no singularity at $\zeta = 0$ because $\phi'(0) = 0$; this is due to the fact that $\phi(\zeta)$ is actually a function of ζ^2 (see equation (2) with the proper variable r).

Numerical calculations of $F(r)$ with the aid of equation (6) and (8) require knowledge not only of $\phi(\zeta)$, but also of $\phi'(\zeta)$. This derivative probably will be difficult to obtain in practice with sufficient accuracy. This limits the usefulness of (6) and (8) for inversion purposes.

3 2-D SLOWNESS FLUCTUATIONS WITH ZERO MEAN

Fluctuating media in different fields of science are often described by exponential rather than Gaussian or other autocorrelation functions. An experience that we have made ourselves is that uniformly distributed impedance fluctuations with zero mean in a stack of layers of similar thicknesses are much better described by an exponential than by a Gaussian autocorrelation function. Therefore, we apply the theory of the foregoing section to the case of a basically exponential autocorrelation function $F(r)$. One modification, however, is made: we require $F(r)$ to correspond to slowness fluctuations $\delta u(x, z)$ with zero mean. For such fluctuations the 2-D Fourier transform $\overline{\delta u}(k_x, k_z)$ vanishes for the wavenumbers $k_x = k_z = 0$. This implies that also the 2-D Fourier transform $\bar{F}(k_\xi, k_\zeta)$ of $F[r = (\xi^2 + \zeta^2)^{1/2}]$ vanishes for $k_\xi = k_\zeta = 0$. Hence, the integral over F in the ξ–ζ plane vanishes. Using polar coordinates, we obtain the following constraint:

$$\int_0^\infty F(r) r\,dr = 0. \tag{9}$$

The modified exponential autocorrelation function

$$F(r) = \varepsilon^2 u_0^2\left(1 - \frac{r^2}{6a^2}\right)e^{-r/a} \tag{10}$$

fulfils this constraint; ε is the standard deviation of the relative slowness fluctuations and a their correlation distance. Fig. 1 shows $F(r)$ and the corresponding autocorrelation function of the traveltime fluctuations, $\phi(\zeta)$, calculated numerically with equation (4). The

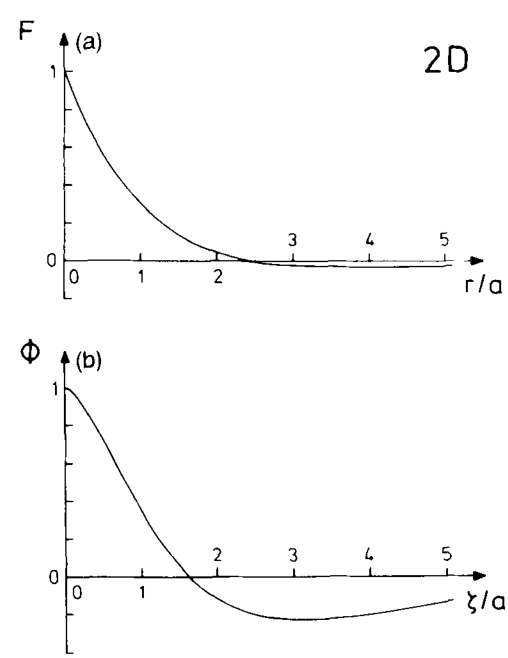

Figure 1. (a) Modified exponential autocorrelation function of the slowness fluctuations of a 2-D random medium, equation (10). (b) The corresponding autocorrelation function of traveltime fluctuations, calculated numerically with equation (4). Both curves are normalized.

variance of the traveltime fluctuations is

$$\phi(0) = \frac{4}{3}\varepsilon^2 u_0^2 La. \tag{11}$$

The 2-D Fourier transform of $F(r)$ in (10) is

$$\bar{F}(k_\xi, k_\zeta) = 2\pi\varepsilon^2 u_0^2 a^2 \left\{ \frac{1}{(1+k^2a^2)^{3/2}} + \frac{\frac{3}{2}k^2a^2 - 1}{(1+k^2a^2)^{7/2}} \right\}, \tag{12}$$

with $k^2 = k_\xi^2 + k_\zeta^2$. Slowness fluctuations $\delta u(x, z)$ with the autocorrelation function (10) are constructed via their Fourier transform $\overline{\delta u}(k_x, k_z)$ whose modulus is $\bar{F}(k_x, k_z)^{1/2}$ from equation (12) and whose phase follows from pseudo random numbers. Such slowness fluctuations are used in the following section.

The autocorrelation function $\phi(\zeta)$ in Fig. 1 has its zero at $\zeta_{\text{zero}} = 1.65\,a$. This value and equation (11) provide a simple interpretation of traveltime fluctuations in synthetic or observed seismograms. The (first) zero of their autocorrelation function yields the correlation-distance estimate

$$a = 0.61\zeta_{\text{zero}}, \tag{13}$$

and equation (11) yields an estimate of the standard deviation ε. Equations (11) and (13) are applied to traveltime fluctuations in synthetic seismograms in the next section.

4 TRAVELTIME INTERPRETATION OF SYNTHETIC SEISMOGRAMS

The acoustic finite-difference computations are performed with a second-order scheme for a heterogeneous medium (Korn & Stöckl 1982). The random medium is constructed, as described in the foregoing section, and occupies a rectangular box with dimensions $L = 1064$ m in x-direction and 2000 m in z-direction. The standard deviation ε is either 3 or 5 per cent, the correlation distance varies from 15 to 240 m, and the grid spacing is 2 m. The average velocity depends on the particular realization of the random structure and is close to 4000 m s^{-1}, and the relative density fluctuations are proportional to the relative velocity fluctuations with the proportionality factor $\kappa = 0.3$.

At one of the long sides of the box ($x = 0$) the input signal

$$s(t) = \sin\frac{2\pi}{T}t - \frac{1}{2}\sin\frac{4\pi}{T}t, \qquad 0 \le t \le T, \tag{14}$$

is prescribed. T is either 30 or 7.5 ms; hence, the dominant wavelength λ is about 120 or 30 m, and the pathlength-to-wavelength ratio L/λ is about 9 or 36. The time step is the critical time step for the maximum velocity of the medium; it is about 0.3 ms.

The seismograms at $x = L$, with the receiver spacing 10 or 20 m, are the results of the computation which are subsequently subjected to a traveltime analysis. Prior to this analysis, numerical grid dispersion, which deforms waves in addition to any physical dispersion and leads to systematic traveltime errors, has to be removed. Since wave propagation is essentially along the grid lines z = constant, the numerical phase velocity is (Alford *et al.* 1974)

$$c(\omega) = \frac{2v}{\Delta t\omega}\arcsin\left(\frac{\Delta t v}{h}\sin\frac{h\omega}{2v}\right) \tag{15}$$

for a homogeneous medium with grid spacing h, time step Δt and material velocity v and for angular frequency ω. According to (15), high frequencies propagate slower than low frequencies.

This dispersion can be removed by filtering with the phase-shift filter

$$f(\omega) = \exp\left[i\omega L\left(\frac{1}{c(\omega)} - \frac{1}{v}\right)\right]. \tag{16}$$

The shift to earlier times by (16) is about 0.1 ms for the dominant frequency of the $T = 30$ ms signal (14), i.e. for the case $L/\lambda = 9$. In this case the dispersion correction is not very essential. For the $T = 7.5$ ms signal and $L/\lambda = 36$, the shift is about 1.3 ms, i.e. about $4\,\Delta t$; in this case the correction is definitely needed.

Application of the filter (16) to finite-difference results for a homogeneous medium yields very accurate traveltimes for both critical and subcritical time steps. Hence, we assume that the filter is also accurate for random media with standard deviations ε of a few per cent because the deviation from the homogeneous case is only small. In this case, the velocity v in (15) is identified with the average velocity, and Δt is the actual time step of the finite-difference computation.

Figs 2–5 present examples of finite-difference seismograms and their traveltime interpretation for the two L/λ ratios and for variable ratio of dominant wavelength to correlation distance, λ/a. The central parts display the traveltimes of the first peaks of the seismograms, their mean value (horizontal continuous line) and the theoretical traveltime of the first peak of the signal (14) according to the average slowness of the grid (dashed line). Additionally, the first arrivals have been determined at times where 1 or 2 per cent of the maximum input amplitude is reached. The similarity of the first-arrival curves and the first-peak curves shows that the traveltimes can be determined safely. At the bottom of Figs 2–5 the autocorrelation function $\phi(\zeta)$ of the traveltime fluctuations is shown. The fluctuations have been related to the mean first-peak time (horizontal continuous line), since equation (1) implies that for slowness fluctuations with zero mean the mean value of the traveltime fluctuations vanishes also. The inferred values, according to equations (11) and (13), of the standard deviation and the correlation distance, $\tilde{\varepsilon}$ and $\tilde{a}$, are also given.

For each λ/a ratio eight different realizations of the random medium have been investigated, both for $\varepsilon = 3$ and 5 per cent. The inferred $\tilde{\varepsilon}$ and $\tilde{a}$ values are given in Fig. 6 for the case $L/\lambda = 9$, $\varepsilon = 5$ per cent, and for four out of six λ/a ratios. The standard deviations of $\tilde{\varepsilon}$ and $\tilde{a}$ decrease with increasing λ/a, but most of this dependence disappears, if they are normalized with the mean values. The results of all computations are compiled in Table 1.

Fig. 7 presents the mean values of the ratios $\tilde{\varepsilon}/\varepsilon$ and $\tilde{a}/a$, i.e. the quotients of reconstructed and original parameters, as functions of λ/a. The mean $\tilde{\varepsilon}$ decreases with increasing λ/a; this is due to the increase of averaging over the wavelength and is an expected feature. At $\lambda/a = 0.5$, $\tilde{\varepsilon}$ is on average 20 per cent lower than ε. The mean $\tilde{a}$ is always larger than a, and $\tilde{a}/a$ increases with λ/a. Both the decay of $\tilde{\varepsilon}$ and the increase of $\tilde{a}$ with increasing λ/a are more pronounced for the longer pathlength ($L/\lambda = 36$).

We conclude from Fig. 7 that the ray-theoretical results

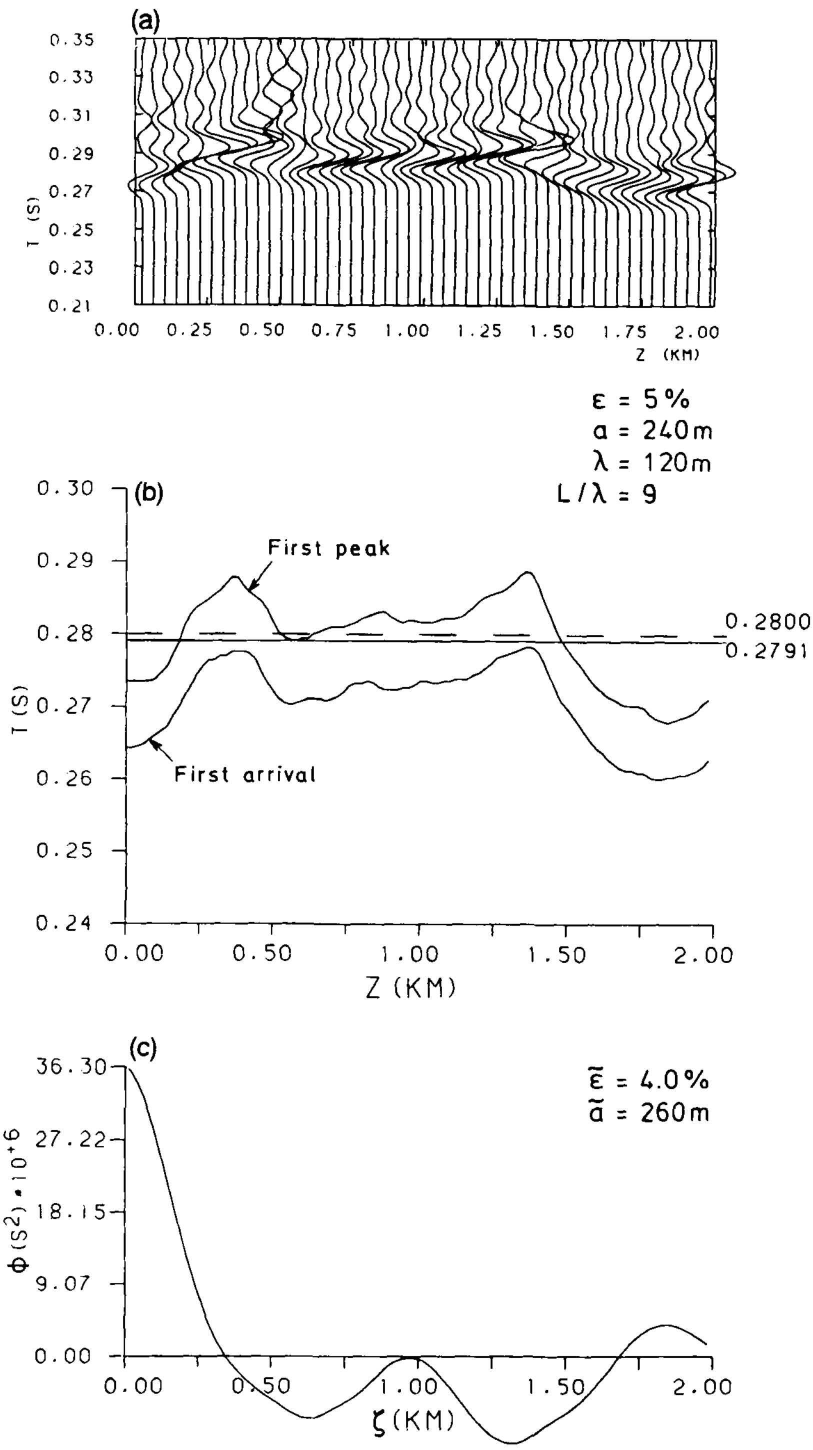

Figure 2. (a) Synthetic finite-difference seismograms for a random medium with standard deviation $\varepsilon = 5$ per cent, correlation distance $a = 240$ m and for wavelength $\lambda = 120$ m (pathlength-to-wavelength ratio $L/\lambda = 9$, $\lambda/a = 0.5$). (b) Traveltime curves of the first arrivals (1 per cent criterion) and the first peaks (solid curves), mean value of first-peak times (horizontal continuous line) and theoretical first-peak time, computed with average slowness (dashed line). (c) Autocorrelation function of the traveltime fluctuations of the first peak. $\tilde{\varepsilon}$ is the inferred standard deviation and $\tilde{a}$ the inferred correlation distance.

for traveltime fluctuations, equations (11) and (13), give reasonable estimates of ε and a, if λ/a is 0.5 or less. For larger ratios (and also for $\lambda/a \leq 0.5$, if desired) Fig. 7 can be used to derive ε and a from $\tilde{\varepsilon}$ and $\tilde{a}$; this requires transformation of the λ/a values on the abscissa into $\lambda/\tilde{a}$ values with the $\tilde{a}/a$ values given.

5 THE VELOCITY SHIFT

The continuous horizontal lines in the central parts of Figs 2–5, which represent the mean values of the first-peak traveltimes in the finite-difference seismograms, are generally earlier than the dashed lines, which represent the

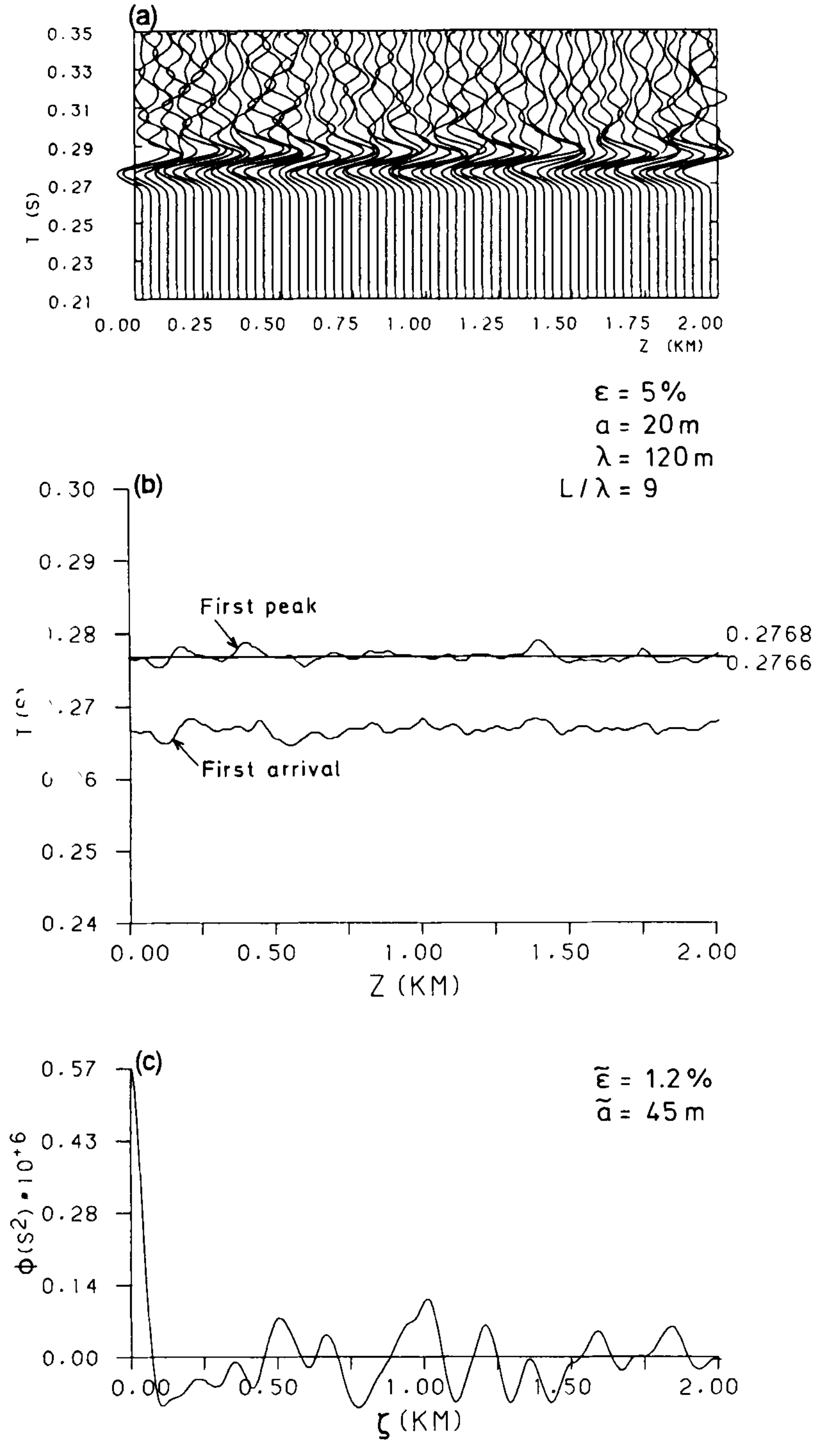

Figure 3. The same as Fig. 2 for $\varepsilon = 5$ per cent, $a = 20$ m, $\lambda = 120$ m ($L/\lambda = 9$, $\lambda/a = 6$). The theoretical first-peak time is 0.2766 s.

first-peak traveltimes according to the average slowness of the finite-difference grid. Early arrivals have been found in almost all models investigated and clearly point to a systematic decrease of the apparent slowness due to the randomness of the medium. According to equation (1), such a shift should not occur for slowness fluctuations with zero mean (therefore, it was disregarded in the interpretations of Section 4). This effect is an extension of Wielandt's (1987) results for one inclusion in a full-space to the case of many inclusions: the wave prefers to propagate over the fast parts of the medium or to diffract around the slow parts. The effect has also been predicted by Soviet authors (see Petersen 1990) from ray theory for media with both deterministic and random velocity variations. In the terminology of scattering theory, the velocity shift is a particular form of multilple scattering, which is faster than

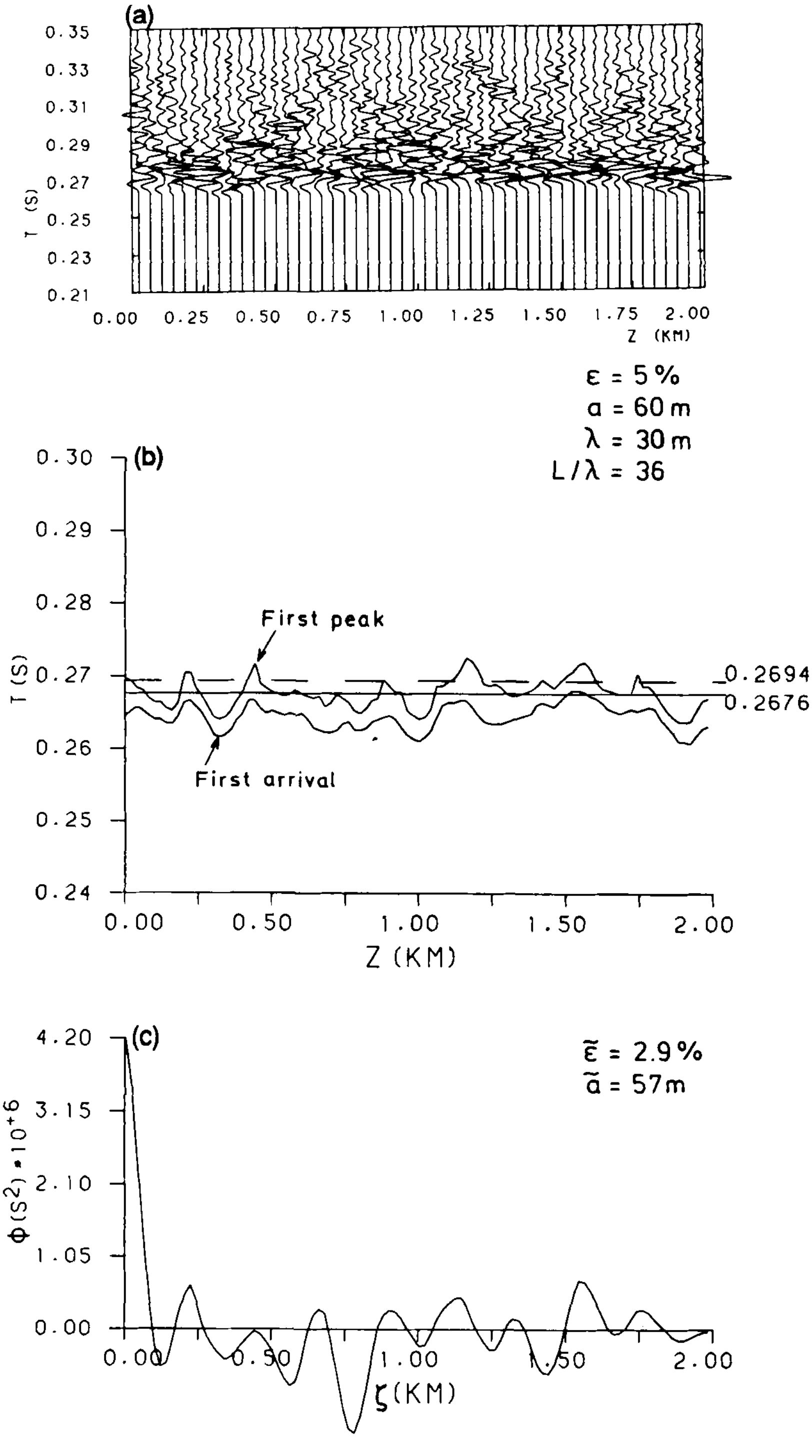

Figure 4. The same as Fig. 2 for $\varepsilon = 5$ per cent, $a = 60$ m, $\lambda = 30$ m ($L/\lambda = 36$, $\lambda/a = 0.5$).

single scattering. An alternative explanation is by leaking-mode propagation in anti-waveguides of variable curvature and cross-section. For $\lambda/a \ll 1$, however, simple ray theory is sufficient, since energy will travel over minimum-time paths which obey Snell's law everywhere.

The relative velocity change $\delta v/v_0$ with respect to the inverse of the average slowness, $v_0 = 1/u_0$, is proportional to ε^2 and depends on the wavelength (Fig. 8). For short wavelengths ($\lambda/a = 0.5$) the velocity increase is

$$\frac{\delta v}{v_0} \approx \begin{cases} 1.5\varepsilon^2 & \text{for } L/\lambda = 9, \\ 4\varepsilon^2 & \text{for } L/\lambda = 36. \end{cases} \tag{17}$$

In the case $L/\lambda = 36$ it is difficult to decide whether the $\delta v/v_0$ value is also representative for $\lambda/a < 0.5$.

The limit of $\delta v/v_0$ for large λ/a appears to be negative, but close to zero. This means that the long-wavelength limit

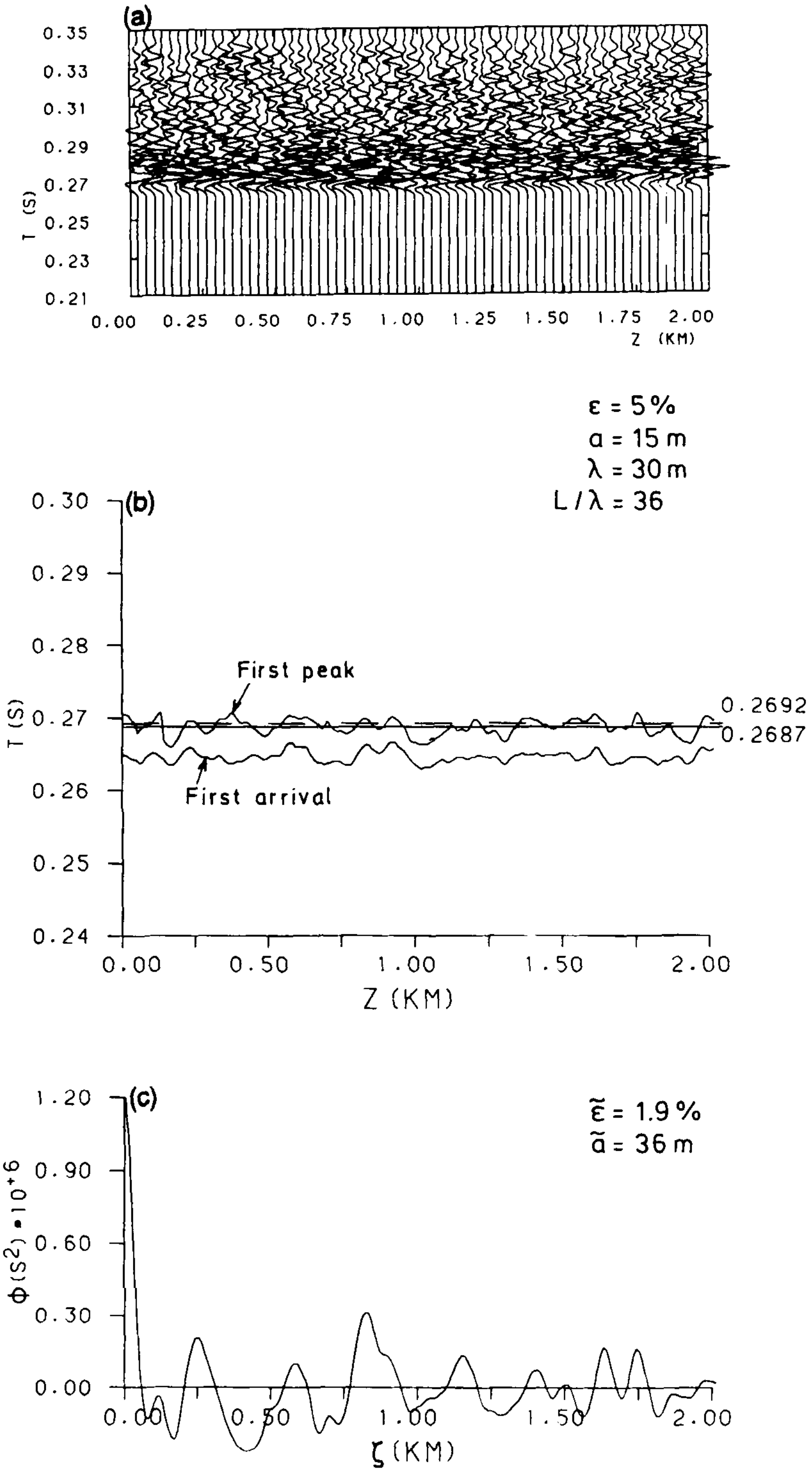

Figure 5. The same as Fig. 2 for $\varepsilon = 5$ per cent, $a = 15$ m, $\lambda = 30$ m ($L/\lambda = 36$, $\lambda/a = 2$).

of the wave velocity is less than, but close to, the velocity v_0. Since v_0 follows from the average velocity $\bar{v}$ by $v_0 = \bar{v}(1 - \varepsilon^2)$, the medium is, with respect to $\bar{v}$, slower at long wavelengths, but still faster at short wavelengths.

The $\delta v/v_0$ values in Fig. 8 are interesting numerical results for the velocity shift of acoustic waves in random media. We continue with a few attempts to explain these results by theoretical arguments or by plausible reasoning.

For instance, one may ask whether there is theoretical support for a long-wavelength velocity limit less than v_0. We have used the elastostatic theory of Hashin (1962) for the effective elastic moduli of two-component media and obtained the limit

$$\frac{\delta v}{v_0} = -\frac{1}{2}(1 + \kappa)^2 \varepsilon^2. \tag{18}$$

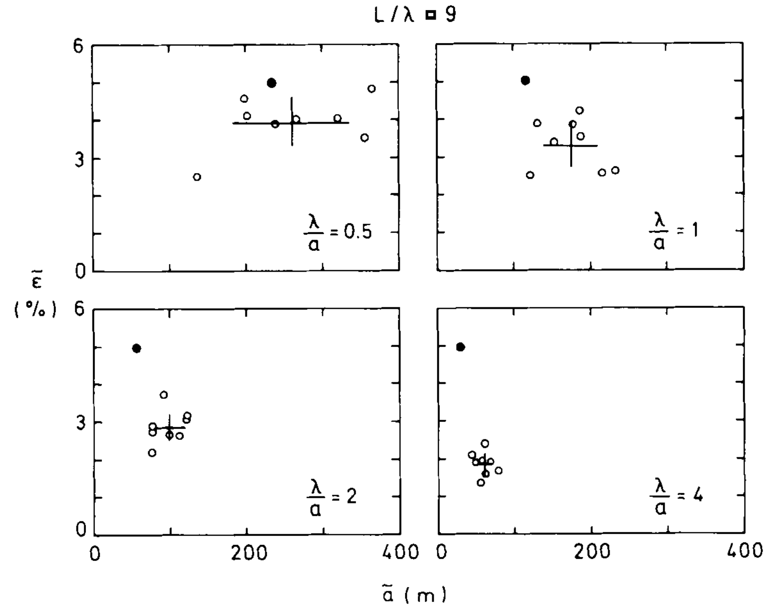

Figure 6. Reconstructed standard deviations $\tilde{\varepsilon}$ and correlation distances $\tilde{a}$ for pathlength-to-wavelength ratio $L/\lambda = 9$ and for different wavelength-to-correlation-distance ratios λ/a. In each case eight different realizations of the statistics were used. The crosses denote mean values and standard deviations of $\tilde{\varepsilon}$ and $\tilde{a}$. The full circles represent the prescribed values ε and a.

In deriving (18), the random medium with a continuous slowness distribution was replaced by a two-component medium with slownesses $(1 \pm \varepsilon)/v_0$, which means a quite pronounced simplification of the structure. For $\kappa = 0.3$, equation (18) yields $\delta v/v_0 = -0.85\ \varepsilon^2$. At least the sign of $\delta v/v_0$ for $\lambda/a = 6$ in Fig. 8 is in agreement with (18), but whether the long-wavelength limit agrees quantitatively with (18) is uncertain in view of the simplification mentioned.

Another question of theoretical interest is whether the weak dispersion in Fig. 8 agrees with the dispersion that can be calculated from attenuation due to scattering by a dispersion relation. Fang & Müller (1991) have shown how

Table 1. Results of the traveltime interpretations.

$\frac{L}{\lambda}$	$\frac{\lambda}{a}$	ε (%)	a (m)	$\tilde{\varepsilon}$ (%)	$\tilde{a}$ (m)	$\delta v/v_0$ (%)
9	0.5	3	240	2.4±0.4	260±80	0.14±0.05
		5		4.0±0.7	260±80	0.39±0.11
9	1	3	120	2.1±0.4	156±30	0.12±0.07
		5		3.4±0.6	180±40	0.39±0.14
9	2	3	60	1.7±0.3	90±15	0.08±0.03
		5		2.9±0.4	100±20	0.23±0.09
9	3	3	40	1.4±0.2	65±10	0.03±0.03
		5		2.3±0.4	77±20	0.05±0.08
9	4	3	30	1.1±0.2	57±13	0.02±0.03
		5		1.9±0.3	60±10	0.03±0.03
9	6	3	20	0.8±0.1	42±3	-0.03±0.01
		5		1.3±0.2	44±2	-0.06±0.01
36	0.5	3	60	2.3±0.4	77±10	0.37±0.10
		5		3.3±0.5	90±18	1.05±0.29
36	1	3	30	1.9±0.2	48±10	0.23±0.05
		5		2.9±0.3	53±9	0.59±0.05
36	2	3	15	1.1±0.1	32±3	0.06±0.02 •
		5		1.8±0.3	34±4	0.20±0.03

L = pathlength (1064 m),
λ = dominant wavelength (120 m and 30 m, respectively),
ε = standard deviation $\}$ of slowness
a = correlation distance $\}$ fluctuations,
$\tilde{\varepsilon}$ =, $\tilde{a}$ = $\}$ reconstructed values,
$\frac{\delta v}{v_0}$ = apparent velocity change with respect to v_0 (v_0^{-1} = average slowness).

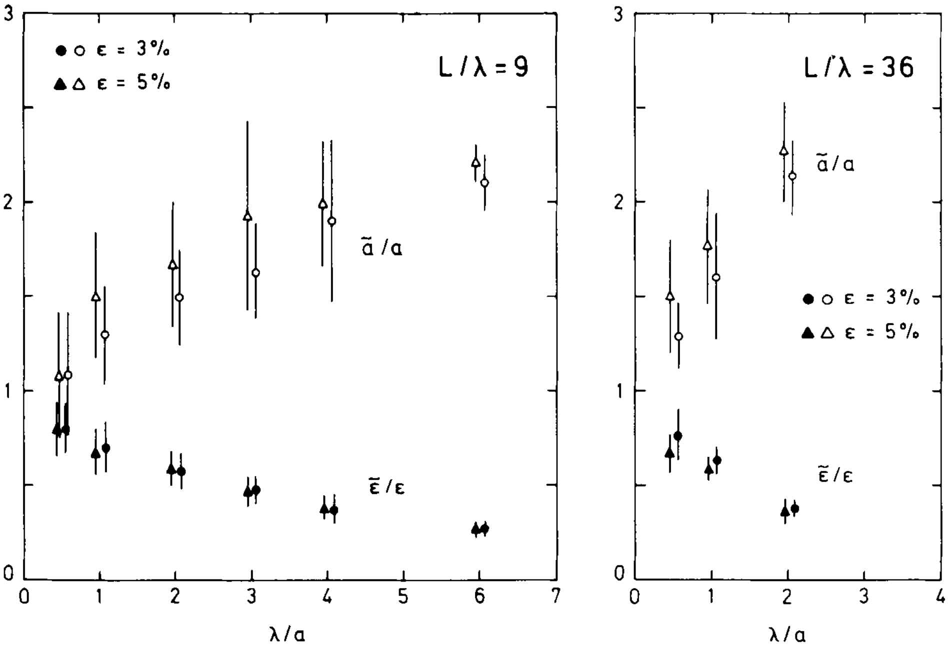

Figure 7. Reconstructed standard deviations $\tilde{\varepsilon}$ and correlation distances $\tilde{a}$, normalized by ε and a, respectively, as functions of the wavelength-to-correlation-distance ratio λ/a. L/λ is the pathlength-to-wavelength ratio. Open symbols represent $\tilde{a}/a$, closed symbols $\tilde{\varepsilon}/\varepsilon$.

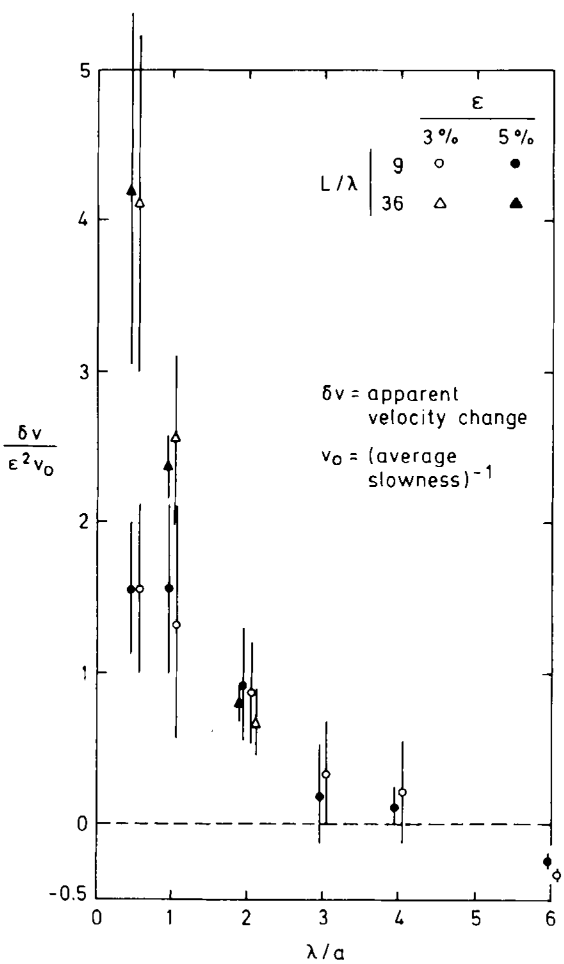

Figure 8. The velocity shift and its wavelength dependence. δv is the apparent velocity change with respect to the inverse v_0 of the average slowness.

attenuation operators for seismic waves can be determined for arbitrary dependence of anelastic or scattering Q on frequency, using the plausible minimum-phase assumption for the operator's phase. Their Fourier method can also be used to determine the phase velocity from Q. Scattering Q for a 2-D random medium with the autocorrelation function (10) and its Fourier transform (12) is given by the following integral over the scattering angle θ (Frankel & Clayton 1986, Appendix C, density term added):

$$Q^{-1} = 2\varepsilon^2 k^2 a^2 \int_{\theta_{\min}}^{\pi} \left(1 + \kappa \sin^2 \frac{\theta}{2}\right) \times \left\{ \frac{1}{\left(1 + 4\sin^2 \frac{\theta}{2} \cdot k^2 a^2\right)^{3/2}} + \frac{6 \sin^2 \frac{\theta}{2} \cdot k^2 a^2 - 1}{\left(1 + 4\sin^2 \frac{\theta}{2} \cdot k^2 a^2\right)^{7/2}} \right\} d\theta. \tag{19}$$

Here, $k = 2\pi/\lambda$ is the wavenumber and $\theta_{\min}$ the minimum scattering angle which is in the range from 10° to 40° (Sato 1982; Frankel & Clayton 1986; Roth 1990). The result of the Fourier method, applied to (19), is shown in Fig. 9 for $\kappa = 0.3$ and two $\theta_{\min}$ values (30° and 10°); both phase velocity c and group velocity U is given in the form of deviations from the velocity value at infinite wavelength which according to Fig. 8 is close to v_0.

The velocities in Fig. 8 represent the phase velocity for the dominant wavelength, since they were obtained by following the first peak from $x = 0$ to $x = L$. There is some general agreement between the phase-velocity curves of Fig. 9 and the results in Fig. 8: the theoretical dispersion is also proportional to ε^2, the magnitudes of the velocity deviations agree, if a correlation of $\theta_{\min}$ and L is accepted (see below),

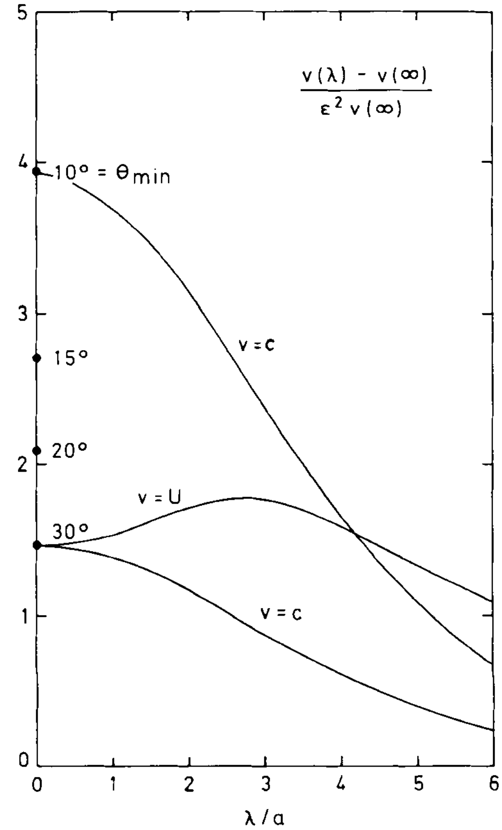

Figure 9. Phase velocity c and group velocity U, related to scattering attenuation according to equation (19) through a dispersion relation. The curves for different minimum scattering angles θ_{min} differ from each other by a constant factor.

and the general form of the wavelength dependence is similar. However, the decay of phase velocity to the long-wavelength limit is slower in Fig. 9 than in Fig. 8. This may indicate that the minimum-phase assumption for attenuation operators is only approximately correct for 2-D media (for 1-D media and plane-wave propagation perpendicular to the interfaces it is exact).

A particularly interesting aspect of the results in Fig. 8, and also of equation (17), is the pathlength dependence of the velocity shift: for $L/\lambda = 36$ the velocity increase with respect to v_0 is much larger than for $L/\lambda = 9$, at least for $\lambda/a \leq 1$. This property can be explained qualitatively as follows. The Fresnel volume of diffraction comprises all points for which single scattering leads to arrival times at the receiver at $x = L$ which agree with the arrival time of the direct wave within a fraction γT of the wave period T ($\gamma = 0.25$ to 0.5). This volume is bounded by a parabola with its focus at the receiver and with the width $(2\gamma\lambda L)^{1/2}$ at $x = 0$, where the random structure begins (Fig. 10). The main contributions to the first arrival come from this volume. For $L/\lambda = 36$ this volume is longer and broader than for $L/\lambda = 9$, such that there exist *more* possibilities for the wave to find fast multiple-scattering paths, which are not much longer than single-scattering paths and therefore contribute to the time interval γT. The consequence is a higher apparent velocity.

Fig. 10 can also explain the dependence of the minimum scattering angle θ_{min} on the pathlength L which is evident from Figs 8 and 9: singly scattered rays in the Fresnel volume have generally smaller scattering angles for $L/\lambda = 36$ than for $L/\lambda = 9$. The same is true for an average scattering angle which can be identified with θ_{min}.

L/λ values larger, even much larger, than 36 would be of interest for seismological applications, e.g. to teleseismic body waves, but on the computer available we cannot go beyond 36. So we conclude that

$$\frac{\delta v}{v_0} = 4\varepsilon^2 \qquad (20)$$

from equation (17) is a *minimum value* for the velocity difference between short and long waves in an acoustic 2-D random medium and for pathlengths in excess of 36 wavelengths.

6 DISCUSSION AND CONCLUSIONS

We have shown in this paper that statistical heterogeneity produces interpretable traveltime fluctuations on profiles. The theory presented is ray-theoretical and hence requires, in principle, waves whose wavelength is less than 0.5 to 1 times the correlation length of the heterogeneities. However, larger wavelengths, up to a few times the correlation length, can also be used since the corresponding

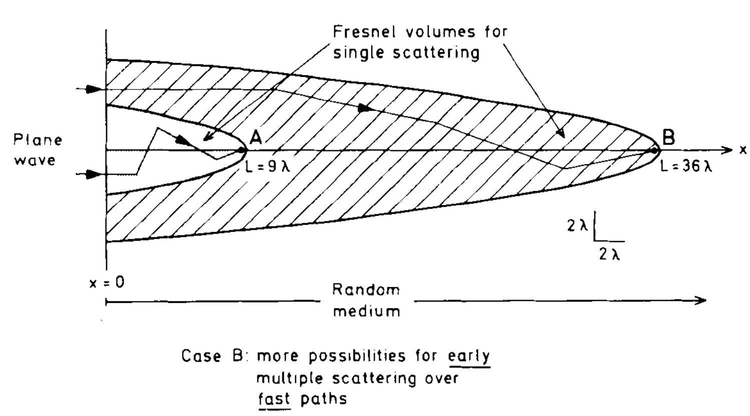

Figure 10. Two receivers in a random medium, A at $L = 9\lambda$ and B at $L = 36\lambda$, and the corresponding Fresnel volumes whose single scattering contributes to the first arrival ($\gamma = 0.5$). Multiple scattering over *fast* paths with *early* arrival times is possible for B, but not for A, because of geometrical reasons. Hence, the apparent velocity of the random medium is higher for B than for A.

results can be corrected with the finite-difference results of this paper (Fig. 7); in the 3-D case these 2-D corrections are only approximations. Application of the interpretation method to observations appears feasible in borehole tomography, but also in modern seismic studies of the Earth's crust with narrow source or receiver spacing, such as the BABEL project in the Baltic Sea. In this project, for instance, very detailed records of the P_g phase were obtained (BABEL Working Group 1991).

For applications a 3-D generalization of the 2-D theory given is needed. In this case one has to consider 2-D traveltime fluctuations $\delta T(y, z)$, observed in a plane perpendicular to the propagation direction of the wave. Their autocorrelation function ϕ depends on two lags, η and ζ; η is the lag corresponding to the y-coordinate. For $\phi(\eta, \zeta)$ equation (4) applies with ζ replaced by $\rho = (\eta^2 + \zeta^2)^{1/2}$. Interpretation of a synthetic or observed $\phi(\eta, \zeta)$ requires replacement or approximation by a ρ-dependent $\phi(\rho)$ which then is treated like the 2-D autocorrelation function $\phi(\zeta)$ above. The 3-D analogues to equations (9), (10), (11) and (13) are:

$$\int_0^\infty F(r) r^2\, dr = 0, \tag{21}$$

$$F(r) = \varepsilon^2 u_0^2 \left(1 - \frac{r^2}{12a^2}\right) e^{-r/a}, \tag{22}$$

$$\phi(0) = \frac{5}{3} \varepsilon^2 u_0^2 L a, \tag{23}$$

$$a = \rho_{\mathrm{half}}. \tag{24}$$

Fig. 11 shows $F(r)$ from (22) and $\phi(\rho)$ from (4); this figure is analogous to Fig. 1. Traveltime fluctuations in the 3-D case would be interpreted with equation (23) and (24); ρ_{half} is the half-width of $\phi(\rho)$. These equations would also be used, if fluctuations were available only along a profile and not in a plane.

We have frequently observed that in seismogram sections such as those in Figs 2–5 early arrivals are connected with low amplitudes, while late arrivals are strong. The explanation is that over fast paths defocusing of the wave (or of rays) prevails and over slow paths focusing. A quantitative description of this correlation is available (Flatté & Wu 1988). It appears worthwhile to look for such a correlation also in data. If correlation is present, a common cause is probable. Lack of correlation may indicate that amplitudes are determined by very local effects which do not influence traveltimes noticeably.

The velocity shift increases the apparent velocity of a random medium over the velocity, corresponding to the average slowness. For slowness fluctuations of a few per cent this shift is so small that no correction of observed velocities is required before they are used for other purposes, e.g. for interpretation in terms of composition. The frequency dependence of the velocity shift is of greater importance, because it is similar to the dispersion due to anelasticity. This latter dispersion, although weak, is widely accepted in seismology and rock physics. For instance, Dziewonski & Anderson (1981), in their construction of the Preliminary Reference Earth Model, found it necessary to incorporate anelasticity in order to make short-period (1 s) body-wave data compatible with long-period (200–1000 s) surface-wave and normal-mode data. For frequency independent anelastic Q the relative phase-velocity increase from the period T ($>1\,s$) to 1 s is

$$\frac{\delta v}{v} = \frac{1}{\pi Q_a} \ln T \ (T \text{ in } s). \tag{25}$$

For $Q_a = 600$, a typical value for P-waves in the Earth's mantle, and $T = 500$ s one obtains from (25) $\delta v/v = 0.33$ per cent. The dispersion of the velocity shift for heterogeneity with correlation lengths larger than the wavelength of a 1 s P-wave, i.e. larger than 10 to 20 km, follows from (20) and is larger than 0.16 per cent for $\varepsilon = 2$ per cent. Therefore, velocity dispersion in the Earth due to statistical heterogeneity can be of similar magnitude as dispersion due to anelasticity, and observed velocity dispersion is not only due to anelasticity, but due to heterogeneity as well. Similar conclusions have been obtained by Nolet & Moser (1991) by ray tracing for S-waves in mantle models with 3-D statistical heterogeneity.

Our discussion of the velocity shift is based on wave-theoretical results, calculated with a finite-difference method. The velocity shift, following from ray theory for random media, has not been discussed. Snieder & Sambridge (1992) present ray perturbation theory for heterogeneous media in a form, which allows calculations of the ray-theoretical velocity shift. Such calculations are under way and will be presented in a future paper.

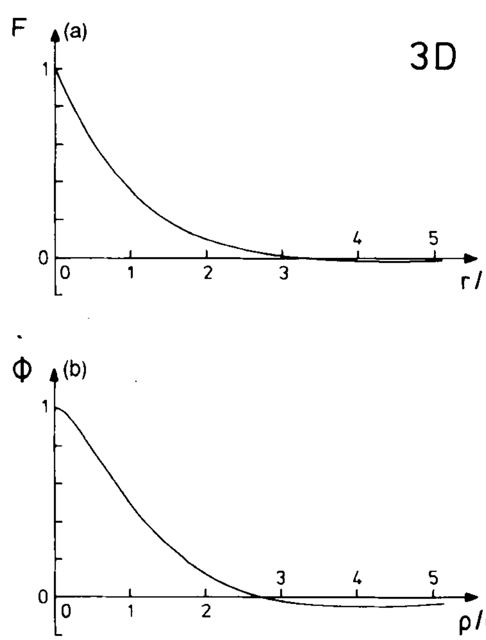

Figure 11. The same as Fig. 1 for the case of a 3-D random medium. $F(r)$ follows from equation (22) and $\phi(\rho)$ from equation (4) through numerical calculation.

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