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Seismic waves in a stratified half space

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Summary. The response of a stratified elastic half space to a general source may be represented in terms of the reflection and transmission properties of the regions above and below the source. For *P–SV* and *SH* waves and both buried sources and receivers, convenient forms of the response may be found in which no loss of precision problems arise from growing exponential terms in the evanescent regime. These expressions have a ready physical interpretation and enable useful approximations to the response to be developed. The reflection representation leads to efficient computational procedures for models composed of uniform layers, which may be extended in an asymptotic development to piecewise smooth models.

1 Introduction

The problem of the excitation and propagation of seismic waves in a stratified elastic half space has been extensively discussed, particularly with regard to seismic surface waves. When the elastic parameter distribution is a function of only one coordinate the stress–strain relations and the elastic equations of motion can be reduced by transform techniques to a set of first-order differential equations, to be solved subject to free surface boundary conditions and excitation at the source (Alterman, Jarosch & Pekeris 1959; Takeuchi & Saito 1972).

The commonest approach has been to consider models of the elastic parameters within the Earth consisting of a number of uniform layers. For such a structure, transfer matrix methods may be developed which relate the stresses and displacements at the top and bottom of these layers. This approach was introduced by Thompson (1950) and corrected and extended by Haskell (1953). The early work was principally concerned with the calculation of surface wave dispersion but both Haskell (1964) and Harkrider (1964) considered the excitation of Love and Rayleigh waves by realistic sources.

A computational difficulty arises in the simple matrix methods associated with loss of precision. In each layer growing exponentials have to be included in the transfer matrix but these cancel in the secular function for surface wave dispersion. In finite accuracy computations the cancellation is not complete since the growing exponentials swamp the significant part of the secular function. This difficulty may be avoided either by a reformulation of the matrix method (Knopoff 1964) or alternatively by considering higher-order minors of the original matrices (Molotkov 1961; Dunkin 1965).

Gilbert & Backus (1966) gave a systematic development of the transfer matrix methods for a general stratification in elastic parameters. They introduced the term 'Propagator matrix' for the transfer operator for stress and displacement between two levels in the stratified medium. In addition they established the general utility of the minor matrix approach.

For smoothly varying models of the elastic parameter distribution, most work has concentrated on the numerical solution of the set of ordinary differential equations (Takeuchi, Saito & Kobayachi 1962; Alterman *et al.* 1959). To avoid numerical problems associated with growing solutions of the differential equations at depth, akin to those already mentioned for uniform layers, the integration is carried in the direction of an increasing solution, i.e. towards the surface.

In this paper we show how the whole response of an elastic half space may be built up in terms of the reflection and transmission properties of the stratified medium, following the approach of Kennett (1974). For both buried sources and buried receivers we are able to derive convenient representations of the seismic wavefield. These expressions do not contain any growing solutions and so completely avoid loss of precision problems.

In the case of a medium composed of uniform layers the representation in terms of the reflection and transmission properties leads to an efficient computational procedure in which the calculation progresses from the base of the layering towards the surface. The method may be extended to a piecewise smooth model and in an asymptotic development retains its computational advantages (*cf.* Woodhouse 1978).

2 A stratified half space

We will consider a horizontally stratified half space with isotropic elastic properties (P wave speed α, S wave speed β, density ρ) depending only on the depth coordinate z (Fig. 1). We assume that the structure is underlain by a uniform half space beneath the level z_L with properties α_L, β_L, ρ_L – this requirement can however be weakened to allow structures in which the elastic parameters ultimately asymptote to a constant value. For simplicity we will also restrict our attention to excitation due to a point source, since more complex sources may easily be generated by superposition.

2.1 THE COUPLED EQUATIONS AND BOUNDARY CONDITIONS

In a cylindrical system of coordinates (x, ϕ, z), with corresponding unit vectors $\hat{\mathbf{x}}$, $\hat{\boldsymbol{\phi}}$, $\hat{\mathbf{z}}$ we may represent the elastic displacement $\mathbf{w}(x, \phi, z, t)$ as a Fourier–Bessel transform

$$\begin{aligned}\mathbf{w}(x, \phi, z, t) &= w_x\hat{\mathbf{x}} + w_\phi\hat{\boldsymbol{\phi}} + w_z\hat{\mathbf{z}}\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \exp(-i\omega t)\int_0^\infty dk\, k \sum_{m=-2}^{2} (U\mathbf{R}_k^m + V\mathbf{S}_k^m + W\mathbf{T}_k^m),\end{aligned} \tag{2.1}$$

in terms of the vector surface harmonics (Takeuchi & Saito 1971)

$$\begin{aligned}\mathbf{R}_k^m &= \hat{\mathbf{z}}y_k^m(x, \phi),\\ \mathbf{S}_k^m &= \frac{1}{k}\nabla_1 y_k^m(x, \phi),\\ \mathbf{T}_k^m &= -\frac{1}{k}\hat{\mathbf{z}}\wedge\nabla_1 y_k^m(x, \phi),\end{aligned} \tag{2.2}$$

where

$$y_k^m(x, \phi) = J_m(kx)\exp(im\phi),$$
$$\nabla_1 = \hat{\mathbf{x}}\partial_x + \hat{\boldsymbol{\phi}}(1/x)\partial_\phi. \tag{2.3}$$

The summation in (2.1) is restricted to $|m| < 2$ by our assumption of a point source. This representation is equivalent to that introduced by Hudson (1969).

The traction vector **T** across a horizontal plane, representing the horizontal and vertical stress components, may also be written in a similar form to (2.1)

$$\mathbf{T}(x, \phi, z, t) = \tau_{xz}\hat{\mathbf{x}} + \tau_{\phi z}\hat{\boldsymbol{\phi}} + \tau_{zz}\hat{\mathbf{z}}$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \exp(-i\omega t)\int_0^{\infty} dk\, k \sum_{m=-2}^{2} (P\mathbf{R}_k^m + S\mathbf{S}_k^m + T\mathbf{T}_k^m), \tag{2.4}$$

with

$$P = \rho\alpha^2\partial_z U - k\rho(\alpha^2 - 2\beta^2)V,$$
$$S = \rho\beta^2(\partial_z V + kU), \tag{2.5}$$
$$T = \rho\beta^2\partial_z W,$$

since the elastic properties are only functions of z. We may note that the harmonic $\mathbf{T}_k^m$ lies wholly within a horizontal plane and for an isotropic medium this part of the displacement and traction separates from the rest to give the *SH* wave part of the seismic field.

When we use the representations (2.1) and (2.4) in the stress–strain relations and the elastic equations of motion we find that the stress and displacement scalars (U, V, P, S) and (W, T) satisfy coupled sets of first-order ordinary differential equations. For notational simplicity it is convenient to introduce the stress-displacement vectors (Woodhouse 1978) (a) for *P–SV* waves

$$\mathbf{B}_P = [U, V, \omega^{-1}P, \omega^{-1}S]^{\mathrm{T}}, \tag{2.6P}$$

and (b) for *SH* waves

$$\mathbf{B}_H = [W, \omega^{-1}T]^{\mathrm{T}}, \tag{2.6H}$$

where T denotes a transpose. These vectors satisfy differential equations of the form (Gilbert & Backus 1966)

$$\partial_z \mathbf{B}(z) = \omega\mathbf{A}(z)\mathbf{B}(z). \tag{2.7}$$

For *P–SV* waves we have

$$\mathbf{A}_P = \begin{pmatrix} 0 & p(1-2\beta^2/\alpha^2) & (\rho\alpha^2)^{-1} & 0 \\ -p & 0 & 0 & (\rho\beta^2)^{-1} \\ -\rho & 0 & 0 & p \\ 0 & \nu p^2 - \rho & -p(1-2\beta^2/\alpha^2) & 0 \end{pmatrix}, \tag{2.8P}$$

where p is the slowness ($\omega p = k$), and $\nu = 4\rho\beta^2(1 - \beta^2/\alpha^2)$, and for *SH* waves

$$\mathbf{A}_H = \begin{pmatrix} 0 & (\rho\beta^2)^{-1} \\ \rho(\beta^2 p^2 - 1) & 0 \end{pmatrix}. \tag{2.8H}$$

A similar set of equations may be obtained for a two-dimensional seismic wavefield where all stresses and displacements are assumed to be independent of one Cartesian horizontal coordinate (see, e.g. Kennett, Kerry & Woodhouse 1978). If we consider the plane $\phi = 0$ and take a Fourier transform $\hat{\mathbf{u}}$ with respect to x rather than a Hankel transform, equations (2.7), (2.8) can be recovered if

$$U = i\hat{u}_z, \quad V = \hat{u}_x, \quad P = i\hat{\tau}_{zz}, \quad S = \hat{\tau}_{xz}, \quad W = \hat{u}_y, \quad T = \hat{\tau}_{yz}. \tag{2.9}$$

The stress-displacement vectors $\mathbf{B}$ have the convenient property that, with the usual conditions of welded contact between elastic media, they are continuous across any horizontal plane. At the free surface ($z = 0$) the stress scalars must vanish so that we have

$$\mathbf{B}(0) = [\mathbf{w}_0, 0]^{\mathrm{T}},$$

or explicitly for P–SV waves

$$\mathbf{B}_P(0) = [U(0), V(0), 0, 0]^{\mathrm{T}}, \tag{2.10}$$

and for SH waves

$$\mathbf{B}_H(0) = [W(0), 0]^{\mathrm{T}}.$$

We will assume that any source lies above the level z_{L} (this may easily be achieved by suitable adjustment of this depth), and then in the underlying half space we require either purely downgoing radiation or that the seismic field be purely evanescent waves decaying with depth, depending on the slowness range being considered.

2.2 DECOMPOSITION OF THE SEISMIC WAVEFIELD

In order to relate the stress-displacement vector (2.6) more directly to the elastic wavefield we follow Kennett *et al.* (1978) and make the transformation

$$\mathbf{B} = \mathbf{D}\mathbf{V}, \tag{2.11}$$

where $\mathbf{D}$ is the eigenvector matrix for $\mathbf{A}$; this procedure appears to have been introduced in the seismic literature by Dunkin (1965). In a uniform medium the new wave vector $\mathbf{V}$ then satisfies

$$\partial_z \mathbf{V} = i\omega\mathbf{\Lambda}\mathbf{V}, \tag{2.12}$$

where $i\mathbf{\Lambda}$ is a diagonal matrix whose entries are the eigenvalues of $\mathbf{A}$. for P–SV waves

$$\Lambda_P = \mathrm{diag}\,\{-q_\alpha, -q_\beta, q_\alpha, q_\beta\},$$

and for SH waves

$$\Lambda_H = \mathrm{diag}\,\{-q_\beta, q_\beta\}, \tag{2.13}$$

with

$$q_\alpha = (\alpha^{-2} - p^2)^{1/2}, \qquad \mathrm{Im}\,\omega q_\alpha, \omega q_\beta \geqslant 0.$$
$$q_\beta = (\beta^{-2} - p^2)^{1/2},$$

The elements of $\mathbf{V}$ may be identified with the amplitudes of upward and downward travelling plane waves

$$\mathbf{V} = [V_{\mathrm{U}}, V_{\mathrm{D}}]^{\mathrm{T}}. \tag{2.14}$$

For P–SV waves, writing ϕ for P-wave amplitude and ψ for S-wave amplitude, we have

$$\mathbf{V}_P = [\phi_U, \psi_U, \phi_D, \psi_D]^T, \tag{2.15}$$

while for SH waves

$$\mathbf{V}_H = [\chi_U, \chi_D]^T.$$

The columns of **D** are the eigenvectors of the matrix **A** and may be identified as 'elementary' stress-displacement vectors corresponding to the different wave types. For P–SV waves

$$\mathbf{D}_P = [b_P^U, b_S^U, b_P^D, b_S^D], \tag{2.16P}$$

and the vectors b take the form

$$b_P^{U,D} = \epsilon_\alpha [\mp iq_\alpha, p, \rho(2\beta^2 p^2 - 1), \mp 2i\rho\beta^2 pq_\alpha]^T,$$

$$b_S^{U,D} = \epsilon_\beta [p, \mp iq_\beta, \mp 2i\rho\beta^2 pq_\beta, \rho(2\beta^2 p^2 - 1)]^T.$$

Since we have a free choice of scaling parameters ϵ_α, ϵ_β we follow Kennett *et al.* (1978) and normalize with respect to energy flux in the z direction, so that

$$\epsilon_\alpha = (2pq_\alpha)^{-1/2}, \qquad \epsilon_\beta = (2pq_\beta)^{-1/2}.$$

In a similar way for SH waves

$$\mathbf{D}_H = [b_H^U, b_H^D] \tag{2.16H}$$

and

$$b_H^{U,D} = \epsilon_\beta [\beta^{-1}, \mp i\rho\beta q_\beta]^T.$$

From the definition of **D** (2.11) we see that its subpartitions play the role of transforming up- and downgoing wave components into stress and displacement. We may display this relation by writing

$$\mathbf{D} = \left(\begin{array}{c|c} \mathrm{M_U} & \mathrm{M_D} \\ \hline \mathrm{N_U} & \mathrm{N_D} \end{array}\right), \tag{2.17}$$

so that $\mathrm{M_U}$, $\mathrm{M_D}$ are the displacement transformations and $\mathrm{N_U}$, $\mathrm{N_D}$ the stress transformations. For the P–SV wave system $\mathrm{M_U}$ etc. will be 2×2 matrices and for SH waves simply scalars.

For a uniform layer the amplitudes of the up- and downgoing waves at different levels are, from (2.12), connected by

$$\mathbf{V}(z) = \exp [i\omega\mathbf{\Lambda}(z - z_0)]\, \mathbf{V}(z_0), \tag{2.18}$$

and thus in terms of the stress-displacement vector **B** we have

$$\mathbf{B}(z) = \mathbf{D} \exp [i\omega\mathbf{\Lambda}(z - z_0)]\, \mathbf{D}^{-1}\mathbf{B}(z_0), \tag{2.19}$$

and the transfer matrix may be identified with the Haskell (1953) layer matrix.

With the choice we have made for the radicals q_α, q_β (2.13) our requirement that the wavefield below z_L should either be travelling in the positive z direction or be evanescent, may be encompassed by requiring that the upgoing wave vector V_U should vanish in $z > z_L$. This means that the stress-displacement field at z_L must take the form

$$\mathbf{B}(z_L) = \mathbf{D}(z_L +)\mathbf{V}(z_L +) = \mathbf{D}(z_L +) \begin{pmatrix} 0 \\ V_D \end{pmatrix} \tag{2.20}$$

where the eigenvector matrix is to be evaluated in the lower uniform half space.

2.3 THE INTRODUCTION OF A SOURCE

Burridge & Knopoff (1964) established that a dislocation source across an arbitrarily oriented plane can be replaced by a system of forces which generate an identical radiation field. Hudson (1969) demonstrated the converse that a point force or dislocation source with arbitrary orientation can be replaced by a point dislocation acting across a horizontal plane to give the same radiation. Thus if the source is confined to a single horizontal plane we may introduce equivalent discontinuities in displacement and traction across that plane, i.e. we have a discontinuity in the stress-displacement vector **B** across the source plane z_s (Fig. 1). We therefore introduce a source vector $\mathscr{S}$ defined by

$$[\mathbf{B}(z_s)]_-^+ = \mathbf{B}(z_s+) - \mathbf{B}(z_s-) = \mathscr{S}(z_s). \tag{2.21}$$

In order to allow the most general form of a point force we make use of the source moment tensor (Gilbert 1971) and consider this in combination with a simple force. Thus we consider a source specified by the force system, on Cartesian axes,

$$f_i = -\partial_j(M_{ij}\delta(\mathbf{x})) + F_i\delta(\mathbf{x}); \quad i, j = x, y, z. \tag{2.22}$$

By a suitable choice of the components of the Moment tensor M_{ij} one may generate an explosion ($M_{ij} = M_0\delta_{ij}$) or a double couple ($M_{ij} = M_0(e_i n_j + e_j n_i)$, $\mathbf{e}\cdot\mathbf{n} = 0$) or other more exotic sources. The explicit form of the discontinuity vector $\mathscr{S}$ (2.21) corresponding to this choice of general source is given in Appendix A.

An alternative approach to the introduction of a source is to regard it as giving rise to a discontinuity in the wave vector **V** and this has been used by Haskell (1964) and Harkrider (1964) to specify their sources. If we can assume the source to lie in a locally homogeneous region about the source plane z_s, then the discontinuity in the wave vector **V** is given by

$$[\mathbf{V}(z_s)]_-^+ = \mathbf{\Sigma}(z_s) = \mathbf{D}^{-1}(z_s)\mathscr{S}(z_s). \tag{2.23}$$

The wavefield discontinuity may be partitioned in a similar way to (2.14)

$$\mathbf{\Sigma} = [\mathbf{\Sigma}_U, \mathbf{\Sigma}_D]^T \tag{2.24}$$

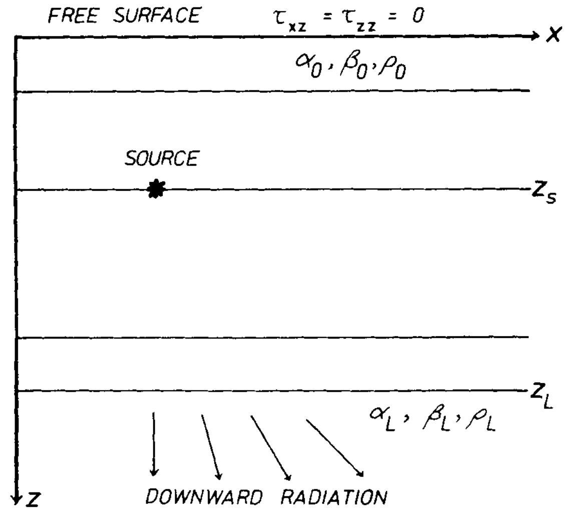

Figure 1. The model of the stratified half space employed in this paper. The half space is assumed to be uniform below z_L and in this region only downgoing radiation will be present.

and we may examine the significance of the terms by considering a source embedded in a uniform medium of infinite extent. By analogy with (2.14) the wavefield solution will be

$$\begin{aligned}\bar{\mathbf{V}}(z) &= [\bar{V}_U(z), 0]^T, \quad z < z_s \\ &= [0, \bar{V}_D(z)]^T, \quad z > z_s\end{aligned}$$

so that the discontinuity $\mathbf{\Sigma}$ has the representation

$$[\mathbf{V}(z_s)]_-^+ = \mathbf{\Sigma} = [-\bar{V}_U(z_s), \bar{V}_D(z_s)]^T. \tag{2.25}$$

Thus when we equate the two forms for $\mathbf{\Sigma}$, (2.24) and (2.25), we see that a source will radiate a wavefield $-\mathbf{\Sigma}_U$ upwards and $\mathbf{\Sigma}_D$ downwards. For the general point source (2.22) the elements of the jump vector $\mathbf{\Sigma}$ are given in Appendix A.

3 The propagator solution

3.1 PROPAGATOR MATRICES

For a horizontally stratified medium we have seen that the stress-displacement vector $\mathbf{B}$ satisfies the system of ordinary differential equations

$$\partial_z \mathbf{B}(z) = \omega \mathbf{A}(z)\mathbf{B}(z). \tag{3.1}$$

The propagator matrix $\mathbf{P}(z, z_0)$ (Gilbert & Backus 1966) is a fundamental matrix solution of the corresponding matrix equation

$$\partial_z \mathbf{P}(z, z_0) = \omega \mathbf{A}(z)\mathbf{P}(z, z_0), \tag{3.2}$$

with linearly independent columns, under the constraint $\mathbf{P}(z_0, z_0) = \mathbf{I}$ (where $\mathbf{I}$ is the identity matrix of appropriate dimensionality). From any fundamental matrix $\mathbf{\Phi}(z)$ for (3.1) we may construct the propagator $\mathbf{P}(z, z_0)$ by

$$\mathbf{P}(z, z_0) = \mathbf{\Phi}(z)\mathbf{\Phi}^{-1}(z_0), \tag{3.3}$$

which will exist since $\mathbf{\Phi}$ is non-singular. For the elastic equations (3.1) the trace of $\mathbf{A}$ vanishes and so det $\mathbf{P} = 1$ everywhere.

In terms of the propagator matrix the solution of (3.1) with the stress-displacement vector specified at some level z_0 is

$$\mathbf{B}(z) = \mathbf{P}(z, z_0)\mathbf{B}(z_0), \tag{3.4}$$

and such an overall propagator may be split at any intermediate level since

$$\mathbf{P}(z, z_0) = \mathbf{\Phi}(z)\mathbf{\Phi}^{-1}(z_0) = \mathbf{\Phi}(z)\mathbf{\Phi}^{-1}(\zeta)\mathbf{\Phi}(\zeta)\mathbf{\Phi}^{-1}(z_0) = \mathbf{P}(z, \zeta)\mathbf{P}(\zeta, z_0),$$

and in particular (3.5)

$$\mathbf{P}(z_2, z_1) = \mathbf{P}(z_1, z_2)^{-1}.$$

These relations also hold in media with discontinuities in the elastic parameters since the continuity of the stress-displacement vector across a horizontal plane (in the absence of sources) ensures the continuity of $\mathbf{P}(z, z_0)$.

For homogeneous layers we have already shown (2.19) that a suitable fundamental matrix is

$$\mathbf{\Phi}(z) = \mathbf{D} \exp [i\omega \mathbf{\Lambda} z], \tag{3.6}$$

and thus the layer matrix (2.19) is a special case of the propagator matrix. Hence if we have a stack of uniform layers the overall propagator will just be a matrix product of the layer matrices between the levels z, z_0 as in the work of Haskell (1953), Harkrider (1964).

3.2 THE RESPONSE OF THE HALF SPACE

We recall the boundary conditions imposed on the seismic wavefield (Fig. 1): from the free surface

$$\mathbf{B}(0) = [w_0, 0]^{\mathrm{T}}, \tag{2.10}$$

and with the requirement of only outgoing or evanescent waves below z_{L},

$$\mathbf{B}(z_{\mathrm{L}}) = \mathbf{D}(z_{\mathrm{L}}+)\mathbf{V}(z_{\mathrm{L}}+) = \mathbf{D}(z_{\mathrm{L}}+)\,[0, V_{\mathrm{D}}]^{\mathrm{T}}. \tag{2.20}$$

Now we may relate the stress-displacement vector just below the source $\mathbf{B}(z_s+)$ to the wave-field in the underlying half space by

$$\mathbf{B}(z_s+) = \mathbf{P}(z_s, z_L)\mathbf{B}(z_{\mathrm{L}}), \tag{3.7}$$

and including the discontinuity term associated with the source (2.15), we may construct the stress-displacement vector just above the source

$$\mathbf{B}(z_s-) = \mathbf{P}(z_s, z_{\mathrm{L}})\mathbf{B}(z_{\mathrm{L}}) - \mathscr{S}. \tag{3.8}$$

The surface displacement is thus given by

$$\begin{aligned}\mathbf{B}(0) &= \mathbf{P}(0, z_s)\mathbf{B}(z_s-)\\ &= \mathbf{P}(0, z_{\mathrm{L}})\mathbf{B}(z_{\mathrm{L}}) - \mathbf{P}(0, z_s)\mathscr{S}\end{aligned} \tag{3.9}$$

We introduce the vector

$$\mathbf{S} = \mathbf{P}(0, z_s)\mathscr{S} = [S_{\mathrm{W}}, S_{\mathrm{T}}]^{\mathrm{T}}, \tag{3.10}$$

which represents the effect of the entire discontinuity due to the source propagated up to the surface, i.e. this represents rather more than just the direct radiation from the source to the surface.

When we include the boundary conditions at the level z_{L},

$$\mathbf{B}(0) = \mathbf{P}(0, z_{\mathrm{L}})\mathbf{D}(z_{\mathrm{L}}+)\mathbf{V}(z_{\mathrm{L}}+) - \mathbf{S}, \tag{3.11}$$

which in the case of a stack of uniform layers is equivalent to equation (55) of Harkrider (1964). We get

$$\mathbf{F}(0, z_{\mathrm{L}}+) = \mathbf{P}(0, z_{\mathrm{L}})\mathbf{D}(z_{\mathrm{L}}+) \tag{3.12}$$

and then the free surface condition requires

$$\begin{pmatrix} w_0 \\ \hline 0 \end{pmatrix} = \left(\begin{array}{c|c} F_{11} & F_{12} \\ \hline F_{21} & F_{22} \end{array}\right) \begin{pmatrix} 0 \\ \hline V_{\mathrm{D}}(z_{\mathrm{L}}+) \end{pmatrix} - \begin{pmatrix} S_{\mathrm{W}} \\ \hline S_{\mathrm{T}} \end{pmatrix} \tag{3.13}$$

where we have introduced the subpartitions of the matrix $\mathbf{F}$, i.e. F_{ij} are 2×2 matrices in the P–SV wave case and scalars for SH waves.

To ensure the vanishing of the surface stress, the wavefield must neutralize the stress S_T induced at the surface by the source discontinuity. Thus formally

$$w_0 = F_{12}F_{22}^{-1}S_T - S_W \tag{3.14}$$

provided that the secular function det F_{22} does not vanish. Once we have found the surface displacement, the stress-displacement field at any other level may be found from

$$\begin{aligned} \mathbf{B}(z) &= \mathbf{P}(z, 0)\,[w_0, 0]^{\mathrm{T}}, \qquad z < z_s \\ &= \mathbf{P}(z, 0)\,[w_0, 0]^{\mathrm{T}} + \mathbf{P}(z, z_s)\mathbf{S}, \qquad z > z_s. \end{aligned} \tag{3.15}$$

The propagator solution thus does allow a complete specification of the seismic wavefield but also suffers from some computational disadvantages.

The propagator matrix $\mathbf{P}(z_1, z_2)$ between two levels includes all the characteristics of the wave propagation in the region between z_1 and z_2, and so both upward and downward travelling waves are considered. This is well illustrated by the propagator for a uniform layer (2.19) which contains exponentials of the form $\exp(i\omega q_\beta h)$, $\exp(-i\omega q_\beta h)$, which appear in the combinations $\cos(\omega q_\beta h)$, $\sin(\omega q_\beta h)$ in the explicit form of the layer matrix (Haskell 1953). This causes little difficulty when waves are propagating in the layer but once the waves become evanescent we encounter terms of the form $\cosh(\omega|q_\beta|h)$, $\sinh(\omega|q_\beta|h)$. Our condition (2.20) means that we are interested in terms with negative exponents which are swamped in the cosh and sinh terms, so that it is difficult to achieve sufficient accuracy, particularly when the frequency is high.

The problem is compounded in the case of *P–SV* waves by the form of the solution. The elements of $F_{12}F_{22}^{-1}$ consist of ratios of minors of $\mathbf{F}$. Thus for a given slowness, even if only part of the structure contains evanescent waves one is faced with the problem of the subtraction of large nearly equal quantities with consequent loss of precision. Molotkov (1961), Dunkin (1965) and Gilbert & Backus (1966) have overcome this difficulty by reformulating the problem *ab initio* in terms of the minors of the propagator matrices, but this procedure does not allow an easy physical interpretation of the results.

In the following sections we present an alternative approach based on the reflection and transmission properties of the half space in which the numerical difficulties are avoided by eliminating the growing solution from the formulation.

4 Reflection and transmission properties of elastic media

4.1 REFLECTION AND TRANSMISSION OF ELASTIC WAVES

We consider an arbitrary vertically inhomogeneous medium in $z_1 < z < z_3$ sandwiched between two uniform half spaces in $z < z_1$, $z_3 < z$, as in Kennett (1974) and Kennett *et al.* (1978). Then the stress-displacement vectors at the top and bottom of the region are related by

$$\mathbf{B}(z_1) = \mathbf{P}(z_1, z_3)\mathbf{B}(z_3), \tag{4.1}$$

and in terms of the wavefields $\mathbf{V}$ in the bounding half spaces we have

$$\begin{aligned} \mathbf{V}(z_1-) &= \mathbf{D}^{-1}(z_1-)P(z_1, z_3)\mathbf{D}(z_3+)\mathbf{V}(z_3+) \\ &= \mathbf{Q}(z_1-, z_3+)\mathbf{V}(z_3+), \end{aligned} \tag{4.2}$$

and by analogy with (4.1) we may term $\mathbf{Q}$ the wave propagator. This wave propagator has similar properties to $\mathbf{P}(z_1, z_3)$, since from (3.5)

$$\begin{aligned}\mathbf{Q}(z_1-, z_3+) &= \mathbf{D}^{-1}(z_1-)\mathbf{P}(z_1, z_2)\mathbf{D}(z_2)\mathbf{D}^{-1}(z_2)\mathbf{P}(z_2, z_3)\mathbf{D}(z_3+)\\ &= \mathbf{Q}(z_1-, z_2)\mathbf{Q}(z_2, z_3+),\end{aligned} \tag{4.3}$$

and in particular

$$\mathbf{Q}(z_1-, z_3+) = \mathbf{Q}(z_3+, z_1-)^{-1}.$$

Although $\mathbf{P}(z_1, z_2)$ is continuous across the level $z = z_2$, $\mathbf{Q}(z_2, z_3+)$ will not be unless there is no discontinuity in the elastic parameters across this plane; hence in (4.2) the +, – indicators are strictly necessary. In a similar way to (3.13) we introduce the subpartitions of $\mathbf{Q}$ so that

$$\begin{pmatrix} V_{\mathrm{U}}(z_1-) \\ \hdashline V_{\mathrm{D}}(z_1-) \end{pmatrix} = \left(\begin{array}{c:c} Q_{11} & Q_{12} \\ \hdashline Q_{21} & Q_{22} \end{array}\right) \begin{pmatrix} V_{\mathrm{U}}(z_3+) \\ \hdashline V_{\mathrm{D}}(z_3+) \end{pmatrix}. \tag{4.4}$$

We may define reflection and transmission matrices $\boldsymbol{R}$, $\boldsymbol{T}$ in terms of the V_{U}, V_{D}; for example if we consider an incident downward wave from $z < z_1$, so that $V_{\mathrm{U}}(z_3+) = 0$, we have

$$V_{\mathrm{U}}(z_1-) = \boldsymbol{R}_{\mathrm{D}} V_{\mathrm{D}}(z_1-), \qquad V_{\mathrm{D}}(z_3+) = \boldsymbol{T}_{\mathrm{D}} V_{\mathrm{D}}(z_1-) \tag{4.5}$$

with a corresponding definition for $\boldsymbol{R}_{\mathrm{U}}$, $\boldsymbol{T}_{\mathrm{U}}$ due to an upward wave from $z > z_3$.

For P–SV waves $\boldsymbol{R}_{\mathrm{D}}$, $\boldsymbol{T}_{\mathrm{D}}$ are 2×2 matrices and we write

$$\boldsymbol{R}_{\mathrm{D}} = \begin{pmatrix} r^{\mathrm{D}}_{PP} & r^{\mathrm{D}}_{PS} \\ r^{\mathrm{D}}_{SP} & r^{\mathrm{D}}_{SS} \end{pmatrix}, \qquad \boldsymbol{T}_{\mathrm{D}} = \begin{pmatrix} t^{\mathrm{D}}_{PP} & t^{\mathrm{D}}_{PS} \\ t^{\mathrm{D}}_{SP} & t^{\mathrm{D}}_{SS} \end{pmatrix}, \tag{4.6}$$

in accordance with the standard indexing of matrix elements, so that, e.g. r^{D}_{PS} is the amplitude of an upward P wave generated from a unit amplitude downward incident S wave. For SH waves $\boldsymbol{R}_{\mathrm{D}}$, $\boldsymbol{T}_{\mathrm{D}}$ are just the reflection coefficients.

In terms of the partitions of $\mathbf{Q}$

$$\begin{aligned} \boldsymbol{T}_{\mathrm{D}} &= Q_{22}^{-1}, \\ \boldsymbol{R}_{\mathrm{D}} &= Q_{12} Q_{22}^{-1}, \\ \boldsymbol{T}_{\mathrm{U}} &= Q_{11} - Q_{12} Q_{22}^{-1} Q_{21}, \\ \boldsymbol{R}_{\mathrm{U}} &= Q_{22}^{-1} Q_{21}, \end{aligned} \tag{4.7}$$

and so the wave propagator takes the form

$$Q(z_1-, z_3+) = \left(\begin{array}{c:c} \boldsymbol{T}_{\mathrm{U}} - \boldsymbol{R}_{\mathrm{D}} \boldsymbol{T}_{\mathrm{D}}^{-1} \boldsymbol{R}_{\mathrm{U}} & \boldsymbol{R}_{\mathrm{D}} \boldsymbol{T}_{\mathrm{D}}^{-1} \\ \hdashline -\boldsymbol{T}_{\mathrm{D}}^{-1} \boldsymbol{R}_{\mathrm{U}} & \boldsymbol{T}_{\mathrm{D}}^{-1} \end{array}\right). \tag{4.8}$$

It is of interest to note that the block matrix form of (4.8) may be inverted explicitly to obtain, via (4.3),

$$Q(z_3+, z_1-) = \left(\begin{array}{c:c} \boldsymbol{T}_{\mathrm{U}}^{-1} & -\boldsymbol{T}_{\mathrm{U}}^{-1} \boldsymbol{R}_{\mathrm{D}} \\ \hdashline \boldsymbol{R}_{\mathrm{U}} \boldsymbol{T}_{\mathrm{U}}^{-1} & \boldsymbol{T}_{\mathrm{D}} - \boldsymbol{R}_{\mathrm{U}} \boldsymbol{T}_{\mathrm{U}}^{-1} \boldsymbol{R}_{\mathrm{D}} \end{array}\right), \tag{4.9}$$

and we see that reflecting the matrix (4.9) blockwise about its centre and exchanging the superscripts U, D we obtain the matrix (4.8). This is equivalent to the statement that the upward matrices are just the downward matrices from the inverted structure. Also from the general symmetry relations (Kennett *et al.* 1978)

$$R_D = R_D^T, \quad R_U = R_U^T, \quad T_D = T_U^T. \tag{4.10}$$

We note that the reflection and transmission matrices are only well defined for $z_1 \leqslant z_3$ so when we wish to represent $\mathbf{Q}(z_A, z_B)$ in terms of reflection and transmission coefficients we will use (4.8) for $z_A \leqslant z_B$ and (4.9) for $z_B < z_A$.

We may recover the propagator **P** from the wave propagator **Q** by (Kennett 1974)

$$\mathbf{P}(z_1, z_3) = \mathbf{D}^{-1}(z_1 -)\mathbf{Q}(z_1 -, z_3 +)\mathbf{D}(z_3 +). \tag{4.11}$$

In the special case of a uniform layer we have the partitioned form

$$\mathbf{Q}(z_3, z_1) = \left(\begin{array}{c|c} E^{-1} & 0 \\ \hline 0 & E \end{array}\right) \tag{4.12}$$

where E is the phase income for downward propagation. For P–SV waves

$$E = \mathrm{diag}\,[\exp(i\omega q_\alpha(z_2 - z_1)), \exp(i\omega q_\beta(z_2 - z_1))], \tag{4.13}$$

and for SH waves

$$E = \exp(i\omega q_\beta(z_2 - z_1)).$$

When we compare (4.12) with (4.9) we see that as we would expect there is no reflection from the layer and

$$T_D = E, \quad T_U = E. \tag{4.14}$$

4.2 REFLECTION FROM A REGION BOUNDED ABOVE BY A FREE SURFACE

Consider a vertically inhomogeneous region $0 < z < z_B$ bounded above by the free surface and below by a uniform half space in $z > z_B$. Then if we consider an incident upward wave-field from $z > z_B$ this will be reflected from the region giving rise to a downward field and we may introduce a reflection matrix $R_U^F(z_B)$ to describe the interaction

$$V_D(z_B +) = R_U^F(z_B)\,V_U(z_B +). \tag{4.15}$$

The free surface displacement is related to the wavefield at $z_B +$ by

$$\mathbf{B}(0) = \mathbf{P}(0, z_B)\mathbf{D}(z_B +)\mathbf{V}(z_B +) = \mathbf{F}(0, z_B +)\mathbf{V}(z_B +) \tag{4.16}$$

using (3.12), and so in terms of the partitions of **F**

$$\left(\begin{array}{c} w_0 \\ \hline 0 \end{array}\right) = \left(\begin{array}{c|c} F_{11} & F_{12} \\ \hline F_{21} & F_{22} \end{array}\right) \left(\begin{array}{c} V_U(z_B +) \\ \hline V_D(z_B +) \end{array}\right). \tag{4.17}$$

Thus from the vanishing of the stress at the surface we may identify

$$R_U^F(z_B) = -F_{22}^{-1}F_{21}. \tag{4.18}$$

In particular if we take the level $z_B = 0-$, i.e. just at the surface

$$R_U^F(0-) = \tilde{R} = -N_D^{-1}N_U, \tag{4.19}$$

in terms of the partitions of $\mathbf{D}(0-)$ (Kennett 1974).

4.3 REFLECTION AND TRANSMISSION COEFFICIENTS FOR SUPERPOSED MEDIA

We consider an inhomogeneous region, as in Section 4.1, in $z_1 < z < z_3$ but now subdivided by some horizontal plane $z = z_2$ such that $z_1+ \leqslant z_2 \leqslant z_3-$. Then from (4.3)

$$\mathbf{Q}(z_1-, z_3+) = \mathbf{Q}(z_1-, z_2)\mathbf{Q}(z_2, z_3+), \tag{4.20}$$

and we can substitute from (4.8) for $\mathbf{Q}(z_1-, z_2)$, $\mathbf{Q}(z_2, z_3+)$ to obtain $\mathbf{Q}(z_1-, z_3+)$ in terms of the reflection and transmission properties of the two inhomogeneous regions $z_1- \leqslant z \leqslant z_2$ and $z_2 \leqslant z \leqslant z_3+$. The overall response for $z_1- \leqslant z \leqslant z_3$ is once again given by (4.8) so that

$$\begin{aligned}
R_D^{13} &= R_D^{12} + T_U^{12}R_D^{23}[I - R_U^{12}R_D^{23}]^{-1}T_D^{12},\\
T_D^{13} &= T_D^{23}\qquad\quad [I - R_U^{12}R_D^{23}]^{-1}T_D^{12},\\
R_U^{13} &= R_U^{23} + T_D^{23}R_U^{12}[I - R_D^{23}R_U^{12}]^{-1}\,T_U^{23},\\
T_U^{13} &= T_U^{12}\qquad\quad [I - R_D^{23}R_U^{12}]^{-1}T_U^{23},
\end{aligned} \tag{4.21}$$

which generalize the relations given by Kennett (1974) for a uniform layer.

A simple heuristic picture helps to explain these relations. We note that expanding the matrix inverse as a power series

$$[I - A]^{-1} = I + A + A^2 + \ldots,$$

so that, e.g.

$$R_D^{13} = R_D^{12} + T_U^{12}R_D^{23}T_D^{12} + T_U^{12}R_D^{23}R_U^{12}R_D^{23}T_D^{12} + \ldots \tag{4.22}$$

which is represented somewhat schematically in Fig. 2. The total response to some incident field V_D can be considered as the sum of contributions from each term in the series. The

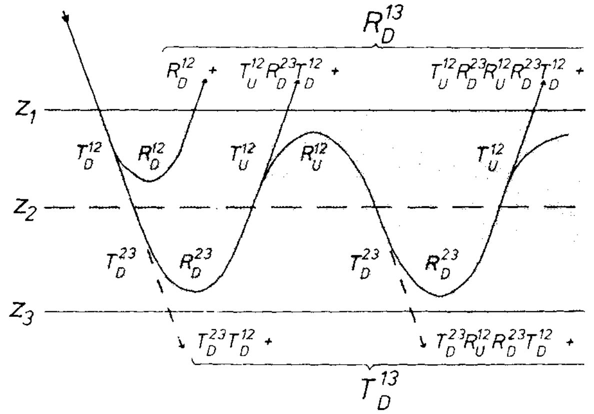

Figure 2. Graphic representation of the first few terms of the expansion (4.22) of the reflection and transmission matrices for superposed media; showing schematically the interactions undergone by the waves with the regions $z_1 < z < z_2$ and $z_2 < z < z_3$.

action of each of these terms can be seen by reading it from right to left. The first term is reflection from the upper region. The second arises from transmission down through the upper zone, reflection by the lower region and transmission back up through the upper zone. In the third term an additional interaction between the two parts of the inhomogeneous region is introduced. The total response includes all reverberations within the central region.

If the series (4.19) is truncated after a finite number of terms then this approximation to R_{D}^{13} only includes a finite number of internal reverberations. Such a device can be very convenient when one is trying to look at the effect of internal multiple reflections and has been used by Kennett (1975, 1978) and Stephen (1977).

4.4 COMPOSITION RELATIONS FOR FREE SURFACE REFLECTION COEFFICIENTS

As in Section 4.2 we consider an inhomogeneous region $0 < z < z_{\mathrm{B}}$ but now divided by the plane $z = z_{\mathrm{A}}(0 < z_{\mathrm{A}} \leqslant z_{\mathrm{B}})$. From the original definition (3.12)

$$\begin{aligned}\mathbf{F}(0, z_{\mathrm{B}}+) &= \mathbf{P}(0, z_{\mathrm{B}})\mathbf{D}(z_{\mathrm{B}}+)\\ &= \mathbf{P}(0, z_{\mathrm{A}})\mathbf{D}(z_{\mathrm{A}})\mathbf{D}^{-1}(z_{\mathrm{A}})\mathbf{P}(z_{\mathrm{A}}, z_{\mathrm{B}})\mathbf{D}(z_{\mathrm{B}}+)\\ &= \mathbf{F}(0, z_{\mathrm{A}})\mathbf{Q}(z_{\mathrm{A}}, z_{\mathrm{B}}+).\end{aligned} \tag{4.23}$$

If we substitute from (4.8) for $\mathbf{Q}(z_{\mathrm{A}}, z_{\mathrm{B}}+)$ in (4.23) and evaluate $R_{\mathrm{U}}^{\mathrm{FB}} = R_{\mathrm{U}}^{\mathrm{F}}(z_{\mathrm{B}})$ from (4.18) we find

$$R_{\mathrm{U}}^{\mathrm{FB}} = R_{\mathrm{U}}^{\mathrm{AB}} + T_{\mathrm{D}}^{\mathrm{AB}}R_{\mathrm{U}}^{\mathrm{FA}}[I - R_{\mathrm{D}}^{\mathrm{AB}}R_{\mathrm{U}}^{\mathrm{FA}}]^{-1}T_{\mathrm{U}}^{\mathrm{AB}}, \tag{4.24}$$

which has the same form as the iterative expression for R_{U} in (4.21). (A similar relation may be deduced for any other linear boundary conditions imposed at $z = 0$.)

4.5 ITERATIVE APPROACH FOR A STACK OF UNIFORM LAYERS

The iterative development (4.21) has been used by Kennett (1975, 1978) and Stephen (1977) for efficient numerical calculation of reflection coefficients for a stack of uniform layers.

If we consider a uniform layer in $z_1 < z < z_2$ overlying an inhomogeneous region in $z_2 < z < z_3$, then we may write

$$\left.\begin{aligned}R_{\mathrm{D}}^{13} &= \bar{R}_{\mathrm{D}}^{12} + \bar{T}_{\mathrm{U}}^{12}\bar{R}_{\mathrm{D}}^{23}[I - \bar{R}_{\mathrm{U}}^{12}\bar{R}_{\mathrm{D}}^{23}]^{-1}\bar{T}_{\mathrm{D}}^{12},\\ T_{\mathrm{D}}^{13} &= \bar{T}_{\mathrm{D}}^{23}\qquad\qquad [I - \bar{R}_{\mathrm{U}}^{12}\bar{R}_{\mathrm{D}}^{23}]^{-1}\bar{T}_{\mathrm{D}}^{12},\\ R_{\mathrm{U}}^{13} &= \bar{R}_{\mathrm{U}}^{23} + \bar{T}_{\mathrm{D}}^{23}\bar{R}_{\mathrm{U}}^{12}[I - \bar{R}_{\mathrm{D}}^{23}\bar{R}_{\mathrm{U}}^{12}]^{-1}\bar{T}_{\mathrm{U}}^{23},\\ T_{\mathrm{U}}^{13} &= \bar{T}_{\mathrm{U}}^{12}\qquad\qquad [I - \bar{R}_{\mathrm{D}}^{23}\bar{R}_{\mathrm{U}}^{12}]^{-1}\bar{T}_{\mathrm{U}}^{23}.\end{aligned}\right\} \tag{4.25}$$

where the coefficients $\bar{R}_{\mathrm{D}}^{12}$ etc. are those for the interface at z_1, and $\bar{R}_{\mathrm{D}}^{23}$ etc. are the coefficients for $z > z_2$ phased relative to the level $z = z_1$, so

$$\bar{R}_{\mathrm{D}}^{23} = ER_{\mathrm{D}}^{23}E, \quad \bar{R}_{\mathrm{U}}^{23} = ER_{\mathrm{U}}^{23}E, \quad \bar{T}_{\mathrm{D}}^{23} = T_{\mathrm{D}}^{23}E, \quad \bar{T}_{\mathrm{U}}^{23} = ET_{\mathrm{U}}^{23}, \tag{4.26}$$

where E is the phase income for downward propagation through the layer (4.13).

Thus by starting at the base of the layers the reflection and transmission coefficients may be calculated in a convenient iterative manner by adding a layer to the stack at each stage.

Further at fixed slowness p the frequency dependence at each layer step only appears through the phase income $\boldsymbol{E}$, since the interface reflection and transmission coefficients are then frequency-independent. Thus if the interface coefficients are stored, calculations may be rapidly performed for many frequencies.

We note that if waves are evanescent, only terms of the form $\exp[-\omega|q_\beta|(z_2-z_1)]$ appear so that there are no problems with growing exponential solutions.

This form of iterative solution may also be easily extended to the free surface reflection matrix, via (4.24), although here the iteration would start with the free surface reflection coefficients at the surface rather from the base of the layering.

4.6 PIECEWISE SMOOTH MODELS

The iterative development in the previous section may be extended to piecewise smooth models in an asymptotic development. Woodhouse (1978) has presented a convenient asymptotic form for fundamental matrices of (3.1), in terms of standing waves characterized by the Airy functions $Ai(x)$, $Bi(x)$. His results may be modified to a travelling wave representation by the replacements

$$\begin{aligned} Ai(x) &\to Ai(x) + iBi(x) \\ Bi(x) &\to Ai(x) - iBi(x) \end{aligned} \tag{4.27}$$

and these are related to the Hankel Functions $H_{1/3}(x)$, used by Richards (1976), but have a simpler behaviour in the complex plane. In a uniform layer the asymptotic fundamental matrix reduces to the previous expression (3.6), but in general can be written as

$$\boldsymbol{\Phi}_s(z) = \mathscr{D}(p,z)\mathscr{E}(\omega,p,z) \tag{4.28}$$

where the phase matrix $\mathscr{E}$ depends on frequency through the Airy functions and gradient terms.

The relations (4.21), (4.24) may be used to calculate reflection and transmission matrices sequentially for a stack of smooth layers if we use an extension of (4.8) and identify, e.g.

$$\begin{aligned} \left(\begin{array}{c|c} \boldsymbol{T}_{\mathrm{U}} - \boldsymbol{R}_{\mathrm{D}}\boldsymbol{T}_{\mathrm{D}}^{-1}\boldsymbol{R}_{\mathrm{U}} & \boldsymbol{R}_{\mathrm{D}}\boldsymbol{T}_{\mathrm{D}}^{-1} \\ \hline -\boldsymbol{T}_{\mathrm{D}}^{-1}\boldsymbol{R}_{\mathrm{U}} & \boldsymbol{T}_{\mathrm{D}}^{-1} \end{array}\right)_{12} &= \boldsymbol{\Phi}_{S1}^{-1}(z_1)\boldsymbol{\Phi}_{S2}(z_1) \\ &= \mathscr{E}_1^{-1}(\omega,p,z)\mathscr{D}_1^{-1}(p,z_1)\mathscr{D}_2(p,z_1)\mathscr{E}_2(\omega,p,z_2). \end{aligned} \tag{4.29}$$

We note that the interface term $\mathscr{D}_1^{-1}(p,z_1)\mathscr{D}_2(p,z_1)$ is independent of frequency so that the computational advantages of the uniform layer case are retained.

If turning points occur within a layer this approach allows a uniform asymptotic connection through the turning point, but the behaviour in this region and below is most effectively expressed in terms of $Ai(x)$, $Bi(x)$. It may therefore prove to be most convenient to adopt different fundamental matrix representations on the two sides of an interface and so generalize the concept of reflection and transmission matrices.

5 The half space response in terms of reflection matrices

5.1 THE RESPONSE VIA A SURFACE SOURCE VECTOR

In Section 3.2 we introduced the vector $\mathbf{S}$ arising from the propagation of the discontinuity in the stress-displacement vector due to the source ($\mathscr{S}$), up to the free surface

$$\mathbf{S} = \mathbf{P}(0,z_s)\mathscr{S} = [S_{\mathrm{W}}, S_{\mathrm{T}}]^{\mathrm{T}}. \tag{5.1}$$

In terms of **S** the free surface displacement took the form

$$w_0 = F_{12}F_{22}^{-1}S_T - S_W, \tag{5.2}$$

where F_{12}, F_{22} are the partitions of $\mathbf{F}(0, z_L+)$. Now

$$\begin{aligned} \mathbf{F}(0, z_L+) &= \mathbf{P}(0, z_L)\mathbf{D}(z_L+) \\ &= \mathbf{D}(0+)\mathbf{Q}(0+, z_L+), \end{aligned} \tag{5.3}$$

from the definition of the wave propagator (4.2). When we calculate the partitions of **F** in terms of those of **D**, **Q**, making use of the representation (2.17) for $\mathbf{D}(0+)$, and (4.8) for $\mathbf{Q}(0+, z_L+)$, we find

$$\begin{aligned} F_{12} &= (\mathrm{M_D} + \mathrm{M_U}R_D)T_D^{-1}, \\ F_{22} &= (\mathrm{N_D} + \mathrm{N_U}R_D)T_D^{-1}. \end{aligned} \tag{5.4}$$

Here the reflection and transmission matrices R_D, T_D refer to the entire half space below $z = 0$.

An alternative representation of the surface displacement (5.2) is thus

$$w_0 = (\mathrm{M_D} + \mathrm{M_U}R_D)(\mathrm{N_D} + \mathrm{N_U}R_D)^{-1}S_T - S_W. \tag{5.5a}$$

Also, from (4.16), the free surface reflection matrix takes the form $\tilde{R} = -\mathrm{N_D^{-1}N_U}$, so that we may equivalently write

$$w_0 = (\mathrm{M_D} + \mathrm{M_U}R_D)[I - \tilde{R}R_D]^{-1}\mathrm{N_D^{-1}}S_T - S_W, \tag{5.5b}$$

as in the treatment of Kennett (1974). The secular function $\det F_{22}$ also has the alternate form $\det(\mathrm{N_D} + \mathrm{N_U}R_D)/\det T_D$, independent of the depth of the source. As we have seen in the previous sections, R_D and T_D may be calculated without needing to introduce the growing exponentials present in the propagator representation of F_{12}, F_{22}. Thus (5.5) contains a rather convenient representation of the entire half space response.

The surface source vector **S** however will still include the possibility of growing terms via the propagator $\mathbf{P}(0, z_s)$. This does not cause much difficulty for very shallow sources and (5.5) has been used successfully by Kennett (1978) in the calculation of complete synthetic seismograms, including surface generated multiples, for small offsets between source and receiver.

For a buried source or a buried receiver at an arbitrary depth in the half space we can overcome the problems with the growing solutions by the method described in the next section, which also allows a convenient physical interpretation of the wavefield.

5.2 BURIED SOURCES AND RECEIVERS

Consider the wavefield at a level z in the medium,

$$\mathbf{V}(z) = [V_U(z), V_D(z)]^T. \tag{5.6}$$

If we take z to lie above the level of the source (i.e. $0 \leqslant z < z_s$) then the vanishing of stress at the free surface requires the up- and downgoing wave parts of the field to be connected by (4.12)

$$V_D(z) = R_U^F(z)V_U(z), \quad 0 \leqslant z < z_s. \tag{5.7}$$

In a similar way if z lies below the source ($z_s < z < z_L$) then our requirement (2.20) that there should be no upcoming wave component below z_L means that we have

$$\begin{pmatrix} 0 \\ V_D(z_L+) \end{pmatrix} = \mathbf{Q}(z_L+, z) \begin{pmatrix} V_U(z) \\ V_D(z) \end{pmatrix}, \tag{5.8}$$

and thus from (4.9) we find

$$[T_D^L(z)]^{-1}\,[V_U(z) - R_D^L V_D(z)] = 0,$$

where $R_D^L(z)$, $T_D^L(z)$ are the reflection and transmission matrices for the structure between z and z_L. The up- and downgoing fields at z are therefore related by

$$V_U(z) = R_D^L(z)\,V_D(z), \qquad z_s < z < z_L. \tag{5.9}$$

The wavefields just below the source and at the level z_L are related by the wave propagator

$$\mathbf{V}(z_s+) = \mathbf{Q}(z_s+, z_L+)\mathbf{V}(z_L+), \tag{5.10}$$

and using the wavefield representation of a source (2.23)

$$\mathbf{V}(z_s+) - \mathbf{V}(z_s-) = \mathbf{\Sigma},$$

we may construct the wavefield just above the source $\mathbf{V}(z_s-)$. In terms of the partitions of $\mathbf{Q}(z_s+, z_L+)$

$$\begin{pmatrix} V_U(z_s-) \\ \hline V_D(z_s-) \end{pmatrix} = \left(\begin{array}{c|c} Q_{11} & Q_{12} \\ \hline Q_{21} & Q_{22} \end{array}\right) \begin{pmatrix} 0 \\ \hline V_D(z_L+) \end{pmatrix} - \begin{pmatrix} \mathbf{\Sigma}_U \\ \hline \mathbf{\Sigma}_D \end{pmatrix} \tag{5.11}$$

and, writing $R_U^F(z_s) = R_U^{FS}$, (5.7) gives the additional relation

$$V_D(z_s-) = R_U^{FS}\,V_U(z_s-). \tag{5.12a}$$

We may solve (5.11) to find the upgoing field $\mathbf{V}_U(z_s-)$, using the relation (4.7) $R_D^L(z_s) = R_D^{SL} = Q_{12}Q_{22}^{-1}$, thus

$$V_U(z_s-) = [I - R_D^{SL}R_U^{FS}]^{-1}(R_D^{SL}\mathbf{\Sigma}_D - \mathbf{\Sigma}_U). \tag{5.12b}$$

An analogous argument, using the free surface boundary condition, yields the wavefield just below the source in the form

$$V_D(z_s+) = [I - R_U^{FS}R_D^{SL}]^{-1}(\mathbf{\Sigma}_D - R_U^{FS}\mathbf{\Sigma}_U)$$

$$V_U(z_s+) = R_D^{SL}\,V_D(z_s+). \tag{5.13}$$

With the aid of the wave propagator we may now construct the wavefield, and hence the stress-displacement field, at any receiver level.

(1) *For a receiver above the source*

$$\mathbf{V}(z_R) = \mathbf{Q}(z_R, z_s-)\mathbf{V}(z_s-). \tag{5.14}$$

We substitute for the partitions of $\mathbf{Q}$ from (4.8) and make use of the composition rule for free surface reflections (4.21), together with the identity

$$I + A\,[I - A]^{-1} = [I - A]^{-1} \tag{5.15}$$

to find the receiver wavefield

$$V_U(z_R) = [I - R_D^{RS} R_U^{FR}]^{-1} T_U^{RS} [I - R_D^{SL} R_U^{FS}]^{-1} (R_D^{SL} \Sigma_D - \Sigma_U),$$

$$V_D(z_R) = R_U^{FR} V_U(z_R), \qquad 0 < z_R < z_s. \tag{5.16}$$

The corresponding receiver displacement may be reconstituted using $\mathbf{B}(z_R) = \mathbf{D}(z_R)\mathbf{V}(z_R)$, and with the representation (2.17) for $\mathbf{D}(z_R)$

$$w(z_R) = (\mathrm{M}_U^R + \mathrm{M}_D^R R_U^{FR}) [I - R_D^{RS} R_U^{FR}]^{-1} T_U^{RS} [I - R_D^{SL} R_U^{FS}]^{-1} (R_D^{SL} \Sigma_D - \Sigma_U). \tag{5.17}$$

(2) *For a receiver below the source*

$$\mathbf{V}(z_R) = \mathbf{Q}(z_R, z_s+)\mathbf{V}(z_s+). \tag{5.18}$$

We now use (4.9) for $\mathbf{Q}$, and the composition relations (4.18) to derive the receiver wavefield

$$V_U(z_R) = R_D^{RL} V_D(z_R),$$

$$V_D(z_R) = [I - R_U^{RS} R_D^{SL}]^{-1} T_D^{RS} [I - R_U^{FS} R_D^{SL}]^{-1} (\Sigma_D - R_U^{FS} \Sigma_U). \tag{5.19}$$

As in the previous case we may find the displacement at the receiver

$$w(z_R) = (\mathrm{M}_D^R + \mathrm{M}_U^R R_D^{RL}) [I - R_U^{RS} R_D^{SL}]^{-1} T_D^{RS} [I - R_U^{FS} R_D^{SL}]^{-1} (\Sigma_D - R_U^{FS} \Sigma_U). \tag{5.20}$$

In all of the expressions (5.12–5.13), (5.16–5.17), (5.19–5.20) we have been able to express the wavefield, or displacement, entirely in terms of the reflection and transmission matrices for various subregions of the structure. Thus we have achieved our objective of developing a formulation which avoids growing exponential terms.

We may use the same heuristic approach as in Section 4.3 to give a physical interpretation to the expressions for the wave or displacement fields. In each case we have a term of the type $[I - R_D^{SL} R_U^{FS}]^{-1}$ – which is a reverberation operator for the whole half space, with waves reflected back by the layer sequence above the source bounded above by the free surface and then interacting with the layering beneath the source. The comparable term $[I - \tilde{R} R_D]^{-1}$ also appears in (5.5). The source terms are arranged to include the upgoing waves generated by the source allowing for the structure beneath it (5.12, 5.16, 5.17) or alternatively the downgoing waves including those arising from the structure above (5.13, 5.19, 5.20).

In the expressions for receiver response we have a direct transmission from source to receiver but then have to allow for possible reverberations between the source level and either the free surface (5.16, 5.17 with term $[I - R_D^{RS} R_U^{FR}]^{-1}$) or the base of the layering (5.19, 5.20 with the term $[I - R_U^{RS} R_D^{SL}]^{-1}$).

If we now consider a surface source with a receiver positioned just beneath it, we find that the displacement field

$$w(0+) = (\mathrm{M}_D + \mathrm{M}_U R_D) [I - \tilde{R} R_D]^{-1} (\Sigma_D - \tilde{R} \Sigma_U), \tag{5.21}$$

which is equivalent to (5.5b) since

$$w(0) = w(0+) - S_W.$$

The expression (5.21) is however more revealing about the nature of the propagator solution, which is based on introducing a source at the surface which is equivalent in its radiation to the original buried source.

For a buried source a more convenient representation of the surface displacement field is to use (5.17) and then

$$w(0+) = (\mathrm{M_U} + \mathrm{M_D}\tilde{\boldsymbol{R}})\,[\boldsymbol{I} - \boldsymbol{R}_\mathrm{D}^{RS}\tilde{\boldsymbol{R}}]^{-1}\boldsymbol{T}_\mathrm{U}^{RS}\,[\boldsymbol{I} - \boldsymbol{R}_\mathrm{D}^{SL}\boldsymbol{R}_\mathrm{U}^{FS}]^{-1}(\boldsymbol{R}_\mathrm{D}^{SL}\boldsymbol{\Sigma}_\mathrm{D} - \boldsymbol{\Sigma}_\mathrm{U}), \tag{5.22}$$

the operator $(\mathrm{M_U} + \mathrm{M_D}\tilde{\boldsymbol{R}})$ is simply that which converts an upgoing wave potential into a free surface displacement.

The representations (5.17), (5.20–5.22) form a convenient starting point for approximations to the full response, as discussed further in Section 6.

5.3 AN OVERLYING FLUID STRATUM

The reflection matrix approach to the calculation of the response of a stratified medium may be extended to the case where a stratified fluid layer overlies an elastic half space. The development of a pressure-displacement vector and its decomposition into up- and downgoing wave components parallels our discussion for the elastic case in Section 2 and is given in Appendix B.

The expressions (5.16–5.17) and (5.19–5.20) are applicable to the fluid solid case provided that care is taken in the definition of the reflection and transmission matrices for the absence of shear waves. Thus we will have to allow for the solid–fluid transition in calculating the free surface reflection terms $\boldsymbol{R}_\mathrm{U}^{FS}$, $\boldsymbol{R}_\mathrm{U}^{FR}$ and possibly also in $\boldsymbol{R}_\mathrm{D}^{RS}$, $\boldsymbol{T}_\mathrm{U}^{RS}$. The simplest consistent formalism is to maintain a 2×2 matrix system throughout the layering and in the fluid just to have a single non-zero entry, e.g.

$$\boldsymbol{R}_\mathrm{D} = \begin{pmatrix} r_{aa}^{\mathrm{D}} & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.23}$$

When a fluid–solid boundary is encountered, e.g. in the iterative approach of Section 4.5 then the interfacial reflection and transmission coefficients used must be those appropriate to such a boundary.

5.4 SURFACE WAVES

In the expressions which we have derived for the displacements in the stratified half space (5.2), (5.5) we have poles when the secular function ($\det \boldsymbol{F}_{22}$ or $\det(\mathrm{N_D} + \mathrm{N_U}\boldsymbol{R}_\mathrm{D})/\det \boldsymbol{T}_\mathrm{D}$) vanishes. The work of Lapwood (1949) and Sezawa (1935) shows that at a large distance from the source the response from these poles corresponds to the surface wave train.

The surface waves exist solely because of the presence of the free surface and correspond, at fixed frequency, to non-trivial solutions of the equations of motion satisfying the conditions of vanishing stress at the surface and decaying displacement at depth ($|w| \to 0$ as $z \to \infty$). This latter condition for our form of structure corresponds to (2.20) for the form of the stress-displacement vector at $z = z_\mathrm{L}$. The surface waves are not excited directly by the source since the polar residue contributions from (5.2) show no discontinuity across the source level (Harkrider 1964) but by the interaction of the entire wavefield with the surface.

The dispersion equation defining the location of the surface wave poles in frequency-slowness space is the vanishing of the secular function, i.e.

$$\Delta = \det \boldsymbol{F}_{22} = 0. \tag{5.24}$$

Now if we consider (5.5b) we may derive an alternative secular function (Kennett 1974)

$$\det\,[\boldsymbol{I} - \tilde{\boldsymbol{R}}\boldsymbol{R}_\mathrm{D}] = 0, \tag{5.25}$$

which we see is related to a condition for constructive interference of waves reflected from the surface back into the structure. In the case of *SH* waves (5.25) takes the particularly simple form

$$R_H = 1, \tag{5.26}$$

i.e. the Love-wave secular relation requires us to seek the combination of frequency and slowness for which a wavefield is reflected from the half space without change of amplitude or phase.

If we visualize a receiver placed at some level z_R within the half space we can obtain a more general representation of the secular function in terms of the reflection properties of the medium. From (4.23)

$$\mathbf{F}(0, z_L) = \mathbf{F}(0, z_R)\mathbf{Q}(z_R, z_L')$$

and so setting $\mathbf{F}(0, z_R) = \mathbf{F}'$ and substituting for the partitions of $\mathbf{Q}$ from (4.8)

$$\Delta = \det \{F'_{21} R_D^{RL} (T_D^{RL})^{-1} + F'_{22} (T_D^{RL})^{-1}\}.$$

From the definition of the free surface reflection matrix (4.18)

$$\Delta = \det (F'_{22}) \det [I - R_U^{FR} R_D^{RL}] / \det (T_D^{RL}),$$

so that, provided we choose z_R so that $\det F'_{22}$ and $\det T_D^{RL}$ are both non-zero, we may redefine the secular function as

$$\tilde{\Delta} = \det [I - R_U^{FR} R_D^{RL}] = 0, \tag{5.27}$$

which generalizes the relation (5.25). The restriction on z_R is to ensure that for a frequency-slowness pair, surface waves are not possible on the structure above z_R or channel waves on the structure below z_R. Equations (5.25) and (5.27) form the basis of an efficient scheme of calculating multimode surface wave dispersion described in greater detail by Kerry (1979).

5.5 CHANNEL WAVES

Channel waves are localized non-trivial solutions of the elastic equations in a stratified space in which the displacement $|w| \to 0$ as $z \to \pm\infty$. If we consider a stratified region $z_k < z < z_L$ bounded above and below by uniform half spaces then for a channel wave to exist

$$\mathbf{V}(z_k-) = [V_U(z_k-), 0]^T,$$
$$\mathbf{V}(z_L+) = [0, V_D(z_L+)]^T, \tag{5.28}$$

and so since $\mathbf{V}(z_k-) = \mathbf{Q}(z_k-, z_L+)\mathbf{V}(z_L+)$, we require the channel wave function

$$\Upsilon = \det Q_{22} = 0, \tag{5.29}$$

and then the reflection and transmission matrices across the zone do not exist. In a similar fashion to our treatment of surface waves we may visualize a receiver at a level z_R within the zone, and then factoring the wave propagator we find

$$\Upsilon = \det (I - R_U^{KR} R_D^{RL}) / \det (T_D^{KR}) \det (T_D^{RL}) = 0.$$

Thus provided the partial transmission terms exist, we may redefine the channel wave secular function as

$$\tilde{\Upsilon} = \det\,(\boldsymbol{I} - \boldsymbol{R}_{\mathrm{U}}^{KR}\boldsymbol{R}_{\mathrm{D}}^{RL}) = 0. \tag{5.30}$$

This represents a constructive interference condition for waves successively reflected above and below the level of the receiver and will only be attainable if there is an inversion in the velocity profile.

5.6 DECOMPOSITION OF THE SURFACE WAVE SECULAR FUNCTION

Dunkin (1965) has demonstrated that at high frequencies in a model composed of a stack of uniform layers the secular function tends to factor into a secular term for the near surface layering and Stoneley functions for the deeper interfaces. In our present treatment we see that the Stoneley functions arise from the denominators of the interface reflection and transmission coefficients ($\bar{\boldsymbol{R}}_{\mathrm{D}}^{12}$ etc. – (4.25)) which are compounded to produce the overall reflection matrix.

We may generalize Dunkin's result to a stratified region by considering the decomposition of the secular function when we take a half space divided into two parts A and B by the level z_{c} (Fig. 3). The overall reflection coefficient matrix $\boldsymbol{R}_{\mathrm{D}}$ may be represented in terms of the reflection and transmission properties of regions A and B as (4.21)

$$\boldsymbol{R}_{\mathrm{D}} = \boldsymbol{R}_{\mathrm{D}}^{\mathrm{A}} + \boldsymbol{T}_{\mathrm{U}}^{\mathrm{A}}\boldsymbol{R}_{\mathrm{D}}^{\mathrm{B}}(\boldsymbol{I} - \boldsymbol{R}_{\mathrm{U}}^{\mathrm{A}}\boldsymbol{R}_{\mathrm{D}}^{\mathrm{B}})^{-1}\boldsymbol{T}_{\mathrm{D}}^{\mathrm{A}}, \tag{5.31}$$

so that the dispersion relation is

$$\det\,\{\boldsymbol{I} - \tilde{\boldsymbol{R}}\boldsymbol{R}_{\mathrm{D}}^{\mathrm{A}} - \tilde{\boldsymbol{R}}\boldsymbol{T}_{\mathrm{U}}^{\mathrm{A}}\boldsymbol{R}_{\mathrm{D}}^{\mathrm{B}}(\boldsymbol{I} - \boldsymbol{R}_{\mathrm{U}}^{\mathrm{A}}\boldsymbol{R}_{\mathrm{D}}^{\mathrm{B}})^{-1}\boldsymbol{T}_{\mathrm{D}}^{\mathrm{A}}\} = 0, \tag{5.32a}$$

and we recognize the first two terms in the matrix as the surface wave secular matrix for the upper part of the layering. A rearrangement of (5.32a) yields

$$\det\,\{(\boldsymbol{I} - \tilde{\boldsymbol{R}}\boldsymbol{R}_{\mathrm{D}}^{\mathrm{A}})(\boldsymbol{T}_{\mathrm{D}}^{\mathrm{A}})^{-1} - \tilde{\boldsymbol{R}}\boldsymbol{T}_{\mathrm{U}}^{\mathrm{A}}\boldsymbol{R}_{\mathrm{D}}^{\mathrm{B}}\}\;\det\,\{(\boldsymbol{I} - \boldsymbol{R}_{\mathrm{U}}^{\mathrm{A}}\boldsymbol{R}_{\mathrm{D}}^{\mathrm{B}})^{-1}\boldsymbol{T}_{\mathrm{D}}^{\mathrm{A}}\} = 0. \tag{5.32b}$$

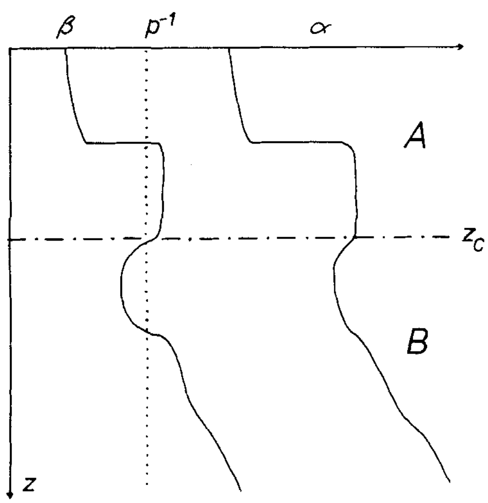

Figure 3. Division of velocity model by the plane $z = z_{\mathrm{C}}$; the slowness considered for the separation into channel and crustal modes is indicated by the dotted line.

If we have a velocity profile for P and S waves which just increases with depth then if we choose the dividing level z_c deep in the evanescent regime for both P and S waves $\boldsymbol{R}_{\mathrm{B}}^{\mathrm{D}}$ will be very small and so (5.32) approximates

$$\det(\boldsymbol{I} - \tilde{\boldsymbol{R}}\boldsymbol{R}_{\mathrm{D}}^{\mathrm{A}}) = 0, \tag{5.33}$$

i.e. the secular function for the truncated structure.

If, however, we have a velocity inversion the situation is rather more complex. For slowness p such that the phase velocity $(1/p)$ is rather greater than the S-wave velocity just outside the inversion, a choice of level z_c in the evanescent regime will give (5.33) again. When the slowness is such that there are propagating waves in the inversion but evanescent waves in a region outside (Fig. 3), then if we choose z_c to lie at the top of the inversion, the coupling between the near surface and the channel is reduced by the factors $\boldsymbol{T}_{\mathrm{D}}^{\mathrm{A}}$, $\boldsymbol{T}_{\mathrm{U}}^{\mathrm{A}}$. At very high frequencies $\boldsymbol{T}_{\mathrm{D}}^{\mathrm{A}}$, $\boldsymbol{T}_{\mathrm{U}}^{\mathrm{A}}$ will be very small and then the first term in (5.32b) will dominate and so approximately

$$\det\{(\boldsymbol{I} - \tilde{\boldsymbol{R}}\boldsymbol{R}_{\mathrm{D}}^{\mathrm{A}})(\boldsymbol{T}_{\mathrm{D}}^{\mathrm{A}})^{-1}(\boldsymbol{I} - \boldsymbol{R}_{\mathrm{U}}^{\mathrm{A}}\boldsymbol{R}_{\mathrm{D}}^{\mathrm{B}})\} = 0, \tag{5.34}$$

which we see factors into the secular equation for surface waves on the near surface structure and the channel wave operator. At intermediate frequencies there will be coupling between the channel and the surface through the transmission matrices $\boldsymbol{T}_{\mathrm{D}}^{\mathrm{A}}$, $\boldsymbol{T}_{\mathrm{U}}^{\mathrm{A}}$; and a given surface wave mode will in certain frequency ranges be mainly confined to the near surface region and in others be mostly a channel wave (Frantsuzova, Levshin & Shkadinskaya 1972; Panza, Schwab & Knopoff 1972).

6 Approximations to the full medium response

6.1 SURFACE REFLECTIONS FOR NEAR SURFACE SOURCE

In Section 5.1 we have shown that the surface displacement response of a stratified half space to a source takes the form (5.5b)

$$\boldsymbol{w}_0 = (\mathrm{M_D} + \mathrm{M_U}\boldsymbol{R}_{\mathrm{D}})(\boldsymbol{I} - \tilde{\boldsymbol{R}}\boldsymbol{R}_{\mathrm{D}})^{-1}\mathrm{N_D^{-1}}\boldsymbol{S}_{\mathrm{T}} - \boldsymbol{S}_{\mathrm{W}}, \tag{6.1}$$

and this expression includes all multiple reflections generated at the free surface, and the return of energy from depth due to the heterogeneity of the half space. We may display this more directly by rewriting (6.1), using the identity (5.15), as

$$\boldsymbol{w}_0 = (\mathrm{M_U} + \mathrm{M_D}\tilde{\boldsymbol{R}})\boldsymbol{R}_{\mathrm{D}}(\boldsymbol{I} - \tilde{\boldsymbol{R}}\boldsymbol{R}_{\mathrm{D}})^{-1}\mathrm{N_D^{-1}}\boldsymbol{S}_{\mathrm{T}} + \mathrm{M_D}\mathrm{N_D^{-1}}\boldsymbol{S}_{\mathrm{T}} - \boldsymbol{S}_{\mathrm{W}}. \tag{6.2}$$

The last two terms are those which would occur in Lamb's problem for a uniform half space with a surface source S. The first term now displays the reverberations between the free surface and the half space layering and we can examine a specified number of surface reflections by truncating the series expansion of the inverse $(\boldsymbol{I} - \tilde{\boldsymbol{R}}\boldsymbol{R}_{\mathrm{D}})^{-1}$. We note that $(\mathrm{M_U} + \mathrm{M_D}\tilde{\boldsymbol{R}})$ is just the displacement operator which appears in (5.22).

If we only consider that part of the response which has been reflected once by the half space and undergone no surface reflections

$$\boldsymbol{w}_0^{(0)} = (\mathrm{M_U} + \mathrm{M_D}\tilde{\boldsymbol{R}})\boldsymbol{R}_{\mathrm{D}}\mathrm{N_D^{-1}}\boldsymbol{S}_{\mathrm{T}}, \tag{6.3}$$

a form which has been used by Kennett (1979) in calculations of theoretical seismograms at small offsets from the source. We note that the neglect of the surface reflections changes the

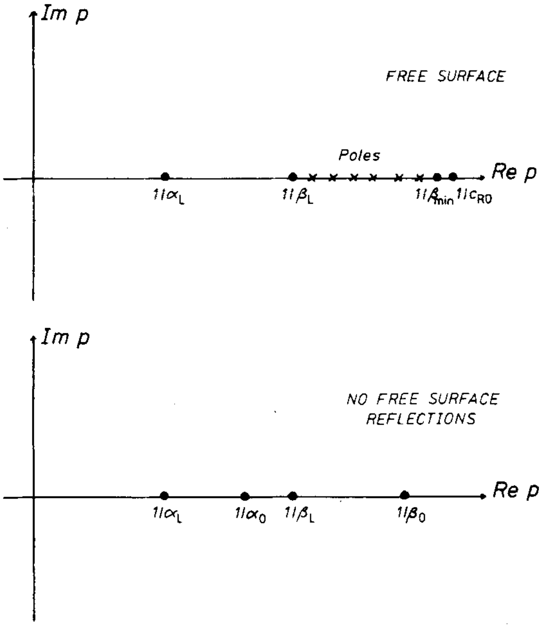

Figure 4. Singularities in the complex slowness plane for the full response and for the approximation neglecting free surface reflections. In the full response there are branch points corresponding to the P- and S-wave slownesses in the underlying half space. In addition there are the surface wave poles with limit points at the Rayleigh-wave slowness for a uniform half space with the surface properties $(\alpha_0, \beta_0, \rho_0)$ for the fundamental Rayleigh mode, and the largest S-wave slowness in the layering for all other modes. In the absence of free surface reflections the surface wave poles are eliminated and further branch points with the surface P- and S-wave slownesses are introduced.

character of the singularities in the slowness plane at fixed frequency, as shown in Fig. 4. In particular there is no longer any surface wave contribution.

For a single surface reflection, we have

$$w_0^{(1)} = (\mathrm{M_U} + \mathrm{M_D}\tilde{\boldsymbol{R}})\boldsymbol{R}_\mathrm{D}(\boldsymbol{I} + \tilde{\boldsymbol{R}}\boldsymbol{R}_\mathrm{D})\mathrm{N_D^{-1}}\boldsymbol{S}_\mathrm{T}, \tag{6.4}$$

and higher-order approximations may be easily developed, but with the surface representation of the source (6.1) and (6.3) are probably the most convenient forms of the response.

6.2 SURFACE REFLECTIONS FOR A BURIED SOURCE

For a buried source the surface displacement takes the form (5.22)

$$w_0 = (\mathrm{M_U} + \mathrm{M_D}\tilde{\boldsymbol{R}})\,(\boldsymbol{I} - \boldsymbol{R}_\mathrm{D}^{RS}\tilde{\boldsymbol{R}})^{-1}\boldsymbol{T}_\mathrm{U}^{RS}(\boldsymbol{I} - \boldsymbol{R}_\mathrm{D}^{SL}\boldsymbol{R}_\mathrm{U}^{FS})^{-1}(\boldsymbol{R}_\mathrm{D}^{SL}\boldsymbol{\Sigma}_\mathrm{D} - \boldsymbol{\Sigma}_\mathrm{U}), \tag{6.5}$$

using the wave vector representation of the source.

If we neglect reverberation in the neighbourhood of the receiver, we may expand the main free surface interaction operator $(\boldsymbol{I} - \boldsymbol{R}_\mathrm{D}^{SL}\boldsymbol{R}_\mathrm{U}^{FS})^{-1}$ to generate approximations with successive free surface reflections included. Of these the most useful is that in which no surface interaction occurs

$$w_0^{(0)} = (\mathrm{M_U} + \mathrm{M_D}\tilde{\boldsymbol{R}})\boldsymbol{T}_\mathrm{U}^{RS}(\boldsymbol{R}_\mathrm{D}^{SL}\boldsymbol{\Sigma}_\mathrm{D} - \boldsymbol{\Sigma}_\mathrm{U}), \tag{6.6}$$

this simulates the direction propagation from a deep earthquake to the surface and has been used for this purpose by Kennett & Simons (1976).

For a deeper source it is convenient to be able to include *pP, sP* phases etc. ('surface ghosts') in addition to the direct phases. We may do this by working carefully to a consistent order in R_D^{SL} so that the surface reflected phases have the same interaction with the deep structure as the original downgoing waves. Thus to this approximation

$$w_0^{(g)} = (\mathrm{M}_U + \mathrm{M}_D\tilde{R})T_U^{RS}[R_D^{SL}(\boldsymbol{\Sigma}_D - R_U^{FS}\boldsymbol{\Sigma}_U) - \boldsymbol{\Sigma}_U], \tag{6.7}$$

and at large ranges it would be appropriate to ignore the direct upward propagation.

The operator $(I - R_D^{RS}\tilde{R})^{-1}$ which we have so far neglected produces near surface reverberations superimposed on more direct propagation, it thus has the character of, e.g. a generator of *PL* coupled shear waves.

6.3 THE REFLECTIVITY METHOD

Since we have consistently considered the response of a stratified medium in terms of its reflection and transmission properties, it is interesting to examine the approximations made in the 'reflectivity method' of Fuchs & Müller (1971).

In their work no free surface reflections are included and the half space is divided into two parts, in the upper region (A) only transmission is allowed for, whilst in the lower part (B) all reflections are included. Thus one is approximating R_D by

$$R_D \simeq T_U^A R_D^B T_D^A, \tag{6.8}$$

rather than (5.31), i.e. all reverberations between the upper and lower parts of the half space are ignored. The final approximation for the displacement is thus

$$w_0^{(R)} = (\mathrm{M}_U + \mathrm{M}_D\tilde{R})T_U^A R_D^B T_D^A \mathrm{N}_D^{-1} S_T, \tag{6.9}$$

which may be compared to the full response (6.1). In Fuchs & Müller's (1971) treatment only an individual wave type was considered.

7 A comparison between the propagator and reflection matrix methods

We have shown in this paper that we are able to produce convenient expressions for the response of an elastic half space in terms of reflection matrices and that these expressions are amenable to physical interpretation. This approach may also be readily extended to consider anisotropic media; the formal structures remain the same but 2 × 2 matrices are replaced by 3 × 3 matrices of reflection and transmission coefficients.

The reflection approach differs strongly from the traditional 'propagator' method and we may illustrate these differences by considering a simple three-layered model.

In terms of the propagators we may relate the stress-displacement vectors at z_0 and z_3 by

$$\mathrm{B}(z_0) = \mathrm{P}(z_0, z_1)\mathrm{P}(z_1, z_2)\mathrm{P}(z_2, z_3)\mathrm{B}(z_3). \tag{7.1}$$

As we have seen in Section 2 even in a piecewise smooth half space we may represent the propagators in each layer in terms of fundamental matrices, i.e.

$$\mathrm{B}(z_0) = \boldsymbol{\Phi}_0(z_0)\boldsymbol{\Phi}_0^{-1}(z_1)\boldsymbol{\Phi}_1(z_1)\boldsymbol{\Phi}_1^{-1}(z_2)\boldsymbol{\Phi}_2(z_2)\boldsymbol{\Phi}_2^{-1}(z_3)\mathrm{B}(z_3). \tag{7.2}$$

Asymptotically, at least, we may choose the fundamental matrix columns to resemble up- and downgoing waves in a propagating region, and such a representation shares the advantage of the uniform layer in only having a frequency dependence, at fixed slowness, in the phase term. In the neighbourhood of turning points the Airy function approach of Chapman (1974), Woodhouse (1978) or the equivalent development of Richards (1976) enables a uniform asymptotic connection to be made to the evanescent regime.

In the propagator approach one takes the grouping

$$\mathbf{P}(z_2, z_3) = \mathbf{\Phi}_2(z_2)\mathbf{\Phi}_2^{-1}(z_3)$$

and so proceeds upward from layer to layer constructing, e.g. $\mathbf{B}(z_2)$ as an intermediate result.

In the reflection matrix method an interfacial grouping is employed. Thus we define

$$\mathbf{C}(z_3 -) = \mathbf{\Phi}_2^{-1}(z_3)\mathbf{B}(z_3) \tag{7.3}$$

and consider successively

$$\mathbf{C}(z_2 -) = \mathbf{\Phi}_1^{-1}(z_2)\mathbf{\Phi}_2(z_2)\mathbf{C}(z_3 -)$$

and (7.4)

$$\mathbf{C}(z_1 -) = \mathbf{\Phi}_0^{-1}(z_1)\mathbf{\Phi}_1(z_1)\mathbf{C}(z_2 -)$$

with finally

$$\mathbf{B}(z_0) = \mathbf{\Phi}_0(z_0)\mathbf{C}(z_1 -). \tag{7.5}$$

The interfacial matrices occurring in (7.4) include the phase terms and interfacial reflection coefficients (allowing for vertical inhomogeneity bordering the interface). In particular for uniform layers

$$\mathbf{\Phi}_0^{-1}(z_1)\mathbf{\Phi}_1(z_1) = \exp(-i\omega\mathbf{\Lambda}_0 z_1)D_0^{-1}D_1 \exp(i\omega\mathbf{\Lambda}_1 z_1). \tag{7.6}$$

When we know the character of the wavefield required at z_3 we may impose this behaviour by the choice of fundamental matrix $\mathbf{\Phi}_2$ and then carry this behaviour up to the level z_0. In the reflection approach we are thus able to select just those parts of the response in which we are interested.

In the propagator method, however, we construct an overall transfer operator by the form of the stress-displacement $\mathbf{B}(z_3)$. We thereby include initially features we do not want, e.g. growing exponentials, which are then cancelled out by the constraints.

Although the reflection matrix method has here been presented for a stratified half space, it is easily extended to a spherical geometry.

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Appendix A: general point source terms

We consider a source specified by the force system, referred to Cartesian axes,

$$f_i = -\partial_j(M_{ij}\delta(\mathbf{x})) + F_i\delta(\mathbf{x}), \qquad i, j = x, y, z, \tag{A1}$$

in terms of the source moment tensor, as in (2.22).

The source may be represented as a discontinuity **S** in the stress-displacement vector with components, for various angular orders

$$
\begin{aligned}
[U]_-^+ &= M_{zz}/\rho\alpha^2, \quad m = 0,\\
[V]_-^+ &= (\pm M_{xz} - iM_{yz})/\rho\beta^2, \quad m = \pm 1,\\
[W]_-^+ &= (\pm M_{yz} - iM_{xz})/\rho\beta^2, \quad m = \pm 1,
\end{aligned}
\tag{A2}
$$

and

$$
\begin{aligned}
[P]_-^+ &= -F_z, \quad m = 0,\\
&= \tfrac{1}{2}\omega p\,[i(M_{zy} - M_{yz}) \pm (M_{xz} - M_{zx})], \quad m = \pm 1,\\
[S]_-^+ &= \tfrac{1}{2}\omega p(M_{xx} + M_{yy}) - \omega p M_{zz}(1 - 2\beta^2/\alpha^2), \quad m = 0,\\
&= \tfrac{1}{2}(\mp F_x + iF_y), \quad m = \pm 1,\\
&= \tfrac{1}{4}\omega p\,[(M_{yy} - M_{xx}) \pm i(M_{xy} + M_{yx})], \quad m = \pm 2,\\
[T]_-^+ &= \tfrac{1}{2}\omega p(M_{xy} - M_{yx}), \quad m = 0,\\
&= \tfrac{1}{2}(iF_x \pm F_y), \quad m = \pm 1,\\
&= \tfrac{1}{4}\omega p\,[\pm i(M_{xx} - M_{yy}) + (M_{xy} + M_{yx})], \quad m = \pm 2.
\end{aligned}
\tag{A3}
$$

From these expressions we see the significance of the following combinations of the Moment tensor components

$$
\begin{aligned}
M_1 &= M_{xx} + M_{yy} - 2M_{zz}, \quad M_2 = M_{xy} + M_{yx}\\
N_1 &= M_{xy} - M_{yx}, \quad N_2 = M_{xz} - M_{zx}\\
N_3 &= M_{zy} - M_{yz}, \quad N_4 = M_{xx} - M_{yy}\\
P_\pm &= \pm M_{xz} - iM_{yz}, \quad Q_\pm = \pm M_{yz} - iM_{xz}.
\end{aligned}
\tag{A4}
$$

Alternatively we look at the source in terms of the wave vector via the jump vector $\mathbf{\Sigma} = \mathbf{D}^{-1}(z_s)\mathbf{S}$. The components of this jump vector are then:

for $m = 0$,

$$
\begin{aligned}
\phi_S^U &= \epsilon_\alpha\,\{-q_\alpha\omega^{-1}F_z + i\,[\tfrac{1}{2}p^2M_1 + M_{zz}/\alpha^2]\},\\
\phi_S^D &= \epsilon_\alpha\,\{-q_\alpha\omega^{-1}F_z - i\,[\tfrac{1}{2}p^2M_1 + M_{zz}/\alpha^2]\},\\
\psi_S^U &= \epsilon_\beta\,\{+ip\omega^{-1}F_z - \tfrac{1}{2}pq_\beta M_1\},\\
\psi_S^D &= \epsilon_\beta\,\{-ip\omega^{-1}F_z - \tfrac{1}{2}pq_\beta M_1\},\\
\chi_S^U &= \epsilon_\beta\beta^{-1}\,\{+\tfrac{1}{2}ip\omega^{-1}N_1\},\\
\chi_S^D &= \epsilon_\beta\beta^{-1}\,\{-\tfrac{1}{2}ip\omega^{-1}N_1\},
\end{aligned}
\tag{A5}
$$

for $m = \pm 1$,

$$
\begin{aligned}
\phi_S^{U} &= \epsilon_\alpha \{+ \tfrac{1}{2} ip\omega^{-1}(\mp F_x - iF_y) + pq_\alpha [P_\pm - \tfrac{1}{2} iN_3 \mp N_2]\}, \\
\phi_S^{D} &= \epsilon_\alpha \{- \tfrac{1}{2} ip\omega^{-1}(\mp F_x - iF_y) + pq_\alpha [P_\pm - \tfrac{1}{2} iN_3 \mp N_2]\}, \\
\psi_S^{U} &= \epsilon_\beta \{- \tfrac{1}{2} q_\beta\omega^{-1}(\mp F_x - iF_y) + \tfrac{1}{2} i(\beta^{-2} - 2p^2)P_\pm + \tfrac{1}{2} ip^2 [iN_1 \pm N_2]\}, \\
\psi_S^{D} &= \epsilon_\beta \{- \tfrac{1}{2} q_\beta\omega^{-1}(\mp F_x - iF_y) - \tfrac{1}{2} i(\beta^{-2} - 2p^2)P_\pm + \tfrac{1}{2} ip^2 [iN_1 \pm N_2]\}, \\
\chi_S^{U} &= \epsilon_\beta\beta^{-1} \{+ \tfrac{1}{2}\omega^{-1}(-F_x \pm iF_y) + \tfrac{1}{2} q_\beta Q_\pm\}, \\
\chi_S^{D} &= \epsilon_\beta\beta^{-1} \{- \tfrac{1}{2}\omega^{-1}(-F_x \pm iF_y) + \tfrac{1}{2} q_\beta Q_\pm\};
\end{aligned}
\tag{A6}
$$

for $m = \pm 2$,

$$
\begin{aligned}
\phi_S^{U} &= \epsilon_\alpha \{+ \tfrac{1}{4} ip^2(N_4 \pm iM_2)\}, \\
\phi_S^{D} &= \epsilon_\alpha \{- \tfrac{1}{4} ip^2(N_4 \pm iM_2)\}, \\
\psi_S^{U} &= \epsilon_\beta \{- \tfrac{1}{4} pq_\beta(N_4 \pm iM_2)\}, \\
\psi_S^{D} &= \epsilon_\beta \{- \tfrac{1}{4} pq_\beta(N_4 \pm iM_2)\}, \\
\chi_S^{U} &= \epsilon_\beta\beta^{-1} \{+ \tfrac{1}{4} p(\pm N_4 + iM_2)\}, \\
\chi_S^{D} &= \epsilon_\beta\beta^{-1} \{- \tfrac{1}{4} p(\pm N_4 + iM_2)\};
\end{aligned}
\tag{A7}
$$

where

$$\epsilon_\alpha = (2\rho q_\alpha)^{-1/2}, \qquad \epsilon_\beta = (2\rho q_\alpha)^{-1/2}.$$

The interface reflection and transmission coefficients corresponding to these source vectors may be derived from (4.8)

$$
\begin{pmatrix} T_U - R_D T_D^{-1} R_U & R_D T_D^{-1} \\ -T_D^{-1} R_U & T_D^{-1} \end{pmatrix}_{12} = \mathbf{D}_1^{-1}(z_1-)\mathbf{D}_2(z_1+). \tag{A8}
$$

Appendix B: fluid media

The differential equations for the pressure-displacement vector in a fluid are

$$
\frac{\partial}{\partial z}\begin{pmatrix} U \\ \omega^{-1}P \end{pmatrix} = \omega \begin{pmatrix} 0 & \rho^{-1}(\alpha^{-2} - p^2) \\ -\rho & 0 \end{pmatrix} \begin{pmatrix} U \\ \omega^{-1}p \end{pmatrix}, \tag{B1}
$$

and the eigenvalue matrix is

$$\mathbf{\Lambda}_f = \mathrm{diag}\,[-q_\alpha, q_\alpha]. \tag{B2}$$

The eigenvector matrix

$$\mathbf{D}_f = [\mathbf{b}_f^{U}, \mathbf{b}_f^{D}], \tag{B3}$$

with

$$\mathbf{b}_f^{U,D} = \epsilon_\alpha [\mp iq_\alpha, \rho]^{T}. \tag{B4}$$