Seismic waves in stratified anisotropic media

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Summary. The response of a structure composed of anisotropic strata can be built up from the reflection and transmission properties of individual interfaces using a slightly modified version of the recursion scheme of Kennett. This scheme is conveniently described in terms of scatterer operators and scatterer products. The effects of a free surface and the introduction of a simple point source at any depth can be accommodated in a manner directly analogous to the treatment for isotropic structures. As in the isotropic case the results so obtained are stable to arbitrary wavenumbers.

For isotropic media, synthetic seismograms can be constructed by computing the structure response as a function of frequency and radial wavenumber, then performing the appropriate Fourier and Hankel transforms to obtain the wavefield in time–distance space. Such a scheme is convenient for any system with cylindrical symmetry (including transverse isotropy). Azimuthally anisotropic structures, however, do not display cylindrical symmetry; for these the transverse component of the wavenumber vector will, in general, be non-zero, with the result that phase, group, and energy velocities may all diverge. The problem is then much more conveniently addressed in Cartesian coordinates, with the frequency–wavenumber to time–distance transformation accomplished by 3-D Fourier transform.

1 Introduction

Seismic velocity anisotropy is a widespread phenomenon in Earth materials. A large variety of mechanisms may give rise to anisotropy: crystal alignments, grain alignments, preferential alignments of cracks (including pore closure under pressure), stress-induced effects, the interleaving of thin sedimentary beds. Until recently, seismic data could be adequately explained by assuming isotropy, so anisotropy could be largely ignored. With the increasing resolution of seismic observations, however, there is a growing awareness that the assumption of isotropy is often violated. Anisotropy has been widely detected in the crust and upper mantle (e.g. Stephen 1981; Fuchs 1977; Anderson & Dziewonski 1982) and laboratory measurements imply that the phenomenon must be widespread in both crystalline and sedimentary rocks (Babuška 1981; Christiansen & Salisbury 1979; Bachman 1979).

There are fundamental differences between wave propagation in isotropic and anisotropic media (Crampin 1977, 1981). In an isotropic medium, *P*-wave particle motion is normal to a wavefront so the *P* polarization vector is coincident with the phase propagation vector. *S* motion may be in any direction orthogonal to *P*. In an anisotropic medium, the *P* polarization vector need not be coincident with the phase propagation vector, hence this phase is denoted q*P* for 'quasi-*P*'. Two quasi-shear polarizations form a mutually orthogonal set with q*P*. Thus for any particular direction of phase propagation, there are three body waves with fixed orthogonal polarizations (Auld 1973, pp. 219–220). In general, the velocities and polarizations vary with direction of phase propagation, causing the transverse component of the wavenumber vector to be non-zero. As a consequence of this behaviour, in an anisotropic medium phase and energy-velocity vectors may diverge (Auld 1973, pp. 223–227) so that a ray may depart from the sagittal plane (the vertical plane through the direction of phase propagation). Further, if the medium is anelastic, energy and group velocity vectors will also diverge (Auld 1973, pp. 227–230).

These departures from the well-understood behaviour of seismic waves in isotropic media mean that effects of anisotropy will be too difficult to comprehend without numerical modelling and the construction of synthetic seismograms. There has been much recent work on wave propagation through anisotropic media, although much of this has been addressed specifically to transverse isotropy (e.g. Sprenke & Kanasewich 1977; Daley & Hron 1977; Levin 1978; Oakley & Vidmar 1983). This form of anisotropy, which displays no azimuthal variation of elastic properties, is too restrictive for many lithospheric materials, especially crystalline rocks (e.g. Christiansen & Salisbury 1979), which typically display a more complicated anisotropy as a consequence of tectonic stresses (either through cracking or crystal orientation). Obviously a means of constructing synthetic seismograms for more general forms of anisotropy is required; so far no method for constructing complete synthetics for arbitrarily anisotropic media has been presented.

The mathematics of wave propagation through stratified media exhibiting azimuthal anisotropy was developed by Crampin (1970) for surface waves and extended to include body waves by Keith & Crampin (1977a, b). The first synthetics were the simple plane-wave seismograms of Keith & Crampin (1977c). Booth & Crampin (1983a, b) have constructed synthetics for point sources, but their procedure considers wave propagation only in the sagittal plane; they assume that the transverse component of the wavenumber vector is approximately zero. This approach yields exact solutions if the sagittal plane is a plane of symmetry but for some arbitrary azimuth results will be approximately correct only if the anisotropy is weak. What 'weak' anisotropy is is not clear; it is probable that some of the waveform subtleties of anisotropy will not be preserved under the assumption of a zero transverse component of the wavenumber vector.

In this paper we describe a scheme which may be used to construct complete synthetics, even for strongly anisotropic media. Like Booth & Crampin (1983a) we use a reflectivity approach, but we include the second wavenumber integration necessary if divergence of phase and group velocity vectors is to be correctly modelled. Unlike Booth & Crampin we choose to use a Cartesian coordinate system. Cylindrical coordinates are the obvious choice for modelling elastic systems with symmetry about a vertical axis (such as isotropy or transverse isotropy), or if propagation in only a single vertical plane is to be considered. If, however, there is any variation of propagation velocity with azimuth, Cartesian coordinates yield much simpler mathematics and are more convenient to use.

We shall follow closely the techniques of Kennett (1974) and Kennett & Kerry (1979) for building up the response of isotropic stratified media from the reflection and transmission properties of individual interfaces. The book by Kennett (1983) must be regarded

as the major reference for this work so we shall, as far as possible, follow Kennett's (1983) notation. For convenience, we refer to Kennett's procedure as the reflection matrix method.

2 The differential system

Consider wave propagation in a medium which is laterally homogeneous but varies as a function of depth. The medium may be anisotropic and anelastic. As in the isotropic case, we begin by multiple Fourier transformation of the equations of motion to accomplish a separation of variables. We define the three-dimensional (3-D) Fourier transform

$$g(p_x, p_y, \omega) = \int\!\!\!\int\!\!\!\int_{-\infty}^{\infty} g(x, y, t) \exp\left[i\omega(t - p_x x - p_y y)\right] dx\, dy\, dt$$

where p_x, p_y are horizontal phase slownesses. By Fourier transformation of the momentum and constitutive equations it is straightforward to show that

$$\partial_z \mathbf{b}(z) = i\omega \mathbf{A}(z)\, \mathbf{b}(z) \tag{2.1}$$

(e.g. Woodhouse 1974), where $\mathbf{b}$ is the vector of those variables continuous across any horizontal plane,

$$\mathbf{b} = \begin{pmatrix} \mathbf{u} \\ \boldsymbol{\tau} \end{pmatrix},$$

$$\mathbf{u} = (u_x, u_y, u_z)^{\mathrm{T}}, \tag{2.2}$$

$$\boldsymbol{\tau} = \frac{-1}{i\omega}(\tau_{xz}, \tau_{yz}, \tau_{zz})^{\mathrm{T}}.$$

Here u_x, u_y, u_z are components of displacement and τ_{xz}, τ_{yz}. τ_{zz} are the z-components of the stress tensor. With this choice of the stress-displacement vector $\mathbf{b}$ the resulting 6×6 system matrix $\mathbf{A}$ in (2.1) is a function only of horizontal slownesses p_x and p_y, density ρ, and elastic parameters c_{ij} (elements of the elasticity tensor). If the elastic parameters are independent of frequency ω, then $\mathbf{A}$ too is independent of frequency. For lossless media, $\mathbf{A}$ is completely real. $\mathbf{A}$ has two other convenient properties: its eigenvalues are simply the permissible vertical slownesses for a given p_x, p_y pair, and, as we shall see, there is a very simple relationship between the associated eigencolumn and eigenrow vectors. The system matrix has the structure

$$\mathbf{A} = \begin{pmatrix} \mathbf{T} & \mathbf{C} \\ \mathbf{S} & \mathbf{T}^{\mathrm{T}} \end{pmatrix} \tag{2.3}$$

(Chadwick & Smith 1977; Chapman & Woodhouse 1981) where $\mathbf{T}$, $\mathbf{S}$ and $\mathbf{C}$ are 3×3 partitions and $\mathbf{C}$ and $\mathbf{S}$ are symmetric. These matrices may be obtained from expressions given by Woodhouse (1974) if due allowance is made for our choice of stress-displacement vector (2.2).

2.1 THE PROPAGATOR

The differential system (2.1) can be solved using classical propagator matrix techniques (Gilbert & Backus 1966; Woodhouse 1974). The propagator of the system (2.1) is the

unique continuous solution $\mathbf{P}(z, z_1)$ of the system

$$\partial_z \mathbf{P}(z, z_1) = i\omega \mathbf{A}(z)\, \mathbf{P}(z, z_1),$$

$$\mathbf{P}(z_1, z_1) = \mathbf{I} \tag{2.4}$$

where $\mathbf{I}$ is the 6×6 identity. If the specific boundary condition applied at z_1, is $\mathbf{b}(z_1)$ then the response vector at z is

$$\mathbf{b}(z) = \mathbf{P}(z, z_1)\, \mathbf{b}(z_1), \tag{2.5}$$

hence the term 'propagator'. The exponential-like properties of propagators have been conveniently summarized by Woodhouse (1974).

The initial value problem (2.4) can be solved numerically; indeed, our first computations of the response of an anisotropic structure were based on such a numerical integration (Fryer & Frazer 1982). However, such an approach suffers from two related problems. First, numerical solution is extremely time-consuming. This is an especially severe problem since (as will become apparent) solutions have to be formed for a volume of (p_x, p_y, ω) space. Second, as slowness is increased beyond evanescence the error tolerances for elements in the propagator have to be progressively reduced if sensible reflectivities are to be computed. Since (2.4) gets progressively more stiff as slowness increases (and therefore more time-consuming to integrate because of a rapidly decreasing step size), this increased accuracy requirement is particularly troublesome. To avoid the expense of direct numerical integration, we follow the same procedure as in the isotropic case (described, for example, by Kennett, Kerry & Woodhouse 1978) and attempt to solve the problem using an eigenvector approach.

3 A uniform anisotropic medium

3.1 EIGENSOLUTIONS

If $\mathbf{D}$ is the local eigenvector matrix of $\mathbf{A}$ then

$$\mathbf{D}^{-1} \mathbf{A} \mathbf{D} = \boldsymbol{\Lambda} \tag{3.1}$$

where $\boldsymbol{\Lambda}$ is diagonal. The diagonal elements of $\boldsymbol{\Lambda}$ are the eigenvalues of $\mathbf{A}$ which are the vertical phase slownesses $q = p_z$. In general we may write

$$\boldsymbol{\Lambda} = \mathrm{diag}\,(q_P^U, q_{S1}^U, q_{S2}^U, q_P^D, q_{S1}^D, q_{S2}^D) \tag{3.2}$$

where superscripts U and D denote upgoing and downgoing disturbances, the subscript P denotes quasi-P and $S1$, $S2$ denote the two types of quasi-S. For an isotropic medium $q^U = -q^D$, as shown in Fig. 1, but for general anisotropy there is no such simple relationship between the vertical slownesses (Keith & Crampin 1977a). However, for our choice of Fourier transform and the definition of $\mathbf{A}$ in (2.1), it follows from the radiation condition that

$$\mathrm{Im}\,(q^D) > 0 \qquad \text{and} \qquad \mathrm{Im}\,(q^U) < 0. \tag{3.3}$$

Given the eigenvector matrix $\mathbf{D}$, we may define a wavevector $\mathbf{v}$ from the transformation

$$\mathbf{b} = \mathbf{D}\mathbf{v}. \tag{3.4}$$

As in the isotropic case the elements of $\mathbf{v}$ may be identified with the amplitudes of upward and downward travelling plane waves,

$$\mathbf{v} = [\mathbf{v}_U, \mathbf{v}_D]^T = [\phi_U, \psi_U, \chi_U, \phi_D, \psi_D, \chi_D]^T \tag{3.5}$$

where ϕ denotes qP amplitude and ψ, χ the two qS amplitudes. As before U and D denote up and down.

If the elastic parameters are locally constant then D is independent of z and substitution of (3.4) and (3.1) into (2.1) yields

$$\partial_z \mathbf{v} = i\omega \mathbf{\Lambda} \mathbf{v} \tag{3.6}$$

with the solution

$$\mathbf{v}(z) = \exp[i\omega\mathbf{\Lambda}(z - z_1)] \mathbf{v}(z_1)$$

$$= \mathbf{Q}(z, z_1) \mathbf{v}(z_1), \tag{3.7}$$

where z_1 is some reference depth. From (3.6) and (3.7) it is apparent that $\mathbf{Q}$ may be regarded as a 'wave propagator' since it is the solution to

$$\partial_z \mathbf{Q}(z, z_1) = i\omega\mathbf{\Lambda}\mathbf{Q}(z, z_1),$$

$$\mathbf{Q}(z_1, z_1) = \mathbf{I}.$$

We note from (3.2) that within the uniform layer, $\mathbf{Q}$ has the structure

$$\mathbf{Q}(z, z_1) = \begin{pmatrix} \mathbf{E}_U & \mathbf{0} \\ \mathbf{0} & \mathbf{E}_D \end{pmatrix} \tag{3.8}$$

with

$$\mathbf{E}_U = \text{diag}\{\exp[i\omega(z - z_1)q_P^U], \exp[i\omega(z - z_1)q_{S1}^U], \exp[i\omega(z - z_1)q_{S2}^U]\}, \tag{3.9}$$

and a similar expression for $\mathbf{E}_D$. Using (3.4) and (3.7) the stress-displacement vector at any level z within the uniform medium is

$$\mathbf{b}(z) = \mathbf{D}\mathbf{Q}(z, z_1) \mathbf{D}^{-1} \mathbf{b}(z_1).$$

By comparison with (2.5) the desired propagator for the uniform interval is

$$\mathbf{P}(z, z_1) = \mathbf{D}\mathbf{Q}(z, z_1) \mathbf{D}^{-1}. \tag{3.10}$$

To find this propagator, it is necessary to find the eigenvalues (vertical slownesses), the eigenvector matrix $\mathbf{D}$, and its inverse $\mathbf{D}^{-1}$. In the isotropic case these are known analytically, so construction of the propagator is straightforward. In the anisotropic case, analytic solutions have been found only for simple symmetries (Fryer & Frazer 1984, in preparation) so in general, solutions will be found numerically. Fortunately, $\mathbf{D}$ and its inverse are very simply related, as we shall show.

3.2 THE INVERSE EIGENVECTOR MATRIX

$\mathbf{D}$ and $\mathbf{\Lambda}$ (and hence $\mathbf{Q}$) can be found using standard numerical techniques for eigensolutions. It remains to find $\mathbf{D}^{-1}$. $\mathbf{D}$ and $\mathbf{D}^{-1}$, being eigencolumn and eigenrow matrices, must be simply related. If $\mathbf{b}_i$ is a column eigenvector (a column of $\mathbf{D}$) and $\mathbf{g}_i^T$ a row eigenvector (a row of $\mathbf{D}^{-1}$) then there exists a transformation $\mathbf{g}_i = \mathbf{J}\mathbf{b}_i$ (Pease 1965, pp. 87–89), so that $(\mathbf{D}^{-1})^T = \mathbf{J}\mathbf{D}$. This means that it is unnecessary to resort to numerical inversion to find $\mathbf{D}^{-1}$. The appropriate transformation matrix $\mathbf{J}$ is given by Chadwick & Smith (1977) and Garmany (1983) but these authors have limited their treatment to lossless media. We present here a simple derivation of $\mathbf{D}^{-1}$ valid for general anelastic anisotropic media.

Each column of $\mathbf{D}$ is an eigenvector of the form

$$\mathbf{b}_i = \epsilon_i \begin{pmatrix} \mathbf{u}_i \\ \mathbf{\tau}_i \end{pmatrix} \tag{3.11}$$
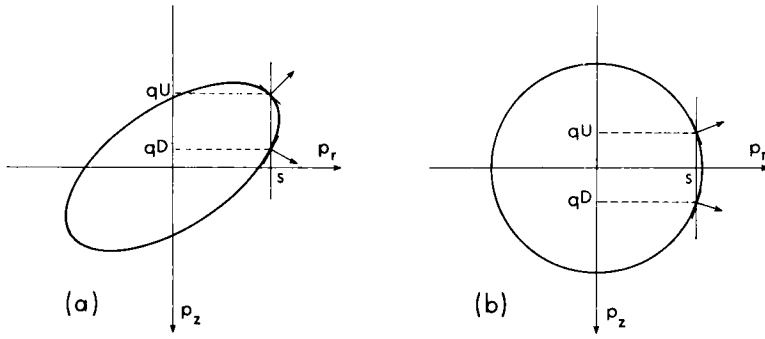
Figure 1. Projections on to the sagittal plane of possible slowness surfaces for (a) general anisotropy and (b) isotropy (p_r is radial slowness, so if the azimuth of the section is ϕ, $p_x = p_r \cos\phi$ and $p_y = p_r \sin\phi$). For any horizontal slowness s there will be two possible vertical slownesses q^U and q^D corresponding to upward and downward propagation. For general anisotropy, (a), there is no simple relationship between q^U and q^D. In the case of isotropy, (b), symmetry demands that $q^U = -q^D$. Energy velocities (equivalent to group velocities in a lossless medium) have a direction normal to the slowness surface (Auld 1973, pp. 223–227) and are indicated by arrows. For anisotropic materials, the energy velocity vector will not necessarily lie in the plane of the figures (modified after Garmany 1983).

where $\mathbf{u}$, τ are the three-element vectors defined in (2.2) and ϵ_i is a normalization constant to be determined. From (3.1)

$$\mathbf{AD} = \mathbf{D\Lambda}$$

and

$$\mathbf{D}^{-1} \mathbf{A} = \mathbf{\Lambda D}^{-1}.$$

The first of these equations can be written

$$\mathbf{A}\mathbf{b}_i = q_i \mathbf{b}_i \qquad (3.12)$$

and the second

$$\mathbf{g}_j^{\mathrm{T}} \mathbf{A} = q_j \mathbf{g}_j^{\mathrm{T}}, \qquad (3.13)$$

where $\mathbf{g}_j^{\mathrm{T}}$ is the jth row of $\mathbf{D}^{-1}$ and is a reciprocal or left-hand eigenvector of $\mathbf{A}$. Since $\mathbf{D}^{-1} \mathbf{D} = \mathbf{I}$, it follows immediately that

$$\mathbf{g}_j^{\mathrm{T}} \mathbf{b}_i = \delta_{ij}. \qquad (3.14)$$

We wish to find the transformation from $\mathbf{b}$ to $\mathbf{g}$ such that

$$\mathbf{g}_i = \mathbf{J}\mathbf{b}_i. \qquad (3.15)$$

Consider the similarity transformation of (3.12),

$$\mathbf{JAJ}^{-1} \mathbf{Jb}_i = q_i \mathbf{Jb}_i,$$

or, using (3.15)

$$\mathbf{A}' \mathbf{g}_i = q_i \mathbf{g}_i, \qquad (3.16)$$

where $\mathbf{A}' = \mathbf{JAJ}^{-1}$. Note that the eigenvalue is invariant under a similarity transformation. Taking the transpose of (3.16) we obtain

$$\mathbf{g}_i^{\mathrm{T}}(\mathbf{A}')^{\mathrm{T}} = \mathbf{g}_i^{\mathrm{T}} q_i.$$

By comparison with (3.13),

$$\mathbf{A}' = \mathbf{JAJ}^{-1} = \mathbf{A}^{\mathrm{T}}$$

or, since $\mathbf{J}$ is non-singular,

$$\mathbf{JA} - \mathbf{A}^{\mathrm{T}}\mathbf{J} = \mathbf{0}. \tag{3.17}$$

To find a $\mathbf{J}$ which has this property we consider the symmetry properties of $\mathbf{A}$. $\mathbf{A}$ has the structure given in (2.3), from which

$$\mathbf{A}^{\mathrm{T}} = \begin{pmatrix} \mathbf{T}^{\mathrm{T}} & \mathbf{S} \\ \mathbf{C} & \mathbf{T} \end{pmatrix}.$$

Using these expressions for $\mathbf{A}$ and $\mathbf{A}^{\mathrm{T}}$ in (3.17) the transformation matrix $\mathbf{J}$ is found by inspection to be

$$\mathbf{J} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix} \tag{3.18}$$

where $\mathbf{I}$ is the 3×3 identity. Hence from (3.11) and (3.15),

$$\mathbf{g}_i = \epsilon_i \begin{pmatrix} \tau_i \\ \mathbf{u}_i \end{pmatrix}.$$

From the orthonormality requirement (3.14),

$$\epsilon_i = (\tau_i^{\mathrm{T}} \mathbf{u}_i + \mathbf{u}_i^{\mathrm{T}} \tau_i)^{-1/2}. \tag{3.19}$$

From (3.15),

$$\mathbf{D}^{-1} = (\mathbf{JD})^{\mathrm{T}} = \mathbf{D}^{\mathrm{T}}\mathbf{J}. \tag{3.20}$$

This suggests a simple algorithm for computing the eigenrow matrix $\mathbf{D}^{-1}$:

(a) Normalize each of the six columns of the raw eigenvector matrix according to (3.19) to form $\mathbf{D}$.
(b) Interchange the stress and displacement 3-vectors (τ_i and $\mathbf{u}_i$) in each column.
(c) Transpose the result.

Garmany (1983) obtained the same results as (3.18)–(3.20) above. However, in trying to relate the normalization to the vertical energy flux, he was forced to limit his treatment to lossless media; the treatment above is not so limited and retains its validity in the anelastic case.

An alternative determination of $\mathbf{D}^{-1}$ may be made by considering the properties of $\mathbf{D}$ implicit in the transformation relationship (3.4). Kennett (1983) has followed this approach to arrive at an expression for the inverse which is independent of the normalization. Our results here are a special case of Kennett's more general result (Kennett 1983, equation 2.63) and correspond to the form given by Kennett (1983, equation 3.40) for isotropic media.

4 A stratified anisotropic structure

We consider a vertically stratified anisotropic medium in $0 < z < z_L$ with a free surface at $z = 0$ and an anisotropic half-space in $z > z_L$, as shown in Fig. 2. The variation between 0 and z_L can be approximated by a series of uniform layers with interfaces at each z_i. We shall initially ignore the free surface and treat the upper layer in $z < z_1$ as a half-space (the free surface will be considered in Section 5). The treatment here essentially follows
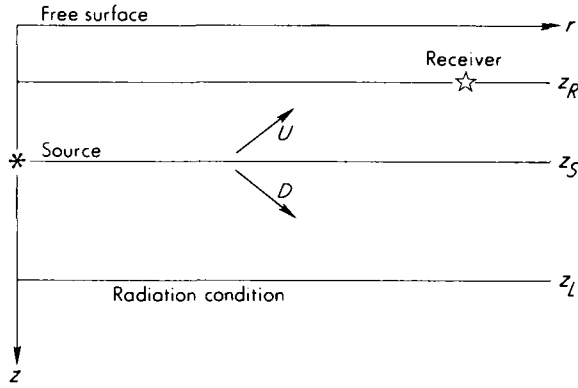
Figure 2. A stratified half-space. Between the free surface and z_L, the medium may vary in a depth-dependent manner. Below z_L the medium is uniform. Any part of the structure may be anisotropic and anelastic. z_S and z_R are source and receiver levels; these may be anywhere between the surface and z_L. The conventions for up and downgoing waves are also indicated.

Kennett & Kerry (1979) and Kennett (1983) except that we have chosen to build up the response of the medium by starting at the uppermost interface then adding the effects of successively deeper and deeper structure. The more traditional procedure is to start at the lowest interface and work upwards. We shall discuss the relative merits of the two approaches in Section 4.3.

4.1 REFLECTION AND TRANSMISSION RESPONSE

If we ignore free-surface effects, from equation (2.5) stresses and displacements at the top and bottom of the structure are related by

$$b(z_L) = P(z_L, 0) \, b(0). \tag{4.1}$$

The propagator $P(z_L, 0)$ is itself a product of individual propagators for each layer, commonly termed layer matrices.

$$P(z_L, 0) = P(z_L, z_{L-1}) \, P(z_{L-1}, z_{L-2}) \ldots P(z_1, 0). \tag{4.2}$$

Each layer matrix can be found by solving the eigenvalue problem for each layer, then assembling the propagator from (3.10). From (4.1) and (3.4) the wavefields at top and bottom of the structure are related by

$$v(z_L +) = D^{-1}(z_L +) \, P(z_L, 0) \, D(0 +) \, v(0 +)$$
$$= Q(z_L +, 0 +) \, v(0 +), \tag{4.3}$$

where, as before, Q is the wave propagator, but now it no longer has the simple structure of (3.6) since the medium between $z = 0$ and $z = z_L$ is not necessarily uniform. Here the modifiers, $+$ and $-$, are necessary as, unlike stress and displacement, wave potentials are not continuous across an interface.

If in (4.3) we partition Q into 3×3 submatrices then we can write

$$\begin{pmatrix} v_U(z_L +) \\ v_D(z_L +) \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} v_U(0 +) \\ v_D(0 +) \end{pmatrix}. \tag{4.4}$$

We may define reflection and transmission matrices in terms of the $\mathbf{v}_U$, $\mathbf{v}_D$. For an incident downward wave, $\mathbf{v}_U(z_L +) = \mathbf{0}$ and we have

$$\mathbf{v}_U(0 +) = \mathbf{R}_D \mathbf{v}_D(0 +), \qquad \mathbf{v}_D(z_L +) = \mathbf{T}_D \mathbf{v}_D(0 +), \tag{4.5}$$

where $\mathbf{R}_D$, $\mathbf{T}_D$ are the 3×3 reflection and transmission coefficient matrices

$$\mathbf{R}_D = \begin{pmatrix} r_{PP}^D & r_{P1}^D & r_{P2}^D \\ r_{1P}^D & r_{11}^D & r_{12}^D \\ r_{2P}^D & r_{21}^D & r_{22}^D \end{pmatrix} \quad \text{and} \quad \mathbf{T}_D = \begin{pmatrix} t_{PP}^D & t_{P1}^D & t_{P2}^D \\ t_{1P}^D & t_{11}^D & t_{12}^D \\ t_{2P}^D & t_{21}^D & t_{22}^D \end{pmatrix}.$$

Here, for example, r_{1P}^D is the amplitude of an upward $qS1$ wave generated by reflection of a downward incident qP wave of unit amplitude. If we also consider energy incident from below so that $\mathbf{v}_D(0 +) = \mathbf{0}$, then from (4.4) all reflection and transmission coefficients are given by

$$\begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{T}_D & \mathbf{R}_U \end{pmatrix} = \begin{pmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{R}_D & \mathbf{T}_U \\ \mathbf{I} & \mathbf{0} \end{pmatrix}. \tag{4.6}$$

From (4.6) we can assemble the scatterer or reflection matrix $\mathbf{R}$ containing all reflection and transmission coefficients,

$$\mathbf{R} = \begin{pmatrix} \mathbf{T}_U & \mathbf{R}_D \\ \mathbf{R}_U & \mathbf{T}_D \end{pmatrix} = \begin{pmatrix} \mathbf{Q}_{11}^{-1} & -\mathbf{Q}_{11}^{-1}\mathbf{Q}_{12} \\ \mathbf{Q}_{21}\mathbf{Q}_{11}^{-1} & \mathbf{Q}_{22} - \mathbf{Q}_{21}\mathbf{Q}_{11}^{-1}\mathbf{Q}_{12} \end{pmatrix} \tag{4.7}$$

(Kennett 1974; Kennett & Kerry 1979).

For constructing synthetics, the desired quantity is usually $\mathbf{R}_D$, the overall reflection coefficient matrix of the structure for energy incident from above. The obvious way to compute $\mathbf{R}_D$ is to solve the eigenvalue problem for each layer and the two half-spaces, perform the multiplications of (4.2) and (4.3) to find $\mathbf{Q}$, then find $\mathbf{R}_D$ from (4.7). Unfortunately, this procedure suffers from severe numerical problems. Since each layer propagator in (4.2) describes both upward and downward propagation, for inhomogeneous waves the propagators contain both growing and decaying exponentials. The disturbances with exponentially increasing amplitude are not of interest, but eventually they swamp the solution, reducing the results to nonsense. This problem has been widely recognized and reported; a description is given by Kennett & Kerry (1979).

The numerical problems can be avoided by building up the overall response in terms of the reflection and transmission properties of each interface (Kennett & Kerry 1979). Before describing the application of this reflection matrix method to problems in anisotropy, it is convenient to introduce the concept of a scatterer product.

4.2 THE SCATTERER OPERATOR AND SCATTERER PRODUCT

Equation (4.7) may be written symbolically

$$\mathbf{R} = \mathscr{S}(\mathbf{Q})$$

where $\mathscr{S}$, the scatterer operator, is a transformation which maps wave propagators $\mathbf{Q}$ into scatterers $\mathbf{R}$. This operator, introduced to seismology by Saastamoinen (1980), allows the matrix algebra of wave propagation in stratified media to be described much more concisely and helps provide insight into the physics of the propagation.

The scatterer matrix $\mathbf{R}$ is given by

$$\mathbf{R}(z_i, z_j) = \begin{pmatrix} \mathbf{T}_U & \mathbf{R}_D \\ \mathbf{R}_U & \mathbf{T}_D \end{pmatrix} \tag{4.8}$$

where the 3×3 reflection and transmission matrices are defined by

$$v_U[\min(z_i, z_j)] = T_U v_U[\max(z_i, z_j)]$$

$$v_U[\min(z_i, z_j)] = R_D v_D[\min(z_i, z_j)]$$

$$v_D[\max(z_i, z_j)] = R_U v_U[\max(z_i, z_j)]$$ (4.9)

$$v_D[\max(z_i, z_j)] = T_D v_D[\min(z_i, z_j)]$$

as shown schematically in Fig. 3. It follows from these definitions that R is independent of the order of its arguments so that

$$R(z_i, z_j) = R(z_j, z_i).$$

The scatterer operator takes different forms depending on the direction of propagation described by Q. We differentiate between these two operators by using subscripts U, D for upward and downward propagation. We can write

$$R(z_i, z_j) = \mathscr{S}_D[Q(z_i, z_j)] \qquad \text{for } z_i > z_j \qquad (4.10)$$

and

$$R(z_i, z_j) = \mathscr{S}_U[Q(z_j, z_i)] \qquad \text{for } z_i > z_j. \qquad (4.11)$$

To define these operators, consider a $2k \times 2k$ matrix M partitioned into $k \times k$ blocks so that

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}.$$

The downward scatterer operator $\mathscr{S}_D$ is defined by

$$\mathscr{S}_D(M) = \begin{pmatrix} M_{11}^{-1} & -M_{11}^{-1} M_{12} \\ M_{21} M_{11}^{-1} & M_{22} - M_{21} M_{11}^{-1} M_{12} \end{pmatrix}, \qquad (4.12)$$

while the upward scatterer operator $\mathscr{S}_U$ is defined by

$$\mathscr{S}_U(M) = \begin{pmatrix} M_{11} - M_{12} M_{22}^{-1} M_{21} & M_{12} M_{22}^{-1} \\ -M_{22}^{-1} M_{21} & M_{22}^{-1} \end{pmatrix}. \qquad (4.13)$$

By forming the product $\mathscr{S}_D(M)\, \mathscr{S}_U(M)$ from (4.12), (4.13) it is apparent that the two operators are related by matrix inversion, i.e.

$$\mathscr{S}_D(M) = [\,\mathscr{S}_U(M)]^{-1}. \qquad (4.14)$$

Applying the definition (4.12), we find that

$$R(z_L +, z_1 -) = \mathscr{S}_D[Q(z_L +, z_1 -)]$$

does indeed reproduce (4.7), as anticipated.

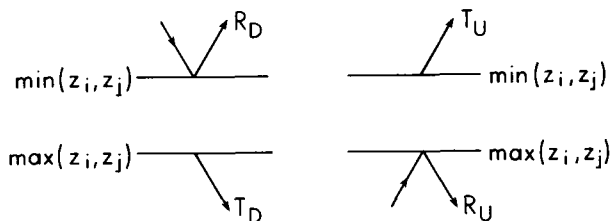**Figure 3.** Definition of reflection and transmission matrices for the depth interval from z_i to z_j. R_D and T_D describe the reflection and transmission of a plane wave initially travelling downward, R_D, T_U of a plane wave initially travelling upward.

Saastamoinen (1980) has listed the properties of scatterer operators, but does not consider the joint existence of complementary operators $\mathscr{S}_U$ and $\mathscr{S}_D$. Two properties, which we shall now discuss, are of central importance to the iterative construction of the response of a stratified medium.

It may readily be verified from (4.12) that $\mathscr{S}_D[\ \mathscr{S}_D(Q)] = Q$, so $\mathscr{S}_D$ and $\mathscr{S}_D^{-1}$ are identical operators. A similar property for $\mathscr{S}_U$ may be found using (4.13). Hence, *a scatterer operator is its own inverse*, i.e.

$$\mathscr{S}_D(M) = \mathscr{S}_D^{-1}(M), \qquad \mathscr{S}_U(M) = \mathscr{S}_U^{-1}(M). \tag{4.15}$$

Since equations (4.9) are valid for arbitrary z_i, z_j, we find that reflection and transmission coefficients for, say, an interface, are extracted from the interface wave propagator $Q(z_i+, z_i-)$ in exactly the same way as the overall coefficients for a stack of layers are extracted from $Q(z_L+, 0+)$. Hence the relationship between a scatterer and the associated wave propagator is always the same. This implicit property of (4.9)–(4.11) is worth stating explicitly: *the form of a scatterer operator is independent of depth*, so that

$$R(z_i, z_j) = \mathscr{S}_D[Q(z_i, z_j)] = \mathscr{S}_U[Q(z_j, z_i)] \qquad \text{for all } z_i > z_j. \tag{4.16}$$

Combining (4.15) and (4.16) we find, for all $z_i > z_j$,

$$Q(z_i, z_j) = \mathscr{S}_D^{-1}[R(z_i, z_j)] = \mathscr{S}_D[R(z_i, z_j)] = \begin{pmatrix} T_U^{-1} & -T_U^{-1}R_D \\ R_U T_U^{-1} & T_D - R_U T_U^{-1} R_D \end{pmatrix} \tag{4.17}$$

and

$$Q(z_j, z_i) = \mathscr{S}_U^{-1}[R(z_i, z_j)] = \mathscr{S}_U[R(z_i, z_j)]$$

$$= \begin{pmatrix} T_U - R_D T_D^{-1} R_U & R_D T_D^{-1} \\ -T_D^{-1}R_U & T_D^{-1} \end{pmatrix}, \tag{4.18}$$

relationships first discovered by Kennett (1974).

To find out how to combine scatterers we first expand a scatterer **R** in terms of the wave propagator **Q**,

$$R(z_i, z_j) = \mathscr{S}_D[Q(z_i, z_j)] = \mathscr{S}_D[Q(z_i, z_k) Q(z_k, z_j)], \qquad z_j < z_i.$$

From (4.15), (4.16) we can write this as a scatterer product

$$R(z_i, z_j) = \mathscr{S}_D\{\mathscr{S}_D[R(z_i, z_k)]\ \mathscr{S}_D[R(z_k, z_j)]\}, \qquad z_j < z_i. \tag{4.19}$$

Alternatively, since $R(z_i, z_j) = R(z_j, z_i)$, we can write

$$R(z_j, z_i) = \mathscr{S}_U\ \mathscr{S}_U[R(z_j, z_k)]\ \mathscr{S}_U[R(z_k, z_i)]\}, \qquad z_j < z_i. \tag{4.20}$$

Note that (4.19), (4.20) must yield identical results. This can readily be verified using the definitions of $\mathscr{S}_U$, $\mathscr{S}_D$. Expansions of the scatterer products, (4.19) or (4.20), are often called recurrence relations, and have been widely published (e.g. Kennett & Kerry 1979, equation 4.21). The scatterer product is conveniently denoted

$$R(z_i, z_j) = R(z_i, z_k) \star R(z_k, z_j). \tag{4.21}$$

We note that the scatterer product incorporates the addition rules for reflection and transmission described by Kennett (1983, p. 127).

4.3 ITERATIVE COMPUTATION OF THE RESPONSE

To avoid the numerical problems suffered in a direct stress-displacement propagator construction of the response of a stratified medium, Kennett (1974) devised the reflection

matrix method in which the response is built up, an interface at a time, in terms of the reflection and transmission properties. A modification by Kennett & Kerry (1979), the factoring out of the wave propagator for a layer, has made the reflection matrix method unconditionally stable; the numerical problems are completely avoided. Although originally derived for isotropic media, the method can readily be extended to include anisotropy and has been used by Booth & Crampin (1983a, b) to compute the response of anisotropic structures. Our approach differs from that of Booth & Crampin in that we develop the response in terms of scatterer products, and, as described earlier, we choose to build up the response by working down through the structure from the surface rather than up from the lower half-space.

For continuity of stress and displacement, the wave propagator for an interface at z_i is simply, from (3.4),

$$Q(z_i+, z_i-) = D^{-1}(z_i+) \, D(z_i-).$$

So the interface scatterer, which includes the reflection and transmission effects, is from (4.16)

$$\begin{pmatrix} T_U & R_D \\ R_U & T_D \end{pmatrix} = R(z_i+, z_i-) = \mathscr{S}_D[D^{-1}(z_i+) \, D(z_i-)]. \tag{4.22}$$

The scatterer for the interval $(z_{i-1}+, z_i-)$, the uniform interval immediately above the interface, is from (3.8),

$$R(z_i-, z_{i-1}+) = \mathscr{S}_D[Q(z_i-, z_{i-1}+)] = \mathscr{S}_D \begin{pmatrix} E_U & 0 \\ 0 & E_D \end{pmatrix} = \begin{pmatrix} E_U^{-1} & 0 \\ 0 & E_D \end{pmatrix}.$$

This expression tells us that there is no reflection from a uniform medium.

To find the overall layer-interface scatterer, which includes all effects of transmission through the layer and reflection/transmission at the interface at the base of the layer, we simply form the scatterer product

$$R(z_i+, z_{i-1}+) = R(z_i+, z_i-) \star R(z_j-, z_{i-1}+).$$

Using (4.19) and the definition of the scatterer operator we obtain

$$T_U(z_i+, z_{i-1}+) = E_U^{-1} T_U(z_i+, z_i-)$$

$$R_U(z_i+, z_{i-1}+) = R_U(z_i+, z_i-)$$

$$R_D(z_i+, z_{i-1}+) = E_U^{-1} R_D(z_i+, z_i-) \, E_D \tag{4.23}$$

$$T_D(z_i+, z_{i-1}+) = T_D(z_i+, z_i-) \, E_D.$$

Here we have used an obvious notation: $T_U(z_j, z_k)$ represents all upward transmission effects of the structure between z_j and z_k. Strictly, E_D should be written $E_D(z_i-, z_{i-1}+)$, etc., but the simplicity of (4.23) is already somewhat obscured by the notation.

The construction of the overall response of the layered structure begins by finding the layer-interface scatterer for the first layer and interface beneath, $R(z_1+, 0+)$, from (4.22) and (4.23). Using the scatterer product, layer-interface scatterers are added in succession, working down through the stack of layers, so that at step i we form the product

$$R(z_i+, 0+) = R(z_i+, z_{i-1}+) \star R(z_{i-1}+, 0+).$$

The iteration is terminated when we reach the base of the structure at z_L. In a similar way, any desired scatterer $R(z_i, z_j)$ can be put together from the individual scatterers for layers and interfaces between z_i and z_j.

This downward iteration scheme is particularly convenient if the response is being computed at large slowness, for which evanescence may prevent significant energy from

penetrating to the base of the structure. Under such circumstances it is often unnecessary to include the complete structure in the computation. While working down through the structure we can monitor the behaviour of the downward transmission matrix of the stack, $T_D(z_i+, 0+)$, and truncate the iteration if all elements of this matrix approach zero (indicating that only negligible energy penetrates to greater depth). By truncating the iteration, considerable computational savings may be realized.

At small values of slowness (the subcritical regime), the complete structure contributes to the response. It is then often more efficient to build the response by working upwards from the lowest interface (using the operator $\mathcal{G}_U$) so that at step i we form the scatterer

$$R(z_L +, z_{L-i} +) = R(z_L +, z_{L-i+1} +) \star R(z_{L-i+1} +, z_{L-i} +).$$

This upward iteration is preferable for those problems which require only the downgoing reflection or transmission matrices, R_D, T_D, of the complete stack (we shall discuss some examples in Section 5). By expanding the upward scatterer product (4.20) it may be verified that upward stack matrices [such as $R_U(z_L +, z_{L-i} +)$] never appear in expressions for downward stack matrices (Kennett 1983, section 6.2.1). This means that if the matrices R_D, T_D, are computed using the upward iteration, the matrices R_U, T_U need not be carried through the iteration and a considerable fraction of the matrix multiplication can be avoided.

The intrinsic stability of the reflection matrix iteration is apparent on examination of (4.23). From (3.3) and (3.9) we know that E_U contains no negative (and E_D no positive) real exponentials. Hence layer-interface scatterers (4.23) do not contain growing exponentials. Further, scatterer products of layer-interface scatterers contain no growing exponentials (Saastamoinen 1980). This means that the reflection matrix iteration is stable to arbitrarily large frequency and slowness, unlike the direct propagator approach.

5 Response of a half-space

5.1 INCLUSION OF A SOURCE

For a source in the structure the original differential equation (2.1) must be replaced by the inhomogeneous equation

$$\partial_z b = i\omega A(z) b(z) + F(z) \tag{5.1}$$

where F is a source term containing the vector f of body forces:

$$F = \frac{-1}{i\omega}\begin{pmatrix} 0 \\ f \end{pmatrix} = \frac{-1}{i\omega}(0, 0, 0, f_x, f_y, f_z)^T. \tag{5.2}$$

We can integrate (5.1) following Gilbert & Backus (1966), to obtain

$$b(z) = P(z, z_0) b(z_0) + \int_{z_0}^{z} P(z, \zeta) F(\zeta) d\zeta \tag{5.3}$$

which should be compared with (2.5) for the source-free case.

Many simple point sources can be modelled as a point force acting in a plane through the source combined with a dipole or couple acting in the same plane (Hudson 1969). If $z = z_S$ is the source depth we therefore take

$$F(z) = F_1 \delta(z - z_S) + F_2 \delta'(z - z_S). \tag{5.4}$$

With this point source excitation substituted in (5.3) the stress-displacement vector b is found to have a discontinuity across the plane z_S,

$$b(z_S +) - b(z_S -) = s(z_S) = F_1 + i\omega A(z_S) F_2 \tag{5.5}$$

(Kennett 1983, p. 44). We consider a point source representation in terms of a force **h** and a moment tensor **M** (Gilbert 1971), for which the equivalent body force (Burridge & Knopoff 1964) is

$$\mathbf{f} = \mathbf{h}\delta(\mathbf{r} - \mathbf{r}_S) - \mathbf{M}(\omega) \cdot \nabla \delta(\mathbf{r} - \mathbf{r}_S). \tag{5.6}$$

We assume the source is located at $(0, 0, z_S)$ as in Fig. 1, so

$$\delta(\mathbf{r} - \mathbf{r}_S) = \delta(x)\,\delta(y)\,\delta(z - z_S),$$

thus

$$\nabla\delta(\mathbf{r} - \mathbf{r}_S) = [ik_x\delta(z - z_S),\ ik_y\delta(z - z_S),\ \partial_2\delta(z - z_S)]^T.$$

From (5.6) we obtain

$$\mathbf{f} = \mathbf{h}\delta(z - z_S) - i\omega[p_x\mathbf{M}_x + p_y\mathbf{M}_y]\,\delta(z - z_S) - \mathbf{M}_z\delta'(z - z_S)$$

where $\mathbf{M}_x, \mathbf{M}_y, \mathbf{M}_z$ are the appropriate columns of the moment tensor $\mathbf{M}$, so that $\mathbf{M}_x = (M_{xx}, M_{yx}, M_{zx})^T$. Substituting this force vector $\mathbf{f}$ into (5.2) we can identify $\mathbf{F}_1$ and $\mathbf{F}_2$ in (5.4). From (5.5) the stress-displacement discontinuity which acts equivalently to $\mathbf{f}$ is

$$
\mathbf{s}(z) = \left\{
\begin{bmatrix}
0 \\
0 \\
0 \\
\dfrac{-1}{i\omega}h_x + p_x M_{xx} + p_y M_{xy} \\
\dfrac{-1}{i\omega}h_y + p_y M_{yx} + p_y M_{yy} \\
\dfrac{-1}{i\omega}h_z + p_x M_{zx} + p_y M_{zy}
\end{bmatrix}
+ \mathbf{A}(z)
\begin{bmatrix}
0 \\
0 \\
0 \\
M_{xz} \\
M_{yz} \\
M_{zz}
\end{bmatrix}
\right\}\delta(z - z_s). \tag{5.7}
$$

If, for example, the source is in an orthotropic medium with symmetry planes parallel to coordinate planes, we can use the system matrix $\mathbf{A}$ given by Fryer & Frazer (1984, in preparation) to obtain

$$
\mathbf{s}(z) =
\begin{bmatrix}
-c_{55}^{-1} M_{xz} \\
-c_{44}^{-1} M_{yz} \\
-c_{33}^{-1} M_{zz} \\
\dfrac{-1}{i\omega}h_x + p_x(M_{xx} - c_{13}c_{33}^{-1}M_{zz}) + p_y M_{xy} \\
\dfrac{-1}{i\omega}h_y + p_x M_{yx} + p_y(M_{yy} - c_{23}c_{33}^{-1}M_{zz}) \\
\dfrac{-1}{i\omega}h_z + p_x(\dot{M}_{zx} - M_{xz}) + p_y(M_{zy} - M_{yz})
\end{bmatrix}
\delta(z - z_S). \tag{5.8}
$$

It is worth noting that equation (5.8), derived in a Cartesian coordinate system, is relatively simple. Had we chosen cylindrical coordinates, (5.8) would have been much less concise [compare with equations (4.59), (4.60) of Kennett 1983 for the equivalent expression for an isotropic medium]. This is a further attraction in working in Cartesians.

Instead of dealing with a jump in stress-displacement, we can equivalently consider the source as a discontinuity in the wave vector. The equivalent wave vector discontinuity is

$$\mathbf{\Sigma}(z_S) = \begin{pmatrix} -\mathbf{\Sigma}_U \\ \mathbf{\Sigma}_D \end{pmatrix} = \mathbf{D}^{-1}(z_S)\,\mathbf{s}(z_S). \tag{5.9}$$

5.2 RESPONSE OF THE HALF-SPACE

With the source specified either by the stress-displacement discontinuity **s** or the wave vector discontinuity $\mathbf{\Sigma}$, the full half-space response, including free-surface effects, may be constructed using the reflection matrix method, exactly as described by Kennett (1983, chapter 7). The reflection matrix development remains valid in the anisotropic case.

Since the reflection matrix method has been described so well elsewhere (Kennett & Kerry 1979; Kennett 1983), we need only give the briefest description here. If we write

$$\mathbf{D}(0+) = \begin{pmatrix} \mathbf{M}_{U0} & \mathbf{M}_{D0} \\ \mathbf{N}_{U0} & \mathbf{N}_{D0} \end{pmatrix}$$

$$\mathbf{R}(0+, z_L+) = \begin{pmatrix} \mathbf{T}_U^{0L} & \mathbf{R}_D^{0L} \\ \mathbf{R}_U^{0L} & \mathbf{T}_D^{0L} \end{pmatrix},$$

with similar notation for $\mathbf{R}(z_S+, z_L+)$ and $\mathbf{R}(0+, z_S-)$, the free surface displacement is given by

$$\mathbf{u}(0) = \tilde{\mathbf{W}}[\mathbf{I} - \mathbf{R}_D^{0L}\tilde{\mathbf{R}}]^{-1}\boldsymbol{\sigma}(z_S) \tag{5.10}$$

(Kennett 1983, equation 7.53). Here $\tilde{\mathbf{W}}$ is the free-surface magnification correction,

$$\tilde{\mathbf{W}} = \mathbf{M}_{U0} + \mathbf{M}_{D0}\tilde{\mathbf{R}},$$

$\boldsymbol{\sigma}$ is a vector including all interactions of the source with the structure excluding the free surface,

$$\boldsymbol{\sigma}(z_S) = \mathbf{T}_U^{0S}[\mathbf{I} - \mathbf{R}_D^{SL}\mathbf{R}_U^{0S}]^{-1}[\mathbf{\Sigma}_U(z_S) + \mathbf{R}_D^{SL}\mathbf{\Sigma}_D(z_s)],$$

and $\tilde{\mathbf{R}}$ is the reflection matrix for the free surface such that $\mathbf{v}_D(0+) = \tilde{\mathbf{R}}\mathbf{v}_U(0+)$. To satisfy conditions of vanishing stress,

$$\tilde{\mathbf{R}} = -\mathbf{N}_{D0}^{-1}\mathbf{N}_{U0}$$

(Kennett 1983, pp. 117−118). If the receiver is not at the surface but at some depth z_R, (5.10) must be replaced by equations (7.35) or (7.38) of Kennett (1983).

Note in (5.10) that the response is given entirely in terms of reflection functions. These can be computed using the iterative construction already described, which is stable to arbitrary slowness and frequency. As a result the response computed via (5.10) is itself unconditionally stable. Such stability is a characteristic of reflection matrix solutions.

5.3 PARTIAL RESPONSE

For many seismological problems, especially long-range refraction studies, reflections from the surface are ignored as they arrive late in the seismogram, long after the arrivals of interest. In modelling the response, surface-reflected phases can be an annoyance as they demand the construction of extremely long synthetics (if only short time series are constructed, aliasing will fold the unwanted arrivals into the time frame of interest and the more important arrivals may be obscured). For example, in marine problems, if the source is

23

in the water it is often convenient to suppress both direct and surface-reflected phases. This can be done by retaining only downgoing radiation from the source and truncating the reverberation operator $[\mathbf{I} - \mathbf{R}_D^{OL} \tilde{\mathbf{R}}]^{-1}$ after the first term in its binomial expansion. This yields the partial solution

$$\mathbf{u}(0) = \mathbf{W}\mathbf{T}_U^{OS}[\mathbf{I} - \mathbf{R}_D^{SL}\mathbf{R}_U^{OS}]^{-1}\mathbf{R}_D^{SL}\mathbf{\Sigma}_D(z_S) \tag{5.11}$$

which further simplifies for a surface source to

$$\mathbf{u}(0) = \mathbf{W}\mathbf{R}_D^{OL}\mathbf{\Sigma}_D(0+). \tag{5.12}$$

Within the reflection matrix formalism there is complete freedom to suppress the response of any part of the structure for sources and receivers at any depth (Kennett 1983, chapter 9).

6 Synthetic seismograms

If the displacement vector at the receiver depth, $\mathbf{u}(z_R)$, is obtained for a range of transformed coordinates (p_x, p_y, ω), to obtain synthetic seismograms it remains to perform the inverse transforms

$$\mathbf{u}(x, y, z_R, t) = \frac{1}{8\pi^3} \iiint_{-\infty}^{\infty} \omega^2 \mathbf{u}(p_x, p_y, z_R, \omega)$$
$$\times \exp\left[-i\omega(t - p_x x - p_y y)\right] dp_x \, dp_y \, d\omega. \tag{6.1}$$

Traditionally, wave propagation problems involving isotropic media have been approached using not Cartesian but cylindrical coordinates (r, ϕ, z). We chose not to use cylindrical coordinates here, but it is instructive to compare the two approaches.

In cylindrical coordinates the Fourier transforms from (x, y) to (p_x, p_y) used in the original separation of variables of Section 2 is replaced by a finite Fourier transform from azimuth ϕ to azimuthal order number m and a Hankel transform from range r to radial slowness p_r. The stress-displacement vector $\mathbf{b}$ of the differential system (2.1) is defined slightly differently (Kennett 1983, pp. 26–29) but the scheme for obtaining $\mathbf{u}(p_r, m, z_R, \omega)$ differs only in detail from that for $\mathbf{u}(p_x, p_y, z_R, \omega)$. The inverse transforms for vertical displacement may be written

$$u_z(r, \phi, z_R, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, \omega^2 \exp(-i\omega t)$$
$$\times \int_0^{\infty} dp_r p_r \sum_m u_z(p_r, m, z_R, \omega) J_m(\omega p_r r) \exp(im\phi) \tag{6.2}$$

(Kennett 1983, equation 2.44) with similar but somewhat more complicated expressions for the other displacement components (Kennett 1983, equation 7.57). The inverse of the finite Fourier transform is a summation over angular order number m, but for point sources composed of force and dipole components the summation can be restricted to $|m| < 2$ because of symmetry (Kennett 1983, p. 173). Equation (6.2) then represents a considerable computational savings over (6.1) as a triple integral has been replaced by two integrals and a (small) finite sum.

For azimuthally anisotropic media, as azimuth ϕ is varied, an incident disturbance senses different elastic properties. In (6.2) convergence of the Fourier series is not then accomplished until m is large, so any advantage over Cartesian coordinates is lost. Further, evaluation of (6.2) for a specified azimuth is not simple. Either those values of m giving a contribution along the desired azimuth must be found, or the coordinate system itself must be rotated so that $\phi = 0$ lies along the desired azimuth (6.2 can then be evaluated by

summing about $m = 0$). The latter method is simpler but the system matrix **A** in (2.1) will have a different form for each azimuth of interest, an inconvenience. In general, when dealing with azimuthal anisotropy, Cartesian coordinates will yield simpler mathematics and result in more efficient codes. This is especially true if any of the coordinate axes can be chosen to be parallel to a symmetry direction. The elasticity tensor will then be sparse, giving **A** a simple form and so simplifying the extraction of eigensolutions (Fryer & Frazer 1984).

If anisotropy is weak so that the transerve component of the wave slowness vector is negligible (i.e. all rays remain close to the sagittal plane), then cylindrical coordinates do offer an advantage as it then becomes legitimate to limit the angular order number to small values. This is obviously true for transversely isotropic media as these display cylindrical symmetry and the transverse component of wave slowness is identically zero. Booth & Crampin (1983a, b) construct synthetics for azimuthally anisotropic media in this way but they admit that such an approach is valid only for weak anisotropy. It is difficult to estimate the error introduced by this procedure. Since wave propagation out of the sagittal plane is a characteristic of anisotropy it seems important to try to retain this phenomenon in any modelling. Certainly propagation out of the sagittal plane must be accommodated if we are to construct complete synthetics.

To compute synthetics, we must find $\mathbf{u}(p_x, p_y, z_R, \omega)$ for a range of frequencies ω and of both horizontal slownesses p_x and p_y, then perform the inverse transformation (6.1). Three-dimensional Fourier transforms involve a daunting amount of computation and so are usually specially designed for the problem at hand. This problem is no different. If we were to execute a complete 3D-FFT of $\mathbf{u}(p_x, p_y, z_R, \omega)$ we would obtain synthetics for every point on the $x-y$ plane, which is obviously in excess of our requirements. To reduce the computation to a reasonable level, we must consider efficiency and avoid computing quantities we do not ultimately want.

Two-dimensional FFTs are now fairly common and are used routinely, for example, in $F-K$ migration of seismic reflection data (e.g. Carter & Frazer 1982). We propose to evaluate (6.1) using a combination of 2-D and 1-D transformations. The obvious choice is for the 2-D operation to transform from (p_x, p_y) to (x, y) and the 1-D from ω to t. We are left with the choice of order of the transforms.

Construction of synthetics using a slowness approach (ω to t transform performed first) is an attractive scheme as intermediate results are plane-wave seismograms in $t-p$ space, which are amenable to interpretation (Fryer 1980). Plane-wave seismograms for anisotropic media have proved quite instructive (Keith & Crampin 1977c). Unfortunately, the slowness approach is wasteful of both computer time and storage. This is illustrated in Fig. 4. When a frequency-to-time transform is executed the number of time samples obtained is the number that will appear in the final seismogram. From our experience with synthetics for isotropic media we expect a 1000-point seismogram typically to be constructed from 100 frequencies. On performing the ω to t transform we can hence expect a 10-fold increase in array size. Each time point will then require a 2-D FFT from (p_x, p_y) to (x, y) space and we shall obtain synthetics for all points in (x, y), exactly the wasteful situation we had hoped to avoid.

The alternative to the slowness approach is the spectral, with the 2-D transform from (p_x, p_y) to (x, y) performed first. This will not increase storage demands and the 2-D transform need be performed typically 100 times rather than the 1000 times of the slowness approach. From the resulting (x, y, ω) data (r, ϕ, ω) data can be interpolated for desired azimuths ϕ and the final synthetics obtained by 1-D FFT. Examples of such analyses will be presented in a future paper.
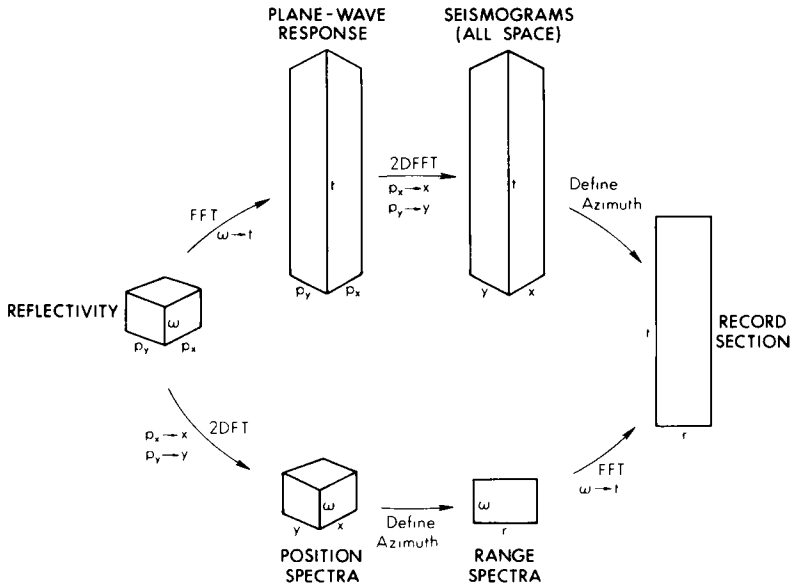
Figure 4. The two possible paths to synthetic seismograms; the goal is to obtain a record section for a given azimuth from the medium response ('reflectivity') which is computed in transformed coordinates. The upper path is the slowness path, in which the initial step is to transform from frequency to time to obtained the transient plane-wave response. This method requires a 2-D Fourier transform for each discrete time, which yields a seismogram for every (x, y). From these the desired record section can be extracted (by interpolation if necessary). In the spectral method (lower path) the 2-D transformation from (p_x, p_y) to (x, y) is performed for each frequency, data for the desired azimuth are extracted, and the final record section obtained by simple 1-D Fourier transformation. Boxes show the dimensionality and relative amounts of storage required at each step. The spectral method is the more economical as it requires much less storage and many fewer transforms.

7 Discussion and conclusions

Solution of the wave equation for vertically stratified media involves three steps, the first purely mathematical: (1) separation of variables by multiple transformation to obtain a system of ordinary differential equations in depth, (2) solution of the differential system, and (3) back transformation to time–distance space. The choice of coordinate system depends on the nature of the problem; the cylindrical symmetry of isotropy and transverse isotropy make cylindrical coordinates the most logical choice; when there is an azimuthal variation of elastic properties, Cartesian coordinates are much more convenient.

The differential system may be solved using Kennett's recursive algorithm. The recursive scheme has to be modified for anisotropic media, but the modifications are straightforward. In problems of $P-SV$ motion in isotropic media the recursive scheme involves 2×2 matrices whereas for general anisotropy it involves 3×3. The recursion relations for isotropy (Kennett 1983, p. 133) implicitly assume that for each wave type, the vertical slownesses for up and downgoing propagation are simply the negative of each other (i.e. that E_U^{-1} and E_D are equal). For anisotropic media this assumption is valid only for those materials with a horizontal plane of elastic symmetry. For general anisotropy E_U^{-1} and E_D are *not* equal and equation (6.16) of Kennett (1983) must be replaced by our equation (4.23). Similarly, Kennett's discussion of piecewise smooth models (Kennett 1983, pp. 136–151) implicitly assumes this symmetry of the vertical slownesses and would have to be modified. We have not considered this problem.

Apart from these modifications to the recursion, Kennett's reflection matrix procedure is directly applicable to problems in anisotropy. All matrix relationships given by Kennett (1983, chapters 7 and 9) describing either the complete or partial response of a layered half-space to a source at any depth, retain their validity in the presence of anisotropy (though the matrices themselves are all 3×3 rather than the 2×2 entities of isotropy).

Construction of synthetic seismograms requires the back transformation of the (transformed) response. For isotropy, this is a 2-D combination of Fourier and Hankel transforms. For anisotropy, a 3-D Fourier transform must be evaluated. The complete process of computing medium response and performing a 3-D inverse transform promises to be extremely time-consuming, but with special-purpose hardware (an array processor) to assist in evaluating the transforms, it should be possible to make the process economical enough for routine use.

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