SELBERG'S ORTHOGONALITY CONJECTURE FOR AUTOMORPHIC L-FUNCTIONS

JIANYA LIU[1] AND YANGBO YE[2]

ABSTRACT. Let π and π' be automorphic irreducible unitary cuspidal representations of $GL_m(\mathbb{Q}_{\mathbb{A}})$ and $GL_{m'}(\mathbb{Q}_{\mathbb{A}})$, respectively. Assume that π and π' are self contragredient. Under the Ramanujan conjecture on π and π', we deduce a prime number theorem for $L(s, \pi \times \tilde{\pi}')$, which can be used to asymptotically describe whether $\pi' \cong \pi$, or $\pi' \cong \pi \otimes |\det(\cdot)|^{i\tau_0}$ for some non-zero $\tau_0 \in \mathbb{R}$, or $\pi' \not\cong \pi \otimes |\det(\cdot)|^{it}$ for any $t \in \mathbb{R}$. As a consequence, we prove the Selberg orthogonality conjecture, in a more precise form, for automorphic L-functions $L(s, \pi)$ and $L(s, \pi')$, under the Ramanujan conjecture. When $m = m' = 2$ and π and π' are representations corresponding to holomorphic cusp forms, our results are unconditional.

1. Introduction. Let π be an irreducible unitary cuspidal representation of $GL_m(\mathbb{Q}_{\mathbb{A}})$. Then the global L-function attached to π is given by products of local factors for $\mathrm{Re}\, s > 1$ (Godement and Jacquet [3]):

$$L(s, \pi) = \prod_p L_p(s, \pi_p),$$
$$\Phi(s, \pi) = L_\infty(s, \pi_\infty) L(s, \pi),$$

where

$$L_p(s, \pi_p) = \prod_{j=1}^{m} \big(1 - \alpha_\pi(p, j) p^{-s}\big)^{-1}$$

and

$$L_\infty(s, \pi_\infty) = \prod_{j=1}^{m} \Gamma_{\mathbb{R}}(s + \mu_\pi(j)).$$

Here $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$, and $\alpha_\pi(p, j)$ and $\mu_\pi(j)$, $j = 1, \ldots, m$, are complex numbers associated with π_p and π_∞, respectively, according to the Langlands correspondence. De-

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note

$$a_\pi(p^k) = \sum_{1\le j\le m} \alpha_\pi(p,j)^k. \tag{1.1}$$

Then for Re $s>1$, we have

$$\frac{d}{ds}\log L(s,\pi) = -\sum_{n\ge1}\frac{\Lambda(n)a_\pi(n)}{n^s}, \tag{1.2}$$

where $\Lambda(n)=\log p$ if $n=p^k$ and $=0$ otherwise. If π' is an automorphic irreducible cuspidal representation of $GL_{m'}(\mathbb{Q}_\mathbb{A})$, we define $L(s,\pi')$, $\alpha_{\pi'}(p,i)$, $\mu_{\pi'}(i)$, and $a_{\pi'}(p^k)$ likewise, for $i=1,\dots,m'$. If π and π' are equivalent, then $m=m'$ and $\{\alpha_\pi(p,j)\}=\{\alpha_{\pi'}(p,i)\}$ for any p. Hence $a_\pi(n)=a_{\pi'}(n)$ for any $n=p^k$, when $\pi\cong\pi'$.

The Selberg orthogonality conjecture for automorphic L-functions $L(s,\pi)$ was proposed in 1989 (Selberg [19]). See also Ram Murty [15] [16].

CONJECTURE 1.1. (i) *For any automorphic irreducible cuspidal representation* π *of* $GL_m(\mathbb{Q}_\mathbb{A})$

$$\sum_{p\le x}\frac{|a_\pi(p)|^2}{p} = \log\log x + O(1). \tag{1.3}$$

(ii) *For any automorphic irreducible cuspidal representations* π *and* π' *of* $GL_m(\mathbb{Q}_\mathbb{A})$ *and* $GL_{m'}(\mathbb{Q}_\mathbb{A})$*, respectively,*

$$\sum_{p\le x}\frac{a_\pi(p)\bar{a}_{\pi'}(p)}{p} \ll 1, \tag{1.4}$$

if π *is not equivalent to* π'*.*

The asymptotic formula in (1.3) was proved by Rudnick and Sarnak [17] under a conjecture on the convergence of a series on prime powers (Hypothesis H below), and unconditionally for $m\le 4$.

HYPOTHESIS H. *For* $k\ge 2$,

$$\sum_p \frac{|a_\pi(p^k)|^2\log^2 p}{p^k} < \infty.$$

This Hypothesis H is trivial for $m=1$, and follows from bounds toward the Ramanujan conjecture for $m=2$. For $m=3$ it was proved by Rudnick and Sarnak [17], while the case of $m=4$ was proved by Kim and Sarnak [9]. For $m>4$, Hypothesis H is an easy consequence of the Ramanujan conjecture. In this paper, we will assume the Ramanujan conjecture for primes p:

CONJECTURE 1.2. *Let π be an irreducible unitary cuspidal representation of $GL_m(\mathbb{Q}_{\mathbb{A}})$. For any unramified p, we have*

$$|\alpha_\pi(p,j)| = 1. \tag{1.5}$$

Note that in Conjecture 1.2 we do not include the Archimedean Ramanujan conjecture, Re $\mu_\pi(j) = 0$.

What we will prove as a consequence of Conjecture 1.2 is the following orthogonality. Denote $\alpha(g) = |\det(g)|$.

THEOREM 1.3. *Let π and π' be irreducible unitary cuspidal representations of $GL_m(\mathbb{Q}_{\mathbb{A}})$ and $GL_{m'}(\mathbb{Q}_{\mathbb{A}})$, respectively, such that at least one of π and π' is self contragredient: $\pi \cong \tilde{\pi}$ or $\pi' \cong \tilde{\pi}'$. Assume the Ramanujan Conjecture 1.2 for both π and π'. Then*

$$\begin{aligned}\sum_{n\le x}(\log n)\Lambda(n)a_\pi(n)\bar{a}_{\pi'}(n) =&\frac{x^{1+i\tau_0}}{1+i\tau_0}\log x - \frac{x^{1+i\tau_0}}{(1+i\tau_0)^2} + O\{x\exp(-c\sqrt{\log x})\}\\ &\textit{if}\ \ \pi' \cong \pi\otimes\alpha^{i\tau_0}\ \ \textit{for}\ \ \textit{some}\ \ \tau_0\in\mathbb{R};\\ =&O\{x\exp(-c\sqrt{\log x})\}\\ &\textit{if}\ \ \pi' \ncong \pi\otimes\alpha^{it}\ \ \textit{for}\ \ \textit{any}\ \ t\in\mathbb{R},\end{aligned}$$

where c is a positive constant.

Note that for $m = m' = 2$ and π and π' being representations corresponding to holomorphic cusp forms, i.e., when their Archimedean local components are discrete series or limits of discrete series, the Ramanujan conjecture was proved by Deligne. Therefore in this case, Theorem 1.3 is an unconditional result.

Theorem 1.3 is indeed a version of prime number theorem for the the Rankin-Selberg L-function $L(s, \pi\times\pi')$. It is stronger than the following Mertens theorem on orthogonality with a weight $(\log n)\Lambda(n)/n$:

COROLLARY 1.4. *Let π and π' be given as in Theorem 1.3. Assume either* (i) *Ramanujan Conjecture 1.2 for both π and π', or* (ii) *that $m = m' = 2$ and π and π' are* *representations corresponding to holomorphic cusp forms. Then*

$$
\begin{aligned}
\sum_{n\le x}\frac{(\log n)\Lambda(n)a_\pi(n)\bar a_{\pi'}(n)}{n} =& \frac{1}{2}\log^2 x + c_1 + O\{\exp(-c\sqrt{\log x})\} \\
& \quad \textit{if } \ \pi'\cong\pi; \\
=& \frac{x^{i\tau_0}}{i\tau_0}\log x + \frac{x^{i\tau_0}-1}{\tau_0^2} + c_2 + O\{\exp(-c\sqrt{\log x})\} \\
& \quad \textit{if } \ \pi'\cong\pi\otimes\alpha^{i\tau_0} \ \ \textit{for some} \ \ \tau_0\in\mathbb{R}^\times; \\
=& c_3 + O\{\exp(-c\sqrt{\log x})\} \\
& \quad \textit{if } \ \pi'\not\cong\pi\otimes\alpha^{it} \ \ \textit{for any} \ \ t\in\mathbb{R},
\end{aligned} \tag{1.6}
$$

where $c, c_1, ..., c_3$ are positive constants.

A remarkable feature of this corollary is that it describes the orthogonality of $a_\pi(n)$ and $a_{\pi'}(n)$ in three cases with different main terms. As we are assuming Ramanujan and hence the Hypothesis H, we can control sums over prime powers and easily get an orthogonality over primes. Selberg's orthogonality conjecture 1.1 is then a consequence of Corollary 1.4 by partial summation.

Corollary 1.5 (Selberg's orthogonality). *Let π and π' be given as in Theorem 1.3. Assume either* (i) *Ramanujan Conjecture 1.2 for both π and π', or* (ii) $m = m' = 2$ *and that π and π' are representations corresponding to holomorphic cusp forms. Then*

$$
\begin{aligned}
\sum_{p\le x}\frac{a_\pi(p)\bar a_{\pi'}(p)}{p} =& \log\log x + c_4 + O\{\exp(-c\sqrt{\log x})\} \\
& \quad \textit{if } \ \pi'\cong\pi; \\
=& c_5 + \mathrm{Ei}(i\tau_0\log x) + O\{\exp(-c\sqrt{\log x})\} \\
& \quad \textit{if } \ \pi'\cong\pi\otimes\alpha^{i\tau_0} \ \ \textit{for some} \ \ \tau_0\in\mathbb{R}^\times; \\
=& c_6 + O\{\exp(-c\sqrt{\log x})\} \\
& \quad \textit{if } \ \pi'\not\cong\pi\otimes\alpha^{it} \ \ \textit{for any} \ \ t\in\mathbb{R}.
\end{aligned} \tag{1.7}
$$

Here Ei *is the exponential integral, and $c, c_4, ..., c_6$ are positive constants.*

Recall that

$$\mathrm{Ei}(i\tau_0\log x) = \frac{x^{i\tau_0}}{i\tau_0\log x}\Big(\sum_{k=0}^{n}\frac{k!}{(i\tau_0\log x)^k}\Big) + O\big((\log x)^{-n-2}\big).$$

Our Corollary 1.5 is thus in a more precise form than Selberg's Conjecture 1.1.

The error terms in our theorem and corollaries are in a form that reflects very much our presesnt knowledge of zero free regions for our Rankin-Selberg L-functions (see §4). The proofs of Corollaries 1.4 and 1.5 proceed along standard arguments, based on variations of Abel summation. We will thus not give these proofs here, but only point out that in the proof of Corollary 1.5, Hypothesis H is used to control sums over prime powers in the expression on the left side of (1.6). This way we can obtain a sum taken over primes as in (1.7).

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2. Rankin-Selberg L-functions. We will use the Rankin-Selberg L-functions $L(s, \pi \times \tilde{\pi}')$ as developed by Jacquet, Piatetski-Shapiro, and Shalika [4], Shahidi [20], and Moeglin and Waldspurger [12], where π and π' are automorphic irreducible cuspidal representations of GL_m and $GL_{m'}$, respectively, over $\mathbb{Q}$ with unitary central characters. This L-function is given by local factors:

$$L(s, \pi \times \tilde{\pi}') = \prod_p L_p(s, \pi_p \times \tilde{\pi}'_p) \tag{2.1}$$

where

$$L_p(s, \pi_p \times \tilde{\pi}'_p) = \prod_{j=1}^{m} \prod_{k=1}^{m'} \left(1 - \alpha_\pi(p, j) \bar{\alpha}_{\pi'}(p, k) p^{-s}\right)^{-1}.$$

The Archimedean local factor $L_\infty(s, \pi_\infty \times \tilde{\pi}'_\infty)$ is defined by

$$L_\infty(s, \pi_\infty \times \tilde{\pi}'_\infty) = \prod_{j=1}^{m} \prod_{k=1}^{m'} \Gamma_{\mathbb{R}}(s + \mu_{\pi \times \tilde{\pi}'}(j, k))$$

where the complex numbers $\mu_{\pi \times \tilde{\pi}'}(j, k)$ satisfy the trivial bound

$$\text{Re } \mu_{\pi \times \tilde{\pi}'}(j, k) > -1. \tag{2.2}$$

Denote

$$\Phi(s, \pi \times \tilde{\pi}') = L_\infty(s, \pi_\infty \times \tilde{\pi}'_\infty) L(s, \pi \times \tilde{\pi}').$$

We will need the following properties of the L-functions $L(s, \pi \times \tilde{\pi}')$ and $\Phi(s, \pi \times \tilde{\pi}')$.

RS1. The Euler product for $L(s, \pi \times \tilde{\pi}')$ in (2.1) converges absolutely for Re $s > 1$ (Jacquet and Shalika [5]).

RS2. The complete L-function $\Phi(s,\pi\times\tilde{\pi}')$ has an analytic continuation to the entire complex plane and satisfies a functional equation

$$\Phi(s,\pi\times\tilde{\pi}')=\varepsilon(s,\pi\times\tilde{\pi}')\Phi(1-s,\tilde{\pi}\times\pi'),$$

with

$$\varepsilon(s,\pi\times\tilde{\pi}')=\tau(\pi\times\tilde{\pi}')Q_{\pi\times\tilde{\pi}'}^{-s}$$

where $Q_{\pi\times\tilde{\pi}'}>0$ and $\tau(\pi\times\tilde{\pi}')=\pm Q_{\pi\times\tilde{\pi}'}^{1/2}$ (Shahidi [20], [21], [22], and [23]).

RS3. Denote $\alpha(g)=|\det(g)|$. When $\pi'\not\cong\pi\otimes\alpha^{it}$ for any $t\in\mathbb{R}$, $\Phi(s,\pi\times\tilde{\pi}')$ is holomorphic. When $m=m'$ and $\pi'\cong\pi\otimes\alpha^{i\tau_0}$ for some $\tau_0\in\mathbb{R}$, the only poles of $\Phi(s,\pi\times\pi')$ are simple poles at $s=i\tau_0$ and $1+i\tau_0$ coming from $L(s,\pi\times\tilde{\pi}')$ (Jacquet and Shalika [5], [6], Moeglin and Waldspurger [12]). We will only consider the latter case in the proof of Theorem 1.3.

RS4. $\Phi(s,\pi\times\tilde{\pi}')$ is meromorphic of order one away from its poles, and bounded in vertical strips (Gelbart and Shahidi [2]).

RS5. $\Phi(s,\pi\times\tilde{\pi}')$ and $L(s,\pi\times\tilde{\pi}')$ are non-zero in Re $s\geq 1$. (Shahidi [20])

3. Estimation of logarithmic derivatives. Let $\mathbb{C}(m)$ be the complex plane with the discs

$$|s-2n+\mu_{\pi\times\tilde{\pi}'}(j,k)|<\frac{1}{8m^2},\quad n\leq 0,\quad 1\leq j,k\leq m,$$

excluded. Here we give a remark about the structure of $\mathbb{C}(m)$. For $j,k=1,\cdots,m$, denote by $\beta(j,k)$ the fractional part of $\mathrm{Re}(\mu_{\pi\times\tilde{\pi}'}(j,k))$. In addition we let $\beta(0,0)=0$ and $\beta(m+1,m+1)=1$. Then all $\beta(j,k)\in[0,1]$, and hence there exist $\beta(j_1,k_1),\beta(j_2,k_2)$ such that $\beta(j_2,k_2)-\beta(j_1,k_1)\geq 1/(3m^2)$ and there is no $\beta(j,k)$ lying between $\beta(j_1,k_1)$ and $\beta(j_2,k_2)$. It follows that the strip $S_0=\{s:\beta(j_1,k_1)+1/(8m^2)\leq \mathrm{Re}\ s\leq\beta(j_2,k_2)-1/(8m^2)\}$ is contained in $\mathbb{C}(m)$. Consequently, for all $n=0,-1,-2,\cdots$, the strips

$$S_n=\{s:n+\beta(j_1,k_1)+1/(8m^2)\leq \mathrm{Re}\ s\leq n+\beta(j_2,k_2)-1/(8m^2)\}\tag{3.1}$$

are subsets of $\mathbb{C}(m)$. This structure of $\mathbb{C}(m)$ will be used later.

In Liu and Ye [10] and [11], we proved the following lemmas. It is believed that a much sharper result than Lemma 3.1 could be obtained using Selberg's explicit formula. This and generalization to a wider class of L-functions will be studied in a subsequent paper.

Lemma 3.1. *Assume $m=m'$ and $\pi'\cong\pi\otimes\alpha^{i\tau_0}$ for some nonzero $\tau_0\in\mathbb{R}$.* (a) *Let $T>2$. The number $N(T)$ of zeros of $L(s,\pi\times\tilde{\pi}')$ in the region* $0\leq \mathrm{Re}\ s\leq 1, |\mathrm{Im}\ s|\leq T$ *satisfies*

$$N(T+1)-N(T)\ll\log(Q_{\pi\times\tilde{\pi}'}T)$$

and

$$N(T) \ll T\log(Q_{\pi\times\tilde{\pi}'}T).$$

(b) *For any* $|T|>2$, *we have*

$$\sum_{|T-\text{Im }\rho|>1}\frac{1}{(T-\text{Im }\rho)^2}\ll\log(Q_{\pi\times\tilde{\pi}'}|T|).$$

(c) *Let* $s=\sigma+it$ *with* $-2\le\sigma\le 2, |t|>2$. *If* $s\in\mathbb{C}(m)$ *is not a zero of* $L(s,\pi\times\tilde{\pi}')$, *then*

$$\begin{aligned}&\frac{d}{ds}\log L(s,\pi\times\tilde{\pi}')\\&\qquad=\sum_{|t-\text{Im }\rho|\le 1}\frac{1}{s-\rho}-\frac{1}{s-1-i\tau_0}-\frac{1}{s-i\tau_0}+O\big(\log(Q_{\pi\times\tilde{\pi}'}|t|)\big).\end{aligned}$$

(d) *For* $|T|>2$, *there exists* τ *with* $T\le\tau\le T+1$ *such that when* $-2\le\sigma\le 2$

$$\frac{d}{ds}\log L(\sigma+i\tau,\pi\times\tilde{\pi}')\ll\log^2(Q_{\pi\times\tilde{\pi}'}|\tau|).$$

(e) *For* $|T|>2$, *there exists* τ *with* $T\le\tau\le T+1$ *such that when* $-2\le\sigma\le 2$

$$\frac{d^2}{ds^2}\log L(\sigma+i\tau,\pi\times\tilde{\pi}')\ll\log^3(Q_{\pi\times\tilde{\pi}'}|\tau|).$$

LEMMA 3.2. *Assume* $m=m'$ *and* $\pi'\cong\pi\otimes\alpha^{i\tau_0}$ *for some nonzero* $\tau_0\in\mathbb{R}$ *as before. If* s *is in some strip* S_n *as in (3.1) with* $n\le -2$, *then*

$$\frac{d^2}{ds^2}\log L(s,\pi\times\tilde{\pi}')\ll_m 1.$$

4. Zero free regions. We need a zero free region for the Rankin-Selberg L-function $L(s,\pi\times\tilde{\pi}')$ which was proved by Moreno [13] and [14]. See also Gelbart, Lapid, and Sarnak [1], and Sarnak [18]. In order for later usage, we formulate the theorem for automorphic L-functions attached to cuspidal representations of GL_m over an algebraic number field F. Similar formulation can also be made to Moreno's zero free region near the possible pole. As in [13] and [14], the constant c' in (4.1) below can be made more precise in terms of the infinite types of the representations.

Denote by q_v the number of elements in the residue field of F_v at a non Archimedean place v of F. Let π and π' be any automorphic irreducible cuspidal representations of $GL_m(F_{\mathbb{A}})$ and $GL_{m'}(F_{\mathbb{A}})$, respectively. Then their Rankin-Selberg L-function is defined by

$$L(s,\pi\times\tilde{\pi}')=\prod_{v<\infty}L_v(s,\pi_v\times\tilde{\pi}'_v)$$

where

$$L_v(s,\pi_v\times\tilde\pi'_v)=\prod_{j=1}^{m}\prod_{k=1}^{m'}\big(1-\alpha_\pi(v,j)\bar\alpha_{\pi'}(v,k)q_v^{-s}\big)^{-1}$$

for Re $s>1$ and by analytic continuation to $\mathbb{C}$. We define its Archimedean local factors in the standard way. This Rankin-Selberg L-function satisfies the same properties **RS1** through **RS5**, and the lemmas in §3 also hold.

THEOREM 4.1. *Let π (resp. π') be any automorphic irreducible cuspidal representation of $GL_m(F_{\mathbb{A}})$ (resp. $GL_{m'}(F_{\mathbb{A}})$). Assume that at least one of π and π' is self-contragredient: $\pi\cong\tilde\pi$ or $\pi'\cong\tilde\pi'$. Then there is an effectively computable constant c' such that there is no zero of $L(s,\pi\times\tilde\pi')$ in the region*

$$\sigma\ge 1-\frac{c'}{\log(Q|t|+1)},\qquad |t|\ge 1. \tag{4.1}$$

Here Q is the largest of $Q_{\pi\times\tilde\pi}$, $Q_{\pi\times\tilde\pi'}=Q_{\pi'\times\tilde\pi}$, and $Q_{\pi'\times\tilde\pi'}$.

5. Proof of Theorem 1.3. We now prove Theorem 1.3 when $\pi'\cong\pi\otimes\alpha^{i\tau_0}$ for some $\tau_0\in\mathbb{R}$. The proof for case of π and π' being not twisted equivalent, in particular, when $m\ne m'$, is the same with all terms related to τ_0 removed. By **RS1**, we have for Re $s>1$ that

$$\frac{d}{ds}\log L(s,\pi\times\tilde\pi')=-\sum_{n=1}^{\infty}\frac{\Lambda(n)a_\pi(n)\bar a_{\pi'}(n)}{n^s},$$

and therefore

$$K(s):=\frac{d^2}{ds^2}\log L(s,\pi\times\tilde\pi')=\sum_{n=1}^{\infty}\frac{(\log n)\Lambda(n)a_\pi(n)\bar a_{\pi'}(n)}{n^s}.$$

By **RS3** and **RS5**, $K(s)$ is holomorphic in Re $s>1$. On Re $s=1$, $L(s,\pi\times\tilde\pi')$ is nonzero (**RS5**) and has only a simple pole at $s=1+i\tau_0$. Thus

$$K(s)=\frac{1}{(s-1-i\tau_0)^2}+G(s) \tag{5.1}$$

has only a double pole in Re $s\ge 1$, where $G(s)$ is analytic for Re $s\ge1$. On $\mathbb{C}$, $K(s)$ also has a double pole at each of the pole at $i\tau_0$, trivial zeros, and nontrivial zeros of $L(s,\pi\times\tilde\pi')$.

By Conjecture 1.2 and (1.1), we have

$$|a_\pi(p^k)|\le m,\quad |a_{\pi'}(p^k)|\le m'.$$

Therefore,

$$|(\log n)\Lambda(n)a_\pi(n)\bar a_{\pi'}(n)|\le mm'\log^2 n,$$

and for $\sigma > 1$,

$$\sum_{n=1}^{\infty} \frac{|(\log n)\Lambda(n)a_\pi(n)\bar{a}_{\pi'}(n)|}{n^\sigma} \ll \frac{1}{(\sigma-1)^2}.$$

Let $T = \exp(\sqrt{\log x})$ and set $b = 1 + 1/\log x$. By Perron's summation formula (see e.g. Theorem 5.1 in [7]), we have

$$\begin{aligned}
&\sum_{n\le x}(\log n)\Lambda(n)a_\pi(n)\bar{a}_{\pi'}(n)\\
=&\frac{1}{2\pi i}\int_{b-iT}^{b+iT} K(s)\frac{x^s}{s}\,ds + O\Big(\frac{x^b}{T(b-1)^2}\Big) + O\Big(\frac{x\log^3 x}{T}\Big)\\
=&\frac{1}{2\pi i}\int_{b-iT}^{b+iT} K(s)\frac{x^s}{s}\,ds + O\Big(\frac{x\log^3 x}{T}\Big). \qquad (5.2)
\end{aligned}$$

Here we used the Ramanujan Conjecture 1.2 to control the size of error terms in (5.2).

Choose a with $-2 < a < -1$ such that the line $\operatorname{Re} s = a$ is contained in the strip $S_{-2} \subset \mathbb{C}(m)$; this is guaranteed by the structure of $\mathbb{C}(m)$. Let T be the τ such that Lemma 3.1(e) holds, by adding a constant d with $0 < d < 1$ to our $T = \exp(\sqrt{\log x})$ if necessary. Now we consider the contour

$$\begin{aligned}
C_1 :&\quad b \ge \sigma \ge a, \quad t = -T;\\
C_2 :&\quad \sigma = a, \quad -T \le t \le T;\\
C_3 :&\quad a \le \sigma \le b, \quad t = T.
\end{aligned}$$

Note that the two poles, some trivial zeros, and certain nontrivial zeros of $L(s, \pi \times \tilde{\pi}')$, as well as the pole at $s = 0$ are passed by the shifting of the contour. The trivial zeros can be determined by the functional equation in **RS2**: $s = -\mu_{\pi\times\tilde{\pi}'}(j,k)$ with $a < -1 - \operatorname{Re}(\mu_{\pi\times\tilde{\pi}'}(j,k)) < 0$. Here we used the facts that $\operatorname{Re}(\mu_{\pi\times\tilde{\pi}'}(j,k)) > -1$ and $-2 < a < -1$.

Then we have

$$\frac{1}{2\pi i}\int_{b-iT}^{b+iT} K(s)\frac{x^s}{s}\ ds \tag{5.3}$$

$$=\frac{1}{2\pi i}\Big(\int_{C_1}+\int_{C_2}+\int_{C_3}\Big)K(s)\frac{x^s}{s}\ ds \tag{5.4}$$

$$+\operatorname*{Res}_{s=0} K(s)\frac{x^s}{s} \tag{5.5}$$

$$+\operatorname*{Res}_{s=i\tau_0,1+i\tau_0} K(s)\frac{x^s}{s} \tag{5.6}$$

$$+\sum_{a+1<-\mathrm{Re}(\mu_{\pi\times\tilde{\pi}'}(j,k))<1}\operatorname*{Res}_{s=-\mu_{\pi\times\tilde{\pi}'}(j,k)} K(s)\frac{x^s}{s} \tag{5.7}$$

$$+\sum_{|\mathrm{Im}\ \rho|\le T}\operatorname*{Res}_{s=\rho} K(s)\frac{x^s}{s}. \tag{5.8}$$

By Lemma 3.1(e), we get

$$\int_{C_1} \ll \int_a^b \log^3(Q_{\pi\times\tilde{\pi}'}T)\frac{x^\sigma}{T}d\sigma \ll \frac{x\log^3(Q_{\pi\times\tilde{\pi}'}T)}{T\log x}, \tag{5.9}$$

and the same upper bound also holds for the integral on C_3. By Lemma 3.2, then

$$\int_{C_2} \ll \int_{-T}^{T}\frac{x^a}{|t|+1}dt \ll \frac{\log T}{x}. \tag{5.10}$$

Obviously, (5.5) is

$$\operatorname*{Res}_{s=0} K(s)\frac{x^s}{s} = K(1). \tag{5.11}$$

Since the poles at $s=i\tau_0$ and $s=1+i\tau_0$ are double poles, the residues in (5.6) give

$$\lim_{s\to i\tau_0}\frac{d}{ds}(s-i\tau_0)^2K(s)\frac{x^s}{s}+\lim_{s\to 1+i\tau_0}\frac{d}{ds}(s-1-i\tau_0)^2K(s)\frac{x^s}{s}.$$

For the second term, we have, by (5.1),

$$\begin{aligned}
&\lim_{s\to 1+i\tau_0}\frac{d}{ds}(s-1-i\tau_0)^2K(s)\frac{x^s}{s}\\
&=\lim_{s\to 1+i\tau_0}\frac{d}{ds}\big(1+(s-1-i\tau_0)^2G(s)\big)\frac{x^s}{s}\\
&=\lim_{s\to 1+i\tau_0}\frac{d}{ds}\frac{x^s}{s}\\
&=\frac{x^{1+i\tau_0}}{1+i\tau_0}\log x-\frac{x^{1+i\tau_0}}{(1+i\tau_0)^2}.
\end{aligned} \tag{5.12}$$

The first term can be estimated similarly, and we get

$$\lim_{s\to i\tau_0}\frac{d}{ds}(s-i\tau_0)^2K(s)\frac{x^s}{s}=\lim_{s\to i\tau_0}\frac{d}{ds}\frac{x^s}{s}\ll\log x.$$

Consequently (5.6) is

$$\frac{x^{1+i\tau_0}}{1+i\tau_0}\log x-\frac{x^{1+i\tau_0}}{(1+i\tau_0)^2}+O(\log x). \tag{5.13}$$

Near a trivial zero $s=-\mu_{\pi\times\tilde{\pi}'}(j,k)$ of order l in (5.7), we can express $K(s)$ as $-l/(s+\mu_{\pi\times\tilde{\pi}'}(j,k))^2$ plus an analytic function, like in (5.1). The residues in (5.7) can therefore computed similar to what we did in (5.12). By (2.2), we know that $\mathrm{Re}(\mu_{\pi\times\tilde{\pi}'}(j,k))\geq\delta-1$ for some $\delta>0$. Consequently (5.7) is bounded

$$\sum_{a<-1-\mathrm{Re}(\mu_{\pi\times\tilde{\pi}'}(j,k))<0}\ \operatorname*{Res}_{s=-\mu_{\pi\times\tilde{\pi}'}(j,k)}K(s)\frac{x^s}{s}\ll x^{1-\delta}\log x. \tag{5.14}$$

To compute the residues corresponding to nontrivial zeros in (5.8), we note that $\Phi(s,\pi\times\tilde{\pi}')$ is of order 1 (**RS4**), and $\Phi(1,\pi\times\tilde{\pi}')\neq0$ (**RS5**). Using a standard argument, we see that

$$\sum_{|\gamma|\leq T}\frac{1}{|\rho|}\ll\log^2T.$$

Consequently, (5.8) becomes

$$\begin{aligned}\sum_{|\mathrm{Im}\ \rho|\leq T}\operatorname*{Res}_{s=\rho}K(s)\frac{x^s}{s}&=-\sum_{|\mathrm{Im}\ \rho|\leq T}\operatorname*{Res}_{s=\rho}\frac{1}{(s-\rho)^2}\frac{x^s}{s}\\&\ll\sum_{|\mathrm{Im}\ \rho|\leq T}\left|\frac{x^\rho\log x}{\rho}\right|.\end{aligned}$$

Using Moreno's zero free region in Theorem 4.1, we get

$$\ll x\exp\left(-c'\frac{\log x}{\log T}\right)(\log x)\log^2T \tag{5.15}$$

By taking $T=\exp(\sqrt{\log x})+d$ for some d with $0<d<1$, we can bound (5.9), (5.10), (5.15), and the error term in (5.2) by $O\{x\exp(-\frac{c'}{2}\sqrt{\log x})\}$ for the c' in Theorem 4.1. Using the main term from (5.13) and other error terms from (5.10), (5.11), (5.14), and (5.15), we get a proof of Theorem 1.3. $\square$

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Jianya Liu: Department of Mathematics, Shandong University
Jinan 250100, China. jyliu@sdu.edu.cn

Yangbo Ye: Department of Mathematics, The University of Iowa
Iowa City, Iowa 52242-1419, USA. yey@math.uiowa.edu