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Review

Selected Concepts of Quantum State Tomography

Artur Czerwinski

Institute of Physics, Faculty of Physics, Astronomy and Informatics, Nicolaus Copernicus University in Torun, ul. Grudziadzka 5, 87-100 Torun, Poland; aczerwin@umk.pl

Abstract: Quantum state tomography (QST) refers to any method that allows one to reconstruct the accurate representation of a quantum system based on data obtainable from an experiment. In this paper, we concentrate on theoretical methods of quantum tomography, but some significant experimental results are also presented. Due to a considerable body of literature and intensive ongoing research activity in the field of QST, this overview is restricted to presenting selected ideas, methods, and results. First, we discuss tomography of pure states by distinguishing two aspects—complex vector reconstruction and wavefunction measurement. Then, we move on to the Wigner function reconstruction. Finally, the core section of the article is devoted to the stroboscopic tomography, which provides the optimal criteria for state recovery by including the dynamics in the scheme. Throughout the paper, we pay particular attention to photonic tomography, since multiple protocols in quantum optics require well-defined states of light.

Keywords: quantum state tomography; photonic tomography; state estimation; open quantum systems; quantum master equations

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1. Introduction

The goal of quantum state tomography (QST) is to reconstruct an accurate representation of a physical system. There are many possible mathematical representations of a quantum system, including the quantum wavefunction, the state vector, the Wigner function, and the density matrix [1,2]. As a result, there is a wide range of methods and techniques related to QST [3]. The importance of QST is associated with the need to manipulate well-characterized quantum objects, which is particularly relevant to quantum key distribution [4] and quantum computing [5], including linear optics with photon counting [6].

Via QST, any state of a microscopic system, including the spin of an electron or the polarization of a photon, can be characterized using an ensemble of identical particles. Quantum measurements of distinct types provide different information about the state. It is analogous to classical tomography that can investigate a three-dimensional object by scanning it from different physical perspectives [7,8].

The simplest algorithm for QST relies on linear inversion. This method leads to an explicit formula for the tomographic reconstruction of the density matrix. However, due to measurement uncertainties, the result of linear inversion may not satisfy the conditions that are necessary for any density matrix of a physical system [9]. To avoid this problem, we can apply methods that can reliably estimate the density matrix based on data burdened with experimental errors. In this context, we often talk about maximum likelihood estimation (MLE), which guarantees positivity and normalization of the result, with the additional benefit of a substantial reduction in statistical errors [10,11]. The concept of MLE has evolved into many specific methods, including superfast MLE [12], hedged MLE [13], and the scalable maximum likelihood algorithm [14]. Apart from MLE, one can also follow the least-squares method [15] or χ^2-estimation to obtain an unknown state [16]. Different estimation methods used in QST can be compared in terms of their efficiency [17].

The stroboscopic approach to quantum tomography, which is discussed in the core section of the paper, originated in 1983 when A. Jamiołkowski published a theorem on the minimal number of distinct observables required for tomography of systems evolving according to the von Neumann equation [18]. The approach was developed in subsequent articles and applied to open quantum systems with evolution given by the Gorini–Kossakowski–Sudarshan–Lindblad (GKSL) generators [19–21]. Among other things, a general formula was proved for the minimal number of distinct observables for quantum tomography of systems with evolution given by the GKSL generator [22].

In the stroboscopic tomography, we assume that the source can repeatedly prepare quantum systems in the same unknown initial state $\rho(0)$. Thus, the researcher has access to a large number of identically prepared copies. Each individual system is measured only once. For this reason, we can neglect the changes in the state due to measurements. This assumption is in line with other models of QST, where an ensemble of multiple equally prepared systems is required to obtain information about a quantum system.

In Section 2, we discuss tomography of pure states, paying special attention to phase retrieval [23]. Then, in Section 3, measurement of the Wigner function is presented. Finally, in Section 4, we focus on stroboscopic tomography while discussing concepts related to mixed state reconstruction. In each section, particular consideration is devoted to the tomography of quantum states encoded on photons.

2. Quantum Tomography of Pure States

A quantum system is said to be pure if we have unambiguous knowledge about the state of the system. In such a case, the properties of the system are encoded in the state vector, i.e., a complex normalized vector $|\psi\rangle$ that belongs to a Hilbert space $\mathcal{H}$ such that $\dim \mathcal{H} = d < \infty$ (in this paper we do not consider Hilbert spaces of infinite dimension). Mathematically, the Hilbert space $\mathcal{H}$ is isomorphic with the space of complex vectors $\mathbb{C}^d$. Throughout the paper, we follow the Dirac notation and denote the inner product by $\langle\psi|\phi\rangle$. By convention, in quantum mechanics, the scalar product is defined to be linear in the second argument [24].

If the basis of the Hilbert space $\mathcal{H}$ is given, we can represent any vector $|\psi\rangle \in \mathcal{H}$ as a linear combination of the basis vectors:

$$|\psi\rangle = \sum_i c_i |k_i\rangle, \tag{1}$$

where $|k_i\rangle$ are the basis elements and c_i are complex coefficients. In physical terms, we say that $|\psi\rangle$ has been expressed as a quantum superposition of the states $|k_i\rangle$. Usually, we operate with an orthonormal basis, which means that $\langle k_i|k_j\rangle = \delta_{ij}$ and $c_i = \langle k_i|\psi\rangle$.

In quantum mechanics, the decomposition (1) plays an important role in the theory of measurement. To be more specific, if we take such basis vectors $\{|k_i\rangle\}$ that are the eigenvectors of a certain observable that is measured on the state $|\psi\rangle$, then the probability that the measurement gives $|k_i\rangle$ can be expressed by $|c_i|^2$. In particular, we are often interested in the position basis, which consists of eigenvectors $|\mathbf{r}\rangle$ (corresponding to eigenvalues $\mathbf{r}$). If the basis elements $|\mathbf{r}\rangle$ are non-degenerate, any state vector $|\psi\rangle$ is equivalent to a complex-valued three-dimensional function:

$$\psi(\mathbf{r}) \equiv \langle \mathbf{r}|\psi\rangle, \tag{2}$$

which is called the wavefunction corresponding to the state $|\psi\rangle$. As a result, we can conclude that under these assumptions, the wavefunction can be considered the accurate representation of a pure quantum system.

Therefore, when it comes to pure states, in this section, we shall review quantum tomography methods concerning the reconstruction of both the state vector and the quantum wavefunction. Although these two quantum representations can be treated as equivalent under specific circumstances, the tomography methods are different because of the distinct mathematical nature of these objects—the first is a vector, and the second is a complex function. The methods and results outlined in the two subsequent parts are only a small fraction of the whole picture.

2.1. Complex Vector Reconstruction

In the case of pure state tomography, we are analyzing the problem of recovering a complex vector from experimentally accessible data—it is commonly referred to as phase retrieval [25]. The very same type of research problem is investigated in many other areas of science, including pure mathematics [26], speech recognition [27], ptychographic microscopy [28], and signal processing [29,30]. Thus, there is vast literature concerning phase retrieval. In recent years, it has been indicated that the problem of complex vector reconstruction can be efficiently studied within the theory of frames; see, for example, references [31,32]. This concept has led to numerous applications in physical sciences. For example, in quantum optics, frames have been proposed as an efficient tool for determining the quantum state of photons [33–35].

Let us start with the following definition.

Definition 1 (Frame)**.** *By an N-element complex frame in $\mathbb{C}^d$ we mean a set of complex vectors that span $\mathbb{C}^d$. A frame is denoted by $\Theta = \{|\theta_1\rangle, \dots, |\theta_N\rangle\}$, where $|\theta_i\rangle \in \mathbb{C}^d$ and $i = 1, \dots, N$.*

In other words, a frame is a collection of vectors that provide a robust and usually non-unique representation of vectors from the analyzed space [33]. We say that a frame Θ generates intensity measurements $\{m_i(|x\rangle)\}$ for any complex vector $|x\rangle$ from the same space. The intensity measurements are represented by $m_i(|x\rangle) := |\langle\theta_i|x\rangle|^2$ for $i = 1, \dots, N$. These figures can be interpreted in two ways. On a purely mathematical level, $m_i(|x\rangle)$ is the squared modulus of the inner product of $|x\rangle$ with the corresponding frame vector. In physics, for any two state vectors, the inner product $\langle\psi|\phi\rangle$ is typically interpreted as the probability amplitude for the state $|\phi\rangle$ to collapse into the state $|\psi\rangle$ [1]. Therefore, on the grounds of quantum physics, $m_i(|x\rangle)$ represents the result of a projective measurement performed on the state $|x\rangle$ with the measurement operator $\Pi_i = |\theta_i\rangle\langle\theta_i|$.

Then, we ask whether the non-linear map $\mathcal{J}_\Theta$ defined by a frame Θ

$$\mathcal{J}_\Theta : \, |x\rangle \quad \rightarrow \quad \left(|\langle\theta_i|x\rangle|^2\right)_{i=1,\dots,N} \tag{3}$$

is sufficient to determine the complex vector $|x\rangle$. General criteria for complex vector reconstruction remain unknown. Let us revise recent theoretical results and their applications to photonic state tomography.

We say that phase retrieval is feasible when for a fixed set of intensity measurements, we can get vectors that differ only by a scalar of norm one. In other words, if $|x\rangle$ and $|x'\rangle$ denote the results of vector reconstruction, then $|x\rangle = e^{i\omega}|x'\rangle$ where $\omega \in \mathbb{R}$ [36]. This statement is in line with the convention of quantum information theory, where the overall phase factor is neglected, since it does not affect the result of a projective measurement. In other words, it is possible to reconstruct a complex vector $|x\rangle$ if and only if the non-linear map $\mathcal{J}_\Theta$ (generated by a frame Θ) is injective. Throughout the paper, we say that the frame Θ generates (or defines) injective measurements to refer to the situations when phase retrieval is feasible.

A relevant research problem of phase retrieval relates to the minimal number of elements in the frame to guarantee the reconstruction of a complex vector. In reference [32], A. S. Bandeira et al. proposed a conjecture according to which a frame that comprises less than $4d - 4$ vectors cannot define injective intensity measurements if we want to reconstruct a vector $|x\rangle$ such that $|x\rangle \in \mathbb{C}^d$. Furthermore, also in reference [32], the authors formulated the second part of the conjecture that a generic frame consisting of $4d - 4$ vectors (or more) generates injective measurements on $\mathbb{C}^d$. The latter part of the conjecture was proved in [36], where it was demonstrated that the map $\mathcal{J}_\Theta$ is injective for a generic frame Θ with at least $4d-4$ elements.

However, the first part of the conjecture from reference [32] has been rejected. In reference [37], the author proved a result that contradicted the hypothesis on the number $4d-4$ vectors being a threshold for phase retrieval. C. Vinzant proposed a frame in $\mathbb{C}^4$ that consisted of 11 vectors and proved that it generated injective measurements on $\mathbb{C}^4$. Consequently, this result is highly significant, as it shows that the figure $4d-4$ cannot be considered the boundary number of intensity measurements.

As a consequence, the current knowledge about phase retrieval does not give a precise answer to the problem: for a complex vector space $\mathbb{C}^d$, what is the minimal number of elements of the frame Θ so that the map $\mathcal{J}_\Theta$ can be injective? For physical applications, it would be desirable to know how many intensity measurements are needed, in general, to reconstruct an unknown complex vector. Nevertheless, in [32], the authors proposed a relatively efficient way to verify whether a frame Θ generates injective measurements. Their approach is presented below as a theorem.

Theorem 1 (Bandeira et al. 2014 [32])**.** *A frame $\Theta = \{|\theta_1\rangle, \dots, |\theta_N\rangle\}$ (where $|\theta_i\rangle \in \mathbb{C}^d$) defines injective measurements, which implies that one can reconstruct an unknown vector $|x\rangle \in \mathbb{C}^d$ based on $|\langle\theta_i|x\rangle|^2$ where $i = 1, \dots, N$, if and only if the linear space*

$$\mathcal{L}_\Theta := \{Q \in \mathbb{C}^{d\times d} : \langle\theta_1|Q|\theta_1\rangle = \ldots = \langle\theta_N|Q|\theta_N\rangle = 0\} \tag{4}$$

does not contain any non-zero Hermitian matrix of the rank ≤ 2.

Theorem 1 precisely states the sufficient condition that has to be satisfied so that a frame Θ defines injective measurements, and as a result, it is possible to reconstruct a complex vector on the basis of the intensity measurements generated by the frame. One can quickly verify for a given frame Θ whether the condition (4) is satisfied or not. However, there has been no proposition concerning the procedure of how to algebraically construct such a sufficient frame.

The theory of frames has been applied to the QST of pure states. In reference [34], two models of qubit tomography were compared—one was constructed based on four frame vectors that correspond to the symmetric, informationally complete, positive operator-valued measure (SIC-POVM) [38], whereas the other scheme utilized an overcomplete frame with six vectors taken from mutually unbiased bases (MUBs) [39]. The latter measurement is particularly important for quantum optics because the vectors from MUBs are usually used to reconstruct the polarization state of light, since they represent vertical/horizontal, diagonal/antidiagonal, and right/left circular polarization states [40]. Such a method can be used not only for qubit tomography but also for polarization-entangled photon pairs. It was demonstrated experimentally that the MUBs significantly improve the fidelity in two-qubit polarization state estimation [41].

In reference [34], both detection models (SIC-POVM and MUBs) were implemented numerically for QST of single photons with simulated measurement results that were distorted by dark counts and the shot noise. The results demonstrated that the overcomplete frame had only a slight advantage over the minimal frame.

Furthermore, in reference [35], the scope of analysis was extended by proposing a similar model for 4-level quantum systems, and particular attention was paid to entangled photon pairs. In that paper, two frames were again compared—one was the minimal frame for $\mathbb{C}^4$ introduced by C. Vinzant [37], and the other contained 20 vectors from the MUBs associated with $\mathbb{C}^4$. The results showed a modest advantage of the overcomplete frame over the minimal one.

In conclusion, in the case of pure state tomography, there is a long-standing debate about the minimal number of projective measurements required to achieve this goal. The 5-bases based pure-state quantum tomographic method (5BB-QT) is one of the proposals for selecting vectors sufficient for QST [42]. The 5BB-QT method requires five observables, or equivalently, $5\,d$ projective measurements. This proposal was subsequently improved by demonstrating that fewer measurements also suffice for the pure state reconstruction. In reference [43], the authors proved that three measurement bases ($3\,d$ projectors) could be effectively implemented for QST. First, the measurement results of $2\,d$ projectors are used to generate a set of 2^{d-1} pure states. Then, the maximum value of the likelihood function is evaluated using the measurement results of the remaining d projectors, which leads to the state that fits optimally to the data. Other modern methods of QST enable approaching the Gill–Massar lower bound, which is a fundamental limit for the estimation accuracy of pure quantum states in high dimensions [44].

2.2. Wavefunction Measurement

One of the first problems of quantum tomography was formulated in 1933 when W. Pauli asked whether the quantum wavefunction of a physical system could be uniquely determined by its position and momentum probability distributions [45,46]. The difficulty stems from the complex nature of the quantum wavefunction—from an experiment, one can get real values, which implies that we need special methods to reconstruct the wavefunction from such data. In this context, we can also talk about phase retrieval, since the key problem in the wavefunction measurement relates to determining the phase factor.

Currently, it is commonly known that in general, Pauli's problem is not uniquely solvable for an arbitrary wavefunction [45,47]. The Gerchberg–Saxton algorithm is one of the tools that allow one to compute the quantum wavefunction when it is feasible [48]. The algorithm is iterative and requires repeatedly performing the Fourier transform and its inverse. This method has been widely applied in all areas of science where the problem of phase retrieval occurs. In optics, not only does it apply to the photonic wavefunction reconstruction [49], but it also applies to image recovery [50].

For a long time, it was believed that there could exist only indirect methods of wavefunction reconstruction. However, in 2011, J. S. Lundeen et al. demonstrated that the quantum wavefunction could be measured in a direct way by implementing the concept of weak measurement [51]. Since 2011, their approach has received much attention, and many other tomography models based on weak measurement have been proposed. Some researchers consider weak measurement as a tool to increase the efficacy of the quantum tomography process, whereas others look at this approach more critically [52].

In spite of the critics, the approach to quantum tomography that is based on the weak measurement has been successfully developed and generalized so that it can be applied to density matrix reconstruction as well [53,54].

3. Wigner Function Measurement

Apart from the state vector and the wavefunction, there is another quantum representation that deserves to get special attention. The Wigner function, which can be defined for both pure and mixed states, was introduced by E. Wigner in 1932 to study quantum corrections to classical statistical mechanics (for a review on the Wigner function, one can see, for example, reference [55]). Due to its ability to describe quantum phenomena by using the classical-like concept of phase space, the Wigner function seems to be an appealing approach to the mathematical description of quantum systems [56].

The definition of the Wigner function combines the distributions of the quantum particle's coordinate and momentum in terms of the wavefunction (in the most basic version, the quantum wavefunction is 1-dimensional) [56]:

$$W(x,p) = \frac{1}{\pi}\int_{-\infty}^{\infty} \psi^*(x+y)\psi(x-y)e^{2ipy}\,dy, \tag{5}$$

where $\psi(x)$ denotes the wavefunction; x and p stand for position and momentum, respectively. $W(x,p)$ is a real-valued function but is not everywhere positive. Therefore, the expression (5) is commonly called the Wigner quasiprobability distribution. It was proved by E. Wigner that the position and momentum distributions are given by the marginals of (5):

$$|\psi(x)|^2 = \int_{-\infty}^{\infty} W(x,p)dp \qquad \text{and} \qquad |\varphi(p)|^2 = \int_{-\infty}^{\infty} W(x,p)dx. \tag{6}$$

In 1949, J. Moyal proved that the Wigner function provides the expectation value of any quantum observable by phase space integration with an appropriate Wigner–Weyl ordered expression [57]. Ever since, the Wigner function has played a major role in the phase space formulation of quantum mechanics [58,59].

The definition of the Wigner function given in (5) can be generalized in such a way that it also relates to a mixed quantum state characterized by the density matrix ρ:

$$W(x,p) = \frac{1}{\pi} \int_{-\infty}^{\infty} \langle x+y|\rho|x-y\rangle e^{2ipy}\, dy, \tag{7}$$

which means that the Wigner function can be understood as a more general representation than the state vector and the wavefunction.

The definition given in (7) is called the Wigner transformation of the density matrix and can be perceived as the inverse of the Weyl transform (which maps functions in the quantum phase space formulation into operators acting in the Hilbert space).

The Wigner function received experimental significance in the 90s due to the introduction of optical homodyne tomography [60]. The researchers proposed a complete quantum procedure to obtain the density matrix of a quantum system from experimental data. The method is not straightforward and requires a sophisticated algorithm to process the data. The experiment provides the probability distributions of quadrature-field amplitude. The Wigner function is reconstructed from its marginal projections, which are related to the statistics of a selected sample of homodyne events. The technique allows one to obtain both the Wigner function and the density matrix of the mode.

The method of optical homodyne tomography was developed in [61]. The authors provided a complete experimental characterization of a family of squeezed states of light. In another significant article [62], the researchers introduced the first method of direct measurement of the Wigner function of a light mode. The main advantage of the technique is the fact that it does not require a complicated numerical algorithm to obtain the Wigner function from experimental data, since it is based on photon counting. The Wigner function at a fixed phase space point can be considered a well-defined quantum observable. What is more, this observable can be measured for optical fields by implementing an arrangement employing an auxiliary coherent probe beam. Then, the Wigner function is measured at the point in the phase space that is indicated by the amplitude and the phase of the probe field. The experiment from reference [62] was limited by the efficiency of the detectors (avalanche photodiodes), since the equipment was not able to resolve the number of simultaneously absorbed photons. However, continuous progress in single-photon detection technology has led to significant improvements in the capabilities of the off-the-shelf detectors [63].

Apart from quantum optics, the Wigner function is also relevant to atomic physics. Clusters of atoms can feature interference and diffraction phenomena, just like waves of light. For a coherent beam of helium atoms, the Wigner function was reconstructed, in a double-slit experiment, from measurements of the quantum-mechanical analog of the classical phase-space distribution [64]. This experiment demonstrated that beams of atoms behave in a strongly non-classical manner.

4. Stroboscopic Quantum Tomography of Mixed States

In the case of mixed states, the goal of quantum tomography is to determine (reconstruct) the density matrix ρ of a quantum system on the basis of data accessible from an experiment. There is a fundamental assumption connected with the methods described in the paper—we expect that the source can repeatedly perform the same procedure of preparing quantum systems in an identical (but unknown) quantum state. Therefore, we are able to have a relatively large number of identical quantum systems at our disposal.

As a result, each physical copy of the quantum system is measured only once, and for that reason, we do not take into account the post-measurement state. In this context, we often talk about an ensemble of identical quantum states that is necessary to reconstruct a quantum state.

For comparison, we distinguish two approaches to the problem of density matrix reconstruction. The first approach to quantum tomography is called static, and in this case, there is no connection between tomography and the evolution of the quantum system; see more in Section 4.1. The other approach, which is the center of attention of this paper, is called stroboscopic tomography and is discussed in Section 4.3. The fundamental assumption behind the latter method claims that we know how the system changes over time. A model of evolution is applied to a set of identically prepared quantum systems, and then we can perform measurements of some selected observables at distinct time instants. Naturally, as it has already been mentioned, each physical copy is measured only once.

The main hypothesis behind the stroboscopic approach to quantum tomography states that the knowledge about evolution can improve the effectiveness of quantum state reconstruction; i.e., performing the same kind of measurement at different time instants (on distinct physical copies but identically prepared) can provide more information about the initial quantum state than a single measurement. In quantum tomography, there is a tendency to look at the problem from the point of view of economy of measurements, which means that one would like to reconstruct the initial density matrix by measuring the minimal number of distinct observables. Each Hermitian operator is associated with a distinct physical quantity, and each measurement requires, in general, preparing a different experimental setup. Therefore, the stroboscopic approach to quantum tomography can be considered more economical, as it aims to determine the optimal criteria for quantum state reconstruction.

4.1. Static Approach to Quantum Tomography of Mixed States

It is well-known that the concept of the Bloch vector can be applied in order to obtain a description of a d-level quantum system [65]. By using a proper set of matrices, we are able to decompose any density matrix in such a way that it depends directly on measurable quantities. Any density matrix associated with the Hilbert space $\mathcal{H}$ such that $\dim \mathcal{H} = d$ contains, in general, $d^2 - 1$ independent parameters. Therefore, in this approach, one needs to find $d^2 - 1$ observables (i.e., Hermitian operators) that can be used as a basis for density matrix decomposition. Let us denote those observables by $\hat{\lambda}_i$, and $\hat{\lambda}_i^*$ stands for the Hermitian conjugate. The observables suitable for such a decomposition of ρ are the generators of the special unitary group of degree d, which is denoted by $\mathrm{SU}(d)$. The $SU(d)$ group consists of $d \times d$ unitary matrices with determinant 1. The dimension of $SU(d)$ as a real manifold equals $d^2 - 1$.Topologically, $SU(d)$ is compact and simply connected; see more in reference [66]. Based on the properties of $SU(d)$, we can enumerate a set of conditions that the observables $\hat{\lambda}_i$ have to satisfy [67]:

1. $\hat{\lambda}_i^* = \hat{\lambda}_i,$
2. $\mathrm{Tr}\, \hat{\lambda}_i = 0,$
3. $\mathrm{Tr}\big(\hat{\lambda}_i \hat{\lambda}_j\big) = 2\delta_{ij},$
4. $\big[\hat{\lambda}_i, \hat{\lambda}_j\big] = 2\,\mathrm{i}\, f_{ijk}\, \hat{\lambda}_k,$
5. $\big\{\hat{\lambda}_i, \hat{\lambda}_j\big\} = \frac{4}{d}\,\delta_{ij}\,\mathbb{I}_d + 2\, g_{ijk}\, \hat{\lambda}_k,$

where f_{ijk} denotes the completely antisymmetric tensor and g_{ijk} the completely symmetric tensor. If the operators $\hat{\lambda}_i$ have been selected so that they satisfy the above conditions, one is able to write a formula for the density matrix in the following way [67]:

$$\rho = \frac{1}{d}\,\mathbb{I}_d + \frac{1}{2}\sum_{i=1}^{d^2-1} \beta_i\,\hat{\lambda}_i, \tag{8}$$

where $\mathbb{I}_d$ denotes the identity operator and $\beta_i = \mathrm{Tr}(\rho\hat{\lambda}_i)$. The vector defined as $\mathbf{s} := (\beta_1, \beta_2, \ldots, \beta_{d^2-1})$, which consists of the expectation values of the operators $\hat{\lambda}_i$, is usually referred to as the Bloch vector (or coherence vector). The decomposition (8) means that if the basis $\hat{\lambda}_i$ is established, any density matrix is fully characterized by the mean values of the observables $\hat{\lambda}_i$. The formula (8) shows a direct link between the concept of the density matrix understood as a mathematical representation of a quantum system and real values coming from experiments.

To reconstruct the unknown density matrix ρ, one needs to determine an informationally complete set of observables, and this role can be fulfilled by $\hat{\lambda}_1, \hat{\lambda}_2, \ldots, \hat{\lambda}_{d^2-1}$. Then, one finds the mean value of each observable measured on the state ρ. We should bear in mind that the number of distinct observables required for quantum tomography in this approach increases quadratically. For this reason, the problem of determining ρ, which may belong to a high-dimensional Hilbert space, appears rather demanding. Therefore, there is a need to develop more effective methods of quantum tomography.

In the static approach to the tomography of 2-level systems, we can take the set of the Pauli matrices, denoted by $\{\sigma_1, \sigma_2, \sigma_3\}$, as the required observables; see, for example, reference [7]. The reconstruction of the density matrix is possible due to the decomposition in the basis $\{\mathbb{I}_2, \sigma_1, \sigma_2, \sigma_3\}$ that takes the form

$$\rho = \frac{1}{2}\left(\mathbb{I}_2 + \sum_{i=1}^{3} s_i\,\sigma_i\right), \tag{9}$$

where s_i is the expectation value of σ_i in state ρ.

Let us clarify that we do not have to directly measure the expectations values $\{s_1, s_2, s_3\}$ related to the Pauli matrices. For example, the polarization state of light can be characterized by the Stokes parameters [7]

$$\begin{aligned} s_1 &= \langle D|\,\rho\,|D\rangle - \langle A|\,\rho\,|A\rangle\,, \\ s_2 &= \langle R|\,\rho\,|R\rangle - \langle L|\,\rho\,|L\rangle\,, \\ s_3 &= \langle H|\,\rho\,|H\rangle - \langle V|\,\rho\,|V\rangle\,, \end{aligned}$$

where $|H\rangle$, $|V\rangle$, $|D\rangle$, $|A\rangle$, $|R\rangle$, and $|L\rangle$ denote the horizontal, vertical, diagonal, anti-diagonal, right-circular, and left-circular polarization states, respectively.

Thus, in the static approach to qubit tomography, we have to perform three different measurements to reconstruct the density matrix of our system. In general, for a d-level system, we need to measure $d^2 - 1$ different observables—more about the general approach can be found in references [67,68]. This observation implies that the standard approach can be challenging, in particular, for high-dimensional quantum states.

Additionally, in a realistic scenario, one usually performs d^2 measurements to ensure proper normalization. As a result, the qubit decomposition (9) can be rewritten [9]:

$$\rho = \frac{1}{2}\sum_{i=0}^{3} \frac{\mathcal{S}_i}{\mathcal{S}_0}\,\sigma_i, \qquad \text{where} \quad \sigma_0 := |H\rangle\,\langle H| + |V\rangle\,\langle V| \equiv \mathbb{I}_2. \tag{10}$$

A realistic representation of the coefficients can be expressed as

$$\begin{aligned} \mathcal{S}_0 &= \mathcal{N}(\langle H|\,\rho\,|H\rangle + \langle V|\,\rho\,|V\rangle), & \mathcal{S}_1 &= \mathcal{N}(\langle D|\,\rho\,|D\rangle - \langle A|\,\rho\,|A\rangle), \\ \mathcal{S}_2 &= \mathcal{N}(\langle R|\,\rho\,|R\rangle - \langle L|\,\rho\,|L\rangle), & \mathcal{S}_3 &= \mathcal{N}(\langle H|\,\rho\,|H\rangle - \langle V|\,\rho\,|V\rangle), \end{aligned}$$

where $\mathcal{N}$ is a constant that depends on the detector efficiency and light intensity.

Due to the quadratic increase in the number of observables needed for state tomography, there is a need to develop methods that operate with fewer measurements. In this paper, we investigate whether we can reconstruct an unknown density matrix ρ from an informationally incomplete set of observables $\{Q_1, \ldots, Q_r\}$ (where $r < d^2 - 1$). This problem has considerable significance from both theoretical and experimental points of view. One way to state explicit conditions required to perform quantum tomography with such a set is to follow the stroboscopic approach to quantum tomography [18].

4.2. Measurement in the Stroboscopic Quantum Tomography

In the paper, we assume that the measurable information about a quantum system is provided from an experiment by mean values of certain observables $Q_1, \ldots, Q_r$. Mathematically, measurement results can be computed through the formula:

$$\langle Q_i \rangle = \mathrm{Tr}(Q_i \, \rho), \tag{11}$$

where Q_i is a self-adjoint operator that represents a particular physical quantity.

The very same approach to quantum measurement was applied to many other physical problems; see, for example, reference [69], where mean values were used in the context of NMR spectroscopy. In reference [69], the authors presented a computational model in which the result of measurement is the expectation value of the observable. This approach is connected with the fact that NMR is a bulk phenomenon—an aggregate signal from an ensemble of particles is necessary for practical observation. Thus, the result of a measurement of an observable is not a random eigenvalue, but it is the expectation value of the observable evaluated on the ensemble. Such a computational model can be realized by NMR spectroscopy on macroscopic ensembles of quantum spins. The model introduced in reference [69] can be understood as an NMR computer—a macroscopic analog of the quantum computer. This approach to quantum computing realization was quickly developed; cf. reference [70].

As it has been already mentioned, in the case of ensemble quantum tomography, we assume that we have a large number of identically prepared systems at our disposal. Similarly to [69], we assume that we can access the knowledge about the mean values of certain observables $\{Q_1, \ldots, Q_r\}$, which mathematically relate to the density matrix ρ according to the formula (11).

Furthermore, the model requires assuming that the expectation values can be measured up to arbitrarily high precision (the same assumption can be found in [69]), which implies that the problem of measurement inaccuracy is not considered. The mean values of selected observables are treated as specific values that can be accessed from an experiment, and the problem of ensemble quantum tomography focuses on determining the relation between the set of mean values of the observables and the unknown density matrix ρ.

4.3. Stroboscopic Approach to Quantum Tomography of Mixed States

In the paper, we elaborate on the stroboscopic approach to quantum tomography, which was first proposed in reference [18] and then expanded in references [22,71]. In the stroboscopic approach, we consider a set of observables $\{Q_i\}_{i=1}^{r}$, where we assume that $r < d^2 - 1$, which means that the set of observables is not informationally complete and a single measurement of the mean value of each observable does not provide sufficient knowledge for the initial density matrix reconstruction.

However, each of the observables can be measured at several time instants $\{t_j\}_{j=1}^{s}$. Every measurement provides the expectation value of the observable that can be denoted by $\mathcal{E}_i(t_j)$. Based on the Born rule, we represent the measurements as $\mathcal{E}_i(t_j) = \mathrm{Tr}(Q_i \rho(t_j))$. Since, in this approach, the measurements are performed at different time instants, it is necessary to assume that we can precisely describe the dynamics of the quantum system. In particular, the stroboscopic tomography is formulated on the assumption that the GKSL quantum generator [19–21] is known or, equivalently, the collection of Kraus operators. Knowledge about the evolution makes it possible to determine not only the initial density matrix $\rho(0)$, but also the complete trajectory of the state since one can compute $\rho(t) = \exp(\mathbb{L}t)[\rho(0)]$, where $\mathbb{L}$ stands for the GKSL quantum generator. To make the problem of state reconstruction clearer, from now on, we assume the following definition; see, for example, reference [22].

Definition 2. *A d-level open quantum system is said to be* $(Q_1, \ldots, Q_r)$*-reconstructible on an interval* $[0, T]$ *if there exists at least one set of time instants* $\{t_j\}_{j=1}^s$ *ordered as* $0 \leq t_1 < \ldots < t_s \leq T$ *such that the trajectory of the state can be uniquely determined by the correspondence*

$$[0, T] \ni t_j \to \mathcal{E}_i(t_j) = \mathrm{Tr}(Q_i \rho(t_j)) \tag{12}$$

for $i = 1, \ldots, r$ *and* $j = 1, \ldots, s$.

The results that we collect from the measurements can be presented in a matrix form as

$$\begin{bmatrix} \mathcal{E}_1(t_1) & \mathcal{E}_1(t_2) & \cdots & \mathcal{E}_1(t_s) \\ \mathcal{E}_2(t_1) & \mathcal{E}_2(t_2) & \cdots & \mathcal{E}_2(t_s) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{E}_r(t_1) & \mathcal{E}_r(t_2) & \cdots & \mathcal{E}_r(t_s) \end{bmatrix}. \tag{13}$$

Then, we consider a fundamental question of the stroboscopic tomography.

What conditions should be satisfied so that we can reconstruct the initial density matrix $\rho(0)$ for a given quantum generator $\mathbb{L}$ based on the measurement record presented in (13)?

Other relevant questions that are studied within the stroboscopic tomography concern the minimal number of observables for a given quantum generator $\mathbb{L}$, the properties of observables, and the minimal number of time instants—and which ones. The general criteria for quantum tomography of systems with a given GKSL generator have been determined and are recapped here by formulating theorems. The proofs can be found in other papers; for examples, see references [18,22,71].

Theorem 2 (The index of cyclicity)**.** *For a quantum system with evolution given by a linear master equation of the form*

$$\frac{d\rho(t)}{dt} = \mathbb{L}\,[\rho(t)], \tag{14}$$

where $\mathbb{L}$ *denotes the generator of evolution in the GKSL form [19–21], the minimal number of distinct observables required to reconstruct the density matrix* $\rho(0)$ *is denoted by* η *and can be computed from [22]:*

$$\eta := \max_{\lambda \in \sigma(\mathbb{L})} \{\dim \mathrm{Ker}(\mathbb{L} - \lambda \mathbb{I})\}, \tag{15}$$

where $\sigma(\mathbb{L})$ *denotes the spectrum of the generator of evolution (i.e., the set of all eigenvalues of* $\mathbb{L}$*) and* $\mathbb{I}$ *is the identity operator. The figure defined in* (15) *is termed the index of cyclicity.*

According to Theorem 2, for every generator of evolution, there always exists a set of η observables such that the system is $(Q_1, \ldots Q_\eta)$-reconstructible. Moreover, if the system is also $(Q_1, \ldots Q_{\eta'})$-reconstructible, then $\eta' \geq \eta$. Naturally, it does not mean that any η observables suffice to reconstruct $\rho(0)$. The necessary condition for the set of observables can be obtained through the algebraic analysis of measurement results.

It is worth noting that the definition of the index of cyclicity is formulated based on the spectrum of the generator of evolution $\mathbb{L}$. Every linear operator can be transformed into its matrix representation, which allows us to study algebraic properties of $\mathbb{L}$. However, starting from 5×5 matrices, we do not have general formulas to compute the eigenvalues. Nevertheless, by making specific assumptions about the structure of the generator, we are able to determine the value of the index of cyclicity even in the case of dim $\mathcal{H} = d$, and for generators that depend on parameters; see, for example, reference [71].

The index of cyclicity is the most important factor that indicates the performance of the stroboscopic tomography because it tells how many different self-adjoint operators we have to measure to be able to reconstruct the initial density matrix. From the experimental point of view, this figure indicates how many distinct experimental setups we would have to prepare to perform state tomography. The index of cyclicity is a natural number from the set $\{1, 2, \ldots, d^2 - 1\}$ (where $d = \dim \mathcal{H}$), and the lower the number, the more advantageous it is to employ the stroboscopic approach instead of the standard static tomography. It is worth noting that generators such that $\eta = 1$ are known as the optimal evolution models for quantum tomography. Specific forms of such generators have been determined for $\dim \mathcal{H} = 2$ and $\dim \mathcal{H} = 3$ [72]. To sum up, the performance of the framework presented in this section depends on the properties of the evolution of an open quantum system that are encoded in the GKSL generator.

Another problem that we investigate relates to the necessary condition that the measurement operators $(Q_1, \ldots Q_\eta)$ have to satisfy so that the system with dynamics given by (14) can be $(Q_1, \ldots Q_\eta)$-reconstructible. First, by $\langle A|B\rangle$, let us denote the Hilbert–Schmidt inner product in the space $B(\mathcal{H})$, which is defined as

$$\langle A|B\rangle = \mathrm{Tr}(A^* B). \tag{16}$$

Furthermore, one can notice that assuming the evolution is given by (14) with the GKSL generator, the formula for $\rho(t)$ at an arbitrary time instant can be expressed in terms of the semigroup:

$$\rho(t) = \exp(\mathbb{L}t)[\rho(0)] = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathbb{L}^k [\rho(0)]. \tag{17}$$

However, the expansion (17) does not bring any progress to the problem of quantum tomography, as it contains an infinite sum. There are two ways to simplify this formula and make it more applicable. One way would be to apply the Cayley–Hamilton theorem, and the other approach would be to introduce the notion of the minimal polynomial. We shall focus on the latter, as it enables one to represent the exponential form of an operator by the lowest number of elements in the sum. There are many mathematical papers presenting algebraic methods that can be employed to analyze the properties of the minimal polynomials of evolution generators—e.g., references [73,74].

Let us denote by $\mu(\zeta, \mathbb{L})$ the minimal polynomial of the generator of evolution $\mathbb{L}$. Let us also assume the structure of the polynomial by:

$$\mu(\zeta, \mathbb{L}) = \sum_{k=0}^{m} d_k \zeta^k, \tag{18}$$

where m, throughout this section, stands for the degree of the minimal polynomial and $d_m = 1$ (the minimal polynomial is always monic). Sometimes, it is written that $m = \deg \mu(\zeta, \mathbb{L})$.

The generator $\mathbb{L}$ has to satisfy its minimal polynomial, which means that

$$\sum_{k=0}^{m} d_k \mathbb{L}^k = 0.$$

Consequently, the mth power of $\mathbb{L}$ (and every higher power) can be represented as a linear combination of $\mathbb{L}^0, \mathbb{L}^1, \ldots, \mathbb{L}^{m-1}$. For the mth power of $\mathbb{L}$, we can easily get the formula by means of the coefficients of the minimal polynomial in the following way:

$$\mathbb{L}^m = -\sum_{k=0}^{m-1} d_k \mathbb{L}^k. \tag{19}$$

Therefore, the semigroup from (17) can be rewritten by using a finite number of elements:

$$\Phi(t) = \exp(\mathbb{L}t) = \sum_{k=0}^{m-1} \alpha_k(t)\mathbb{L}^k, \tag{20}$$

where $\alpha_k(t)$ are some time-dependent functions, and it can be proved that they are mutually linearly independent [75].

Interestingly, there exists an explicit relation between the set of functions $\{\alpha_k(t)\}$ and the coefficients $\{d_k\}$ of the minimal polynomial of $\mathbb{L}$. If we calculate the time derivative of the map from (20), we obtain:

$$\frac{d\,\Phi(t)}{d\,t} = \sum_{k=0}^{m-1} \frac{d\,\alpha_k(t)}{d\,t}\,\mathbb{L}^k, \tag{21}$$

but, on the other hand, the same derivative can be expressed as:

$$\begin{aligned} \frac{d\,\Phi(t)}{d\,t} &= \mathbb{L}\exp(\mathbb{L}t) = \sum_{k=0}^{m-1} \alpha_k(t)\,\mathbb{L}^{k+1} \\ &= \sum_{k=0}^{m-2} \alpha_k(t)\,\mathbb{L}^{k+1} + \alpha_{m-1}(t)\,\mathbb{L}^m \\ &= \sum_{k=0}^{m-2} \alpha_k(t)\,\mathbb{L}^{k+1} - \left(\sum_{k=0}^{m-1} d_k\mathbb{L}^k\right)\alpha_{m-1}(t), \end{aligned} \tag{22}$$

where the last expression has been obtained by substituting $\mathbb{L}^m$ with the formula based on the minimal polynomial (19). If we compare the expressions (21) and (22), we obtain a set of differential equations that demonstrate the interdependence between the coefficients of the minimal polynomial of $\mathbb{L}$ and the functions $\{\alpha_k(t)\}$ [71]

$$\begin{aligned} \frac{d\,\alpha_0(t)}{d\,t} &= -d_0\,\alpha_{m-1}(t), \\ \frac{d\,\alpha_1(t)}{d\,t} &= \alpha_0(t) - d_1\,\alpha_{m-1}(t), \\ \frac{d\,\alpha_2(t)}{d\,t} &= \alpha_1(t) - d_2\,\alpha_{m-1}(t), \\ &\vdots \\ \frac{d\,\alpha_{m-1}(t)}{d\,t} &= \alpha_{m-1}(t) - d_{m-1}\,\alpha_{m-1}(t). \end{aligned} \tag{23}$$

The functions $\{\alpha_k(t)\}$ can be computed from the above set of differential equations, provided the structure of the minimal polynomial of the generator $\mathbb{L}$ is known. Thus, the ability to determine, for a given generator of evolution, the structure of its minimal polynomial, appears to be one of the crucial algebraic methods that are required to implement the stroboscopic approach to quantum tomography.

Assuming that we can calculate the functions $\{\alpha_k(t)\}$, which constitute the map $\Phi(t)$ according to (20), we can expand the formula for the results of measurements (12) in the following way:

$$\begin{aligned} \mathcal{E}_i(t_j) = \mathrm{Tr}(Q_i\rho(t_j)) = \langle Q_i|\rho(t_j)\rangle = \langle Q_i|\Phi(t_j)[\rho(0)]\rangle &= \sum_{k=0}^{m-1} \alpha_k(t_j)\langle Q_i|\mathbb{L}^k[\rho(0)]\rangle \\ &= \sum_{k=0}^{m-1} \alpha_k(t_j)\langle(\mathbb{L}^*)^k[Q_i]|\rho(0)\rangle, \end{aligned} \tag{24}$$

where $\mathbb{L}^*$ is the dual operator to $\mathbb{L}$, or in other words, $\mathbb{L}$ in the Heisenberg representation.

Therefore, the measurement record generated in the time domain (13) allows us to calculate the projections $\langle(\mathbb{L}^*)^k[Q_i]|\rho(0)\rangle$ for $k=0,1,\ldots,m-1$ and $i=1,2,\ldots,\eta$. It can be observed that the initial state $\rho(0)$ (and consequently the trajectory $\rho(t)\equiv\exp(\mathbb{L}t)[\rho(0)]$) can be uniquely determined if and only if the operators $(\mathbb{L}^*)^k[Q_i]$ (for $k=0,1,\ldots,m-1$ and $i=1,\ldots,\eta$) span the vector space of all self-adjoint operators on $\mathcal{H}$. This space shall be denoted by $B_*(\mathcal{H})$ and will be referred to as the Hilbert–Schmidt space. Now, if the evolution of the system is given by (14), this observation can be presented as a formal theorem.

Theorem 3. *The quantum system is $(Q_1,\ldots Q_\eta)$-reconstructible if and only if the operators $\{Q_1,\ldots,Q_\eta\}$ satisfy the condition [22]*

$$\bigoplus_{i=0}^{\eta} K_m(\mathbb{L},Q_i)=B_*(\mathcal{H}), \tag{25}$$

where $\bigoplus$ denotes the Minkowski sum of subspaces, m is the degree of the minimal polynomial of $\mathbb{L}$, and $\mathrm{K}_m(\mathbb{L},Q_i)$ denotes the Krylov subspace that is defined as

$$\mathrm{K}_m(\mathbb{L},Q_i):=\mathrm{Span}\Big\{Q_i,\mathbb{L}^*[Q_i],(\mathbb{L}^*)^2[Q_i],\ldots,(\mathbb{L}^*)^{\eta-1}[Q_i]\Big\}. \tag{26}$$

Remark 1. *In the Theorem 3 by Q_0, we denote an identity matrix of the appropriate dimension. One can notice that for any generator of evolution $\mathbb{L}$, we have $K_m(\mathbb{L},\mathbb{I})=\mathbb{I}$.*

When discussing the usefulness of the stroboscopic tomography, it is important to notice that if we consider a Hermitian operator $\tilde{Q}$ that belongs to the invariant subspace of the Heisenberg generator $\mathbb{L}^*$, then $K_m(\mathbb{L},\tilde{Q})=\tilde{Q}$. Therefore, repeated measurements of the same observable $\tilde{Q}$ do not lead to projections of $\rho(0)$ into distinct operators. To benefit from the stroboscopic approach, we can allow for only such observables that do not belong to the invariant subspace of $\mathbb{L}^*$. Thus, if one considers the implementation of the stroboscopic tomography in an experiment, its performance depends on whether we can implement such measurement operators that do not belong to the invariant subspace of the Heisenberg generator.

The last theorem that will be presented in this section gives the condition for the choice of time instants. Assuming that for a given generator of evolution $\mathbb{L}$, we can determine the index of cyclicity η and a set of observables $\{Q_1,\ldots,Q_\eta\}$ that satisfy the condition in the Theorem 3, then the last question that should be answered relates to the number and the choice of time instants $\{t_1,\ldots,t_s\}$. If the mean value of each observable Q_i is measured at s time instants $\{t_1,\ldots,t_s\}$, then we get a set of s equations:

$$\begin{aligned}
\mathcal{E}_i(t_1)&=\sum_{k=0}^{m-1}\alpha_k(t_1)\langle(\mathbb{L}^*)^k[Q_i]|\rho(0)\rangle,\\
\mathcal{E}_i(t_2)&=\sum_{k=0}^{m-1}\alpha_k(t_2)\langle(\mathbb{L}^*)^k[Q_i]|\rho(0)\rangle,\\
&\vdots\\
\mathcal{E}_i(t_s)&=\sum_{k=0}^{m-1}\alpha_k(t_s)\langle(\mathbb{L}^*)^k[Q_i]|\rho(0)\rangle,
\end{aligned} \tag{27}$$

which can be combined into one matrix equation:

$$\begin{bmatrix} \mathcal{E}_i(t_1) \\ \mathcal{E}_i(t_2) \\ \vdots \\ \mathcal{E}_i(t_s) \end{bmatrix} = \begin{bmatrix} \alpha_0(t_1) & \alpha_1(t_1) & \dots & \alpha_{m-1}(t_1) \\ \alpha_0(t_2) & \alpha_1(t_2) & \dots & \alpha_{m-1}(t_2) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_0(t_s) & \alpha_1(t_s) & \dots & \alpha_{m-1}(t_s) \end{bmatrix} \begin{bmatrix} \langle (\mathbb{L}^*)^0[Q_i] | \rho(0) \rangle \\ \langle (\mathbb{L}^*)[Q_i] | \rho(0) \rangle \\ \vdots \\ \langle (\mathbb{L}^*)^{m-1}[Q_i] | \rho(0) \rangle \end{bmatrix}. \tag{28}$$

On the left-hand side of the matrix Equation (28), we have a vector that contains the data accessible from an experiment, whereas on the right-hand side there is a matrix $[\alpha_k(t_j)]$ multiplied by the vector that comprises the projections $\langle (\mathbb{L}^*)^k[Q_i] | \rho(0) \rangle$. From the physical point of view, the optimal situation takes place when the matrix $[\alpha_k(t_j)]$ is square and invertible. Then, one can solve the Equation (28) by multiplying it by the matrix inverse to $[\alpha_k(t_j)]$. Thus, we can conclude that in the optimal case, the number of time instants is equal to the degree of the minimal polynomial of $\mathbb{L}$, i.e., $s = m$. We can formulate the following theorem.

Theorem 4. *For a quantum system that is $(Q_1, \dots, Q_\eta)$-reconstructible and evolves according to the GKSL Equation (14), we can determine the initial density matrix $\rho(0)$ if the time instants $\{t_j\}_{j=1}^m$ satisfy the condition [71]:*

$$\det\left[\alpha_k(t_j)\right] \neq 0, \tag{29}$$

where $\left[\alpha_k(t_j)\right]$ is a $m \times m$ matrix. In the above relation, $\alpha_k(t_j)$ denotes the functions that appear in the polynomial representation of the semigroup $\Phi(t) = exp(\mathbb{L}t)$; see (20).

In this section, the theoretical foundations of the stroboscopic tomography have been revised. To sum up, we may conclude that to construct a complete quantum tomography model for a specific generator of evolution, we can follow a four-step procedure.

1. Calculate the index of cyclicity for the generator $\mathbb{L}$ according to Theorem 2.
2. Select η distinct observables such that the condition from Theorem 3 is satisfied.
3. Determine the degree and the coefficients of the minimal polynomial of $\mathbb{L}$; then, select m time instants in such a way that $\det\left[\alpha_k(t_j)\right] \neq 0$ (i.e., follow Theorem 4).
4. Write η matrix equations of the form (28), and by solving them, calculate the projections $\langle (\mathbb{L}^*)^k[Q_i] | \rho(0) \rangle$ (where $k = 0, 1, \dots, m-1$ and $i = 1, \dots, \eta$).

If we select a very specific generator of evolution, it may be possible to construct a complete quantum tomography model, which means that at every step we will be able to obtain a concrete result, and at the end, we will get an explicit formula for the unknown initial density matrix $\rho(0)$. However, for high-dimensional cases or for parametric-dependent generators of evolution, we may have to narrow the analysis only to formulating existential theorems or performing only one step of the procedure (such as determining the minimal number of distinct observables required for quantum tomography).

There are two well-known GKSL generators of evolution that were thoroughly studied in the context of the stroboscopic tomography.

In 1983, the optimal criteria for quantum tomography of systems with evolution given by the von Neumann equation were introduced in reference [18]. The author assumed that the quantum system is isolated, and the evolution of its density matrix can be expressed by the equation:

$$\frac{d\,\rho(t)}{d\,t} = \mathbb{L}[\rho(t)] = -i[H, \rho(t)], \tag{30}$$

where H is the system's Hamiltonian. Then, the minimal number of distinct observables required for quantum tomography was determined on the basis of the algebraic properties of the Hamiltonian. In general, one would have to distinguish between the geometric and the algebraic multiplicity of the eigenvalues of H. In the stroboscopic tomography, we take into account the geometric one; see (15). However, in the case of self-adjoint operators (for example, the Hamiltonian), both multiplicities are equal. If we assume that the spectrum of

H consists of ξ unique eigenvalues, i.e., $\sigma(H) = \{\lambda_1, \dots, \lambda_\xi\}$, and to each eigenvalue λ_i we can assign its geometric multiplicity n_i, then we can formulate the following theorem.

Theorem 5 (Index of cyclicity for the von Neuman generator [18])**.** *Let $\mathcal{S}$ denote an isolated quantum system described by a Hamiltonian H. The minimal number of distinct observables $\{Q_1, \dots, Q_\eta\}$ such that the system $\mathcal{S}$ can be $(Q_1, \dots, Q_\eta)$-observable is given by the formula:*

$$\eta = \sum_{i=1}^{\xi} n_i^2, \tag{31}$$

where

$$n_i = \dim \operatorname{Ker} (\lambda_i \mathbb{I} - H) \qquad \text{and} \qquad i = 1, \dots, \xi. \tag{32}$$

Apart from the explicit formula for the index of cyclicity of the generator in the von Neumann form (30), in reference [18] the author also formulated the necessary and sufficient conditions concerning the choice of the observables.

In 2004, the stroboscopic approach was applied in order to introduce a dynamic quantum tomography model for d-level systems governed by the Gaussian semigroup [71]. The GKSL generator of evolution, in this case, can be expressed by means of a double commutator:

$$\begin{aligned} \frac{d\rho(t)}{dt} = \mathbb{L}[\rho(t)] = \frac{1}{2}\{[H, \rho(t)H] + [H\rho(t), H]\} = \\ = -\frac{1}{2}[H, [H, \rho(t)]]. \end{aligned} \tag{33}$$

Assuming again that the spectrum of H consists of ξ unique eigenvalues, i.e., $\sigma(H) = \{\lambda_1, \dots, \lambda_\xi\}$, and each eigenvalue has a corresponding multiplicity n_i, we can notice that the spectrum of the generator $\mathbb{L}$ is expressed by:

$$\sigma(\mathbb{L}) = \{u_{ij} \in \mathbb{R} : \, u_{ij} = (\lambda_i - \lambda_j)^2, \quad \text{for} \quad i, j = 1, \dots, \xi\}. \tag{34}$$

In this case, it is not possible to uniquely determine the multiplicities of the eigenvalues of $\mathbb{L}$ without making further assumptions. In reference [71], the author analyzed the worst-case scenario such that the eigenvalues of H constituted an arithmetic sequence:

$$\lambda_k = \lambda_1 + (k-1)c, \tag{35}$$

where $k = 1, \dots, \xi$ and c is a positive constant. This assumption concerning the eigenvalues of H was necessary to obtain a formula for the index of cyclicity of the generator (33).

Theorem 6 (Index of cyclicity for the Gaussian semigroup [71])**.** *Let $\mathcal{S}$ denote a d-level open quantum system with evolution given by the GKSL generator of the form* (33)*. The index of cyclicity of the generator $\mathbb{L}$ can be computed from the formula:*

$$\eta = \max\{\kappa, \gamma_1, \dots, \gamma_r\}, \tag{36}$$

where $r = \frac{\xi-1}{2}$ if ξ is odd or $r = \frac{\xi-2}{2}$ when ξ is even. Figures κ and γ_k are defined based on the multiplicities of the eigenvalues of H:

$$\kappa = \sum_{i=1}^{\xi} n_i^2,$$

$$\gamma_k = \sum_{i=1}^{\xi-k} n_i n_{i+k}.$$

Apart from the theorem on the minimal number of observables for quantum tomography of d-level systems with evolution given by (33), in reference [71], the author also proved a theorem regarding the choice of the moments of measurement.

Finally, let us notice that the stroboscopic tomography, which was originally formulated for systems with dynamics given by a GKSL generator, inspired a dynamic approach to state reconstruction of systems subject to pure decoherence (phase-damping channels). In this case, the dynamical map representing the trajectory of the state is expressed by the Hadamard product of the initial state with a time-dependent matrix that contains information about the interactions. By implementing algebraic properties of the Hadamard product, it was possible to propose a comprehensive model for dynamic state recovery [76]. The model was utilized to solve several QST problems, including systems governed by the Gaussian semigroup (33), which is also an example of pure decoherence. In this way, an alternative theoretical approach confirmed the result presented in Theorem 6; see more in reference [76].

The research into the stroboscopic approach to quantum tomography is strictly connected with other modern methods of quantum tomography. The postulate to measure the ensemble average of quantum states for certain observables is widespread and has been applied in numerous models of quantum tomography; see, for example, reference [77], Chapter 2 of reference [78], or reference [79]. In the latter reference, a noteworthy QST scheme was introduced—the observables whose expectation values were assumed to be achievable from an experiment were constructed as projectors. This method was devised as a minimalistic and experimentally feasible scheme for the reconstruction of a density operator describing the state of a single optical field mode.

5. Conclusions and Outlook

Quantum tomography has become a key component of emerging quantum technology, since it allows one to characterize quantum states, processes, and devices. A wide scope of applications drives the search for tomographic methods that achieve better efficiency and precision. In this paper, we have reviewed some results on quantum state reconstruction. These methods provide complete information about a microscopic system by determining its mathematical representation. In particular, we focused on the tomography of state vectors, the quantum wavefunction, the Wigner function, and the density operator. Many specific examples of photonic systems characterization were discussed. Finally, stroboscopic tomography was revised in connection with the possibility of reducing the number of measurements needed for state recovery. Utilizing the knowledge of the system's dynamics appears to be one of the ways to perform economical state reconstruction.

Multiple quantum tomography schemes start from the assumption that we can use a set of informationally complete measurement operators; see, for example, references [80,81]. The stroboscopic approach differs considerably from such models, as with it we assume that the initial set of observables is not complete. However, knowledge about the evolution of the system allows us to generate a set of data that is sufficient for state identification. In this regard, stroboscopic tomography is not the only framework that implements quantum dynamics. For example, it was demonstrated that suitable unitary dynamics could facilitate quantum state distinguishability [82]. Moreover, quantum state reconstruction can be conducted from a measurement record obtained as a sequence of expectation values of an observable evolving under repeated application of a single dynamical map [83]. Furthermore, continuous measurements on an ensemble of evolving cesium atomic spins have been performed to obtain the quantum state from incomplete data [84]. In the future, we can expect that the importance of QST frameworks enhanced by quantum dynamics will increase due to the ability to characterize quantum states with incomplete measurements. In addition, measurements generated in the time domain can open up future research fields for quantum tomography. This relates to situations where the state is well defined, but its dynamics remains unidentified. In such cases, the measurement record in the time domain can provide information about the system's Hamiltonian or Kraus operators, which is known as quantum process tomography.

Rapid experimental progress of quantum-enhanced technologies leads to an increased demand for efficient methods of quantum state reconstruction. Recent proposals have indicated that machine learning methods can shape the future of quantum tomography. In particular, Bayesian models have been intensively studied, since they can optimize the data collection process by adaptive measurements in state reconstruction; see, for example, references [85,86]. Finally, neural networks were proposed to facilitate QST in high dimensions. Recent experiments have proved that a neural network architecture can provide a reliable tool for QST [87]. We can forecast that if dynamic QST models are combined with machine learning algorithms, it will lead to further improvement in the subfield of quantum state reconstruction.

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