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SELECTED PROBLEMS AND RESULTS OF TOPOLOGICAL ALGEBRA

Mitrofan M. Choban, Liubomir L. Chiriac

Tiraspol State University, Chişinău, Republic of Moldova

mmchoban@gmail.com, llchiriac@gmail.com

Abstract In the class of topological algebras of a given signature the notion of totally boundedness and distinct notions of compactness are studied. The general properties of free topological algebras and compactifications of topological algebras are investigated too. We discuss some old and new results and open problems.

Keywords: topological universal algebra, totally bounded algebra, pseudocompact space, Mal'cev algebra, homogeneous algebra, pseudocompact space, rezolvable space.
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1. INTRODUCTION

We use the terminology from [12, 53, 33]. Let $\mathbb{N} = \{0, 1, 2, ...\}$ and E be the discrete sum of topological spaces $\{E_n : n \in \mathbb{N}\}$. We say that E_n is the space of symbols of n-ary operations on topological E-algebras. A topological universal algebra of signature E or a topological E-algebra is a non-empty topological space G on which there are given the continuous mappings $\{e_{nG} : E_n \times G^n \to G : n \in \mathbb{N}\}$. The mappings e_{nG} form the algebraical structure on G.

Let G be a topological E-algebra, $n \in \mathbb{N}$ and $u \in E_n$. If $n = 0$, then $u(G^0) = e_{0G}(\{0\} \times G^0)$ is a singleton and $u : G^0 \to G$ is a mapping. If $n \geq 1$, then we consider the n-ary operation $u : G^n \to G$, where $u(x_1, ..., x_n) = e_{nG}(u, x_1, ..., x_n)$.

The polynomials are constructed in the following way:

- E are polynomials;

- if $n \in N$, $n \geq 1$, $u \in E_n$, p_i is an m_i-ary polynomial, then $p = u(p_1, ..., p_n)$ is an m-ary polynomial, where

$$m = m_1 + m_2 + ... + m_n \text{ and}$$
$$p(x_1, ..., x_m) = u(p_1(x_1, ..., x_{m1}), ..., p_n(x_{m_{n-1}+1}, ..., x_m)).$$

Let $n \geq m \geq 1$, p be an n-ary polynomial and $q : \{1, 2, ..., n\} \to \{1, ..., m\}$ be a mapping. Then $v(x_1, ..., x_m) = p(x_{q(1)}, x_{q(2)}, ..., x_{q(n)})$ is an m-ary term. The polynomials are terms too. If u is an n-ary term and v is an m-ary term, then $u(x_1, ..., x_n) = v(y_1, ..., y_m)$ is an identity on E-algebras.

Denote by $|X|$ the cardinality of the set X. Any space is considered to be a T_{-1}-space.

Let $i \in \{-1, 0, 1, 2, 3, 3\frac{1}{2}\}$.

A class $\mathbb{K}$ of topological E-algebra is called a T_i-*quasivariety* if:
- any algebra $G \in \mathbb{K}$ is a T_i-space,
- if $G \in \mathbb{K}$ and B is a subalgebra of G, then $B \in \mathbb{K}$,
- the topological product of algebras from $\mathbb{K}$ is a topological algebra from $\mathbb{K}$,
- if $(G, \mathcal{T}) \in \mathbb{K}$, $\mathcal{T}'$ is a T_i-topology on G and $(G, \mathcal{T}')$ is a topological E-algebra, then $(G, \mathcal{T}') \in \mathbb{K}$.

If Ω is a set of identities and $V(E, \Omega, i)$ is the class of all topological E-algebras with identities Φ, which are T_i-spaces, then $V(E, \Omega, i)$ is a T_i-*variety*. Any T_i-variety is a T_i-quasivariety.

A class V of E-algebras is non-trivial if $|G| \geq 2$ for some $G \in V$.

The investigations of topological algebras are effected in the following directions.

DP. *Investigation of the relationship between the algebraic and topological properties of the topological E-algebras G from* $V(E, \Omega, i)$.

The afore named Problem DP is examined in light of the following problems.

DT. *Let G be an E-algebra. Determine the kinds of topologies, which can be considered on the E-algebra G that makes it a topological E-algebra.*

DA. *Let G be a topological space. Determine the types of algebraic structures that can be considered on the space G, which makes it a topological E-algebra.*

DC. *Application of the Theory of Topological Algebras.*

2. COMPATIBILITY AND INCOMPATIBILITY

Fix a signature $E = \oplus\{E_n : n \in N\}$ and a set Ω of identities. One of the general problems, determined by the direction DA, is the next.

Problem 2.1. *Let G be a topological non-empty space, E be a signature and* Ω *be a set of identities. Is it true that G admits a structure of topological E-algebra for which* $G \in V(E, \Omega, -1)$*?*

One of the first results in this direction is the Pontryagin variant of the Frobenius theorem in the abstract algebra (see [89, 90]).

Theorem 2.1. *(Frobenius - Pontryagin). Let D be a connected locally compact division ring. Then:*

1. If D is associative and commutative, then either D is the ring of reals $\mathbb{R}$*, or the ring* $\mathbb{C}$ *of complex numbers.*

2. If D is associative and non-commutative, then D is the ring of quaternions $\mathbb{H}$*.*

3. If D is non-associative, then D is the ring of octonions $\mathbb{D}$*.*

The algebra of quaternions was discovered by Hamilton in 1843 and the algebra of the octonions - by J. T. Graves in 1843. The Cayley-Diskson construction produces a sequence of topological algebras over the given topological field (in particular over the reals). In the case of reals, we obtain the algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{D}$ (see [14]).

Really, let R be a topological ring with involution $x \to x^*$. Denote by $A(R, *)$ the set $R^2 = R \times R$ with the operations:

- $(x, y) + (u, v) = (x + u, y + v)$;
- $(x, y) \cdot (u, v) = (xu - v^* y, vx + yu^*)$;
- $(x, y)^* = (x^*, -y)$.

Then $A(R, *)$ is a topological ring with the involution and a topological R-module. The mapping $x \to (x, 0)$ is the natural embedding of the ring R into $A(R, *)$. As a rule, the point $x \in R$ is identified by the point $(x, 0) \in A(R, *)$ and one may consider that $R \subseteq A(R, *)$.

If on the field $\mathbb{R}$ of reals the identical mapping $x \to x^* = x$ is the given involution, then $\mathbb{C} = A(\mathbb{R}, *)$ is the algebra of complex numbers, $\mathbb{H} = A(\mathbb{C}, *)$ is the algebra of quaternions (hypercomplex) number and $\mathbb{D} = A(\mathbb{H}, *)$ is the algebra of octonions. The algebras $\mathbb{H}_1 = A(\mathbb{H}, *)$ and $\mathbb{H}_{n+1} = A(\mathbb{H}_n, *)$ relatively to the multiplication are not with division for all n.

Corollary 2.1. *Let G be an infinite connected and locally compact space. If dim $G \notin \{1, 2, 4, 8\}$, then G does not admit the structure of the topological division ring.*

Obviously, any topological space G admits structures of topological E-algebras. For this it is sufficient to fix some continuous mapping $e_{nG} : E_n \times G^n \to G$ for any $n \in N$. In particular, the operation $xy = x$ determines on G the structure of a topological semigroup with a right identity: the element $e \in G$ is a right (respectively, left) identity if $xe = x$ (respectively, $ex = x$) for any $x \in G$.

Remark 2.1. *There exists a metrizable connected compact space A such that if xy is a structure of a topological groupoid with right identity, then $xy = x$ for all $x, y \in A$. In this case any continuous mapping $\varphi : A \times A \to A$ is one of the projections or a constant mapping. The space A is called the Cook continuum (see [89, 90]).*

Theorem 2.2. *(L. M. James, [63, 64]) If $n \notin \{0, 1, 3, 7\}$, then on the sphere S^n from the $(n + 1)$-dimensional Euclidean space E^{n+1} does not exist the structure of a topological groupoid xy with the identity $e \in S^n$.*

Theorem of L.M.James and the fixed point principle have many applications.

Corollary 2.2. *Let $n \geq 1$, $B^n = \{x \in E^n : \|x\| \leq 1\}$, and $e \in S^{n-1} \subseteq B^n \subseteq E^n$. The following assertions are equivalent:*

1. On the sphere S^{n-1} there exists the structure xy of a topological groupoid with the identity $e \in S^{n-1}$.

2. On Euclidean space E^n there exists the structure xy of a topological groupoid with the identity $e \in S^{n-1}$ such that S^{n-1} and B^n are subgroupoids.

3. On Euclidean space E^n there exists the structure xy of a topological groupoid with the identity $e \in S^{n-1}$ such that $B^n \setminus \{xy : x, y \in S^{n-1}\} \neq \emptyset$.

4. $n \in \{1, 2, 4, 8\}$

Proof. Implications $1 \to 4 \to 1$ immediately follows from the James' Theorem 2.2.

Assume that $x \cdot y$ is a structure of a topological groupoid on S^{n-1} with the identity $e \in S^{n-1}$. Let $0 = (0, ..., 0)$ be the neutral element of the Euclidean space E^n. If $x \in E^n$ and $x \neq 0$, then there exists a unique point $p(x) \in S^{n-1}$ such that $p(x) = \frac{1}{\|x\|}x$. The mapping $h : E^n \setminus \{0\} \longrightarrow S^{n-1}$ is continuous. Now we put $x * 0 = 0 * x = 0$ for each $x \in E^n$ and $y * z = \|y\| \cdot \|z\| \cdot h(y) \cdot h(z)$ for all $y, z \in E^n \setminus \{0\}$. Then $(E^n, *)$ is a topological groupoid with the identity e and $(S^{n-1}, \cdot)$, $(B^n, *)$ are subroupoids. Obviously $x \cdot y = x * y$ for $x, y \in S^{n-1}$. Implication $1 \to 2$ is proved. Implication $2 \to 3$ is obvious.

Assume that xy is a structure of a topological groupoid on E^n with the identity $e \in S^{n-1}$ and $B^n \setminus \{xy : x, y \in S^{n-1}\} \neq \emptyset$. We can suppose that $0 \in B^n \setminus \{xy : x, y \in S^{n-1}\}$. Then $x \circ y = h(xy)$ is a structure of a topological groupoid on S^{n-1} with the identity $e \in S^{n-1}$. Implication $3 \to 1$ is proved. The proof is complete. ∎

We need some definitions. A topological quasigroup is a non-empty space G with three binary operations $\{\cdot, r, l\}$ and identities $x \cdot l(x, y) = r(y, x) \cdot x = l(x, x \cdot y) = l(r(x, y) \cdot x)) = r(y \cdot x, x) = y$.

A homogeneous algebra is a non-empty space G with two binary operations $\{+, \cdot\}$ and the identities $x + x \cdot y = x \cdot (x + y) = y, x \cdot x = y \cdot y$.

A biternary Mal'cev [72] algebra is a non-empty space with two ternary operations $\{p, q\}$ and identities $p(y, y, x) = q(p(x, y, z), y, z) = p(q(x, y, z), y, z) = x$.

A Mal'cev algebra is a non-empty space with one ternary operation $\{p\}$ and identities $p(x, x, y) = p(y, x, x) = y$.

A topological quasigroup with the identity is a loop. Every topological group is a loop. A space admits a structure of a topological quasigroup if an only if it admits a structure of a topological loop (A. I. Mal'cev, 1956, [72]). Any biternary Mal'cev algebra is a Mal'cev algebra (A. I. Mal'cev, 1956, [72]). Any topological quasigroup admits a structure of a biternary Mal'cev algebra (A. I. Mal'cev, 1956, [72]). A space admits a structure of a homogeneous algebra if and only if admits a structure of a biternary Mal'cev algebra (M. M. Choban [28]). A space X admits a structure of a homogeneous algebra if and only if X is a rectifiable space, i.e. there exist a homeomorphism $h : X \times X \to X \times X$ and a point $c \in X$ such that $h(x \times X) = x \times X$ and $h(x, x) = (x, c)$ for any $x \in X$ (M. Choban [28]). The mapping h is called a rectification on X.

A space X is homogeneous if for any two points $a, b \in X$ there exists a homeomorphism $h_{ab} : X \to X$ such that $h_{ab}(a) = b$.

Let $\{+, \cdot\}$ be a structure of a homogeneous algebra on a space G, $a, b \in G$ and $x \cdot y = c$ for all $x \in G$. Then $P_a(x) = a \cdot x$, $Q_a(u) = a + x$ are homeomorphisms, $P_a^{-1} = Q_a, P_a(a) = c$ and $Q_a(c) = a$. On G there exists a structure $\{+, \cdot\}$ of homogeneous algebra such that c is the a priori given point. The mapping $\Psi(x, y) = (x, x \cdot y)$ is a homeomorphism of $G \times G$ onto $G \times G$ such that $\Psi(x, x) = (x, c)$ and $\Psi(\{x\} \times G) = \{x\} \times G$ for any $x \in G$.

Assume now that $c \in X$ and $h : G \times G \to G \times G$ is a homeomorphism such that $h(x, x) = (x, c)$ and $h(\{x\} \times G) = \{x\} \times G$ for any $x \in G$. Let $p : G \times G \to G$ be the projection $p(x, y) = y$. We put $p(h(x, y)) = x \cdot y$ and $p(h^{-1}(x, y)) = x + y$ for all $x, y \in G$. Then $\{+, \cdot\}$ is a structure of a homogeneous algebra on a space G.

Let now $f_b(x) = x + b$. Since $h^{-1}(G \times c) = \{(x, x) : x \in G\}$ and $h^{-1}(G \times \{b\})$ is the graphic of the mapping f_b, then for $b \neq c$ we have $f_b(x) \neq x$ for any $x \in G$. Thus the mapping f_b does not contains fixed points for any $b \neq c$. In particular, G is not a fixed point space. This simple fact was observed by A. S. Gul'ko ([58], Proposition 4.1). From this fact it follows.

Corollary 2.3. *Any homogeneous algebra G is a homogeneous space. If $|G| \geq 2$, then G is not a fixed point space.*

Let $X \subseteq Y$. The mapping $r : Y \to X$ is a retraction if $r(x) = x$ for all $x \in X$. If $p : Y^3 \to Y$ is a Mal'cev ternary operation on Y, then $q(x, y, z) = r(p(x, y, z))$ is a ternary Mal'cev operation on X. Thus a retract of a Mal'cev algebra is a Mal'cev algebra. In particular, any AR-space admits a structure of a Mal'cev algebra.

Corollary 2.4. *For any cardinal $\tau \geq 1$ the cube I^τ is a Mal'cev algebra and it does not admit a structure of homogeneous algebra. For τ infinite the space I^τ is homogeneous.*

Corollary 2.5. *Any AR-space is a fixed point space, admits a structure of a Mal'cev space and does not admit a structure of a homogeneous algebra.*

If a compact space X admits a structure of a Mal'cev algebra, then X is a Dugundji space (see [31, 32, 33, 79, 95]). In [8] it was proved that for a Hausdorff compactification bX of a rectifiable space X the remainder $bX \setminus X$ is a pseudocompact or a Lindelöf space. The last assertion is not true for Mal'cev algebras [8].

The next questions are open.

Problem 2.2. *Is it true that any Mal'cev algebra is a retract of some homogeneous algebra, or of some topological quasigroup?*

Problem 2.3. *(A.V.Arhangel'skii). Is it true that any compact Mal'cev algebra is a retract of some compact group?*

Problem 2.4. *Let X be a first-countable completely regular space, the Souslin number $c(X^\tau)$ is countable for any cardinal τ and X^m admits a structure of a homogeneous algebra for some cardinal m. Is it true that the space $X^{\aleph_0}$ admits a structure of a homogeneous algebra?*

The minimal infinite cardinal number τ for which $|\gamma| \leq \tau$ for any disjoint family γ of open subsets of a space X is called the *Souslin number of the space X* and it is noted by $c(X)$.

Remark 2.2. *Let X be a space and A be a non-empty set. Fix a point $0 \in X$ and an element $\alpha \in A$. For any $x \in X$ we put $e(x) = (x_\beta : \beta \in A) \in X^A$, where $x_\alpha = x$ and $x_\beta = 0$ for all $\beta \neq \alpha$. Then $h : X \longrightarrow X^A$ is an embedding. We identify X and $h(X)$ and consider that $X = h(X) \subseteq X^A$. Then the mapping $r : X^A \longrightarrow X$, where $r(x_\beta : \beta \in A) = h(x_\alpha)$, is a retraction. Thus the following assertions are equivalent:*

1. The space X admits a structure of a Mal'cev algebra.

2. The space X^τ admits a structure of a Mal'cev algebra for any cardinal number τ.

3. The space X^τ admits a structure of a Mal'cev algebra for some cardinal number $\tau \geq 1$.

3. PRECOMPACT TOPOLOGIES ON ALGEBRAS

Fix a discrete signature $E = \oplus\{E_n : n \in N\}$. A topological E-algebra G is precompact if G is a topological E-algebra of some Hausdorff compact E-algebra. In this section any space is considered to be completely regular.

Let G be a topological E-algebra. A pair (B, φ) is an a-compactification or an almost periodic compactification of G if B is a compact E-algebra, $\varphi : G \to B$ is a continuous homomorphism and the set $\varphi(G)$ is dense in B.

If (B, φ) and (H, ψ) are a-compactifications of G, then $(H, \psi) \leq (B, \varphi)$ if there exists a continuous homomorphism $g : B \to H$ such that $\psi = g \circ \varphi$. For any topological E-algebra the class $AC(G)$ of all a-compactifications of G is a complete lattice. The maximal a-compactification (bhG, b_G) of G is called the Bohr-Holm compactification of G. The mapping $b_G : G \to bhG$ is an embedding if and only if G is precompact. The Bohr-Holm compactifications were studied in [62, 60, 61, 37, 38, 42, 43, 76, 86].

Let G be a topological E-algebra and G_d be the algebra G with the discrete topology. A pair (H, φ) is called an ap-extension of G if (H, φ) is an a-compactification of G_d and $(bhG, b_G) \leq (H, \varphi)$. Thus the class $EP(G)$ of all ap-extensions of G is a complete lattice with the maximal element (apG, a_G) and minimal element (bhG, b_G). If the space G is discrete, then $apG = bhG$.

Let $\mathbb{C}$ be the field of complex numbers and $C(X)$ be the Banach algebra of all continuous bonded complex-valued functions on the space X. By $B(X)$ denote the Banach-algebra of bounded Baire-measurable complex-valued functions on X. The algebra $Ba(X)$ of Baire-measurable sets of the space X is the σ-algebra generated by the class of functionally closed sets $\{f^{-1}(0) : f \in C(X)\}$ of the space X. A function $g : X \longrightarrow \mathbb{C}$ is Baire-measurable if $g^{-1}(U) \in Ba(X)$ for each open subset U of $\mathbb{C}$. The algebra of functional-measurable sets $Fun(X)$ of the space X is the σ-algebra generated by the class of functionally sets $\{f^{-1}(H) : f \in C(X), H \subseteq \mathbb{C}\}$ of the space X. A function $g : X \longrightarrow \mathbb{C}$ is functionally-measurable if $g^{-1}(U) \in Fun(X)$ for each open subset $U \subseteq \mathbb{C}$. By $\Phi(X)$ denote the Banach-algebra of bounded functional-measurable complex-valued functions on X. By $F(X)$ denote the Banach-algebra of all bounded complex-valued functions on X. Obviously, $C(X) \subseteq B(X) \subseteq \Phi(X) \subseteq F(X)$.

If G is a topological E-algebra and (H, φ) is an a-compactification of G, then $APC_{(H,\varphi)}(G) = \{f \circ \varphi : f \in C(H)\}$. If (H, φ) is an ap-extension of G, then $AP_{(H,\varphi)}(G) = \{f \circ \varphi : f \in C(H)\}$. Let $AP(G) = AP_{(apG,a_G)}(G)$ and $APC(G) = APC_{(bhG,b_g)}(G)$. Then $AP(G)$ is the Banach algebra of all almost periodic functions on G and $APC(G)$ is the Banach algebra of all almost periodic continuous functions on G.

If G is a topological group, then the function $f \in F(G)$ is almost periodic if the closure of the set $\{f_a : a \in G\}$, where $f_a(x) = f(ax)$ for all $a, x \in G$, in $F(G)$ is a compact set.

Remark 3.1. *For a subalgebra $L \subseteq AP(G)$ the following assertions are equivalent:*

AP1. *$L = AP_{(H,\varphi)}(G)$ for some ap-extension (H, φ) of G.*

AP2. *The algebra L has the next properties:*

- $APC(G) \subseteq L$;

- L is closed in $AP(G)$;

- if $f \in L$, then $\overline{f} \in L$.

Theorem 3.1. *Let X be a pseudocompact space. Then there exists a one-to-one mapping $\Psi : \Phi(\beta X) \to \Phi(X)$ with the properties:*

1. $\Psi(f) = f|X$ and $\|f\| = \|\Psi(f)\|$.

2. $\Psi(f + g) = \Psi(f) + \Psi(g)$ and $\Psi(f \cdot g) = \Psi(f) \cdot \Psi(g)$.

3. If the sequence $\{f_n \in \Phi(\beta X) : n \in N\}$ converges pointwise to the function $f \in F(X)$, then $f \in \Phi(\beta X)$ and the sequence $\{\Psi(f_n) : n \in N\}$ converges pointwise to $\Psi(f)$.

4. $\Psi(C(\beta X) = C(X)$, $\Psi(B(\beta X) = B(X)$ and $\Psi(\Phi(\beta X) = \Phi(X)$.

5. If X is a topological group, then the function $f \in B(\beta X)$ is almost periodic on βX if and only if the function $\Psi(f)$ is almost periodic on X.

Proof. Assertions 1 - 4 were proved in [27]. Really, for any bounded continuous function $f \in C(X)$ there exists a unique continuous function βf on βX such that $f = \beta f|X$. Thus for each functionally-measurable set L of the space X there exists a functionally-measurable set L_β of the space βX such that $L = L_\beta \cap X$. For the set L_β and any point $x \in L_\beta$ there exists a G_δ-subset E of βX such that $x \in E \subseteq L_\beta$. Hence, since the space X is pseudocompact, the set L_β is unique. Therefore, for each function $g \in \Phi(X)$ there exists a unique function $\beta g \in \Phi(\beta X)$ such that $g = \beta g|X$ and the operator $\Psi(f) = f|X$ is a one-to-one mapping of $\Phi(\beta X)$ onto $\Phi(X)$. This fact proves the assertions 1 - 4. Assertion 5 is obvious. The proof is complete.

Let G be a pseudocompact E-algebra. If (H, φ) is an ap-extension of G, then denote by $G_H = \varphi(G)$ the algebra G as a topological subalgebra of the compact algebra H. The ap-extension (H, φ) is called B-measurable if $AP_{H,\varphi} \subseteq B(G)$. The ap-extension (H, φ) is called ap-pseudocompact if the space G_H is pseudocompact. ∎

Theorem 3.2. *Let G be a pseudocompact group and (H, φ) be an ap-pseudo-compact ap-extension of G. Then $AP_{(H,\varphi)}(G) \cap \Phi(G) = C(G)$.*

Proof. Let βG be the Stone-Čech compactification of the pseudocompact group. Then βG is a topological group and G be a dense subgroup of βG (see [12]). There exists a continuous homomorphism $\phi : H \longrightarrow \beta G$ such that $\phi(x) = x$ for any $x \in G \subseteq H$. Assume that $f \in (AP_{(H,\varphi)}(G) \cap \Phi(G))$. Then, by virtue of Theorem 3.1, there exist $g \in \Phi(\beta G))$ and $g_1 \in C(H)$ such that $f = g|G$ and $g(\phi(z)) = g_1(z)$ for each $z \in H$. If B is a closed subset of $\mathbb{C}$, then, since the function g_1 is continuous, the set $g_1^{-1}(B)$ is closed in H. Since the mapping ϕ is closed, the set $\phi(g_1^{-1}(B)) = g^{-1}(B)$ is closed in βG. Hence the functions g and f are continuous. The proof is complete. ∎

Therefore the almost periodicity of the functional-measurable function is in opposite with pseudocompactness. In this context it is interesting to mention the next three results.

Theorem 3.3. *(P. Kirku [70]) Let G be a divisible torsion-free Abelian group of the uncountable cardinality $|G| = 2^{\alpha} = \tau$. Then G admits exactly 2^{τ}-many compact group topologies.*

Theorem 3.4. *(W. W. Comfort and D. Remus [48, 46]). Let (G, T) be a compact Abelian group. Then G has a pseudocompact group topology $W \supseteq T$ such that the weight $w(G, W) \geq 2^{w(G,T)}$.*

Existence of compact and pseudocompact topologies on groups and rings were studied in [44, 68, 69, 94].

Let (G, T) be a compact group and W be a pseudocompact group topology on G such that $T \subseteq W$. Then the Stone-Čech compactification H of the group (G, W) is an ap-pseudocompact ap-compactification of the group G.

Theorem 3.5. *(W. Comfort, S. U. Ruczkowski and F. J. Trigos-Arrieta [47]). Every infinite Abelian group G admits a family $\mathcal{A}$ of totally bounded group topologies with $|A| = 2^{2^{|G|}}$ and the spaces (G, T), (G, w) are not homeomorphic for distinct $(T, W) \in \mathcal{A}$.*

A cardinal number τ is a strong limit cardinal if $2^m < \tau$ provided $m < \tau$. By virtue of Theorem 9.11.2 from ([12], p. 672) it follows:

Corollary 3.1. *Let τ be a sequential strong limit cardinal. Then no group of cardinality τ admits a pseudocompact group topology.*

There exist many sequential strong limit cardinals. Let $\tau \geq 2$. We put $1(\tau) = 2^{\tau}, (n + 1)(\tau) = 2^{n(\tau)}$ and $\omega(\tau) = sup\{n(\tau) : n \in N\}$. Then $\omega(\tau)$ is a sequential strong limit cardinal.

Under Martin's Axiom MA the infinite Abelian group G admits a pseudocompact group topology if and only if G admits a countable compact group topology without non-trivial convergent sequence. ([12], Theorem 9.12.9, D. Dikranjan and M. G. Tkachenko).

The following questions are intriguing.

Problem 3.1. *Let G admits some totally bounded topology and consider G as a subspace of the space apG.*

a. Is it true that any bounded subset of G is finite?

b. Is G as a subspace of apG a Dieudonné complete space?

c. Is G closed in apG relatively to the G_δ-topology on apG?

A space X is Dieudonné complete if it is complete relatively to the maximal uniformity. A subset L of a space X is bounded if any continuous function $f : X \to R$ is bounded on L. For Abelian groups the answer to the question in Problem 3.1.a is "Yes" ([12],Theorem 9.9.42 of F. J. Trigos-Arrieta). The finiteness of compact subsets $F \subseteq G$ of apG for Abelian G was established by H. Leptin [71] and I. Glicksberg [55].

If H is a measurable subgroup of the compact group G with the Haar measure λ, then or H is open in G or $\lambda(H) = 0$. Let λ be the Haar measure on apG, where G is a group with some precompact topology. Then or $\lambda(G) = 0$, or G is not measurable in apG and $\lambda(U) = 1$ for any measurable set U of apG which contains G. For example $\lambda(G) = 0$, if $|G| < 2^{\aleph_0}$, and $\lambda(U) = 1$ for any measurable set U of apG which contains G, if G admits pseudocompact group topologies. Under which conditions $\lambda(G) = 0$?

4. PARATOPOLOGICAL AND SEMITOPOLOGICAL ALGEBRAS

Fix a discrete signature $E = \oplus\{E_n : n \in N\}$ and the subspaces $S \subseteq E$ and $P \subseteq E$. An E-algebra G with the topology T is called:

- an S-semitopological E-algebra if the operation $u : G^n \to G$ is separately continuous for all $n \in N$ and $u \in S \cap E_n$;

- a P-paratopological E-algebra if the operation $u : G^n \to G$ is continuous for all $n \in N$ and $u \in P \cap E_n$;

- a (P, S)-quasitopological E-algebra if G is an S-semitopological and a P-paratopological E-algebra.

Any P-paratopological E-algebra is a topological P-algebra. In natural way the notion of a T_i-quasivariety of (P, S)-quasitopological E-algebra is defined.

Theorem 4.1. *(M. Choban [28]) Let V be a T_i-quasivariety of (P, S)-quasitopological E-algebras. Then for any non-empty space X there exists an algebra $F(X, V) \in V$ and a continuous mapping $\varphi_X : X \to F(X, V)$ such that:*

1. The set $\varphi_X(X)$ algebraically generates the E-algebra $F \to (X, V)$.

2. For any continuous mapping $g : X \to G \in V$ there exists a continuous homomorphism $\overline{g} : F(X, V) \to G$ such that $g = \overline{g} \circ \varphi_X$.

The pair $(F(X,V),\varphi_X)$ is called a free (P,S)-quasitopological E-algebra of the space X in the class V.

The algebra $F(X,V)$ is abstract free if for any mapping $g : X \to G \in V$ there exists a homomorphism $\overline{g} : F(X,V) \to G$ such that $g = \overline{g} \circ \varphi_X$.

Problem 4.1. *Assume that there exists a space $G \in V$ with a proper open subset.*
a. Under which conditions the mapping $\varphi_X : X \to F(X,V)$ is an embedding?
b. Under which conditions the algebra $F(X,V)$ is abstract free?

For varieties of topological E-algebras the Problems 4.1 were formulated by A. I. Mal'cev [72]. The answers are positive for any completely regular Hausdorff space [28].

Let $\{\cdot, {}^{-1}, e\}$ be the signature of groups. If $S = P = \{\cdot\}$ then an S-semitopological group is called a semitopological group and a P-paratopological group is called a paratopological group.
Let $\mathbb{Z}$ be the discrete group of integers.

If V is a T_i-quasivariety of semitopological groups and $V_p = \{G \in V : G \textit{ is a paratopological group}\}$, $V_g = \{G \in V : G \textit{ is a topological group}\}$, then:

1. $V_g \subseteq V_p \subseteq V$;
2. If $G \in V$ and G_d is the group G with the discrete topology, then $G_d \in V_g$;
3. If $(F(X,V),\varphi_X), (F(X,V_p),\varphi_{pX})$ and $(F(X,V_g),\varphi_{gX})$ are the free objects of a space X, then there exist the continuous homomorphisms $\psi_X : F(X,V) \to F(X,V_p)$ and $\theta_X : F(X,V_p) \to F(X,V_g)$ such that $\varphi_{pX} = \psi_X \circ \varphi_X$ and $\varphi_{gX} = \theta_X \circ \varphi_{pX}$;
4. For any completely regular space X the mappings ψ_X and θ_X are continuous isomorphisms.

Theorem 4.2. *Let $i \in \{-1, 0, 1, 3\frac{1}{2}\}$, V be a T_i-quasivariety of semitopological groups and $\mathbb{Z} \in V$. Then for any T_i-space X:*
1. $\varphi_X : X \to F(X,V)$ and $\varphi_{pX} : X \to F(X,V_p)$ are embeddings.
2. The groups $F(X,V)$ and $F(X,V_p)$ are abstract free in V and V_p respectively.

Proof. Consider the following four cases.

Case 1. $i = 3\frac{1}{2}$.

This case was proved in [28].

Case 2. $i = 1$.

On any set X consider the cofinite topology $\mathcal{T}_{cf} = \{X\} \cup \{X \setminus F : F \textit{ is a finite set}\}$. Then $(X,\mathcal{T}_{cf})$ is a compact T_1-space. If G is a group, then $(G,\mathcal{T}_{cf})$ is a semitopological compact group. We can assume that $X = \varphi_{X_d}(X) \subseteq F(X_d,V)$ as a set. Fix a T_1-space X. The group $F(X_d,V)$ is the abstract free group of the set X in the class V.

Since $(F(X_d,V),\mathcal{T}_{cf}) \in V$, there exists a unique continuous homomorphism $g : F(X,V) \longrightarrow (F(X_d,V),\mathcal{T}_{cf})$ such that $g(\varphi_X(x)) = x$ for each $x \in X$. Then g is an isomorphism and the object $F(X,V)$ is abstract free in V. Obviously, that φ_X is an embedding for the space $(X,\mathcal{T}_{cf})$. Since any T_1-space X for some cardinal number τ admits an embedding in $(F(X_d,V),\mathcal{T}_{cf})^\tau$, the mapping φ_X is an embedding.

Case 3. $i = 0$.

Let D_ω be the group $\mathbb{Z}$ with the topology $\{\emptyset\} \cup \{U_n = \{m \in \mathbb{Z} : m \geq n\} : n \in \}$. Then $D_\omega \in V_p \subseteq V$. Let a, b be two distinct points of a T_0-space X. Assume that U is open in X, $a \notin U$ and $b \in U$. Then the mapping $g : X \to D_\omega$, where $g^{-1}(1) = U$ and $g^{-1}(0) = X \setminus U$ is continuous. Thus $\varphi_{pX} : X \to F(X, V_p)$ is an embedding. The assertion 1 is proved. The proof of the assertion 2 is proved in [36].

Case 4. $i = -1$.

Let X be a space. Let G_0 be the group $\mathbb{Z} \times \mathbb{Z}$ with the topology $\{\emptyset\} \cup \{V_n = \{m \in \mathbb{Z} : m \geq n\} \times \mathbb{Z} : n \in \mathbb{Z}\}$. The space X admits an embedding in $G_0^{w(X)}$. Thus φ_X is an embedding and we can assume that $X = \varphi_X(X) \subseteq F(X, V)$. Let G_X be the group $F(X_d, V)$ with the anti-discrete topology $\{\emptyset, F(X_d, V)\}$ and $X = \varphi_{X_d}(X) \subseteq F(X_d, V)$ as a set. Then the identical mapping $f : X \longrightarrow G_X$, where $f(x) = x$ for each $x \in X$ is a continuous mapping and there exists a continuous homomorphism $g : F(X, V) \longrightarrow G_X$ such that $f = g|X$. Since g is an isomorphism, the group $F(X, V)$ is abstract free in the class V.

The proof is complete. ∎

Theorem 4.3. *Let $i \in \{1, 3\frac{1}{2}\}$, V be a non-trivial T_i-quasivariety of semitopological groups and $\mathbb{Z} \notin V$. Then for any T_i-space X:*

1. $\varphi_X : X \to F(X, V)$ and $\varphi_{pX} : X \to F(X, V_p)$ are embeddings.

2. The groups $F(X, V)$ and $F(X, V_p)$ are abstract free in V and V_p respectively.

Proof. Consider the following two cases.

Case 1. $i = 3\frac{1}{2}$.

This case is proved in [28].

Case 2. $i = 1$.

This case is similar to the case 2 in the proof of the previous theorem. ∎

A group G with a topology is called a *left* (respectively, *right*) topological group if the left translation $L_a(x) = ax$ (respectively, the right translation $R_a(x) = xa$) is continuous for any $a \in G$.

A class $\mathcal{V}$ of left topological groups is called a T_i-quasivariety of left topological groups if:

(LF1) the class $\mathcal{V}$ is multiplicative;

(LF2) if $G \in \mathcal{V}$ and A is a subgroup of G, then $A \in \mathcal{V}$;

(LF3) every space $G \in \mathcal{V}$ is a T_i-space;

(LF4) if $G \in \mathcal{V}$, $\mathcal{T}$ is a compact T_i-topology on G and $(G, \mathcal{T})$ is a left topological group, then $(G, \mathcal{T}) \in \mathcal{V}$;

From Theorems 4.2 and 4.3 it follows

Corollary 4.1. *Let $i \in \{-1, 0, 1, 3\frac{1}{2}\}$, V be a T_i-quasivariety of left topological groups and $\mathbb{Z} \in V$. Then for any T_i-space X:*

1. $\varphi_X : X \to F(X, V)$ and $\varphi_{pX} : X \to F(X, V_p)$ are embeddings.

2. The groups $F(X, V)$ and $F(X, V_p)$ are abstract free in V and V_p respectively.

Corollary 4.2. *Let $i \in \{1, 3\frac{1}{2}\}$, V be a non-trivial T_i-quasivariety of left topological groups and $\mathbb{Z} \in V$. Then for any T_i-space X:*
1. $\varphi_X : X \to F(X, V)$ and $\varphi_{pX} : X \to F(X, V_p)$ are embeddings.
2. The groups $F(X, V)$ and $F(X, V_p)$ are abstract free in V and V_p respectively.

The following assertion completes Theorem 4.3 and Corollary 4.2.

Lemma 4.1. *Let G be a left topological group and for any $x \in G$ there exists $n(x) \in \mathbb{N}$ such that $x^{n(x)} = e$. Then G is a T_1-space.*

Proof. Any finite T_0-space contains a closed one-point subset. Thus any finite left topological group is a T_1-space. By hypothesis, any point $a \in G$ is contained in the finite subgroup $G(a) = \{a^i : 0 \leq i \leq n(a)\}$. Thus $\{e\}$ is a closed subset of the group G and G is a T_1-space. ∎

Remark 4.1. *The similar assertions are true for classes of right topological groups.*

Remark 4.2. *Let V be the class of all paratopological groups, or of all paratopological Abelian groups. In [88] it was proved that the answers to the questions from Problems 4.1 are positive for any T_0-space X. For this the authors of [88] use the method of left (right) invariant pseudo-quasi-metrics. Since topology generated by the left (right) invariant pseudo-quasi-metrics may not be a paratopological topology [74, 12, 17], this point of view may create dangerous moments. Nevertheless, the extensions of the quasi-metrics from [88] are invariant quasi-metrics. For this in [36] we use the method of invariant pseudo-quasi-metrics. The method of left (right) invariant pseudo-metrics was proposed in [74] and [17]. The method of invariant pseudo-metrics on free objects was developed in [57, 30].*

Let $S \subseteq E$, G be an E-algebra, $n \geq 1$, $j \in \{1, 2, ..., n\}$, $u \in E_n \cap S$ and $a_1, a_2, ..., a_n \in G$. We put $R(G, j, u, a_1, ..., a_n) = \{x \in G : u(a_1, ..., a_{j-1}, x, a_{j+1}, ..., a_n) = a_j\}$.

The E-algebra G is called an S-simple E-algebra if for all $n \geq 1$, $j \in \{1, 2, ..., n\}$, $u \in E_n \cap S$ and $a_1, a_2, ..., a_n \in G$ we have $R(G, j, u, a_1, ..., a_n) = G$ or the set $R(G, j, u, a_1, ..., a_n)$ is finite.

All quasigroups are simple algebras.

Theorem 4.4. *Let $S \subseteq E$, $i \in \{-1, 0, 1\}$ and V be a non-trivial T_i-quasivariety of S-semitopological S-simple E-algebras. Then for any T_1-space X:*
1. the mapping $\varphi_X : X \to F(X, V)$ is an embedding.
2. the algebra $F(X, V)$ is abstract free.

Proof. Let $G \in V$. Denote by $\mathcal{T}_{cf} = \{\emptyset\} \cup \{G \setminus F : F \text{ is a finite set}\}$ the co-finite topology on G. Since G is an S-semitopological S-simple E-algebra the operation $u : G^n \to G$ is separately continuous for all $n \in N$ and $u \in S \cap E_n$. Thus $(G, \mathcal{T}_{cf}) \in V$.

Fix a non-empty T_1-space X. Denote by X_d the set X with the discrete topology. Then the E-algebra $(F(X_d, V), \varphi_{X_d})$ is the abstract free algebra of the space X in the class V. Let G_X be the algebra $F(X_d, V)$ with the co-finite topology $\mathcal{T}_{cf}$. Then $G_X \in V$, the mapping $g = \varphi_{X_d} : X \longrightarrow G_X$ is continuous and an injection. There exists a continuous homomorphism $h : F(X, V) \longrightarrow G_X$ such that $h(\varphi_X(x)) = g(x)$ for each $x \in X$. Hence g is an isomorphism and the algebra $F(X, V)$ is abstract free. Since $|X| \leq |G_X|$, then for some cardinal τ the space X admits an embedding in G_X^τ. Thus the mapping $\varphi_X : X \to F(X, V)$ is an embedding. The proof is complete ∎

Now we mention the following open problems.

Problem 4.2. *a. Let $i \in \{2, 3\}$ and V be a non-trivial T_i-quasivariety of semitopological groups. Are Theorems 4.2 and 4.3 true?*

b. Let $i \in \{2, 3\}$ and V be a non-trivial T_i-quasivariety of left topological groups. Are Corollaries 4.1 and 4.6 true?

5. THEOREMS OF MONTGOMERY AND ELLIS

In 1936 D. Montgomery [75] set the following problems.

Problem 1G. *Under which conditions a semitopological group is a paratopological group?*

Problem 2G. *Under which conditions a paratopological group is a topological group?*

D. Montgomery [75] has proved that every complete matrizable semitopological group is a paratopological group and every complete metrizable separable semitopological group is a topological group. In 1957 R. Ellis (see [52, 12]) showed that any locally compact semitopological group is a topological group.

In 1960, W. Zelazko [100] established that any complete metrizable semitopological group is a topological group. Then in 1982 N. Brand [22] proved that a Čech complete paratopological groups is a topological group. A. Bouziad [19, 20, 21] proved this assertion for semitopological groups. Many interesting results were obtained in [7, 6, 10, 65, 23, 54, 59, 80, 83, 87, 97].

We mention the following two result.

Theorem 5.1. *(P. Kenderov, I. S. Kortezov and W. B. Moors [65, 10]) If a regular semitopological group G contains a dense Čeeh complete subspace, then G is a topological group.*

Theorem 5.2. *(A. Arhangelskii and M. M. Choban [6, 7, 10]) If a regular paratopological group G contains a dense subspace which is a dense G_δ-subspace of some pseudocompact space, then G is a topological group and a dense G_δ-subspace of some pseudocompact space.*

Let $\{\cdot, r, l\}$ be the signature of quasigroups.

A quasigroup G with a topology is called:

- a paratopological quasigroup if the multiplicative operation $\{\cdot\}$ and the translations $l_a = l(a, x)$, $r_a = r(x, a)$, $a \in G$, are continuous;

- a semitopological quasigroup if the translations $L_a(x) = a \cdot x$, $R_a(x) = x \cdot a$, $l_a = l(a, x)$, $r_a = r(x, a)$, $a \in G$, are continuous.

Any paratopological (respectively, semitopological) group is a paratopological (respectively, semitopological) group. Any paratopological quasigroup is a semitopological quasigroup. In a semitopological quasigroup all translations $L_a(x) = a \cdot x$, $R_a(x) = x \cdot a$, $l_a = l(a, x)$, $r_a = r(x, a)$, $a \in G$, are homeomorphisms. Moreover, $l_a = L_a^{-1}$ and $r_a = R_a^{-1}$ for each $a \in G$.

The next problems are similar to the Montgomery's problems.

Problem 5.1. *Under which conditions a semitopological quasigroup is a paratopological quasigroup?*

Problem 5.2. *Under which conditions a paratopological quasigroup is a topological quasigroup?*

Let $(G, \cdot)$ be a groupoid. Denote by $P(G, \cdot)$ the minimal semigroup of mappings $g : G \longrightarrow G$ such that $L_a, R_a \in P(G, \cdot)$ for each $a \in G$.

A T-groupoid (or a Toyoda groupoid) is a non-empty set G with one binary operation $\{\cdot\}$ and four unary operations $\{a_1, a_2, b_1, b_2\}$ such that:

- if $x \circ y = a_1(x) \cdot b_1(y))$, then $(G, \circ)$ is a group;
- $a_1(a_2(x)) = b_1(b_2(x)) = x$ for each $x \in G$;
- $\{a_1, a_2\} \cap P(G, \cdot) \neq \emptyset$ and $\{b_1, b_2\} \cap P(G, \cdot) \neq \emptyset$.

In this case we say that $(G, \circ)$ is the group associated to the T-groupoid $(G, \cdot, a_1, a_2, b_1, b_2)$. By definitions, $a_2 = a_1^{-1}$ and $b_2 = b_1^{-1}$.

Any T-groupoid is a quasigroup.

Let $(G, \circ)$ be the topological group associated to a topological T-groupoid $(G, \cdot, a_1, a_2, b_1, b_2)$. By virtue of Albert's theorem [2, 3], all topological groups $(G, \circ)$ associated to the given T-groupoid are topologically isomorphic. In this sens that group is unique. Hence, if the topological quasigroup $(G, \cdot, r, l)$ for some mappings $\{a_1, a_2, b_1, b_2, c_1, c_2\}$ is a topological T-groupoid, then:

- we have $x \cdot y = a_2(x) \circ b_2(y)$, $l(x, y) = b_1(a_2(x)^{-1} \circ y)$ and $r(x, y) = a_1(x \circ b_2(y)^{-1})$;

- there exists many structures of the kind $\{a_1, a_2, b_1, b_2\}$ on G;

- all topological groups associated to the T-groupoids $(G, \cdot, a_1, a_2, b_1, b_2)$ are topologically isomorphic.

Therefore any topological T-groupoid is considered a topological quasigroup, too. Moreover,we assume that the T-groupoid $(G, \cdot)$ as a universal algebra is the quasigroup $(G, \cdot, r, l)$. Distinct classes of T-quasigroups were introduced and studied in [66, 67, 15, 16, 41]. For this general case we use the notion of a "T-groupoid". Since any Hausdotff topological group is a completely regular space, then the space of a topological T-groupoid is completely regular provided it is a T_0-space.

A T-groupoid $(G, \cdot, a_1, a_2, b_1, b_2)$ with a topology is called:

- a topological T-groupoid if the operation $(G, \cdot, a_1, a_2, b_1, b_2)$ are continuous and G is a topological quasigroup;

- a paratopological T-groupoid if the operation $\{\cdot, a_1, a_2, b_1, b_2\}$ are continuous;

- a semitopological T-groupoid if the multiplicative operation $\{\cdot\}$ is separate continuous and the operation $\{a_1, a_2, b_1, b_2\}$ are continuous.

If a $(G, \cdot, a_1, a_2, b_1, b_2)$ is a semitopological T-groupoid, then the operations $\{a_1, a_2, b_1, b_2\}$ are homeomorphisms. Moreover, if a T-groupoid $(G, \cdot, a_1, a_2, b_1, b_2)$ is a semitopological quasigroup, then the operation $\{a_1, a_2, b_1, b_2\}$ are homeomorphisms.

We mention that a T-groupoid $(G, \cdot, a_1, a_2, b_1, b_2)$ with topology:

- is a topological T-groupoid if and only if $(G, \cdot, r, l)$ $(G, \cdot r, l)$ is a topological quasigroup;

- is a paratopological T-groupoid if and only if $(G, \cdot, r, l)$ $(G, \cdot r, l)$ is a paratopological quasigroup;

- is a semitopological T-groupoid if and only if $(G, \cdot, r, l)$ $(G, \cdot r, l)$ is a semitopological quasigroup.

Any group with the identical mappings $\{a_1, a_2, b_1, b_2\}$ is considered a T-groupoid too. Therefore:

- any semitopological group is a a semitopological T-groupoid;
- any paratopological group is a a paratopological T-groupoid;
- any topological group is a a topological T-groupoid.

By virtue of K. Toyoda theorem [93] it follows that:

- any medial quasigroup is a T-groupoid;
- any semitopological medial quasigroup is a a semitopological T-groupoid;
- any paratopological medial quasigroup is a a paratopological T-groupoid;
- any topological medial quasigroup is a a topological T-groupoid.

Theorem 5.3. *Let $\mathcal{K}$ be a class of topological spaces. Then:*

1. Any semitopological T-groupoid $G \in \mathcal{K}$ is a topological quasigroup if and only if any semitopological group $H \in \mathcal{K}$ is a topological group.

2. Any paratopological T-groupoid $G \in \mathcal{K}$ is a topological quasigroup if and only if any paratopological group $H \in \mathcal{K}$ is a topological group.

Proof. Let $H = (G, \circ)$ be the associated group at the T-groupoid $(G, \cdot, a_1, a_2, b_1, b_2)$ with the topology and $\{a_1, a_2, b_1, b_2\}$.

Then:

- H is a semitopological group if and only if G is a semitopological T-groupoid ;
- H is a paratopological group if and only if G is a paratopological T-groupoid;
- H is a topological group if and only if G is a topological quasigroup, i.e a topological T-groupoid.

The proof is complete. ∎

Hence, Theorems 5.1 and 5.2 are true for medial quasigroups and for paramedial quasigroups.

Problem 5.3. *Is Theorem 5.2 true for any quasigroups? In particular, is Theorem 5.2 true for IP-quasigroups?*

Problem 5.4. *Is Theorem 5.2 true for any quasigroup? In particular, is Theorem 5.2 true for IP-quasigroups?*

Distinct classes of spaces and algebras were studied in [5, 6, 8, 9, 10, 11, 13, 29, 18, 24, 45, 49, 73, 77, 78, 84, 85, 91, 96, 98, 99].

6. SOLVABILITY OF ALGEBRAS

Let X be a space and τ be a cardinal. The space X is called τ-solvable if there exists a family $\{X_\alpha : \alpha \in A\}$ of pairewise disjoint dense subspaces such that $|A| \geq \tau$. A 2-solvable space is called solvable. A $|X|$-solvable space is called totally solvable.

Let $\mathcal{T}$ be a topology on a quasigroup G. The topology $\mathcal{T}$ is weakly bounded if for any non-empty set $U \in \mathcal{T}$ there exists a finite set $L \subseteq G$ such that $G = L \cdot U$. We do not suppose that $(G, \mathcal{T})$ is a topological, or a semitopological quasigroup.

Example 6.1. *Denote by $T_1(G) = \{X\} \cup \{X \setminus F : F$ is a finite subset of $G\}$ the minimal T_1-topology on the quasigroup G, i.e. the cofinite topology on G. If $b \in G$, then $T_0(G, b) = \{U \in T_1(G) : b \in U\}$ is a T_0-topology on G. Then:*

- if $\mathcal{T} \subseteq T_1(G)$, then $\mathcal{T}$ is a weakly bounded topology on G;

- $(G, T_1(G))$ is a semitopological quasigroup;

- if the set G is infinite, then $(G, T_1(G))$ is not a paratopological quasigroup;

- let G contains two distinct points and $b \in G$, then $(G, T_0(G, b))$ is not a semitopological quasigroup.

Theorem 6.1. *(M. Choban and L. Chiriac [39]) Let G be an infinite group of cardinality τ. Then there exists a disjoint family $\{B_\mu : \mu \in M\}$ of subsets of G such that:*

1. $|M| = |G|$.

2. $G = \cup\{B_\mu : \mu \in M\}$.

3. $(G \setminus B\mu) \cdot K \neq G$ for all $\mu \in M$ and every finite subset K of G.

4. The sets $\{B_\mu : \mu \in M\}$ are dense in all totally bounded topologies on G.

This theorem generalized a result of I. Protasov [81]. In [39] Theorem 6.1 is proved for IP-quasigroups. More general result was proved in [26].

Problem 6.1. *Let G be a topological quasigroup (or IP-quasigroup). Is it true that $G \times G$ is a solvable space?*

The answer is positive for groups (I. P. Protasov).

7. ON ALGEBRAS WITH DIVISIONS

Let E be a signature. If $n \geq 1, g \in E_n$ and $1 \leq i \leq n$, then

$$g(a_1, ..., a_{i-1}, x, a_{i+1}, ..., a_n) = a_i$$

is an equation on E-algebras. Denote by $e(g, n, i)$ this equation.

Let φ be a set of equations on E-algebras. By $V(E, \varphi)$ we denote the class of all topological E-algebras on which the equations $e(g, n, i) \in \varphi$ are solutions, i.e. for any $a_1, a_2, ..., a_n \in G$ there exists $b \in G$ such that $g(a_1, ..., a_{i-1}, b, a_{i+1}, a_n) = a_i$. By $V(E, u\varphi)$ we denote the class of all algebras $G \in V(E, \varphi)$ on which the equations $e(g, n, i) \in \varphi$ are unique solutions. Let $e(g, n, i)$ be an equation from φ. We say that there exists a primitive solution on G of this equation if there exists a term $h(y_1, y_2, ..., y_n)$ such that $g(a_1, ..., a_{i-1}, h(a_1, ..., a_i, ..., a_n), a_{i+1}, ..., a_n) = a_i$ for any $a_1, a_2, ..., a_n \in G$. Let $V(E, \varphi, \Pi)$ be the class of E-algebras $G \in V(E, \varphi)$ with the primitive solutions for all equations from φ. Obviously $V(E, \varphi, \Pi) \subseteq V(E, \varphi)$. In some cases we may extend the signature E and consider that the solutions from φ are operations from the signature.

There exists E-algebras in which some equations are solutions but does not exist primitive continuous solutions. From this point of view it seems to be important the next notions.

Definition 7.1. *The equation $g(a_1, ..., a_{i-1}, x, a_{i+1}, a_n) = a_i$ is with continuous division on G if for any $b \in G$ for which $g(a_1, ..., a_{i-1}, b, a_{i+1}, a_n) = a_i$ and any open set $U \ni b$ there exist the open sets $U_1 \ni a_1$, $U_2 \ni a_2$,..., $U_n \ni a_n$ such that for all $c_1 \in U_1$, $c_2 \in U_2$,..., $c_n \in U_n$ there exits $c \in V$ such that $g(c_1, ..., c_{i-1}, c, c_{i+1}, c_n) = c_i$.*

There exists equations with continuous division and without primitive continuous division.

Example 7.1. *Let $G = \{(x, y) : x^2 + y^2 = 1\}$ and $(x, y) \cdot (u, v) = (xy - yv, xv + yu)$. Then $(G, \cdot)$ is a compact group. We put $z \circ w = z \cdot w \cdot w$ for any $z, w \in G$. Then $(G, \circ)$ is a topological groupoid. Consider on G the equations $a \circ x = b$ and $y \circ a = b$. The equation $y \circ a = b$ has a primitive continuous $h(a, b) = b \cdot a^{-1} \cdot a^{-1} = b \circ a^{-1}$, where a^{-1} is the inverse element of a in the group $(G, \cdot)$. If $u = (cos(\varphi), sin(\varphi)) \in G$ and $0 \leq \varphi \leq 2\pi$, then $r(u) = (cos(\varphi/2), sin(\varphi/2))$. In this case $\lambda(a, b) = r(a^{-1} \cdot b)$ is a primitive solution of the equation $a \circ x = b$. But for the equation $a \circ x = b$ does not exist some continuous primitive solution. The equation $a \circ x = b$ is with continuous solution. The equation $a \circ x = b$ has two distinct solutions for any pair (a, b).*

Definition 7.2. *A pair $(F(X, E, \varphi), \theta_X)$ is a topological free E-algebra of a space X in the class $V(E, \varphi)$ if the following conditions hold:*

1. $F(X, E, \varphi) \in V(E, \varphi)$ and $\theta_X : X \to F(X, E, \varphi)$ is a continuous mapping.

2. If $\theta_X(X) \subseteq G \subseteq F(X, E, \varphi)$ and $G \in V(E, \varphi)$, then $F(X, E, \varphi) = G$.

3. For any continuous mapping $g : X \to G \in V(E, \varphi)$ there exists a continuous homomorphism $\overline{g} : F(X, E, \varphi) \to G$ such that $g = \overline{g} \circ \theta_X$.

Theorem 7.1. *For any non-empty space X the free object $(F(X, E, \varphi), \theta_X)$ exists and is unique.*

Proof. For any $G \in V(E, \varphi)$ and any equation $e(g, n, i)$ from φ we consider the mappings $h_{(gi)} : G^n \to G$ for which $g(a_1, ..., a_{i-1}, h_{(gi)}(a_1, ..., a_i, ..., a_n), a_{i+1}, ..., a_n) = a_i$.

Consider that h_{gi} is a symbol of a new operation. Now we put $H = \{h_{(gi)} : e(g, n, i) \in \varphi\}$ and $\varphi E = E \cup H$. Then any algebra $G \in V(E, \varphi)$ states an φE-algebra. In this case, the operations from E are continuous. Thus the φE-algebras $G \in V(E, \varphi)$ are E-paratopological φE-algebras. Let $V(\varphi E)$ be the class of all E-paratopological φE-algebras with the primitive solution from H for all equations φ. Obviously, any $G \in V(\varphi E)$ as a topological E-algebra is from $V(E, \varphi)$. Reversely, any $G \in V(E, \varphi)$ as a topological E-algebra with some fixed operations of the type h_{gi}, $e(g, n, i) \in \varphi$, is from $V(\varphi E)$. The object $(F(X, E, \varphi), \theta_X) = (F(X, V(\varphi E)), \theta_X)$ is the free object of the space X in the quasivariety $V(\varphi E)$ of φE-algebras and the desired free object in the class $V(E, \varphi)$. The proof is complete. ∎

For abstract algebra the following Theorem 7.2 was proved in [40] (see also [26], Theorem 5.3.5).

Theorem 7.2. *Let X be a non-empty space and the free object $(F(X, E, \varphi), \theta_X)$ is abstract free in the class $V(E, \varphi)$. Then any equation $e(g, n, i) \in \varphi$ has no more than two solutions in the free algebra $F(X, E, \varphi)$.*

Corollary 7.1. *Let X be a non-empty completely regular space. Then any equation $g \in \varphi$ has no more than two solutions in the free algebra $F(X, \varphi E)$.*

Remark 7.1. *Let Ω be a set of identities and φ be a set of equations on E-algebras. By $V(E, \Phi, \varphi, i) = V(E, \Phi, i) \cap V(E, \varphi)$ we denote the class of all topological E-algebras with identities Φ, which are T_i-spaces, and on which the equations $e(g, n, i) \in \varphi$ are solutions. The definition of the free object $(F(X, E, \Phi, \varphi, i), \theta_X)$ of a space X in the class $V(E, \Phi, \varphi, i)$ is as in Definition 7.2. Then, as in Theorem 7.1, one can establish that for any non-empty space X the free object $(F(X, E, \Phi, \varphi, i), \theta_X)$ exists and is unique. The Theorem 7.2 remain true for the identities of commutativity and associativity types. For any set Φ of identities that assertion is an open question.*

8. TOPOLOGICAL BIGROUPOIDS

A topological bigroupoid is a topological space G with two binary continuous operations $\{\circ, *\}$ for which there exists an element $e \in G$ such that $x \circ e = x$ for each $x \in G$.

A bigroupoid G is a bigroupoid with a division or, briefly a d-bigroupoid if for each two elements $a, b \in G$ there exist two elements $c, p \in G$ such that $a \circ c = b$ and $p \circ a = b$.

A bigroupoid G is called an a-bigroupoid if $x * (y \circ z) = (x \circ y) * z$ for all $x, y, z \in G$.

There exists a general construction of bigroupoids.

Construction 8.1, [35]. Let $(G_1, +)$ be a topoloical groupoid with a right unity e_1 and $(G_2, +)$ be a topological groupoid with a right unity e_2. We put $G = G_1 \times G_2, e =$

(e_1, e_2) and $\pi : G_1 \times G_2 \to G_1$ is the projection $\pi(x, y) = x$. Fix a continuous mapping $g : G_1 \to G$. Now we consider on G the next two binary operations:

- $(x_1, x_2) \circ (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$;
- $(x_1, x_2) * (y_1, y_2) = g(x_1 + y_1) = g(\pi((x_1, x_2) \circ (y_1, y_2)))$.

By construction:
A8.1. The operations $\{\circ, *\}$ are continuous.
A8.2. $(G, \circ, *, e)$ is a topological bigroupoid.
A8.3. G is a d-bigroupoid if and only if G_1 and G_2 are d-bigroupoids.
A8.4. If G_1 is a group, then $(G, \circ, *, e)$ is an a-bigroupoid.
These properties of the groupoid G constructed above are completed by the next general fact.

Theorem 8.1. *Let $(G, \circ, *, e)$ be an a-groupoid, $x * e = y * e$ if and only if $x = y$ and $g(x) = x * e$ for each $x \in G$. Then:*
1. $x * y = g(x \circ y)$ for all $x, y \in G$.
2. $(G, \circ)$ is a semigroup.
3. If G is a division a-groupoid, then $(G, \circ)$ is a group.

Proof. By definition, $x * y = x * (y \circ e) = (x \circ y) * e = g(x \circ y)$. Fix $x, y, z \in G$. Then $((x \circ y) \circ z) * e = (x \circ y) * (z \circ e) = (x \circ y * z) = x * (y \circ z)$, $(x \circ (y \circ z)) * e = x * ((y \circ z) \circ e)$ $= x * (y \circ z)$ and $((x \circ y) \circ z) * e = (x \circ (y \circ z)) * e$, i.e. $(x \circ y) \circ z = x \circ (y \circ z)$. The assertions 1 and 2 are proved. The assertion 3 follows from the assertion 2. ∎

Corollary 8.1. *Let $(G, \circ, *, e)$ be an a-groupoid and $(G, *)$ be a quasigroup. Then $(G, \circ)$ is a semigroup.*

9. TOPOLOGICAL E-AUTOMATA

Fix a signature E of topological algebras and a set Ω of identities on the class of all topological E-algebras. Let $V = V(E, \Omega)$ be the class of all E-algebras with the identities Ω.

Definition 9.1. *A topological E-automaton is a seven-tuple $M = (A, S, B, \sigma, \delta, a_0, F)$, where:*
- A and B are topological E-algebras, A is the space of states and B is the output space;
- S is a topological semigroup and it is the space of inputs;
- F is a closed subset of A and is called the subspace of accepting states;
- $a_0 \in F$ is the initial state;
- $\delta : A \times S \to A$ and $\sigma : A \times S \to B$ are continuous mappings, δ is the translation function and σ is the output function;

- $\delta(x, \alpha\cdot\beta) = \delta(\delta(x, \alpha), \beta)$ and $\sigma(x, \alpha\cdot\beta) = \sigma(\delta(x, \alpha), \beta)$ for all $x \in A$ and $\alpha, \beta \in S$;
- for any $s \in S$ the mappings $\delta_s : A \to A$ and $\sigma_s : A \to B$, where $\delta_s(x) = \delta(x, s)$ and $\sigma_s(x) = \sigma(x, s)$, are homomorphisms.

For $F = A$ and $E = \emptyset$ the automata M is called a Mealy machine. A topological E-automaton $M = (A, S, B, \sigma, \delta, a_0, F)$ is called a Meally topological E-automaton (or E-machine) if there exists a continuous homomorphism $\mu : A \to B$ such that $\sigma(x, s) = \varphi(\delta(x, s))$ for all $x \in A$ and $s \in S$.

Assume that the topological semigroup S is fixed and $F = A$ for any automaton $M = (A, S, B, \sigma, \delta, a_0, F)$. Denote by $M(E, \Omega)$ the class of all topological E-automaton $M = (A, S, B, \sigma, \delta, a_0, F)$ for which $A, B \in V(E, \Omega)$.
We say that $\varphi = (\varphi_1, \varphi_2) : M_1 \to M_2$ is a homomorphism of an E-automaton $M_1 = (A_1, S, B_1, \sigma_1, \delta_1, a_1, F_1)$ into an E-automaton $M_2 = (A_2, S, B_2, \sigma_2, \delta_2, a_2, F_2)$ if $\varphi_1 : A_1 \to A_2$ and $\varphi_2 : B_1 \to B_2$ are continuous homomorphisms, $\varphi_1(a_1) = a_2, \varphi_1(F_1) \subseteq F_2$ and $\varphi_1(\delta_1(x, s)) = \delta_2(\varphi(x), s)$, $\varphi_2(\sigma_1(x, s)) = \sigma_2(\varphi_1(x), s)$ for all $x \in A_1$ and $s \in S$.

There exists a Mealy automaton which is not a Moore automaton. In this context the next assertion is interesting .

Proposition 9.1. *Any topological E-automaton is a continuous homomorphic image of some topological Moore E-automaton.*

Proof. Fix the topological E-automaton $M = (A, S, B, \delta, \sigma, a_0, F)$. As in ([82], Proposition 3.5) we put $A' = A \times B$, $B' = B$, $\delta'((x, y), \alpha) = (\delta(x, \alpha)\sigma(y, \alpha))$ and $\sigma'((x, y), \alpha) = \sigma(x, \alpha)$ for all $x \in A, y \in B, \alpha \in S$. Fix $b \in B$. We put $a_1 = (a_0, b)$ and $F' = F \times B$. Then $M' = (A', S, B', \delta', \sigma', a_1, F')$ is a Moore E-automaton for the homomorphism $\mu : A \times B \to B$, where $\mu(x, y) = y$. We put $\varphi_1(x, y) = x$ and $\varphi_2(y) = y$. Then $\varphi(\varphi_1, \varphi_2)$ is a continuous homomorphism of the automaton M' onto M. The class $M(E, \Omega)$ is closed under the topological product and on a subautomata. ∎

Theorem 9.1. *Let $M = (X, S, Y, \delta, \sigma, a_0, F)$ be a topological automaton, S be a discrete space, $F(X)$ and $F(Y)$ be the free topological E-algebras of the spaces X and Y in the class $V(E, \Omega)$, $\delta : F(X)\times S \to F(X)$ and $\sigma : F(X)\times S \to F(Y)$ be the homomorphisms generated by the mappings δ and σ. Then $F(M) = (F(X), S, F(Y), \delta, \sigma, a_0, F)$ is a topological E-automaton.*

Proof. Is obvious.

If a topological E-algebra A is a bigroupoid, then we say that A is an E-bigroupoid. For $E'_2 = E_2 \cup \{\circ, *\}$ and $E' = E_2 \cup \{\circ, *\}$, any topological E-bigroupoid is a topological E'-algebra. Let $M = (A, S, B, \delta, \sigma, a_0, F)$ be a topological E-automaton, Q be a topological E-bigroupoid, a and B be E-subalgebras of the E-algebra Q, S be a subgoupoid of the groupoid $(Q, \circ)$ and $\delta(x, \alpha) = x \circ \alpha, \sigma(x, \alpha) = x * \alpha$ for all $x \in A$ and $\alpha \in S$. Then we say that the topological E-automaton is an automaton in the category of E-bigroupoids. ∎

Theorem 9.2. *For every topological E-automaton $M = (A, S, B, \delta, \sigma, a_0, F)$ there exists a topological E-bigroupoid such that M is an automaton in topological E-bigroupoid Q.*

Universal algebras and automata represent an important field of research in modern mathematics and computer science. Interesting results in this field were obtained in [1, 4, 25, 35, 50, 51, 56, 82].

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OSCILLATION OF FIXED POINTS OF SOLUTIONS AND THEIR DERIVATIVES OF SOME HIGHER LINEAR DIFFERENTIAL EQUATIONS

Abdallah El Farissi[1], Zoubir Dahmani[2]

[1] *Department of Mathematics, University of Bechar, Algeria*

[2] *Laboratory LPAM, Faculty SEI, UMAB, Mostaganem, Algeria*

zzdahmani@yahoo.fr

Abstract In this paper, we investigate the relationship between solutions and their derivatives for the differential equation $f^{(k)}+A_{k-1}f^{(k-1)}+...+A_0f = 0$ with $k \geq 2$ and entire functions of finite iterated $p-$order, when A_j $(j = 0, 1, ..., k-1)$ are entire functions of finite iterated $p-$order in order to generalize and extend the results given by Wang and Lü, Liu and Zhang and Belaïdi.

Keywords: linear differential equations, entire solutions, iterated $p-$order, iterated exponent of convergence of the sequence of distinct zeros.

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1. INTRODUCTION AND MAIN RESULT

In this paper, we shall assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions (see [3], [8]). For the definition of the iterated order of a meromorphic function, we use the same definition as in [4], [2, p. 317], [5, p. 129]. For all $r \in \mathbb{R}$, we define $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp\left(\exp_p r\right)$, $p \in \mathbb{N}$. We also define for all r sufficiently large $\log_1 r := \log r$ and $\log_{p+1} r := \log\left(\log_p r\right)$, $p \in \mathbb{N}$. Moreover, we denote by $\exp_0 r := r$, $\log_0 r := r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$.

Definition 1.1. ([4], [5]) *Let f be a meromorphic function. The iterated $p-$order $\rho_p(f)$ of f is defined by*

$$\rho_p(f) = \overline{\lim_{r\to+\infty}} \frac{\log_p T(r,f)}{\log r} \quad (p \geq 1 \ \text{is an integer}), \tag{1.1}$$

where $T(r, f)$ is the Nevanlinna characteristic function of f (see [3], [8]). For $p = 1$, this notation is called: order, and for $p = 2$: hyper-order.

Definition 1.2. ([4], [5]) *The finiteness degree of the order of a meromorphic function f is defined by*

$$i(f)=\begin{cases} 0, & \textit{for } f \textit{ rational,}\\ \min\left\{j\in\mathbb{N}:\rho_j(f)<+\infty\right\}, & \textit{for } f \textit{ transcendental for which}\\ & \textit{some } j\in\mathbb{N} \textit{ with } \rho_j(f)<+\infty \textit{ exists,}\\ +\infty, & \textit{for } f \textit{ with } \rho_j(f)=+\infty \textit{ for all } j\in\mathbb{N}.\end{cases} \tag{1.2}$$

Definition 1.3. ([4]) *Let f be a meromorphic function. The iterated exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined by*

$$\overline{\lambda}_p(f)=\overline{\lim_{r\to+\infty}}\frac{\log_p \overline{N}\left(r,\frac{1}{f}\right)}{\log r}\ ;\ p\geq 1 \ \textit{ is an integer,} \tag{1.3}$$

where $\overline{N}\left(r,\frac{1}{f}\right)$ is the counting function of distinct zeros of $f(z)$ in $\{|z|<r\}$. For $p=1$, this notation is called: exponent of convergence of the sequence of distinct zeros, and for $p=2$, we get the hyper-exponent of convergence of the sequence of distinct zeros.

Definition 1.4. ([6]) *Let f be a meromorphic function. Then the iterated exponent of convergence of the sequence of distinct fixed points of $f(z)$ is defined by*

$$\overline{\tau}_p(f)=\overline{\lambda}_p(f-z)=\overline{\lim_{r\to+\infty}}\frac{\log_p \overline{N}\left(r,\frac{1}{f-z}\right)}{\log r};\ p\geq 1 \ \textit{ is an integer.} \tag{1.4}$$

For $p=1$, this notation is called: exponent of convergence of the sequence of distinct fixed points. However, for $p=2$, we get the hyper-exponent of convergence of the sequence of distinct fixed points (see [7]*). Thus $\overline{\tau}_p(f)=\overline{\lambda}_p(f-z)$ is an indication of oscillation of distinct fixed points of $f(z)$.*

Definition 1.5. *The growth index of the iterated convergence exponent of the sequence of zero points of a meromorphic function f with iterated order is defined by*

$$i_\lambda(f)=\begin{cases} 0 & \textit{if } n\left(r,\frac{1}{f}\right)=O(\log r)\\ \min\{n\in\mathbb{N}:\lambda_n(f)<\infty\} & \textit{if } \lambda_n(f)<\infty \textit{ for some } n\in\mathbb{N}.\\ \infty & \textit{if } \lambda_n(f)<\infty \textit{ for all } n\in\mathbb{N}\end{cases}$$

Similarly, we can define the growth index $i_{\overline{\lambda}}(f)$ of $\overline{\lambda}_p(f)$ and $i_\tau(f), i_{\overline{\tau}}(f)$ of $\tau_p(f), \overline{\tau}_p(f)$.

For $k\geq 2$, we consider the linear differential equation

$$f^{(k)}+A(z)f=0, \tag{1.5}$$

where $A(z)$ is a transcendental meromorphic function of finite iterated order $\rho_p(A)=\rho>0$. Many important results have been obtained on the fixed points of general transcendental meromorphic functions for almost four decades (see [11, 13]). However, there are a few studies on the fixed points of solutions of differential equations. In

[15] , Wang and Lü have investigated the fixed points and hyper-order of solutions of second order linear differential equations with meromorphic coefficients and their derivatives. They have obtained the following result:

Theorem A ([15]) *Suppose that* $A(z)$ *is a transcendental meromorphic function satisfying* $\delta(\infty, A) = \underline{\lim}_{r\to+\infty} \frac{m(r,A)}{T(r,A)} = \delta > 0$, $\rho(A) = \rho < +\infty$. *Then every meromorphic solution* $f(z) \not\equiv 0$ *of the equation*

$$f'' + A(z) f = 0 \tag{1.6}$$

is such that f, f' *and* f'' *have infinitely many fixed points and*

$$\overline{\tau}(f) = \overline{\tau}\left(f'\right) = \overline{\tau}\left(f''\right) = \rho(f) = +\infty, \tag{1.7}$$

$$\overline{\tau}_2(f) = \overline{\tau}_2\left(f'\right) = \overline{\tau}_2\left(f''\right) = \rho_2(f) = \rho. \tag{1.8}$$

Theorem A has been generalized to higher order differential equations by Liu and Zhang as follows (see [13]):

Theorem B ([13]) *Suppose that* $k \geq 2$ *and* $A(z)$ *is a transcendental meromorphic function satisfying* $\delta(\infty, A) = \underline{\lim}_{r\to+\infty} \frac{m(r,A)}{T(r,A)} = \delta > 0$, $\rho(A) = \rho < +\infty$. *Then every meromorphic solution* $f(z) \neq 0$ *of* (1.4), *has the property:* f *and* $f', f'', ..., f^{(k)}$ *all have infinitely many fixed points and*

$$\overline{\tau}(f) = \overline{\tau}\left(f'\right) = \overline{\tau}\left(f''\right) = ... = \overline{\tau}\left(f^{(k)}\right) = \rho(f) = +\infty, \tag{1.9}$$

$$\overline{\tau}_2(f) = \overline{\tau}_2\left(f'\right) = \overline{\tau}_2\left(f''\right) = ... = \overline{\tau}_2\left(f^{(k)}\right) = \rho_2(f) = \rho. \tag{1.10}$$

Theorem A and B have been generalized by B. Belaidi for iterated p-order (see [2]):

Theorem C ([2]) *Let* $k \geqslant 2$ *and* $A(z)$ *be transcendental meromorphic function of finite iterated order* $\rho_p(A) = \rho > 0$ *such that* $\delta(\infty, A) = \underline{\lim}_{r\to+\infty} \frac{m(r,A)}{T(r,A)} = \delta > 0$. *Suppose, moreover, that either:*

(i) *all poles of* f *are of uniformly multiplicity or that*

(ii) $\delta(\infty, f) > 0$.

If $\varphi \neq 0$ *is a meromorphic function with finite* iterated $p-$order $\rho_p(\varphi) < +\infty$, *then every meromorphic solution* $f(z) \neq 0$ *of* (1.5), *satisfies*

$$\overline{\lambda}_p(f - \varphi) = \overline{\lambda}_p\left(f' - \varphi\right) = ... = \overline{\lambda}_p\left(f^{(k)} - \varphi\right) = \rho_p(f) = +\infty, \tag{1.11}$$

and

$$\overline{\lambda}_{p+1}(f-\varphi)=\overline{\lambda}_{p+1}\left(f^{'}-\varphi\right)=...=\overline{\lambda}_{p+1}\left(f^{(k)}-\varphi\right)=\rho_{p+1}(f)=\rho. \tag{1.12}$$

For $k\geq 2$, we consider the linear differential equation

$$f^{(k)}+A_{k-1}f^{(k-1)}+...+A_0 f=0,\ k\geq 2, \tag{1.13}$$

where A_j $(j=0,1,...,k-1)$ are entire functions of finite iterated $p-$order.

The main purpose of this paper is to study the relation between solutions and their derivatives of the differential equation (1.13) and entire functions of finite iterated $p-$order where we generalize and extend the results of Wang and Lü, Liu and Zhang and Belaidi. In fact, we prove the following result:

Theorem 1.1. *Let $k\geq 2$ and $(A_j)_{j=0,1,2,...k-1}$ be entire functions of finite iterated p-order such that $i(A_0)=p; 0<p<\infty$. Assume that*

$$\max\left\{i\left(A_j\right),(j=1,...,k-1)\right\}<i(A_0)$$

or

$$\max\left\{\rho_p\left(A_j\right),(j=1,...,k-1)\right\}<\rho_p(A_0)<+\infty.$$

If $\varphi(z)\neq 0$ is an entire function with $i(\varphi)<p+1$ or $\rho_{p+1}(\varphi)<\rho_p(A_0)$, then every solution $f(z)\neq 0$ of (1.13) satisfies

$$i_{\overline{\lambda}}\left(f^{(i)}-\varphi\right)=i_{\lambda}\left(f^{(i)}-\varphi\right)=i(f)=p+1,\ i\in\mathbb{N} \tag{1.14}$$

and

$$\overline{\lambda}_{p+1}\left(f^{(i)}-\varphi\right)=\overline{\lambda}_{p+1}\left(f^{(i)}-\varphi\right)=\rho_{p+1}(f)=\rho_p(A_0),\ i\in\mathbb{N}. \tag{1.15}$$

For $\varphi(z)=z$ in Theorem 1.1, we obtain the following corollaries:

Corollary 1.1. *Let $k\geq 2$ and $(A_j)_{j=0,1,2,...k-1}$ be entire functions of finite iterated p-order such that $i(A_0)=p(0<p<\infty)$. Assume that*

$$\max\left\{i\left(A_j\right),(j=1,...,k-1)\right\}<i(A_0)$$

or

$$\max\left\{\rho_p\left(A_j\right),(j=1,...,k-1)\right\}<\rho_p(A_0)<+\infty.$$

Then every solution $f(z)\neq 0$ of (1.13), is such that all the derivatives $f^{(i)}$ $(i\in\mathbb{N})$ have infinitely many fixed points and we have

$$i_{\overline{\tau}}\left(f^{(i)}\right)=i_{\tau}\left(f^{(i)}\right)=i(f)=p+1,\ i\in\mathbb{N} \tag{1.16}$$

and

$$\overline{\tau}_{p+1}\left(f^{(i)}\right) = \tau_{p+1}\left(f^{(i)}\right) = \rho_{p+1}(f) = \rho_p(A_0) = \rho,\ i \in \mathbb{N}. \tag{1.17}$$

Corollary 1.2. *Suppose that* $k \geq 2$ *and* $A(z)$ *is a transcendental entire function such that* $0 < \rho_p(A) = \rho < +\infty$. *If* $\varphi(z) \not\equiv 0$ *is an entire function with* $i(\varphi) < p+1$ *or* $\rho_{p+1}(\varphi) < \rho$*, then every solution* $f(z) \not\equiv 0$ *of* (1.5) *satisfies* (1.14) *and* (1.15).

2. AUXILIARY LEMMATA

To prove our main results, we need the following lemmata.

Lemma 2.1. [6] *Suppose that* $A_0, A_1, ..., A_{k-1}, F\,(\not\equiv 0)$ *are meromorphic functions and let* f *be a meromorphic solution of the equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + ... + A_1 f' + A_0 f = F, \tag{2.1}$$

such that $i(f) = \rho + 1\ (0 < p < \infty)$. *If either*

$$\max\left\{i(F), i\left(A_j\right)\ j = 0, 1, ..., k-1\right\} < p+1$$

or

$$\max\left\{\rho_{p+1}(F), \rho_{p+1}\left(A_j\right)\ j = 0, 1, ..., k-1\right\} < \rho_{p+1}(f),$$

then we have $i_{\overline{\lambda}}(f) = i_\lambda(f) = i(f) = p+1$ *and* $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{p+1}(f)$.

Lemma 2.2. *(see Remark 1.3 of* [10]*). If* f *is a meromorphic function with* $i(f) = p$*, then* $\rho_p\left(f'\right) = \rho_p(f)$.

Lemma 2.3. ([10]) *Let* $k \geq 2$ *and* $A_j\ (j = 0, 1, ..., k-1)$ *be entire functions of finite iterated p-order such that* $i(A_0) = p, (0 < p < \infty)$. *Assume that*

$$\max\left\{i\left(A_j\right), (j = 1, ..., k-1)\right\} < i(A_0)$$

or

$$\max\left\{\rho_p\left(A_j\right), (j = 1, ..., k-1)\right\} < \rho_p(A_0) < +\infty.$$

Then every solution $f(z) \not\equiv 0$ *of* (1.13) *satisfies* $i(f) = p+1$ *and* $\rho_{p+1}(f) = \rho_p(A_0)$.

Let $A_j\ (j = 0, 1, ..., k-1)$ be entire functions. We define the following sequence of functions:

$$\begin{cases} A_j^0 = A_j, & j = 0, 1, ..., k-1 \\ A_{k-1}^i = A_{k-1}^{i-1} - \dfrac{\left(A_0^{i-1}\right)'}{A_0^{i-1}}, & i \in \mathbb{N} \\ A_j^i = A_j^{i-1} + A_{j+1}^{i-1}\dfrac{\left(\Psi_{j+1}^{i-1}\right)'}{\Psi_{j+1}^{i-1}}, & j = 0, 1, ..., k-2,\ i \in \mathbb{N}, \end{cases} \tag{2.2}$$

where $\Psi_{j+1}^{i-1} = \dfrac{A_{j+1}^{i-1}}{A_0^{i-1}}$.

Remark 2.1. *In the case where one of functions* A_j^i $(j = 0, 1, ..., k-1)$ *is equal to zero then* $A_j^{i+1} = A_{j-1}^i$ $(j = 0, 1, ..., k-1)$.

Lemma 2.4. *Assume that* f *is an entire solution of* (1.13). *Then* $g_i = f^{(i)}$ *is an entire solution of the equation*

$$g_i^{(k)} + A_{k-1}^i g_i^{(k-1)} + ... + A_0^i g_i = 0, \tag{2.3}$$

where A_j^i $(j = 0, 1, ..., k-1)$ *are given by* (2.2).

Proof. Assume that f is a solution of equation (1.13) and let $g_i = f^{(i)}$. We prove that g_i is an entire solution of the equation (2.11). Our proof is by induction: For $i = 1$, differentiating both sides of (1.13), we obtain

$$f^{(k+1)} + A_{k-1} f^{(k)} + \left(A_{k-1}^{'} + A_{k-2}\right) f^{(k-1)} + ... + \left(A_1^{'} + A_0\right) f^{'} + A_0^{'} f = 0, \tag{2.4}$$

and replacing f by

$$f = -\frac{(f^{(k)} + A_{k-1} f^{(k-1)} + ... + A_1 f^{'})}{A_0},$$

we get

$$f^{(k+1)} + \left(A_{k-1} - \frac{A_0^{'}}{A_0}\right) f^{(k)} + \left(A_{k-1}^{'} + A_{k-2} - A_{k-1}\frac{A_0^{'}}{A_0}\right) f^{(k-1)} ... + \left(A_1^{'} + A_0 - A_1 \frac{A_0^{'}}{A_0}\right) f^{'} = 0.$$

That is

$$g_1^{(k)} + A_{k-1}^1 g_1^{(k-1)} + A_{k-2}^1 g_1^{(k-2)} ... + A_0^1 g_1 = 0.$$

Suppose that the assertion is true for the values which are strictly smaller than a certain i. We suppose g_{i-1} is a solution of the equation

$$g_{i-1}^{(k)} + A_{k-1}^{i-1} g_{i-1}^{(k-1)} + A_{k-2}^{i-1} g_{i-1}^{(k-2)} ... + A_0^{i-1} g_{i-1} = 0. \tag{2.5}$$

Differentiating (2.5), we can write

$$\begin{aligned} &g_{i-1}^{(k+1)} + A_{k-1}^{i-1} g_{i-1}^{(k)} + \left(\left(A_{k-1}^{i-1}\right)^{'} + A_{k-2}\right) g_{i-1}^{(k-1)} + ... \\ &+ \left(\left(A_1^{i-1}\right)^{'} + A_0^{i-1}\right) g_{i-1}^{'} + A_0^{'} g_{i-1} = 0. \end{aligned} \tag{2.6}$$

In (2.6), replacing g_{i-1} by

$$g_{i-1} = -\frac{(g_{i-1}^{(k)} + A_{k-1}^{i-1} g_{i-1}^{(k-1)} + A_{k-2}^{i-1} g_{i-1}^{(k-2)} ... + A\,(g_{i-1})^{'})}{A_0^{i-1}},$$

yields

$$g_{i-1}^{(k+1)}+\left(A_{k-1}^{i-1}-\frac{\left(A_0^{i-1}\right)'}{A_0^{i-1}}\right)g_{i-1}^{(k)}+\left(\left(A_{k-1}^{i-1}\right)'+A_{k-2}-A_{k-1}^{i-1}\frac{\left(A_0^{i-1}\right)'}{A_0^{i-1}}\right)g_{i-1}^{(k-1)}...+$$

$$+\left(\left(A_1^{i-1}\right)'+A_0^{i-1}-A_1^{i-1}\frac{\left(A_0^{i-1}\right)'}{A_0^{i-1}}\right)g_{i-1}'=0. \tag{2.7}$$

That is

$$g_{i-1}^{(k)}+A_{k-1}^{i-1}g_{i-1}^{(k-1)}+A_{k-2}^{i-1}g_{i-1}^{(k-2)}...+A_0^{i-1}g_{i-1}=0.$$

Lemma 2.4 is thus proved. ∎

Lemma 2.5. *Let A_j $(j=0,1,...,k-1)$ be entire functions of finite order. Assume that*

$$\max\left\{i\left(A_j\right),(j=1,...,k-1)\right\}<i\left(A_0\right)$$

or

$$\max\left\{\rho_p\left(A_j\right),(j=1,...,k-1)\right\}<\rho_p\left(A_0\right)<+\infty,$$

and let A_j^i, $(j=0,1,...,k-1)$ be defined as in (2.2)*. Then all nontrivial meromorphic solution g of the equation*

$$g^{(k)}+A_{k-1}^i g^{(k-1)}+...+A_0^i g=0,\ k\geq 2 \tag{2.8}$$

satisfy : $i(g)=p+1$ and $\rho_{p+1}(g)=\rho$.

Proof. Let $\{f_1,f_2,...,f_k\}$ be a fundamental system of solutions of (1.13). We show that $\left\{f_1^{(i)},f_2^{(i)},...,f_k^{(i)}\right\}$ is a fundamental system of solutions of (2.8). By Lemma 2.4, it follows that $f_1^{(i)},f_2^{(i)},...,f_k^{(i)}$ is a solutions (2.8) . Let $\alpha_1,\alpha_2,...,\alpha_k$ be constants such that

$$\alpha_1 f_1^{(i)}+\alpha_2 f_2^{(i)}+...+\alpha_k f_k^{(i)}=0.$$

Then, we have

$$\alpha_1 f_1+\alpha_2 f_2+...+\alpha_k f_k=P(z),$$

where $P(z)$ is a polynomial of degree less than i. Since $\alpha_1 f_1+\alpha_2 f_2+...+\alpha_k f_k$ is a solution of (1.13), then P is a solution of (1.13), and by Lemma 2.3, we conclude that P is an infinite solution of (1.13); this leads to a contradiction. Therefore, P is a trivial solution. We deduce that $\alpha_1 f_1+\alpha_2 f_2+...+\alpha_k f_k=0$. Using the fact that $\{f_1,f_2,...,f_k\}$ is a fundamental system of solutions of (1.13), we get $\alpha_1=\alpha_2=...=\alpha_k=0$. Now, let g be a non trivial solution of (2.8). Then, using the fact that $\left\{f_1^{(i)},f_2^{(i)},...,f_k^{(i)}\right\}$ is a fundamental solution of (2.8) , we claim that there exist constants $\alpha_1, \alpha_2, ..., \alpha_k$ not all equal to zero, such that $g = \alpha_1 f_1^{(i)} + \alpha_2 f_2^{(i)} + ... + \alpha_k f_k^{(i)}$. Let $h = \alpha_1 f_1 + \alpha_2 f_2 + ... + \alpha_k f_k$, h be a solution of (1.13) and $h^{(i)} = g$. Hence, by Lemma 2.2, we have $\rho_{p+1}(h) = \rho_{p+1}(g)$, and by Lemma 2.3, we have $i(h) = i(g) = p + 1$ and $\rho_{p+1}(h) = \rho_{p+1}(g) = \rho$. ∎

3. PROOF OF THEOREM 1.1

Assume that f is a solution of equation (1.13). By Lemma 2.3, we can write $i(f) = p + 1, \rho_{p+1}(f) = \rho_p(A_0)$. Taking $g_i = f^{(i)}$, then, using Lemma 2.2, we have $i(g_i) = p+1, \rho_{p+1}(g_i) = \rho_p(A_0)$. Now, let $w(z) = g_i(z) - \varphi(z)$, where φ is an entire function with $\rho_{p+1}(\varphi) < \rho_p(A_0)$.
Then $i(w) = i(g_i) = p + 1$, and $\rho_{p+1}(w) = \rho_{p+1}(g_i) = \rho_{p+1}(f) = \rho(A_0)$.
In order to prove $i_{\overline{\lambda}}(g_i - \varphi) = i_\lambda(g_i - \varphi) = p + 1$ and $\overline{\lambda}_{p+1}(g_i - \varphi) = \lambda_{p+1}(g_i - \varphi) = \rho(A_0)$, we need to prove only $i_{\overline{\lambda}}(w) = i_\lambda(w) = p + 1$ and $\overline{\lambda}_{p+1}(w) = \rho(A_0)$. Using the fact that $g_i = w + \varphi$, and by Lemma 2.4 we get

$$w^{(k)} + A_{k-1}^i w^{(k-1)} + ... + A_0^i w = -\left(\varphi^{(k)} + A_{k-1}^i \varphi^{(k-1)} + ... + A_0^i \varphi\right) = F. \qquad (3.1)$$

By $\rho_p\left(A_j^i\right) < \infty, \rho_{p+1}(\varphi) < \rho_p(A_0)$ and Lemma 2.3, we get $F \not\equiv 0$ and $\rho_{p+1}(F) < \infty$. By Lemma 2.4 $i_{\overline{\lambda}}(w) = i_\lambda(w) = p + 1$ and $\overline{\lambda}_{p+1}(w) = \lambda_{p+1}(w) = \rho_{p+1}(w) = \rho(A_0)$. The proof of theorem 1.1 is complete.

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ON THE SECOND BRYANT SCHNEIDER GROUP OF UNIVERSAL OSBORN LOOPS

Temitope Gbolahan Jaiyéolá[1], John Olusola Adéníran[2],
Adesina Abdul Akeem Agboolá [2]
[1] *Department of Mathematics, Obafemi Awolowo University, Ile Ife, Nigeria*
[2] *Department of Mathematics, Federal University of Agriculture, Abeokuta, Nigeria*
tjayeola@oauife.edu.ng, adeniranoj@unaab.edu.ng, aaaola2003@yahoo.com

Abstract A group, called the second Bryant Schneider group (2^{nd} BSG), is naturally assigned to any Osborn loop. An Osborn loop has the properties: it is universal if and only if its 2^{nd} BSG contains a tri-mapping and it is left (right) universal if and only if its 2^{nd} BSG contains a bi-mapping. An Osborn loop in which the tri-mapping is of exponent 2 is shown to be an abelian group. Consequently, a universal Osborn loop like a Moufang loop, an extra loop, a CC-loop, a VD-loop, a universal WIPL that is non-associative and non-abelian has the tri-mapping (not of exponent 2) in its 2^{nd} BSG. The conjugate of this tri-mapping for a universal Osborn loop that is a G-loop (e.g VD-loops, CC-loops, extra loops and some classes of Moufang loops) or which belongs to a family of commutative Moufang loops is shown to be in the 2^{nd} BSG of its loop isotope. A necessary and sufficient condition for a loop to be a universal Osborn loop in which an arbitrary principal isotope is isomorphic to some principal isotope under the identity map is established.

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1. INTRODUCTION

A loop is called an Osborn loop if it obeys the identity below.

$$OS_0 \; : \; x(yz \cdot x) = x(yx^{\lambda} \cdot x) \cdot zx \tag{1}$$

where x^{λ} denotes the left inverse element of x.

For a comprehensive introduction to Osborn loops and universal Osborn loops as well as a detailed literature review on it, readers should check Jaiyéolá [7, 6], Jaiyéolá and Adéníran [9, 8], and Jaiyéolá , Adéníran and Sòlárìn [10]. In this present paper, we shall follow the style and notations used in Jaiyéolá and Adéníran [8], and Jaiyéolá , Adéníran and Sòlárìn [10]. Some concepts and notions, and results which will be introduced and stated here are those that were not defined or stated in Jaiyéolá and Adéníran [8], and Jaiyéolá , Adéníran and Sòlárìn [10].

Let x be an arbitrarily fixed element in a loop $(G,\cdot)$. For any $x\in G$, the left and right translation maps are denoted by L_x and R_x respectively. The inverses of L_x and R_x will be denoted respectively by $\mathbb{L}_x$ and $\mathbb{R}_x$.

Theorem 1.1. *(Basarab, [2]) If an Osborn loop is of exponent 2, then it is an abelian group.*

We state an easy result that will later be of use.

Theorem 1.2. *Let $(G,\cdot)$ be a "certain" loop where "certain" is an isomorphic invariant property. $(G,\cdot)$ is a universal "certain" loop if and only if every f,g-principal isotope $(G,*)$ of $(G,\cdot)$ has the "certain" loop property.*

Let $SYM(G,\cdot)$ represent the symmetric group of any loop $(G,\cdot)$.

Definition 1.1. *(Robinson [13])*
Let $(G,\cdot)$ be a loop.

1 . *A mapping $\theta\in SYM(G,\cdot)$ is said to be a right special map if there exists $f\in G$ so that $(\theta,\theta\mathbb{L}_f,\theta)\in AUT(G,\cdot)$.*

2 *A mapping $\theta\in SYM(G,\cdot)$ is said to be a left special map for G if that there exists $g\in G$ so that $(\theta\mathbb{R}_g,\theta,\theta)\in AUT(G,\cdot)$.*

3 *A mapping $\theta\in SYM(G,\cdot)$ is named a special map for G if there exist $f,g\in G$ so that $(\theta\mathbb{R}_g,\theta\mathbb{L}_f,\theta)\in AUT(G,\cdot)$.*

From Definition 1.1, it can be observed that θ is a left or right special map for a loop $(G,\cdot)$ with identity element e if and only if θ is an isomorphism of $(G,\cdot)$ onto some e,g- or f,e- principal isotope $(G,\circ)$ of $(G,\cdot)$. Moreso, θ is a special map for a loop $(G,\cdot)$ if and only if θ is an isomorphism of $(G,\cdot)$ onto some f,g-principal isotope $(G,\circ)$ of $(G,\cdot)$.

Robinson [13] went further to show that if

$$BS(G,\cdot)=\{\theta\in SYM(G,\cdot)\ :\ \exists\, f,g\in G\ \ni\ (\theta\mathbb{R}_g,\theta\mathbb{L}_f,\theta)\in AUT(G,\cdot)\}$$

i.e the set of all special maps in a loop, then $BS(G,\cdot)\le SYM(G,\cdot)$ is called the Bryant-Schneider group of the loop $(G,\cdot)$ (because its importance and motivation stem from the work of Bryant and Schneider [3]). Since the definition of the Bryant-Schneider group, some studies by Adeniran [1] and Chiboka [5] have been done on it relative to CC-loops and extra loops. This group will now be called the first Bryant-Schneider group (1^{st} BSG) and represented by $BS_1(G,\cdot)=BS_1(G)$ for a loop $(G,\cdot)$. Let

$$BS_2(G,\cdot)=\{\theta\in SYM(G)\ :\ G(a,b)\overset{\theta}{\cong}G(c,d)\text{ for some }a,b,c,d\in G\}.$$

As shown in Bryant and Schneider [3], $BS_2(G,\cdot)$ forms a group for a loop $(G,\cdot)$ and it shall be called the second Bryant-Schneider group(2^{nd} BSG) of the loop. It is easy to see that $BS_1(G,\cdot) \le BS_2(G,\cdot) \le SYM(G)$. The 2nd BSG will be more useful than the 1st BSG in this study. This is as a result of some mappings that are in the 2nd BSG and not in the 1st BSG.

Results of Bryant and Schneider [3].

Theorem 1.3. *Let $(G,\cdot)$ and $(H,\odot)$ be quasigroups. If $(G,\cdot)$ is isomorphic to $(H,\odot)$ under θ, then $BS_2(H,\odot) = \theta^{-1}BS_2(G,\cdot)\theta$.*

Theorem 1.4. *If $(Q,\cdot)$ is a quasigroup, then $Q(a,b,\circ)$ is trivially isomorphic to $Q(c,d,*)$ if and only if $c\cdot b, a\cdot d \in N_\mu(Q(a,b,\circ))$ and $a\cdot b = c\cdot d$.*

Corollary 1.1. *If $(Q,\cdot)$ is a loop with identity e, then $(Q,\cdot)$ is trivially isomorphic to $Q(c,d)$ if and only if $c,d \in N_\mu(Q,\cdot)$ and $c\cdot d = e$.*

Results of Robinson [13]. Let $(Q,\cdot)$ be a loop and $ROB(Q,\cdot) = ROB(Q)$ be the set of autotopisms $\Re = (\delta\mathbb{R}_g, \delta\mathbb{L}_f, \delta)$ for $f, g, \in Q$. The author observed that $ROB(Q) \le AUT(Q)$ and we shall call it the Robinson group (ROBG) of a loop. Furthermore, he mentioned that the mapping $\Theta : ROB(Q,\cdot) \longrightarrow BS_1(Q,\cdot)$ defined by $\Theta : (\delta\mathbb{R}_g, \delta\mathbb{L}_f, \delta) \longrightarrow \delta$ is an homomorphism and proved the following results about its kernel.

Theorem 1.5. *Let $(Q,\cdot)$ be a loop with identity e, let $f, g \in Q$ and let $\delta \in SYM(Q)$. Then, $\Re = (\delta\mathbb{R}_g, \delta\mathbb{L}_f, \delta) \in \ker\Theta$ if and only if $\delta = I$, $g\cdot f = e$ and $g \in N_\mu(Q)$.*

In this study, the group called the second Bryant Schneider group (2nd BSG) is investigated in universal Osborn loops. An Osborn loop is shown to be universal if and only if its 2nd BSG contains a tri-mapping and is left (right) universal if and only if its 2nd BSG contains a bi-mapping. An Osborn loop in which the tri-mapping is of exponent 2 is shown to be an abelian group. Consequently, a universal Osborn loop like a Moufang loop, an extra loop, a CC-loop, a VD-loop or a universal WIPL that is non-associative and non-abelian has the tri-mapping (not of exponent 2) in its 2nd BSG. The conjugate of this tri-mapping for a universal Osborn loop that is a G-loop (e.g VD-loops, CC-loops, extra loops and some classes of Moufang loops) or which belongs to a family of commutative Moufang loops is shown to be in the 2nd BSG of its loop isotope. A necessary and sufficient condition for a loop to be a universal Osborn loop in which an arbitrary principal isotope is isomorphic to some principal isotope under the identity map is established.

2. MAIN RESULTS

Theorem 2.1. *Let $(Q,\cdot,\backslash,/)$ be an Osborn loop. Let $\phi(x,u,v) = (u\backslash([(uv)/(u\backslash(xv))]v))$ and $\gamma(x,u,v) = \mathbb{R}_v R_{[u\backslash(xv)]}\mathbb{L}_u L_x$ for all $x,u,v \in Q$, then $(Q,\cdot,\backslash,/)$ is a universal Osborn loop if and only if the composition*

$$(Q,\cdot) \xrightarrow[\text{principal isotopism}]{(R_{\phi(x,u,v)},L_u,I)} (Q,*) \xrightarrow[\text{isomorphism}]{(\gamma,\gamma,\gamma)} (Q,\circ) \xrightarrow[\text{principal isotopism}]{(\mathbb{R}_v,\mathbb{L}_x,I)} (Q,\cdot) \qquad (2)$$

holds, where $(Q,\circ)$ *is an arbitrary principal isotope of* $(Q,\cdot)$ *and* $(Q,*)$ *some principal isotope of* $(Q,\cdot)$.

Proof. Let $\mathfrak{Q}=(Q,\cdot,\backslash,/)$ be an Osborn loop with any arbitrary principal isotope $\mathfrak{Q}=(Q,\blacktriangle,\nwarrow,\nearrow)$ such that

$$x\blacktriangle y=xR_v^{-1}\cdot yL_u^{-1}=(x/v)\cdot(u\backslash y)\ \forall\ u,v\in Q. \tag{3}$$

If $\mathfrak{Q}$ is a universal Osborn loop then, $\mathfrak{Q}$ is an Osborn loop. $\mathfrak{Q}$ obeys identity OS_0 implies

$$x\blacktriangle[(y\blacktriangle z)\blacktriangle x]=\{x\blacktriangle[(y\blacktriangle x^{\lambda'})\blacktriangle x]\}\blacktriangle(z\blacktriangle x) \tag{4}$$

where $x^{\lambda'}=xJ_{\lambda'}$ is the left inverse of x in $\mathfrak{Q}$. The identity element of the loop $\mathfrak{Q}$ is uv. So,

$$x\blacktriangle y=xR_v^{-1}\cdot yL_u^{-1}\text{ implies }y^{\lambda'}\blacktriangle y=y^{\lambda'}R_v^{-1}\cdot yL_u^{-1}=uv\text{ implies}$$

$$y^{\lambda'}R_v^{-1}R_{yL_u^{-1}}=uv\text{ implies }yJ_{\lambda'}=(uv)R^{-1}_{yL_u^{-1}}R_v=(uv)R^{-1}_{(u\backslash y)}R_v=[(uv)/(u\backslash y)]v.$$

Thus, by using (3), $\mathfrak{Q}$ is an Osborn loop if and only if

$$(x/v)\cdot u\backslash\{[(y/v)\cdot(u\backslash z)]/v\cdot(u\backslash x)\}=$$

$$=((x/v)\cdot u\backslash\{[(y/v)(u\backslash([(uv)/(u\backslash x)]v))]/v\cdot(u\backslash x)\})/v\cdot u\backslash[(z/v)(u\backslash x)].$$

Do the following replacements:

$$x'=x/v\Rightarrow x=x'v,\ z'=u\backslash z\Rightarrow z=uz',\ y'=y/v\Rightarrow y=y'v$$

we have

$$x'\cdot u\backslash\{(y'z')/v\cdot[u\backslash(x'v)]\}=$$

$$=(x'\cdot u\backslash\{[y'(u\backslash([(uv)/(u\backslash(x'v))]v))]/v\cdot[u\backslash(x'v)]\})/v\cdot u\backslash[((uz')/v)(u\backslash(x'v))].$$

This is precisely identity OS_0' below, obtained by replacing x', y' and z' by x, y and z respectively,

$$x\cdot u\backslash\{(yz)/v\cdot[u\backslash(xv)]\}=$$

$$=(x\cdot u\backslash\{[y(u\backslash([(uv)/(u\backslash(xv))]v))]/v\cdot[u\backslash(xv)]\})/v\cdot u\backslash[((uz)/v)(u\backslash(xv))]. \qquad \mathrm{OS}_0'$$

Writing identity OS_0' in autotopic form, we will obtain the fact that the triple $(\alpha(x,u,v),\beta(x,u,v),\gamma(x,u,v))\in AUT(\mathfrak{Q})$ for all $x,u,v\in Q$ where $\alpha(x,u,v)=R_{(u\backslash([(uv)/(u\backslash(xv))]v))}\mathbb{R}_vR_{[u\backslash(xv)]}\mathbb{L}_uL_x\mathbb{R}_v$, $\beta(x,u,v)=L_u\mathbb{R}_vR_{[u\backslash(xv)]}\mathbb{L}_v$ and $\gamma(x,u,v)=\mathbb{R}_vR_{[u\backslash(xv)]}\mathbb{L}_uL_x$ are elements of $\mathcal{M}ult(Q)$. The triple

$$(\alpha(x,u,v),\beta(x,u,v),\gamma(x,u,v))=\left(R_{(u\backslash([(uv)/(u\backslash(xv))]v))}\gamma\mathbb{R}_v,L_u\gamma\mathbb{L}_x,\gamma\right)$$

can be written as the following compositions $\Big(R_{(u\backslash([(uv)/(u\backslash(xv))]v))}, L_u, I\Big)(\gamma,\gamma,\gamma)(\mathbb{R}_v,\mathbb{L}_x,I)$. Let $(Q,\circ)$ be an arbitrary principal isotope of $(Q,\cdot)$ and $(Q,*)$ a particular principal isotope of $(Q,\cdot)$. Let $\phi(x,u,v) = (u\backslash([(uv)/(u\backslash(xv))]v))$, then the composition above can be expressed as:

$$(Q,\cdot)\xrightarrow[\text{principal isotopism}]{(R_{\phi(x,u,v)},L_u,I)}(Q,*)\xrightarrow[\text{isomorphism}]{(\gamma,\gamma,\gamma)}(Q,\circ)\xrightarrow[\text{principal isotopism}]{(\mathbb{R}_v,\mathbb{L}_x,I)}(Q,\cdot).$$

The proof of the converse is as follows. Let $\mathfrak{Q} = (Q,\cdot,\backslash,/)$ be an Osborn loop. Assuming that the composition in equation (2) holds, then doing the reverse of the proof of necessity, $(\alpha(x,u,v),\beta(x,u,v),\gamma(x,u,v)) \in AUT(\mathfrak{Q})$ for all $x,u,v\in Q$ which means that $\mathfrak{Q}$ obeys identity OS_0' hence, it will be observed that equation (3) is true for any arbitrary u,v-principal isotope $\mathfrak{Q} = (Q,\blacktriangle,\nwarrow\!\!\backslash,\nearrow\!\!/)$ of $\mathfrak{Q}$. So, every f,g-principal isotope $\mathfrak{Q}$ of $\mathfrak{Q}$ is an Osborn loop. Following Theorem 1.2, $\mathfrak{Q}$ is a universal Osborn loop if and only if $\mathfrak{Q}$ is an Osborn loop. This completes the proof ∎

Corollary 2.1. *Let $(Q,\cdot,\backslash,/)$ be a loop. Q is a universal Osborn loop if and only if the tri-mapping $\gamma(x,u,v) = \mathbb{R}_vR_{[u\backslash(xv)]}\mathbb{L}_uL_x \in BS_2(Q)$ for all $x,u,v\in Q$.*

Proof. This is obtained from Theorem 2.1 as a consequence of the composition in equation (2). ∎

Lemma 2.1. *A loop $(Q,\cdot,\backslash,/)$ in which $\big|\gamma(x,u,v)\big| = 2$, $\gamma(x,u,v) = \mathbb{R}_vR_{[u\backslash(xv)]}\mathbb{L}_uL_x$ for all $x,u,v\in Q$ is a loop of exponent 2. Hence, if Q is an Osborn loop, then it is an abelian group.*

Proof. The fact that Q is a loop of exponent 2 can be deduced from the fact that $\gamma=\gamma^{-1}$ implies $x = x^\rho$ which gives $x^2 = 2$ by taken $u=v=e$. When Q is an Osborn loop, then following Theorem 1.1, it is an abelian group. ∎

Corollary 2.2. *In any non-associative non-abelian Moufang loop or extra loop or CC-loop or VD-loop or universal WIPL $(Q,\cdot,\backslash,/)$, the tri-mapping $\gamma(x,u,v) = \mathbb{R}_vR_{[u\backslash(xv)]}\mathbb{L}_uL_x \in BS_2(Q)$ for all $x,u,v\in Q$ and $\big|\gamma(x,u,v)\big| \neq 2$.*

Proof. The fact that $\gamma(x,u,v)\in BS_2(Q)$ for all $x,u,v\in Q$ follows from Corollary 2.1 since a Moufang loop or extra loop or CC-loop or VD-loop or universal WIPL is a universal Osborn loop. If $\big|\gamma(x,u,v)\big| = 2$, then by Lemma 2.1, it is associative and commutative which are contradictions. So, $\big|\gamma(x,u,v)\big| \neq 2$. ∎

Lemma 2.2. *Let $(G,\cdot,\backslash,/)$ be an Osborn loop that is a G-loop with arbitrary isotope $(H_i,\circ_i)$ $i\in\Omega$. There exists a bijection $\theta_i : G\to H_i$, $i\in\Omega$ such that $\theta_i^{-1}\gamma(x,u,v)\theta_i \in BS_2(H_i,\circ_i)$ $i\in\Omega$ for all $x,u,v\in G$ where $\gamma(x,u,v) = \mathbb{R}_vR_{[u\backslash(xv)]}\mathbb{L}_uL_x$.*

Proof. Assuming that G is isomorphic to H_i under θ_i, $i\in\Omega$, then the proof of the lemma follows by using Theorem 1.3 and Corollary 2.1. ∎

Lemma 2.3. *Let $(G, \cdot, \backslash, /)$ be a universal Osborn loop with an arbitrary isotope $(H, \circ)$ such that $(G, \cdot) \overset{\theta}{\cong} (H, \circ)$. Then, $\theta^{-1}\gamma(x, u, v)\theta \in BS_2(H, \circ)$ for all $x, u, v \in G$ where $\gamma(x, u, v) = \mathbb{R}_v R_{[u\backslash(xv)]}\mathbb{L}_u L_x$.*

Proof. This is a direct consequence of Theorem 1.3. ∎

Corollary 2.3. *Let $(G, \cdot, \backslash, /)$ be a CC-loop or VD-loop or extra loop with arbitrary isotope $(H_i, \circ_i)$ $i \in \Omega$. There exists a bijection $\theta_i : G \to H_i$, $i \in \Omega$ such that $\theta_i^{-1}\gamma(x, u, v)\theta_i \in BS_2(H_i, \circ_i)$ $i \in \Omega$ for all $x, u, v \in G$ where $\gamma(x, u, v) = \mathbb{R}_v R_{[u\backslash(xv)]}\mathbb{L}_u L_x$.*

Proof. This follows from Lemma 2.2 and the fact that a CC-loop or VD-loop or extra loop is a G-loop. ∎

Corollary 2.4. *Let $M = (M, \cdot, \backslash, /)$ be*

- *a Moufang loop such that $M = N(M)M^3$ or*
- *a simple Moufang loop with identity such that $M^3 \neq e$ or*
- *Moufang loop which satisfies an M_k-law for $k \not\equiv 1 \bmod 3$*

with arbitrary isotope $(H_i, \circ_i)$ $i \in \Omega$. There exists a bijection $\theta_i : G \to H_i$, $i \in \Omega$ such that $\theta_i^{-1}\gamma(x, u, v)\theta_i \in BS_2(H_i, \circ_i)$ $i \in \Omega$ for all $x, u, v \in M$ where $\gamma(x, u, v) = \mathbb{R}_v R_{[u\backslash(xv)]}\mathbb{L}_u L_x$.

Proof. This follows from Lemma 2.2 and the fact that such a Moufang loop is a G-loop in each case as shown in Corollary IV.4.7, Corollary IV.4.8 and Theorem IV.4.10 of [12] ∎

Corollary 2.5. *Let $(G, \cdot, \backslash, /)$ be any commutative Moufang loop which belongs to a family of isotopic commutative Moufang loops $\mathfrak{F}$. For every arbitrary $H_i \in \mathfrak{F}$ $i \in \Omega$, there exists a bijection $\theta_i : G \to H_i$, $i \in \Omega$ such that $\theta_i^{-1}\gamma(x, u, v)\theta_i \in BS_2(H_i, \circ_i)$ $i \in \Omega$ for all $x, u, v \in G$ where $\gamma(x, u, v) = \mathbb{R}_v R_{[u\backslash(xv)]}\mathbb{L}_u L_x$.*

Proof. This follows from Lemma 2.2 and the fact that a family of isotopic commutative Moufang loops forms an isomorphic family as shown in Theorem IV.5.6 of [12]. ∎

Lemma 2.4. *Let $(Q, \cdot, \backslash, /)$ be a Moufang loop with an arbitrary isotope $(H, \circ)$ such that $(Q, \cdot)$ is isomorphic to $(H, \circ)$ under θ. Then, $\theta^{-1}\gamma(x, u, v)\theta \in BS_2(H, \circ)$ for all $x, u, v \in G$ where $\gamma(x, u, v) = \mathbb{R}_v R_{[u\backslash(xv)]}\mathbb{L}_u L_x$.*

Proof. This follows from Lemma 2.3 ∎

Lemma 2.5. *A loop $Q = (Q, \cdot, \backslash, /)$ is a universal Osborn loop in which an arbitrary principal isotope is isomorphic to some principal isotope under the identity map if and only if Q is an abelian group of exponent 2.*

Proof. By equation (2) of Theorem 2.1, it can be deduced that if $(Q, \circ)$ and $(Q, *)$ are principal isotopes $(Q, \cdot)$ and $\gamma(x, u, v) = \mathbb{R}_v R_{[u\backslash(xv)]}\mathbb{L}_u L_x$, then $(Q, x, v, \circ)$ is isomorphic to $(Q, u, \phi(x, u, v), *)$ under γ^{-1},

$$\text{where } \phi(x, u, v) = (u\backslash([(uv)/(u\backslash(xv))]v)) \text{ for all } x, u, v \in Q.$$

We now switch to Theorem 1.4. If $\gamma^{-1} = I$ then $\gamma = I$ if and only if $\gamma(x, u, v) = \mathbb{R}_v R_{[u\backslash(xv)]}\mathbb{L}_u L_x = I$ if and only if $R_{[u\backslash(xv)]} = R_v \mathbb{L}_x L_u$ which implies $y[u\backslash(xv)] = u[x\backslash(yv)]$ for all $x, y, u, v \in Q$. Taking $u = v = y = e$, we get $x^2 = e$. By Theorem 1.1, G is an abelian group. This fact can also be proved by using the sufficient part of Theorem 1.4. The converse is easy. ∎

Lemma 2.6. *A loop $Q = (Q, \cdot, \backslash, /)$ is a universal Osborn loop which is isomorphic to some principal isotope under the identity map if and only if Q is an abelian group of exponent 2.*

Proof. The procedure of the proof is similar to that of Lemma 2.5 i.e. using $\gamma(e, u, e)$. This fact can also be proved by using the sufficient part of Corollary 1.1. The converse is easy. ∎

Corollary 2.6. *A loop $Q = (Q, \cdot, \backslash, /)$ is a Moufang loop or extra loop or VD-loop or CC-loop or universal WIPL which is isomorphic to some principal isotope under the identity map if and only if Q is an abelian group of exponent 2.*

Proof. Consequence of Lemma 2.6. ∎

Theorem 2.2. *Let $(Q, \cdot, \backslash, /)$ be a universal Osborn loop, $(Q, *)$ an arbitrary right principal isotope of $(Q, \cdot)$ and $(Q, \circ)$ some principal isotope of $(Q, \cdot)$. Let $\psi(x, u, v) = (u\backslash[(u/v)(u\backslash(xv))])$ and $\gamma(x, u, v) = \mathbb{R}_v R_{[u\backslash(xv)]}\mathbb{L}_u L_x$ for all $x, u, v \in Q$, then the composition*

$$(Q, \cdot) \xrightarrow[\text{right principal isotopism}]{(I, L_u, I)} (Q, *) \xrightarrow[\text{isomorphism}]{(\gamma, \gamma, \gamma)} (Q, \circ) \xrightarrow[\text{principal isotopism}]{\left(\mathbb{R}_{\psi(x,u,v)}, \mathbb{L}_x, I\right)} (Q, \cdot) \qquad (5)$$

holds.

Proof. Theorem 2.1 will be employed. Let $z = e$ in identity OS_0', then

$$x \cdot u\backslash\{y/v \cdot [u\backslash(xv)]\} =$$
$$= (x \cdot u\backslash\{[y(u\backslash([(uv)/(f\backslash(xv))]v))]/v \cdot [u\backslash(xv)]\})/v \cdot u\backslash[((u/v)(u\backslash(xv))].$$

So, identity OS_0' can now be written as

$$x \cdot u\backslash\{(yz)/v \cdot [u\backslash(xv)]\} =$$
$$= \Big\{\{x \cdot u\backslash[y/v \cdot (u\backslash(xv))]\}/\{u\backslash[((u/v)(u\backslash(xv))]\}\Big\} \cdot u\backslash[((uz)/v)(u\backslash(xv))].$$

Putting this in autotopic form, we have

$$\Big(\gamma(x,u,v)\mathbb{R}_{(u\backslash[(u/v)(u\backslash(xv))])},\beta(x,u,v),\gamma(x,u,v)\Big)\in AUT(Q).$$

$$\Big(\gamma(x,u,v)\mathbb{R}_{\psi(x,u,v)},\beta(x,u,v),\gamma(x,u,v)\Big)=\Big(\gamma(x,u,v)\mathbb{R}_{\psi(x,u,v)},L_u\gamma\mathbb{L}_x,\gamma(x,u,v)\Big)\in AUT(Q)$$

for all $x,u,v\in Q$. Writing

$$\Big(\gamma(x,u,v)\mathbb{R}_{\psi(x,u,v)},L_u\gamma(x,u,v)\mathbb{L}_x,\gamma(x,u,v)\Big)=$$
$$(I,L_u,I)(\gamma(x,u,v),\gamma(x,u,v),\gamma(x,u,v))\Big(\mathbb{R}_{\psi(x,u,v)},\mathbb{L}_x,I\Big)$$

such that

$$(Q,\cdot)\xrightarrow[\text{right principal isotopism}]{(I,L_u,I)}(Q,*)\xrightarrow[\text{isomorphism}]{(\gamma,\gamma,\gamma)}(Q,\circ)\xrightarrow[\text{principal isotopism}]{\left(\mathbb{R}_{(u\backslash[(u/v)(u\backslash(xv))])},\mathbb{L}_x,I\right)}(Q,\cdot)$$

where $(Q,*)$ is an arbitrary right principal isotope of $(Q,\cdot)$ and $(Q,\circ)$ are some particular principal isotope of $(Q,\cdot)$, the conclusion of the theorem follows. ∎

Theorem 2.3. *Let $(Q,\cdot,\backslash,/)$ be a loop, $(Q,\circ)$ an arbitrary principal isotope of $(Q,\cdot)$ and $(Q,*)$ some left principal isotope of $(Q,\cdot)$. Let $\phi(x,v)=([v/(xv)]v)$ and $\gamma(x,v)=\mathbb{R}_vR_{(xv)}\mathbb{L}_uL_x$ for all $x,v\in Q$, then $(Q,\cdot,\backslash,/)$ is a left universal Osborn loop if and only if the composition*

$$(Q,\cdot)\xrightarrow[\textit{left principal isotopism}]{(R_{\phi(x,v)},I,I)}(Q,*)\xrightarrow[\textit{isomorphism}]{(\gamma,\gamma,\gamma)}(Q,\circ)\xrightarrow[\textit{principal isotopism}]{(\mathbb{R}_v,\mathbb{L}_x,I)}(Q,\cdot)\qquad(6)$$

holds.

Proof. The method of the proof of this theorem is similar to the method used to prove Theorem 2.1 by just using the arbitrary left principal isotope $\mathfrak{Q}=(Q,\blacktriangle,\nwarrow,\nearrow)$ such that

$$x\blacktriangle y=xR_v^{-1}\cdot y=(x/v)\cdot y\ \forall\ v\in Q.$$

In the process of the proof, it will be observed that a loop $\mathfrak{Q}=(Q,\cdot,\backslash,/)$ is a left universal Osborn loop if and only if it obeys the identity

$$x\cdot[(y\cdot zv)/v\cdot(xv)]=(x\cdot\{[y([v/(xv)]v)]/v\cdot(xv)\})/v\cdot[z\cdot xv]\qquad \mathrm{OS}_0^\lambda.$$

Writing identity OS_0^λ in autotopic form, we can conclude that $\mathfrak{Q}$ is a left universal Osborn loop if and only if the triple $(\alpha(x,v),\beta(x,v),\gamma(x,v))\in AUT(Q)$ for all $x,v\in Q$ where $\alpha(x,v)=R_{([v/(xv)]v)}\mathbb{R}_vR_{[xv]}L_x\mathbb{R}_v$, $\beta(x,v)=\mathbb{R}_vR_{[xv]}\mathbb{L}_v$ and $\gamma(x,v)=\mathbb{R}_vR_{[xv]}L_x$ are elements of $\mathcal{M}\mathrm{ult}(Q)$. Breaking this into compositions like we did in the proof of Theorem 2.1, we shall get equation (6). ∎

Corollary 2.7. *Let $(Q, \cdot, \backslash, /)$ be a loop. Q is a left universal Osborn loop if and only if the bi-mapping $\gamma(x, v) = \mathbb{R}_v R_{(xv)} \mathbb{L}_u L_x \in BS_2(Q)$ for all $x, v \in Q$.*

Proof. This is gotten from Theorem 2.3 as a consequence of the composition in equation (6). ∎

Lemma 2.7. *Let $(G, \cdot, \backslash, /)$ be an Osborn loop that is a G_λ-loop with arbitrary left isotope $(H_i, \circ_i)$ $i \in \Omega$. There exists a bijection $\theta_i : G \to H_i$, $i \in \Omega$ such that $\theta_i^{-1}\gamma(x, v)\theta_i \in BS_2(H_i, \circ_i)$ $i \in \Omega$ for all $x, v \in G$ where $\gamma(x, v) = \mathbb{R}_v R_{[xv]} L_x$.*

Proof. Assuming that G is isomorphic to H_i under θ_i, $i \in \Omega$, then the proof of the lemma follows by using Theorem 1.3 and Corollary 2.7. ∎

Lemma 2.8. *Let $(G, \cdot, \backslash, /)$ be a left universal Osborn loop with an arbitrary left isotope $(H, \circ)$ such that $(G, \cdot)$ is isomorphic to $(H, \circ)$ under θ. Then, $\theta^{-1}\gamma(x, v)\theta \in BS_2(H, \circ)$ for all $x, v \in G$ where $\gamma(x, v) = \mathbb{R}_v R_{[xv]} L_x$.*

Proof. This is a direct consequence of Theorem 1.3. ∎

Corollary 2.8. *Let $(G, \cdot, \backslash, /)$ be a CC-loop or VD-loop or extra loop with arbitrary left isotope $(H_i, \circ_i)$ $i \in \Omega$. There exists a bijection $\theta_i : G \to H_i$, $i \in \Omega$ such that $\theta_i^{-1}\gamma(x, v)\theta_i \in BS_2(H_i, \circ_i)$ $i \in \Omega$ for all $x, v \in G$ where $\gamma(x, v) = \mathbb{R}_v R_{[xv]} L_x$.*

Proof. This follows from Lemma 2.7 and the fact that a CC-loop or VD-loop or extra loop is a G_λ-loop. ∎

Corollary 2.9. *Let $M = (M, \cdot, \backslash, /)$ be*

- *a Moufang loop such that $M = N(M)M^3$ or*
- *a simple Moufang loop with identity such that $M^3 \neq e$ or*
- *Moufang loop which satisfies an M_k-law for $k \not\equiv 1 \bmod 3$*

with arbitrary left isotope $(H_i, \circ_i)$ $i \in \Omega$. There exists a bijection $\theta_i : G \to H_i$, $i \in \Omega$ such that $\theta_i^{-1}\gamma(x, v)\theta_i \in BS_2(H_i, \circ_i)$ $i \in \Omega$ for all $x, v \in M$ where $\gamma(x, v) = \mathbb{R}_v R_{[xv]} L_x$.

Proof. This follows from Lemma 2.7 and the fact that such a Moufang loop is a G_λ-loop in each case as shown in Corollary IV.4.7, Corollary IV.4.8 and Theorem IV.4.10 of [12]. ∎

Corollary 2.10. *Let $(G, \cdot, \backslash, /)$ be any commutative Moufang loop which belongs to a family of left isotopic commutative Moufang loops $\mathfrak{F}$. For every arbitrary $H_i \in \mathfrak{F}$ $i \in \Omega$, there exists a bijection $\theta_i : G \to H_i$, $i \in \Omega$ such that $\theta_i^{-1}\gamma(x, v)\theta_i \in BS_2(H_i, \circ_i)$ $i \in \Omega$ for all $x, v \in G$ where $\gamma(x, v) = \mathbb{R}_v R_{[xv]} L_x$.*

Proof. This follows from Lemma 2.7 and the fact that a family of left isotopic commutative Moufang loops forms an isomorphic family as shown in Theorem IV.5.6 of [12]. ∎

Lemma 2.9. *Let $(Q, \cdot, \backslash, /)$ be a Moufang loop with an arbitrary left isotope $(H, \circ)$ such that $(Q, \cdot)$ is isomorphic to $(H, \circ)$ under θ. Then, $\theta^{-1}\gamma(x, v)\theta \in BS_2(H, \circ)$ for all $x, v \in G$ where $\gamma(x, v) = \mathbb{R}_v R_{[xv]} L_x$.*

Proof. This follows from Lemma 2.8. ∎

Lemma 2.10. *A loop $Q = (Q, \cdot, \backslash, /)$ is a left universal Osborn loop in which an arbitrary principal isotope is isomorphic to some left principal isotope under the identity map if and only if Q is an abelian group of exponent 2.*

Proof. By equation (6) of Theorem 2.3, it can be deduced that if $(Q, \circ)$ is an arbitrary principal isotope of $(Q, \cdot)$, $(Q, *)$ a left principal isotope of $(Q, \cdot)$ and $\gamma(x, v) = \mathbb{R}_v R_{[xv]} L_x$, then $(Q, x, v, \circ)$ is isomorphic to $(Q, e, \phi(x, v), *)$ under γ^{-1},

$$\text{where } \phi(x, v) = ([v/(xv)]v) \text{ for all } x, v \in Q.$$

We now switch to Theorem 1.4. If $\gamma^{-1} = I$ then $\gamma = I$ if and only if $\gamma(x, v) = \mathbb{R}_v R_{[xv]} L_x = I$ if and only if $R_{(xv)} = R_v \mathbb{L}_x$ which implies $y(xv) = [x\backslash(yv)]$ for all $x, y, v \in Q$. Taking $v = y = e$, we get $x^2 = e$. By Theorem 1.1, G is an abelian group. This fact can also be proved by using the sufficient part of Theorem 1.4. The converse is easy. ∎

Corollary 2.11. *A loop $Q = (Q, \cdot, \backslash, /)$ is a Moufang loop or extra loop or VD-loop or CC-loop or universal WIPL in which an arbitrary principal isotope is isomorphic to some left principal isotope under the identity map if and only if Q is an abelian group of exponent 2.*

Proof. Consequence of Lemma 2.10 ∎

Theorem 2.4. *Let $(Q, \cdot, \backslash, /)$ be a left universal Osborn loop and $(Q, \circ)$ some principal isotope of $(Q, \cdot)$. Let $\psi(x, v) = (v^\lambda \cdot xv)$ and $\gamma(x, v) = \mathbb{R}_v R_{[xv]} L_x$ for all $x, v \in Q$, then the composition*

$$(Q, \cdot) \xrightarrow[\textit{isomorphism}]{(\gamma,\gamma,\gamma)} (Q, \circ) \xrightarrow[\textit{principal isotopism}]{\left(\mathbb{R}_{\psi(x,v)}, \mathbb{L}_x, I\right)} (Q, \cdot) \tag{7}$$

holds.

Proof. This follows by using identity OS_0^λ of Theorem 2.3 the way identity OS_0' of Theorem 2.1 was used in to prove Theorem 2.2. ∎

Theorem 2.5. *Let $(Q, \cdot, \backslash, /)$ be a non-associative left universal Osborn loop, let $\gamma(x, v) = \mathbb{R}_v R_{[xv]} L_x$ for all $x, v \in Q$ and let the mapping $\Theta : ROB(Q, \cdot) \longrightarrow BS_1(Q, \cdot)$ be defined by $\Theta : (\delta R_g^{-1}, \delta L_f^{-1}, \delta) \longrightarrow \delta$.*
Then, the autotopism $(\gamma(x, v)\mathbb{R}_{(v^\lambda \cdot xv)}, \gamma(x, v)\mathbb{L}_x, \gamma(x, v)) \notin \ker \Theta$ for all $x, v \in Q$.

Proof. Going by Theorem 2.4, $\mathfrak{R} = (\gamma(x,v)\mathbb{R}_{(v^\lambda \cdot xv)}, \gamma(x,v)\mathbb{L}_x, \gamma(x,v)) \in ROB(Q,\cdot)$. Assuming that the autotopism $(\gamma(x,v)\mathbb{R}_{(v^\lambda \cdot xv)}, \gamma(x,v)\mathbb{L}_x, \gamma(x,v)) \in \ker\Theta$ for all $x, v \in Q$, then using Theorem 1.5, $\gamma(x,v) = I$ which means Q is a group. Which will be a contradiction. ∎

Theorem 2.6. *Let $(Q,\cdot,\backslash,/)$ be a loop, $(Q,\circ)$ an arbitrary right principal isotope of $(Q,\cdot)$ and $(Q,*)$ some principal isotope of $(Q,\cdot)$. Let $\phi(x,u) = (u\backslash[u/(u\backslash x)])$ and $\gamma(x,u) = R_{[u\backslash x]}\mathbb{L}_u L_x$ for all $x, u \in Q$, then $(Q,\cdot,\backslash,/)$ is a right universal Osborn loop if and only if the composition*

$$(Q,\cdot) \xrightarrow[\text{principal isotopism}]{(R_{\phi(x,u)},L_u,I)} (Q,*) \xrightarrow[\text{isomorphism}]{(\gamma,\gamma,\gamma)} (Q,\circ) \xrightarrow[\text{right principal isotopism}]{(I,\mathbb{L}_x,I)} (Q,\cdot) \quad (8)$$

holds.

Proof. The method of the proof of this theorem is similar to the method used to prove Theorem 2.1 by just using the arbitrary right principal isotope $\mathfrak{Q} = (Q, \blacktriangle, \nwarrow, \nearrow)$ such that
$$x\blacktriangle y = x\cdot yL_u^{-1} = x\cdot(u\backslash y)\ \forall\ u \in Q.$$
In the process of the proof, it will be observed that a loop $\mathcal{Q} = (Q,\cdot,\backslash,/)$ is a right universal Osborn loop if and only if it obeys the identity

$$\underbrace{(ux)\cdot u\backslash\{yz\cdot x\} = ((ux)\cdot u\backslash\{[y(u\backslash[u/x])]\cdot x\})\cdot u\backslash[(uz)x].}_{\mathrm{OS}_0^\rho}$$

Writing identity OS_0^ρ in autotopic form, we can conclude that $\mathcal{Q}$ is a right universal Osborn loop if and only if the triple $(\alpha(x,u),\beta(x,u),\gamma(x,u)) \in AUT(Q)$ for all $x, u \in Q$ where $\alpha(x,u) = R_{(u\backslash[u/(u\backslash x)])}R_{[u\backslash x]}\mathbb{L}_u L_x$, $\beta(x,u) = L_u R_{[u\backslash x]}$ and $\gamma(x,u) = R_{[u\backslash x]}\mathbb{L}_u L_x$ are elements of $\mathcal{M}\mathrm{ult}(Q)$. Breaking this into compositions like we did in the proof of Theorem 2.1, we shall get equation (8). ∎

Corollary 2.12. *Let $(Q,\cdot,\backslash,/)$ be a loop. Q is a right universal Osborn loop if and only if the bi-mapping $\gamma(x,u) = R_{[u\backslash x]}\mathbb{L}_u L_x \in BS_2(Q)$ for all $x, u \in Q$.*

Proof. This is gotten from Theorem 2.6 as a consequence of the composition in equation (8). ∎

Lemma 2.11. *Let $(G,\cdot,\backslash,/)$ be an Osborn loop that is a G_ρ-loop with arbitrary right isotope $(H_i,\circ_i)$ $i \in \Omega$. There exists a bijection $\theta_i : G \to H_i$, $i \in \Omega$ such that $\theta_i^{-1}\gamma(x,v)\theta_i \in BS_2(H_i,\circ_i)$ $i \in \Omega$ for all $x, u \in G$ where $\gamma(x,u) = R_{[u\backslash x]}\mathbb{L}_u L_x \in BS_2(Q)$.*

Proof. Assuming that G is isomorphic to H_i under θ_i, $i \in \Omega$, then the proof of the lemma follows by using Theorem 1.3 and Corollary 2.12. ∎

Lemma 2.12. *Let $(G, \cdot, \backslash, /)$ be a right universal Osborn loop with an arbitrary right isotope $(H, \circ)$ such that $(G, \cdot)$ is isomorphic to $(H, \circ)$ under θ. Then, $\theta^{-1}\gamma(x, u)\theta \in BS_2(H, \circ)$ for all $x, u \in G$ where $\gamma(x, u) = R_{[u\backslash x]}\mathbb{L}_u L_x \in BS_2(Q)$.*

Proof. This is a direct consequence of Theorem 1.3. ∎

Corollary 2.13. *Let $(G, \cdot, \backslash, /)$ be a CC-loop or VD-loop or extra loop with arbitrary right isotope $(H_i, \circ_i)$ $i \in \Omega$. There exists a bijection $\theta_i : G \to H_i$, $i \in \Omega$ such that $\theta_i^{-1}\gamma(x, u)\theta_i \in BS_2(H_i, \circ_i)$ $i \in \Omega$ for all $x, u \in G$ where $\gamma(x, u) = R_{[u\backslash x]}\mathbb{L}_u L_x \in BS_2(Q)$.*

Proof. This follows from Lemma 2.11 and the fact that a CC-loop or VD-loop or extra loop is a G_ρ-loop. ∎

Corollary 2.14. *Let $M = (M, \cdot, \backslash, /)$ be*

- *a Moufang loop such that $M = N(M)M^3$ or*
- *a simple Moufang loop with identity such that $M^3 \neq e$ or*
- *Moufang loop which satisfies an M_k-law for $k \not\equiv 1 \bmod 3$*

with arbitrary right isotope $(H_i, \circ_i)$ $i \in \Omega$. There exists a bijection $\theta_i : G \to H_i$, $i \in \Omega$ such that $\theta_i^{-1}\gamma(x, u)\theta_i \in BS_2(H_i, \circ_i)$ $i \in \Omega$ for all $x, u \in M$ where $\gamma(x, u) = R_{[u\backslash x]}\mathbb{L}_u L_x \in BS_2(Q)$.

Proof. This follows from Lemma 2.11 and the fact that such a Moufang loop is a G_ρ-loop in each case as shown in Corollary IV.4.7, Corollary IV.4.8 and Theorem IV.4.10 of [12]. ∎

Corollary 2.15. *Let $(G, \cdot, \backslash, /)$ be any commutative Moufang loop which belongs to a family of right isotopic commutative Moufang loops $\mathfrak{F}$. For every arbitrary $H_i \in \mathfrak{F}$ $i \in \Omega$, there exists a bijection $\theta_i : G \to H_i$, $i \in \Omega$ such that $\theta_i^{-1}\gamma(x, u)\theta_i \in BS_2(H_i, \circ_i)$ $i \in \Omega$ for all $x, u \in G$ where $\gamma(x, u) = R_{[u\backslash x]}\mathbb{L}_u L_x \in BS_2(Q)$.*

Proof. This follows from Lemma 2.11 and the fact that a family of right isotopic commutative Moufang loops forms an isomorphic family as shown in Theorem IV.5.6 of [12]. ∎

Lemma 2.13. *Let $(Q, \cdot, \backslash, /)$ be a Moufang loop with an arbitrary right isotope $(H, \circ)$ such that $(Q, \cdot)$ is isomorphic to $(H, \circ)$ under θ. Then, $\theta^{-1}\gamma(x, u)\theta \in BS_2(H, \circ)$ for all $x, u \in G$ where $\gamma(x, u) = R_{[u\backslash x]}\mathbb{L}_u L_x \in BS_2(Q)$.*

Proof. This follows from Lemma 2.12. ∎

Lemma 2.14. *A loop $Q = (Q, \cdot, \backslash, /)$ is a right universal Osborn loop in which an arbitrary right isotope is isomorphic to some principal isotope under the identity map if and only if Q is an abelian group of exponent 2.*

Proof. By equation (8) of Theorem 2.6, it can be deduced that if $(Q, \circ)$ is an arbitrary right principal isotope of $(Q, \cdot)$, $(Q, *)$ a principal isotope of $(Q, \cdot)$ and $\gamma(x, u) = R_{[u\backslash x]}\mathbb{L}_u L_x \in BS_2(Q)$, then $(Q, x, e, \circ)$ is isomorphic to $(Q, u, \phi(x, u), *)$ under γ^{-1},

$$\text{where } \phi(x, u) = (u\backslash[u/(u\backslash x)]) \text{ for all } x, u \in Q.$$

We now switch to Theorem 1.4. If $\gamma^{-1} = I$ then $\gamma = I$ if and only if $\gamma(x, u) = R_{[u\backslash x]}\mathbb{L}_u L_x = I$ if and only if $R_{(u\backslash x)} = \mathbb{L}_x L_u$ which implies $y(u\backslash x) = u \cdot x\backslash y$ for all $x, y, u \in Q$. Taking $u = y = e$, we get $x^2 = e$. By Theorem 1.1, G is an abelian group. This fact can also be proved by using the sufficient part of Theorem 1.4. The converse is easy. ∎

Lemma 2.15. *A loop $Q = (Q, \cdot, \backslash, /)$ is a right universal Osborn loop which is isomorphic to some principal isotope under the identity map if and only if Q is an abelian group of exponent 2.*

Proof. The procedure of the proof is similar to that of Lemma 2.14 i.e. using $\gamma(e, u)$. This fact can also be proved by using the sufficient part of Corollary 1.1. The converse is easy. ∎

Corollary 2.16. *A loop $Q = (Q, \cdot, \backslash, /)$ is a Moufang loop or extra loop or VD-loop or CC-loop or universal WIPL which is isomorphic to some principal isotope under the identity map if and only if Q is an abelian group of exponent 2.*

Proof. Consequence of Lemma 2.15 ∎

Theorem 2.7. *Let $(Q, \cdot, \backslash, /)$ be a right universal Osborn loop, $(Q, *)$ an arbitrary right principal isotope of $(Q, \cdot)$ and $(Q, \circ)$ some principal isotope of $(Q, \cdot)$. Let $\gamma(x, u) = R_{[u\backslash x]}\mathbb{L}_u L_x$ for all $x, u \in Q$, then the composition*

$$(Q, \cdot) \xrightarrow[\text{right principal isotopism}]{(I, L_u, I)} (Q, *) \xrightarrow[\text{isomorphism}]{(\gamma, \gamma, \gamma)} (Q, \circ) \xrightarrow[\text{principal isotopism}]{\left(\mathbb{R}_{(u\backslash x)}, \mathbb{L}_x, I\right)} (Q, \cdot) \qquad (9)$$

holds.

Proof. This follows by using identity OS_0^ρ of Theorem 2.6 the way identity OS_0' of Theorem 2.1 was used in to prove Theorem 2.2 . ∎

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SOME CLASSES OF ANALYTIC FUNCTIONS WITH RESPECT TO (J, K) - SYMMETRIC POINTS

Kadhavoor Ragavan Karthikeyan

Mathematics Section, Department Of Information Technology, Salalah College Of Tecnology, Salalah, Sultanate of Oman

kr_karthikeyan1979@yahoo.com

Abstract The author introduces new classes of analytic functions with respect to $(j,\ k)$-symmetric points. Integral representation, interesting conditions for starlikeness and inclusion relations for functions in these classes are obtained.

Keywords: univalent, analytic, $(j,\ k)$- symmetrical functions.
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1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad a_n \geq 0, \tag{1}$$

which are analytic in the open disc $\mathcal{U} = \{z \in \mathbb{C} \setminus \mid z \mid < 1\}$ and $\mathcal{S}$ be the class of functions $f \in \mathcal{A}$ which are univalent in $\mathcal{U}$.

We denote by $\mathcal{S}^*$, $\mathcal{C}$, $\mathcal{K}$ and $\mathcal{C}^*$ the familiar subclasses of $\mathcal{A}$ consisting of functions which are respectively starlike, convex, close-to-convex and quasi-convex in $\mathcal{U}$. Our favorite references of the field are [2, 3] which covers most of the topics in a lucid and economical style.

Let $f(z)$ and $g(z)$ be analytic in $\mathcal{U}$. Then we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathcal{U}$, if there exists an analytic function $w(z)$ in $\mathcal{U}$ such that $|w(z)| < |z|$ and $f(z) = g(w(z))$, denoted by $f(z) \prec g(z)$. If $g(z)$ is univalent in $\mathcal{U}$, then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

Let k be a positive integer and $j = 0,\ 1,\ 2, \ldots (k-1)$. A domain D is said to be $(j,\ k)$-fold symmetric if a rotation of D about the origin through an angle $2\pi j/k$ carries D onto itself. A function $f \in \mathcal{A}$ is said to be $(j,\ k)$-symmetrical if for each $z \in \mathcal{U}$

$$f(\varepsilon z) = \varepsilon^j f(z), \tag{2}$$

where $\varepsilon = \exp(2\pi i/k)$. The family of $(j,\ k)$-symmetrical functions will be denoted by $\mathcal{F}_k^j$. We observe that $\mathcal{F}_2^1$, $\mathcal{F}_2^0$ and $\mathcal{F}_k^1$ are well-known families of odd functions, even functions and k-symmetrical functions respectively.

Also let $f_{j,k}(z)$ be defined by the following equality

$$f_{j,k}(z) = \frac{1}{k}\sum_{\nu=0}^{k-1}\frac{f(\varepsilon^{\nu}z)}{\varepsilon^{\nu j}}, \quad (f \in \mathcal{A};\ k = 1, 2, \ldots;\ j = 0, 1, 2, \ldots (k-1)), \tag{3}$$

where ν is an integer.

The notion of $(j,\ k)$-symmetric functions was introduced and studied by P. Liczberski and J. Połubiński in [4].

The following identities follow directly from (3):

$$\begin{aligned} f_{j,k}(\varepsilon^{\nu}z) &= \varepsilon^{\nu j} f_{j,k}(z), \\ f'_{j,k}(\varepsilon^{\nu}z) &= \varepsilon^{\nu j-\nu} f'_{j,k}(z) = \frac{1}{k}\sum_{\nu=0}^{k-1}\frac{f'(\varepsilon^{\nu}z)}{\varepsilon^{\nu j-\nu}}, \\ f''_{j,k}(\varepsilon^{\nu}z) &= \varepsilon^{\nu j-2\nu} f''_{j,k}(z) = \frac{1}{k}\sum_{\nu=0}^{k-1}\frac{f''(\varepsilon^{\nu}z)}{\varepsilon^{\nu j-2\nu}}. \end{aligned} \tag{4}$$

Motivated by the concept introduced by K. Sakaguchi in [7], recently several subclasses of analytic functions with respect to k-symmetric points were introduced and studied by various authors. In this paper, new subclasses of analytic functions with respect to $(j,\ k)$-symmetric points are introduced.

We now define the following:
A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_s^{(j,k)}$ if and only if it satisfies the condition

$$Re\left(\frac{zf'(z)}{f_{j,k}(z)}\right) > 0 \quad (z \in \mathcal{U}). \tag{5}$$

We call the functions $f \in \mathcal{A}$ that satisfy the condition (5) to be starlike with respect to $(j,\ k)$-symmetric points.

Similarly, we define the class $\mathcal{C}_s^{(j,k)}$ of convex functions with respect to $(j,\ k)$-symmetric points if and only if

$$Re\left(\frac{(zf'(z))'}{f'_{j,k}(z)}\right) > 0 \quad (z \in \mathcal{U}). \tag{6}$$

The different subclasses of $\mathcal{S}_s^{(j,k)}$ can be obtained by replacing condition (5) by

$$\frac{zf'(z)}{f_{j,k}(z)} \prec \phi(z), \tag{7}$$

$$(z \in \mathcal{U};\ k = 1, 2, \ldots;\ j = 0, 1, 2, \ldots (k-1)),$$

where $\phi \in \mathcal{P}$, the class of functions with positive real part. We denote by $\mathcal{S}_s^{(j,k)}(\phi)$, the class of functions $f \in \mathcal{A}$ that satisfies the condition (7).
Similarly let $\mathcal{C}_s^{(j,k)}(\phi)$ denote the class of functions in $\mathcal{S}$ satisfying the condition

$$\frac{\left(zf'(z)\right)'}{f'_{j,k}(z)} \prec \phi(z),$$

$$(z \in \mathcal{U};\ k = 1, 2, \ldots;\ j = 0, 1, 2, \ldots (k-1)),$$

where $\phi \in \mathcal{P}$.

Remark 1.1. *For different choices of the parameters* j, k *and the function* $\phi(z)$*, the classes* $\mathcal{S}_s^{(j,k)}(\phi)$ *and* $\mathcal{C}_s^{(j,k)}(\phi)$ *reduce to various other well-known and new subclasses of analytic functions. For details see [8].*

2. INCLUSION RELATIONSHIPS AND INTEGRAL REPRESENTATIONS OF THE CLASSES $\mathcal{S}_s^{(J,K)}(\phi)$ AND $\mathcal{C}_s^{(J,K)}(\phi)$

Let us begin with the following:

Theorem 2.1. *If* $f \in \mathcal{C}_s^{(j,k)}(\phi)$*, then* f *is univalent in* $\mathcal{U}$*.*

Proof. From the definition of $\mathcal{C}_s^{(j,k)}(\phi)$,

$$Re\left(\frac{(zf'(z))'}{f'_{j,k}(z)}\right) > 0, \tag{8}$$

since $Re\{\phi(z)\} > 0$. If we replace z by $\varepsilon^\nu z$ in (8), then (8) will be of the form

$$Re\left\{\frac{f'(\varepsilon^\nu z) + \varepsilon^\nu z f''(\varepsilon^\nu z)}{f'_{j,k}(\varepsilon^\nu z)}\right\} > 0, \quad (z \in \mathcal{U};\ \nu = 0, 1, 2, \ldots, k-1). \tag{9}$$

Using (4) in (9), we get

$$Re\left\{\frac{f'(\varepsilon^\nu z) + \varepsilon^\nu z f''(\varepsilon^\nu z)}{\varepsilon^{\nu j - \nu} f'_{j,k}(z)}\right\} > 0, \quad (z \in \mathcal{U}). \tag{10}$$

Let $\nu = 0, 1, 2, \ldots, k-1$ in (10) respectively and summing them, we get

$$Re\left\{\frac{\sum_{\nu=0}^{k-1} \varepsilon^{\nu - \nu j} f'(\varepsilon^\nu z) + z \sum_{\nu=0}^{k-1} \varepsilon^{2\nu - \nu j} f''(\varepsilon^\nu z)}{f'_{j,k}(z)}\right\} > 0, \quad (z \in \mathcal{U}).$$

Or equivalently,

$$Re\left(\frac{f'_{j,k}(z) + z f''_{j,k}(z)}{f'_{j,k}(z)}\right) > 0, \quad (z \in \mathcal{U}),$$

that is $f_{j,k}(z) \in \mathcal{C}$. Using this together with the condition (6) shows the functions in $\mathcal{C}_s^{(j,k)}$ are quasi-convex. It is well-known that the class of quasi-convex functions are univalent, hence functions which are convex with respect to (j, k)-symmetric points are univalent. ∎

By using the same method as that of Theorem 2.1, we may obtain the following result.

Theorem 2.2. *If $f \in \mathcal{S}_s^{(j,k)}(\phi)$, then $f_{j,k}(z) \in \mathcal{S}^*$.*

Remark 2.1. *Using the condition (5) together with Theorem 2.2 shows that the functions in $\mathcal{S}_s^{(j,k)}$ are close-to-convex. It is well-known that the class of close-to-convex functions are univalent, hence functions which are starlike with respect to (j, k)-symmetric points are univalent.*

Theorem 2.3. *Let $f \in \mathcal{S}_s^{(j,k)}(\phi)$, then we have*

$$f_{j,k}(z) = z \exp\left\{\frac{1}{k}\sum_{\nu=0}^{k-1}\int_0^{\varepsilon^\nu z} \frac{\phi(w(t)) - 1}{t}\, dt\right\} \tag{11}$$

where $f_{j,k}(z)$ defined by equality (3), $w(z)$ is analytic in $\mathcal{U}$ and $w(0) = 0$, $\mid w(z) \mid < 1$.

Proof. Let $f \in \mathcal{S}_s^{(j,k)}(\phi)$, from the definition of $\mathcal{S}_s^{(j,k)}(\phi)$, we have

$$\frac{zf'(z)}{f_{j,k}(z)} = \phi(w(z)), \tag{12}$$

where $w(z)$ is analytic in $\mathcal{U}$ and $w(0) = 0$, $\mid w(z) \mid < 1$. Substituting z by $\varepsilon^\nu z$ in the equality (12) respectively ($\nu = 0, 1, 2, \ldots, k-1$, $\varepsilon^k = 1$), we have

$$\frac{\varepsilon^\nu z f'(\varepsilon^\nu z)}{f_{j,k}(\varepsilon^\nu z)} = \phi(w(\varepsilon^\nu z)) \tag{13}$$

Using (4) in (13), we get

$$\frac{z\varepsilon^{\nu-\nu j} f'(\varepsilon^\nu z)}{f_{j,k}(z)} = \phi(w(\varepsilon^\nu z)). \tag{14}$$

Let $\nu = 0, 1, 2, \ldots, k-1$ in (14) respectively and summing them we get,

$$\frac{zf'_{j,k}(z)}{f_{j,k}(z)} = \frac{1}{k}\sum_{\nu=0}^{k-1}\phi(w(\varepsilon^\nu z)),$$

From this equality, we get

$$\frac{f'_{j,k}(z)}{f_{j,k}(z)} - \frac{1}{z} = \frac{1}{k}\sum_{\nu=0}^{k-1}\frac{\phi(w(\varepsilon^\nu z)) - 1}{z}.$$

Integrating this equality, we get

$$\begin{aligned}\log\left\{\frac{f_{j,k}(z)}{z}\right\} &= \frac{1}{k}\sum_{\nu=0}^{k-1}\int_0^z \frac{\phi\left(w(\varepsilon^\nu\zeta)\right)-1}{\zeta}\,d\zeta\\ &= \frac{1}{k}\sum_{\nu=0}^{k-1}\int_0^{\varepsilon^\nu z} \frac{\phi\left(w(t)\right)-1}{t}\,dt,\end{aligned}$$

or equivalently,

$$f_{j,k}(z) = z\ \exp\left\{\frac{1}{k}\sum_{\nu=0}^{k-1}\int_0^{\varepsilon^\nu z} \frac{\phi\left(w(t)\right)-1}{t}\,dt\right\}.$$

This completes the proof of Theorem 2.3. ∎

Theorem 2.4. *Let $f \in \mathcal{C}_s^{(j,k)}(\phi)$, then we have*

$$f_{j,k}(z) = \int_0^z \exp\left\{\frac{1}{k}\sum_{\nu=0}^{k-1}\int_0^{\varepsilon^\nu \zeta} \frac{\phi\left(w(t)\right)-1}{t}\,dt\right\}d\zeta \tag{15}$$

where $f_{j,k}(z)$ defined by equality (3), $w(z)$ is analytic in $\mathcal{U}$ and $w(0) = 0$, $|\, w(z) \,| < 1$.

Remark 2.2. *Several well-known and new results can be obtained as a special case of the results stated in this section for different choice of the parameters. For example see [8].*

3. CONDITIONS FOR STARLIKENESS WITH RESPECT TO SYMMETRIC POINTS

We now state the following result which will be used in the sequel.

Lemma 3.1. *[5, 1] Let the function q be univalent in the open unit disc $\mathcal{U}$ and θ and ϕ be analytic in a domain D containing $q(\mathcal{U})$ with $\phi(w) \neq 0$ when $w \in q(\mathcal{U})$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$. Suppose that*

1. *Q is starlike univalent in $\mathcal{U}$, and*
2. *$Re\left(\frac{zh'(z)}{Q(z)}\right) > 0$ for $z \in \mathcal{U}$.*

If

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)),$$

then $p(z) \prec q(z)$ and q is the best dominant.

Theorem 3.1. *Let the function $g(z)$ be convex univalent in $\mathcal{U}$ and also let*

$$Re\left\{\alpha\left(\frac{g(z)}{zg'(z)}(g(z)-1)+1\right)+\beta\frac{g(z)}{zg'(z)}\right\}>0 \tag{16}$$

and

$$h(z)=\alpha zg'(z)+\alpha g^2(z)+(\beta-\alpha)g(z),$$

where $\alpha>0$, $\alpha+\beta>0$.
If $f\in\mathcal{A}$ with $\frac{f_{j,k}(z)}{z}\neq 0$ satisfies the condition

$$\alpha\left\{\frac{z^2f''(z)}{f_{j,k}(z)}-\frac{z^2f'(z)f'_{j,k}(z)}{\left(f_{j,k}(z)\right)^2}+\frac{z^2(f'(z))^2}{\left(f_{j,k}(z)\right)^2}\right\}+\beta\frac{zf'(z)}{f_{j,k}(z)}\prec h(z), \tag{17}$$

then $f\in\mathcal{S}_s^{(j,k)}(g)$ and g is the best dominant.

Proof. Let the function p be defined by

$$p(z)=\frac{zf'(z)}{f_{j,k}(z)}\qquad(z\in\mathcal{U};\ z\neq 0;\ f\in\mathcal{A}),$$

then $p(z)=1+p_1z+p_2z^2+\cdots\in\mathcal{P}$. By a straight forward computation, we have

$$zp'(z)=\frac{zf'(z)}{f_{j,k}(z)}+\frac{z^2f''(z)}{f_{j,k}(z)}-\frac{z^2f'(z)f'_{j,k}(z)}{\left(f_{j,k}(z)\right)^2}.$$

Thus by (17), we have

$$\alpha zp'(z)+\alpha p^2(z)+(\beta-\alpha)p(z)\prec h(z). \tag{18}$$

By setting

$$\theta(w):=\alpha w^2+(\beta-\alpha)w\qquad\text{and}\qquad\phi(w):=\alpha,$$

it can be easily verified that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}$ with $\phi(0)\neq 0$ in the w-plane.
If we let $Q(z)=zg'(z)\phi(g(z))$ and $h(z)=\theta(g(z))+Q(z)$, then

$$Q(z)=\alpha zg'(z)$$

and

$$h(z)=\alpha(g(z))^2+(\beta-\alpha)g(z)+\alpha zg'(z).$$

Since $g(z)$ is convex univalent in $\mathcal{U}$ it implies that $Q(z)$ is starlike univalent in $\mathcal{U}$. Further, we have

$$Re\frac{zh'(z)}{Q(z)}=Re\left\{\alpha\left(\frac{g(z)}{zg'(z)}(g(z)-1)+1\right)+\beta\frac{g(z)}{zg'(z)}\right\}>0.$$

The assertion of the Theorem 3.1 now follows by applying Lemma 3.1. ∎

Corollary 3.1. *If $f \in \mathcal{A}$ with $\frac{f_{j,k}(z)}{z} \neq 0$ satisfies the condition*

$$\alpha\left\{\frac{z^2 f''(z)}{f_{j,k}(z)} - \frac{z^2 f'(z) f'_{j,k}(z)}{\left(f_{j,k}(z)\right)^2} + \frac{z^2 (f'(z))^2}{\left(f_{j,k}(z)\right)^2}\right\} + \beta \frac{z f'(z)}{f_{j,k}(z)} \prec h(z),$$

where

$$h(z) = \frac{a[\alpha(a-b) + \beta b]z^2 + [2\alpha(a-b) + \beta(a+b)]z + \beta}{(1+bz)^2},$$

$$-1 \leq b < a \leq 1 \qquad \text{and} \qquad \beta \geq 2\alpha^2\left(\frac{|b|}{1+|b|} - \frac{1-a}{1-b}\right)$$

then $f \in \mathcal{S}_s^{(j,k)}\left(\frac{1+az}{1+bz}\right)$.

Proof. We let $g(z) = \frac{1+az}{1+bz}$, in Theorem 3.1. Clearly $g(z)$ is convex univalent in $\mathcal{U}$. Hence the proof of the Corollary follows from Theorem 3.1. ∎

If we let $j = k = 1$ in the Corollary 3.1, we get the following interesting result.

Corollary 3.2. *[9] If $f \in \mathcal{A}$ with $\frac{f(z)}{z} \neq 0$ satisfies the condition*

$$\frac{z^2 f''(z)}{f(z)} + \beta \frac{z f'(z)}{f(z)} \prec h(z),$$

where

$$h(z) = \frac{a[a - b + \beta b]z^2 + [2(a-b) + \beta(a+b)]z + \beta}{(1+bz)^2},$$

$$-1 \leq b < a \leq 1 \qquad \text{and} \qquad \beta \geq 2\left(\frac{|b|}{1+|b|} - \frac{1-a}{1-b}\right),$$

then $f \in \mathcal{S}\left(\frac{1+az}{1+bz}\right)$.

Corollary 3.3. *If $f \in \mathcal{A}$ with $\frac{f_{j,k}(z)}{z} \neq 0$, $z \in \mathcal{U}$ and*

$$D = \mathbb{C} \setminus \left\{z \in \mathbb{C} : \, Re\, z \leq -\frac{1}{2}, \, Im\, z = 0\right\},$$

then

$$\frac{z^2 f''(z)}{f_{j,k}(z)} - \frac{z^2 f'(z) f'_{j,k}(z)}{\left(f_{j,k}(z)\right)^2} + \frac{z^2 (f'(z))^2}{\left(f_{j,k}(z)\right)^2} + \frac{z f'(z)}{f_{j,k}(z)} \in D \quad \Longrightarrow f \in \mathcal{S}_s^{(j,k)}.$$

Proof. If we let $\alpha = 1$, $\beta = 1$ and $g(z) = \dfrac{1+z}{1-z}$ in Theorem 3.1. It follows that $h(z)$ is convex with respect to the point $u = -1/2$. Hence the proof of the Corollary. ▮

If we let $j = k = 1$ in the Corollary 3.3, we get the following well-known result.

Corollary 3.4. *[6] If $f \in \mathcal{A}$ with $\dfrac{f(z)}{z} \neq 0$, $z \in \mathfrak{U}$ and*

$$D = \mathbb{C} \setminus \left\{ z \in \mathbb{C} : \, Re\, z \leq -\frac{1}{2}, \, Im\, z = 0 \right\},$$

then

$$\frac{z^2 f''(z)}{f(z)} + \frac{z f'(z)}{f(z)} \in D \quad \Longrightarrow Re\left(\frac{z f'(z)}{f(z)}\right) > 0.$$

Corollary 3.5. *If $f \in \mathcal{A}$ with $\dfrac{f_{j,k}(z)}{z} \neq 0$, $z \in \mathfrak{U}$, satisfy the condition*

$$\Phi_k^j(z) = \alpha \left\{ \frac{z^2 f''(z)}{f_{j,k}(z)} - \frac{z^2 f'(z) f'_{j,k}(z)}{\left(f_{j,k}(z)\right)^2} + \frac{z^2 (f'(z))^2}{\left(f_{j,k}(z)\right)^2} \right\} + \frac{z f'(z)}{f_{j,k}(z)} \prec 1 + \delta z,$$

where $\delta = \mu(2\alpha + 1 - \alpha\mu)$ and $0 < \mu \leq \left(1 + \dfrac{1}{2\alpha}\right)$, then

$$\frac{z f'(z)}{f_{j,k}(z)} \prec 1 + \mu z.$$

Proof. If we let $\beta = 1$ and $g(z) = 1 + \mu z$ in Theorem 3.1, then $h(z)$ will be of the form $h(z) = 1 + (2\alpha + 1)\mu z + \alpha\mu^2 z^2$. For $|\, z \,|= 1$,

$$|h(z) - 1| = \mu\, |2\alpha + 1 + \alpha\mu z| \geq \mu\, (2\alpha + 1 - \alpha\mu)\,.$$

If we put $\delta = (2\alpha + 1 - \alpha\mu)$, then from the above inequality it follows that $h(z)$ is superordinate to $1 + \delta z$. Hence the proof of the Corollary. ▮

If we let $\alpha = 1$ and $\mu = 1$ in the Corollary 3.5, then we have the following result.

Corollary 3.6. *If $f \in \mathcal{A}$ with $\dfrac{f_{j,k}(z)}{z} \neq 0$, $z \in \mathfrak{U}$, then*

$$\left| \frac{z f'(z)}{f_{j,k}(z)} \left(1 + \frac{f''(z)}{f'(z)} - \frac{z f'_{j,k}(z)}{f_{j,k}(z)} + \frac{z f'(z)}{f_{j,k}(z)} \right) - 1 \right| < 2 \quad (z \in \mathfrak{U})$$

implies $\left| \dfrac{z f'(z)}{f_{j,k}(z)} - 1 \right| < 1$, *for all* $z \in \mathfrak{U}$.

If we let $j = k = 1$ in the Corollary 3.6, we get the following interesting result.

Corollary 3.7. *If* $f \in \mathcal{A}$, *then*

$$\left| \frac{zf'(z)}{f(z)} \left(1 + \frac{f''(z)}{f'(z)} \right) - 1 \right| < 2 \quad (z \in \mathcal{U}), \quad \Longrightarrow \quad \left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 \quad (z \in \mathcal{U}).$$

It is well-known that a function $f \in \mathcal{A}$ is called strongly-starlike of order λ, $0 < \lambda \leq 1$, if

$$\left| \arg \frac{zf'(z)}{f(z)} \right| < \lambda \frac{\pi}{2}, \quad (z \in \mathcal{U})$$

and we denote by $\mathcal{SS}^*(\lambda)$ the class of such functions. Similarly, we denote the class of strongly-starlike functions of order λ with respect to (j, k)-symmetric points by $\mathcal{SS}_s^{(j,k)}(\lambda)$.

Now, we give the sufficient conditions for strongly-starlike of order λ with respect to (j, k)-symmetric points

Corollary 3.8. *Let* $0 < \lambda < 1$, *and let*

$$h(z) = \left(\frac{1+z}{1-z} \right)^{\lambda} \left[\frac{2\lambda z}{1-z^2} + \left(\frac{1+z}{1-z} \right)^{\lambda} \right].$$

If $f \in \mathcal{A}$ *with* $\dfrac{f_{j,k}(z)}{z} \neq 0$, $z \in \mathcal{U}$, *satisfies the condition*

$$\frac{z^2 f''(z)}{f_{j,k}(z)} - \frac{z^2 f'(z) f'_{j,k}(z)}{\left(f_{j,k}(z)\right)^2} + \frac{z^2 (f'(z))^2}{\left(f_{j,k}(z)\right)^2} + \frac{zf'(z)}{f_{j,k}(z)} \prec h(z),$$

then $f \in \mathcal{SS}_s^{(j,k)}(\lambda)$.

If we let $j = k = 1$ in the Corollary 3.8, we get the following interesting result.

Corollary 3.9. *Let* $0 < \lambda < 1$, *and let*

$$h(z) = \left(\frac{1+z}{1-z} \right)^{\lambda} \left[\frac{2\lambda z}{1-z^2} + \left(\frac{1+z}{1-z} \right)^{\lambda} \right].$$

If $f \in \mathcal{A}$ *with* $\dfrac{f_{j,k}(z)}{z} \neq 0$, $z \in \mathcal{U}$, *satisfies the condition*

$$\frac{zf'(z)}{f(z)} \left[\frac{zf''(z)}{f'(z)} + 1 \right] \prec h(z),$$

then $f \in \mathcal{SS}^*(\lambda)$.

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FIRST PASSAGE TO A SEMI-INFINITE LINE FOR A TWO-DIMENSIONAL WIENER PROCESS

Mario Lefebvre
Département de Mathématiques et de Génie Industriel, École Polytechnique, Montréal, Canada
mlefebvre@polymtl.ca

Abstract Assume that $X(t)$ and $Y(t)$ are independent Wiener processes with drift -1 and 0, respectively, and diffusion coefficient equal to 2 (in both cases). Let $I(x,y)$ be the indicator function of the event $\{\tau(x,y) < \infty\}$, where $\tau(x,y) = \inf\{t > 0 : Y(t) = 0, X(t) \geq 0 \mid X(0) = x, Y(0) = y\}$, in which $y \neq 0$ or $x < 0$. We obtain an explicit expression for $\phi(x,y) = E\left[e^{-aX[\tau(x,y)]}I(x,y)\right]$. An application to an optimal control problem is also presented.

Keywords: Brownian motion, first exit time, Kolmogorov backward equation, optimal stochastic control.
2010 MSC: 60J70.

1. INTRODUCTION

We consider the two-dimensional Wiener process $(X(t), Y(t))$ defined by the system of stochastic differential equations

$$\begin{aligned} dX(t) &= -dt + \sqrt{2}\, dB_1(t), \\ dY(t) &= \sqrt{2}\, dB_2(t), \end{aligned}$$

where $B_1(t)$ and $B_2(t)$ are independent standard Brownian motions. Let

$$\tau(x,y) = \inf\{t > 0 : Y(t) = 0, X(t) \geq 0 \mid X(0) = x, Y(0) = y\},$$

where $y \neq 0$ or $x < 0$. We define

$$I(x,y) = \begin{cases} 1 & \text{if } \tau(x,y) < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

That is, $I(x,y)$ is the indicator function of the event $\{\tau(x,y) < \infty\}$.

The function

$$\phi(x,y) := E\left[e^{-aX[\tau(x,y)]}I(x,y)\right], \tag{1}$$

where $a > 0$, satisfies the Kolmogorov equation

$$\phi_{yy} + \phi_{xx} - \phi_x = 0,$$

and is subject to the conditions $\phi(x,0) = e^{-ax}$ if $x \geq 0$, and $\phi(x,y) \to 0$ if $x^2+y^2 \to \infty$.

Since the stochastic processes $X(t)$ and $Y(t)$ are independent, if we replace the first-passage time $\tau(x, y)$ by

$$\tau_0(x, y) = \inf\{t > 0 : Y(t) = 0 \mid X(0) = x, Y(0) = y\},$$

where $y \neq 0$ and $x \in \mathbb{R}$, then the function $\phi_0(x, y)$ that corresponds to $\phi(x, y)$ is easy to obtain. Indeed, first we can state that $\tau_0(x, y)$ actually does not depend on the variable x. Moreover, it is well known that $P[\tau_0(y) < \infty] = 1$. Therefore, we can write that

$$\phi_0(x, y) = E\left[e^{-aX[\tau_0(y)]} \,\middle|\, X(0) = x, Y(0) = y\right].$$

Next, making use of the fact that $X(t)$ has a Gaussian distribution with mean $x - t$ and variance $2t$, and of the following formula for the probability density function of the random variable $\tau_0(y)$ (see Lefebvre [3], for instance):

$$f_{\tau_0(y)}(t) = \frac{|y|}{\sqrt{4\pi t^3}} \exp\left\{-\frac{y^2}{4t}\right\} \quad \text{for } t > 0, \tag{2}$$

we can derive an explicit (and exact) expression for $\phi_0(x, y)$ by conditioning on the random variable $\tau_0(y)$. That is, we write that

$$\begin{aligned}
\phi_0(x, y) &= E\left[e^{-aX[\tau_0(y)]} \,\middle|\, X(0) = x, Y(0) = y\right] \\
&= E\left[E\left[e^{-aX[\tau_0(y)]} \,\middle|\, \tau_0(y), X(0) = x, Y(0) = y\right]\right] \\
&= \int_0^\infty E\left[e^{-aX[\tau_0(y)]} \,\middle|\, \tau_0(y) = t, X(0) = x, Y(0) = y\right] f_{\tau_0(y)}(t) dt \\
&= \int_0^\infty \int_{-\infty}^\infty \frac{e^{-au}}{\sqrt{4\pi t}} \exp\left\{-\frac{(u - x + t)^2}{4t}\right\} \frac{|y|}{\sqrt{4\pi t^3}} \exp\left\{-\frac{y^2}{4t}\right\} du dt \\
&= \frac{|y|}{4\pi} \int_0^\infty \int_{-\infty}^\infty \frac{e^{-au}}{t^2} \exp\left\{-\frac{(u - x + t)^2 + y^2}{4t}\right\} du dt.
\end{aligned}$$

The main difficulty in computing the function $\phi(x, y)$ stems from the fact that it is discontinuous on the boundary $y = 0$. A related problem for which the function is discontinuous on the boundary has been considered by the author and Whittle (see Lefebvre and Whittle [4]) in an optimization context. They defined the two-dimensional diffusion process $(X(t), Y(t))$ by

$$dX(t) = Y(t)\, dt, \tag{3}$$

$$dY(t) = bu(t)dt + \sigma\, dB(t), \tag{4}$$

where $b \neq 0$ is a constant, $u(t)$ is the control variable and $B(t)$ is a standard Brownian motion. Hence, $X(t)$ is a controlled integrated Brownian motion. They looked for the control u^* that minimizes the expected value of the cost function

$$J(x) = \int_0^{T_d(x)} \left(\frac{1}{2} q u^2(t) - \lambda\right) dt,$$

in which

$$T_d(x) = \inf\{t > 0 : |X(t)| = d \mid X(0) = x\},$$

with $-d < x < d$, and q and λ are positive constants. By appealing to a theorem proved in Whittle [5], the authors were able to express the value of u^* in terms of the following mathematical expectation for the uncontrolled process $(\xi(t), \eta(t))$ that corresponds to $(X(t), Y(t))$:

$$\phi_1(x) := E\left[e^{\lambda \tau_d(x)/\alpha} \,\middle|\, \xi(0) = x\right],$$

where

$$\alpha = \frac{\sigma^2 q}{b^2}$$

and $\tau_d(x)$ is the same as $T_d(x)$, but for the process $(\xi(t), \eta(t))$ obtained by setting $u(t)$ equal to 0 in (4).

The function $\phi_1(x)$ is also discontinuous on the boundaries $x = d$ and $x = -d$, because the process $X(t)$ cannot hit the boundary $x = d$ for the first time with $y < 0$ or, equivalently, the boundary $x = -d$ with $y > 0$.

The authors were not able to derive an exact expression for $\phi_1(x)$. Instead, they used a technique that enabled them to obtain an approximate solution for the optimal control.

Actually, a few years later, Lachal [2] considered, in particular, the problem of computing the probability density function of the random variable

$$\tau_b(x, y) := \inf\{t > 0 : X(t) = b \mid X(0) = x, Y(0) = y\}$$

for the two-dimensional diffusion process $(X(t), Y(t))$ defined by

$$\begin{aligned} dX(t) &= Y(t)\, dt, \\ dY(t) &= dB(t). \end{aligned}$$

That is, $X(t)$ is the integral of the standard Brownian motion $Y(t)$. He derived the following exact expression:

$$\begin{aligned} f_{\tau_b(x,y)}(t) &= \epsilon \left[\sqrt{\frac{3}{2\pi}} \left(\frac{3}{2} \frac{b - x}{t^{5/2}} - \frac{1}{2} \frac{y}{t^{3/2}} \right) \exp\left\{ -\frac{3(b - x - ty)^2}{2t^3} \right\} \right. \\ &\quad \left. + \int_0^\infty z dz \int_0^t f_{\tau_0(0,-z)}(s) q(x, y; b, z; t - s) ds \right], \end{aligned}$$

where

$$f_{\tau_0(0,-z)}(s) = \int_0^\infty \frac{3\mu}{\sqrt{2}\pi s^2} \exp\left\{-\frac{2}{s}(z^2 - \mu z + \mu^2)\right\} d\mu \int_0^{4\mu z/s} e^{-3\theta/2} \frac{d\theta}{\sqrt{\pi\theta}},$$

in which ϵ is the sign of $(b-x)$, $z > 0$ and

$$q(x, y; u, v; t) = p(x, y; u, v; t) - p(x, y; u, -v; t),$$

the function $p(x, y; u, v; t)$ being the joint density function of the random vector $(X(t), Y(t))$, which is known to be

$$\begin{aligned} p(x, y; u, v; t) \quad = \quad & \frac{\sqrt{3}}{\pi t^2} \exp\left\{-\frac{6}{t^3}(u - x - ty)^2 + \frac{6}{t^2}(u - x - ty)(v - y)\right. \\ & \left. -\frac{2}{t}(v - y)^2\right\}. \end{aligned}$$

We see that the exact solution to such a one-boundary problem is quite complicated, and we can expect the solution in the case of a two-boundary problem to be even more complicated. In the context of an optimization problem, such as in Lefebvre and Whittle [4], this exact solution would not have been very useful, at any rate, because one must be able to give an expression for the optimal control that the optimizer can actually implement.

In Section 2, by making use of the Wiener-Hopf technique, we will calculate the Fourier transform of $\phi(x, y)$. We will invert this transform in the case when $y = 0$. Finally, with the help of probabilistic arguments, we will obtain an explicit expression for $\phi(x, y)$.

In Section 3, an application to an optimal control problem will be presented, and we will conclude this work with a few remarks in Section 4.

2. COMPUTATION OF THE FUNCTION $\phi(X, Y)$

To obtain an exact expression for the function $\phi(x, y)$, we will first compute its Fourier transform, with the help of the Wiener-Hopf technique. Let

$$\Phi(\omega, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x, y) e^{i\omega x} dx.$$

We find that $\Phi(\omega, y)$ satisfies the ordinary differential equation

$$\frac{d^2\Phi(\omega, y)}{dy^2} - (\omega^2 - i\omega)\Phi(\omega, y) = 0. \tag{5}$$

The Wiener-Hopf technique consists in assuming that $\phi(x, 0)$ is known for all $x \in \mathbb{R}$, and not only for $x \geq 0$. We write that

$$\phi(x, 0) = \begin{cases} e^{-ax} & \text{if } x \geq 0, \\ u(x) & \text{if } x < 0, \end{cases}$$

where $u(x)$ is a function that will need to be determined later.

Next, the solution of Eq. (5) that tends to 0 as $|y|$ increases to ∞ is

$$\Phi(\omega, y) = \left[U(\omega) + \frac{1}{\sqrt{2\pi}}\frac{1}{1-i\omega}\right]\exp\left(-|y|\sqrt{\omega^2 - i\omega}\right),$$

where

$$U(\omega) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} u(x)e^{i\omega x}dx$$

is the Fourier transform of $u(x)$.

It can be shown (see Zwillinger [6], pp. 383-386) that, when $a = 1$,

$$U(\omega) = -\frac{1}{\sqrt{\omega - i}}\left[\frac{\sqrt{\omega - i} - \sqrt{-2i}}{\sqrt{2\pi}(1 - i\omega)}\right],$$

from which we deduce that

$$\Phi(\omega, y) = \frac{1}{\sqrt{2\pi}}\frac{\sqrt{-2i}}{\sqrt{\omega - i}(1 - i\omega)}\exp\left(-|y|\sqrt{\omega^2 - i\omega}\right). \tag{6}$$

Remark 2.1. *There are a few misprints in Zwillinger's book. In particular, in Eq. (104.4), p. 384, it should be ϕ_x instead of ϕ_y. Moreover, the formula for the function $U(\omega)$ should be as above, rather than as in Eq. (104.17). That is, it is $(1 - i\omega)$ in the denominator, instead of $\sqrt{1 - i\omega}$.*

Next, the formula for $\Phi(\omega, y)$ in the case when $a > 0$ can be found in Davies [1], p. 281:

$$\Phi(\omega, y) = \frac{1}{\sqrt{2\pi}}\frac{e^{-i\pi/4}\sqrt{1+a}}{\sqrt{\omega - i}(a - i\omega)}\exp\left(-|y|\sqrt{\omega^2 - i\omega}\right). \tag{7}$$

Remark 2.2. *In Davies [1], the Fourier transform of $f(x)$ is defined as follows:*

$$\mathcal{F}(\omega) = \int_{-\infty}^{\infty} f(x)e^{i\omega x}dx.$$

Therefore, we must multiply the formula on p. 281 by $1/\sqrt{2\pi}$. It is easy to check that if we set a equal to 1 in (7), then we indeed retrieve Eq. (6).

In order to obtain the function $\phi(x, y)$ that we are looking for, we must invert the Fourier transform $\Phi(\omega, y)$. However, it turns out to be a very difficult task in the general case when $y \in \mathbb{R}$. We can, however, invert this transform when $y = 0$. Indeed, making use of the mathematical software *Maple*, we find that

$$\phi(x, 0) = \begin{cases} e^{-ax} & \text{if } x \geq 0, \\ e^{-ax}\left[1 - \operatorname{erf}(\sqrt{-x}\sqrt{1+a})\right] & \text{if } x < 0, \end{cases} \tag{8}$$

in which *erf* is the *error function*.

Remark 2.3. *In Maple, the Fourier transform of $f(x)$ is defined as follows:*

$$\mathcal{F}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx.$$

Now, with the help of Eq. (8) and probabilistic arguments, we can obtain an explicit expression for $\phi(x, y)$ for any real y. First, we define (as in the Introduction)

$$\tau_0(y) = \inf\{t > 0 : Y(t) = 0 \mid Y(0) = y\},$$

where $y \neq 0$. That is, $\tau_0(y)$ is the first-passage time to 0 for the process $Y(t)$, independently of the value of $X[\tau_0(y)]$.

Next, we condition on $\tau_0(y)$ and $X[\tau_0(y)]$:

$$\begin{aligned} \phi(x, y) &= \int_0^{\infty} \int_{-\infty}^{\infty} E\left[e^{-aX[\tau(x,y)]} I(x, y) \mid X[\tau_0(y)] = x_1, \tau_0(y) = t\right] \\ &\quad \times f_{X(\tau_0)|\tau_0}(x_1 \mid t) f_{\tau_0}(t) dx_1 dt. \end{aligned}$$

We can write that

$$\begin{aligned} \phi(x, y) &= \int_0^{\infty} \int_{-\infty}^{0} \phi(x_1, 0) f_{X(\tau_0)|\tau_0}(x_1 \mid t) f_{\tau_0}(t) dx_1 dt \\ &\quad + \int_0^{\infty} \int_0^{\infty} e^{-ax_1} f_{X(\tau_0)|\tau_0}(x_1 \mid t) f_{\tau_0}(t) dx_1 dt. \end{aligned}$$

Finally, we mentioned in the Introduction that $X(\tau_0) \mid \{\tau_0 = t\} \sim \mathrm{N}(x-t, 2t)$ and the probability density function of the random variable $\tau_0(y)$ is given in Eq. (2). Hence, we can now state the following proposition.

Proposition 2.1. *The function $\phi(x, y)$ defined in (1) is given by*

$$\begin{aligned} \phi(x, y) &= \int_0^{\infty} \int_{-\infty}^{0} e^{-ax_1} \left[1 - \operatorname{erf}(\sqrt{-x_1}\sqrt{1+a})\right] \\ &\quad \times \frac{1}{2\sqrt{\pi t}} \exp\left\{-\frac{1}{4t}(x_1 - x + t)^2\right\} \frac{|y|}{2\sqrt{\pi t^3}} \exp\left\{-\frac{y^2}{4t}\right\} dx_1 dt \\ &\quad + \int_0^{\infty} \int_0^{\infty} e^{-ax_1} \frac{1}{2\sqrt{\pi t}} \exp\left\{-\frac{1}{4t}(x_1 - x + t)^2\right\} \\ &\quad \times \frac{|y|}{2\sqrt{\pi t^3}} \exp\left\{-\frac{y^2}{4t}\right\} dx_1 dt. \end{aligned}$$

In the next section, we will briefly mention a possible application of the previous proposition in stochastic optimal control.

3. AN OPTIMAL CONTROL APPLICATION

In Lefebvre and Whittle [4], the authors used the process defined by (3), (4) as a rudimentary model for an airplane. The process $X(t)$ denoted the height of the airplane, the value $x = -d$ represented ground level and $x = d$ was a height at which the airplane was likely to be detected by a radar. The aim of the optimizer was to try to make $X(t)$ remain in the interval $(-d, d)$ for as long as possible.

A possible application of the model considered in this paper is the following: assume that an airplane is moving from right to left, from $X(0) = x > 0$, as it approaching the runway. The initial height of the airplane is $Y(0) = y > 0$. The optimizer wants the plane to reach the ground, represented by the value $y = 0$, at time $\tau(x, y)$, with $X[\tau(x, y)] \geq 0$. That is, the value $x = 0$ denotes here the end of the runway.

Consider the controlled two-dimensional diffusion process defined by the system of stochastic differential equations

$$\begin{aligned} dX_1(t) &= -dt + b_1 u_1(t)dt + \sqrt{2}\, dB_1(t), \\ dX_2(t) &= b_2 u_2(t)dt + \sqrt{2}\, dB_2(t), \end{aligned}$$

where the constants b_1 and b_2 are different from zero.

Assume that the cost function, whose expected value we want to minimize, is given by

$$J_0(x, y) = \int_0^{\tau(x,y)} \frac{1}{2}[q_1 u_1^2(t) + q_2 u_2^2(t)]dt + X[\tau(x, y)] - \gamma \ln I(x, y),$$

where q_1, q_2 and γ are positive constants. Thus, the pilot should try to land his/her airplane as close as possible to the end of the runway, taking the quadratic control costs into account. Notice that we give an infinite penalty if the landing does not take place in finite time. In practice, we could replace $I(x, y)$ by

$$I_0(x, y) = \begin{cases} 1 & \text{if } \tau(x, y) < t_0, \\ 0 & \text{otherwise,} \end{cases}$$

where $t_0 \in [0, \infty)$.

If the constant γ is such that

$$2 = \gamma \frac{b_i^2}{q_i} \quad \text{for } i = 1, 2,$$

then we can use the theorem in Whittle [5] to express the optimal control u_i^*, for $i = 1, 2$, in terms of the function $\phi(x, y)$ given in Proposition 2.1. More precisely, the optimal control would be given by

$$u_i^* = \gamma \frac{b_i}{q_i} \frac{\phi_{x_i}(x, y)}{\phi(x, y)} = \frac{2}{b_i} \frac{\phi_{x_i}(x, y)}{\phi(x, y)},$$

with a replaced by $1/\gamma$ in Proposition 2.1.

4. CONCLUSION

Thanks to the Fourier transform of the mathematical expectation $\phi(x, y)$ defined in (1) that was computed in Zwillinger [6] and Davies [1], we were able to obtain an explicit and exact expression for the function $\phi(x, y)$. In Section 3, we presented a possible application of the results to an optimization problem.

As we have already mentioned, the main difficulty in the computation of the function $\phi(x, y)$ is the fact that it is discontinuous on the boundary $y = 0$. We saw that the solution to such a problem, like the one found by Lachal [2], is generally quite complicated. The expression that we have given in Proposition 2.1 is rather involved, but it is still usable in an optimization context.

As a sequel, we could consider other first-passage problems for two-dimensional diffusion processes for which there is a discontinuity on the boundary. The Wiener-Hopf technique is well adapted to compute the Fourier transform of the function we want to determine in such a case. Then, the problem of inverting this Fourier transform will generally be very difficult. Therefore, we could again appeal to probabilistic arguments to solve this type of problems.

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NONEXISTENCE OF POSITIVE SOLUTIONS FOR A SYSTEM OF HIGHER-ORDER MULTI-POINT BOUNDARY VALUE PROBLEMS

Rodica Luca, Ciprian Deliu

Department of Mathematics, ,,Gh. Asachi" Technical University, Iaşi, Romania

rlucatudor@yahoo.com, cipriandeliu@gmail.com

Abstract We determine intervals for two eigenvalues for which there exists no positive solution of a system of nonlinear higher-order ordinary differential equations subject to multi-point boundary conditions.

Keywords: higher-order differential system, multi-point boundary conditions, positive solutions, nonexistence.

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1. INTRODUCTION

In the last decades, nonlocal boundary value problems (including multi-point boundary value problems) for ordinary differential or difference equations/systems have become a rapidly growing area of research. Several phenomena in engineering, physics and life sciences can be modelled in this way. These problems have been studied by many authors by using different methods, such as fixed point theorems in cones, the Leray-Schauder continuation theorem, nonlinear alternatives of Leray-Schauder and coincidence degree theory.

In this paper, we consider the system of nonlinear higher-order ordinary differential equations

$$(S) \qquad \begin{cases} u^{(n)}(t) + \lambda a(t) f(u(t), v(t)) = 0, & t \in (0, T), \\ v^{(m)}(t) + \mu b(t) g(u(t), v(t)) = 0, & t \in (0, T), \end{cases}$$

with the multi-point boundary conditions

$$(BC) \qquad \begin{cases} u(0) = \sum_{i=1}^{p-2} a_i u(\xi_i), \ \ u'(0) = \cdots = u^{(n-2)}(0) = 0, \ \ u(T) = 0, \\ v(0) = \sum_{i=1}^{q-2} b_i v(\eta_i), \ \ v'(0) = \cdots = v^{(m-2)}(0) = 0, \ \ v(T) = 0, \end{cases}$$

where n, m, p, $q \in \mathbb{N}$, n, $m \geq 2$, p, $q \geq 3$, $0 < \xi_1 < \cdots < \xi_{p-2} < T$ and $0 < \eta_1 < \cdots < \eta_{q-2} < T$. In the case $n = 2$ or $m = 2$ the above conditions are of the form

$u(0) = \sum_{i=1}^{p-2} a_i u(\xi_i)$, $u(T) = 0$, or $v(0) = \sum_{i=1}^{q-2} b_i v(\eta_i)$, $v(T) = 0$, respectively, that is without conditions on the derivatives of u and v in the point 0.

We establish intervals for the eigenvalues λ and μ such that there exists no positive solution for problem $(S)-(BC)$. By a positive solution of $(S)-(BC)$ we mean a pair of functions $(u,v) \in C^n([0,T]) \times C^m([0,T])$ satisfying (S) and (BC) with $u(t) \geq 0$, $v(t) \geq 0$ for all $t \in [0,T]$ and $(u,v) \neq (0,0)$. The existence of positive solutions for the above problem was investigated in [4] by using the Guo-Krasnosel'skii fixed point theorem. Some particular cases of the problem from [4] have been studied in [1], [5], [11]. We also mention the paper [13], where we investigated the existence and nonexistence of positive solutions ($u(t) > 0$, $v(t) > 0$ for all $t \in [0,T)$) of the system (S) with $\lambda = \mu = 1$ and $f(u,v) = \widetilde{f}(v)$, $g(u,v) = \widetilde{g}(u)$ and the boundary conditions $u(0) = \sum_{i=1}^{p-2} a_i u(\xi_i) + a_0$, $u'(0) = \cdots = u^{(n-2)}(0) = 0$, $u(T) = 0$, $v(0) = \sum_{i=1}^{q-2} b_i v(\eta_i) + b_0$, $v'(0) = \cdots = v^{(m-2)}(0) = 0$, $v(T) = 0$, ($a_0, b_0 > 0$), by using the Schauder fixed point theorem. The system (S) with $n = m = 2$ subject to various boundary conditions was studied in [2], [3], [6]-[9], [12].

In Section 2, we present some auxiliary results which investigate a boundary value problem for a n-th order differential equation (problem (1)−(2) below), and in Section 3, we give our main results.

2. AUXILIARY RESULTS

In this section, we present some auxiliary results from [10] related to the following n-th order differential equation with p-point boundary conditions

$$u^{(n)}(t) + y(t) = 0, \quad t \in (0,T), \tag{1}$$

$$u(0) = \sum_{i=1}^{p-2} a_i u(\xi_i), \quad u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(T) = 0. \tag{2}$$

We present these results for the interval $[0,T]$ of the t-variable. Their proofs are similar to those from [10] where $T = 1$.

Lemma 2.1. *If* $d = T^{n-1} - \sum_{i=1}^{p-2} a_i (T^{n-1} - \xi_i^{n-1}) \neq 0$, $0 < \xi_1 < \cdots < \xi_{p-2} < T$ *and* $y \in C([0,T])$*, then the solution of* (1)-(2) *is given by*

$$u(t) = -\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} y(s)\, ds + \frac{t^{n-1}}{d} \left[\sum_{i=1}^{p-2} a_i \int_0^{\xi_i} \frac{(\xi_i - s)^{n-1}}{(n-1)!} y(s)\, ds + \right.$$
$$\left. + \left(1 - \sum_{i=1}^{p-2} a_i\right) \int_0^T \frac{(T-s)^{n-1}}{(n-1)!} y(s)\, ds \right] + \frac{1}{d} \sum_{i=1}^{p-2} a_i \xi_i^{n-1} \int_0^T \frac{(T-s)^{n-1}}{(n-1)!} y(s)\, ds -$$

$$-\frac{T^{n-1}}{d}\sum_{i=1}^{p-2}a_i\int_0^{\xi_i}\frac{(\xi_i-s)^{n-1}}{(n-1)!}y(s)\,ds.$$

Lemma 2.2. *Under the assumptions of Lemma 2.1, the Green's function for the boundary value problem* (1)-(2) *is*

$$G_1(t,s)=g_1(t,s)+\frac{T^{n-1}-t^{n-1}}{d}\sum_{i=1}^{p-2}a_ig_1(\xi_i,s),$$

where

$$g_1(t,s)=\frac{1}{(n-1)!T^{n-1}}\begin{cases} t^{n-1}(T-s)^{n-1}-T^{n-1}(t-s)^{n-1}, & 0\le s\le t\le T,\\ t^{n-1}(T-s)^{n-1}, & 0\le t\le s\le T.\end{cases}$$

Using the above Green's function the solution of problem (1)-(2) is expressed as $u(t)=\displaystyle\int_0^T G_1(t,s)y(s)\,ds$.

Lemma 2.3. *The function g_1 has the properties*

a) g_1 is a continuous function on $[0,T]\times[0,T]$ and $g_1(t,s)\ge 0$ for all $(t,s)\in[0,T]\times[0,T]$;

b) $g_1(t,s)\le g_1(\theta_1(s),s)$, for all $(t,s)\in[0,T]\times[0,T]$;

c) For any $c\in\left(0,\frac{T}{2}\right)$,

$\displaystyle\min_{t\in[c,T-c]}g_1(t,s)\ge\frac{c^{n-1}}{T^{n-1}}g_1(\theta_1(s),s)$, *for all $s\in[0,T]$,*

where $\theta_1(s)=s$ if $n=2$ and $\theta_1(s)=\begin{cases}\dfrac{s}{1-\left(1-\frac{s}{T}\right)^{\frac{n-1}{n-2}}}, & s\in(0,T],\\ \frac{T(n-2)}{n-1}, & s=0,\end{cases}$ *if $n\ge 3$.*

In the case $n\ge 3$, we choose the values of θ_1 in $s=0$ and $s=T$ such that θ_1 be a continuous function on $[0,T]$.

Lemma 2.4. *Assume that $a_i\ge 0$ for all $i=1,\dots,p-2$, $0<\xi_1<\cdots<\xi_{p-2}<T$ and $d>0$. Then the Green's function G_1 of problem* (1)-(2) *has the properties*

a) G_1 is a continuous function on $[0,T]\times[0,T]$ and $G_1(t,s)\ge 0$ for all $(t,s)\in[0,T]\times[0,T]$;

b) $G_1(t,s)\le J_1(s)$ for all $(t,s)\in[0,T]\times[0,T]$ and for any $c\in(0,T/2)$ we have $\displaystyle\min_{t\in[c,T-c]}G_1(t,s)\ge\frac{c^{n-1}}{T^{n-1}}J_1(s)$ *for all $s\in[0,T]$,*

where $J_1(s)=g_1(\theta_1(s),s)+\dfrac{T^{n-1}}{d}\displaystyle\sum_{i=1}^{p-2}a_ig_1(\xi_i,s),\quad\forall\, s\in[0,T].$

Lemma 2.5. *If $a_i \geq 0$ for all $i = 1, \ldots, p-2$, $0 < \xi_1 < \cdots < \xi_{p-2} < T$, $d > 0$, $y \in C([0,T])$ and $y(t) \geq 0$ for all $t \in [0,T]$, then the solution of problem* (1)-(2) *satisfies $u(t) \geq 0$ for all $t \in [0,T]$.*

Lemma 2.6. *Assume that $a_i \geq 0$ for all $i = 1, \ldots, p-2$, $0 < \xi_1 < \cdots < \xi_{p-2} < T$, $d > 0$, $y \in C([0,T])$, $c \in (0, T/2)$ and $y(t) \geq 0$ for all $t \in [0,T]$. Then the solution of problem* (1)-(2) *satisfies $\min\limits_{t \in [c, T-c]} u(t) \geq \dfrac{c^{n-1}}{T^{n-1}} \max\limits_{t' \in [0,T]} u(t')$.*

We can also formulate similar results as Lemma 2.1 - Lemma 2.6 above for the boundary value problem

$$v^{(m)}(t) + h(t) = 0, \quad t \in (0,T), \tag{3}$$

$$v(0) = \sum_{i=1}^{q-2} b_i v(\eta_i), \quad v'(0) = \cdots = v^{(m-2)}(0) = 0, \quad v(T) = 0, \tag{4}$$

where $0 < \eta_1 < \cdots < \eta_{q-2} < T$, $b_i \geq 0$ for all $i = 1, \ldots, q-2$ and $h \in C([0,T])$. If $e = T^{m-1} - \sum\limits_{i=1}^{q-2} b_i \left(T^{m-1} - \eta_i^{m-1}\right) \neq 0$, we denote by G_2 the Green's function associated to problem (3)-(4) and defined in a similar manner as G_1. We also denote by g_2, θ_2 and J_2 the corresponding functions for (3)-(4) defined in a similar manner as g_1, θ_1 and J_1, respectively.

3. MAIN RESULTS

We present the assumptions that we shall use in the sequel:
$(H1)$ $0 < \xi_1 < \cdots < \xi_{p-2} < T$, $0 < \eta_1 < \cdots < \eta_{q-2} < T$, $a_i \geq 0$, $i = 1, \ldots, p-2$, $b_i \geq 0$, $i = 1, \ldots, q-2$, $d = T^{n-1} - \sum\limits_{i=1}^{p-2} a_i (T^{n-1} - \xi_i^{n-1}) > 0$, $e = T^{m-1} - \sum\limits_{i=1}^{q-2} b_i (T^{m-1} - \eta_i^{m-1}) > 0$.
$(H2)$ The functions a, $b \in C([0,T], [0,\infty))$ and there exist t_1, $t_2 \in (0,T)$ such that $a(t_1) > 0$, $b(t_2) > 0$.
$(H3)$ The functions f, $g \in C([0,\infty) \times [0,\infty), [0,\infty))$.

From assumption $(H2)$, there exists $c \in (0, T/2)$ such that t_1, $t_2 \in (c, T-c)$. We shall work in this section with this number c. This implies that

$$\int_c^{T-c} J_1(s) a(s)\, ds > 0, \quad \int_c^{T-c} J_2(s) b(s)\, ds > 0,$$

where J_1 and J_2 are defined in Section 2 (Lemma 2.4).

We introduce the following extreme limits

$$\begin{aligned} f_0^s &= \limsup_{u+v\to 0^+} \frac{f(u,v)}{u+v}, \quad g_0^s = \limsup_{u+v\to 0^+} \frac{g(u,v)}{u+v},\\ f_0^i &= \liminf_{u+v\to 0^+} \frac{f(u,v)}{u+v}, \quad g_0^i = \liminf_{u+v\to 0^+} \frac{g(u,v)}{u+v},\\ f_\infty^s &= \limsup_{u+v\to \infty} \frac{f(u,v)}{u+v}, \quad g_\infty^s = \limsup_{u+v\to \infty} \frac{g(u,v)}{u+v},\\ f_\infty^i &= \liminf_{u+v\to \infty} \frac{f(u,v)}{u+v}, \quad g_\infty^i = \liminf_{u+v\to \infty} \frac{g(u,v)}{u+v}. \end{aligned}$$

By using the Green's functions G_1 and G_2 from Section 2 (Lemma 2.2), our problem $(S)-(BC)$ can be written equivalently as the following nonlinear system of integral equations

$$\begin{cases} u(t) = \lambda \displaystyle\int_0^T G_1(t,s)a(s)f(u(s),v(s))\,ds, & 0 \le t \le T,\\ v(t) = \mu \displaystyle\int_0^T G_2(t,s)b(s)g(u(s),v(s))\,ds, & 0 \le t \le T. \end{cases}$$

We consider the Banach space $X = C([0,T])$ with supremum norm $\|\cdot\|$, and the Banach space $Y = X \times X$ with the norm $\|(u,v)\|_Y = \|u\| + \|v\|$. We define the cone $P \subset Y$ by

$$P = \{(u,v) \in Y;\ \ u(t) \ge 0,\ v(t) \ge 0,\ \forall\, t \in [0,T] \text{ and } \inf_{t\in[c,T-c]} (u(t)+v(t)) \ge \gamma \|(u,v)\|_Y\},$$

where $\gamma = \min\{c^{n-1}/T^{n-1}, c^{m-1}/T^{m-1}\}$.

For λ, $\mu > 0$, we introduce the operators $Q_1, Q_2 : Y \to X$ and $\mathcal{Q} : Y \to Y$ defined by

$$\begin{aligned} Q_1(u,v)(t) &= \lambda \int_0^T G_1(t,s)a(s)f(u(s),v(s))\,ds, \quad 0 \le t \le T,\\ Q_2(u,v)(t) &= \mu \int_0^T G_2(t,s)b(s)g(u(s),v(s))\,ds, \quad 0 \le t \le T, \end{aligned}$$

and $\mathcal{Q}(u,v) = (Q_1(u,v), Q_2(u,v))$, $(u,v) \in Y$. The solutions of our problem $(S)-(BC)$ are the fixed points of the operator $\mathcal{Q}$. By using standard arguments, we can easily show that, under assumptions $(H1)-(H3)$, the operator $\mathcal{Q}$ is completely continuous.

Theorem 3.1. *Assume that $(H1)-(H3)$ hold. If f_0^s, f_∞^s, g_0^s, $g_\infty^s < \infty$, then there exist positive constants λ_0, μ_0 such that for every $\lambda \in (0,\lambda_0)$ and $\mu \in (0,\mu_0)$, the boundary value problem $(S)-(BC)$ has no positive solution.*

Proof. Since f_0^s, $f_\infty^s < \infty$, we deduce that there exist M_1', M_1'', r_1, $r_1' > 0$, $r_1 < r_1'$ such that

$$\begin{aligned} f(u,v) &\le M_1'(u+v), \quad \forall\, u,\ v \ge 0, \quad u+v \in [0,r_1],\\ f(u,v) &\le M_1''(u+v), \quad \forall\, u,\ v \ge 0, \quad u+v \in [r_1',\infty). \end{aligned}$$

We consider $M_1 = \max\left\{M_1', M_1'', \max\limits_{r_1 \le u+v \le r_1'} \dfrac{f(u,v)}{u+v}\right\} > 0$. Then, we obtain

$$f(u,v) \le M_1(u+v), \quad \forall\, u,\ v \ge 0.$$

Since g_0^s, $g_\infty^s < \infty$, we deduce that there exist M_2', M_2'', r_2, $r_2' > 0$, $r_2 < r_2'$ such that

$$\begin{aligned} g(u,v) &\le M_2'(u+v), \quad \forall\, u,\ v \ge 0, \quad u+v \in [0, r_2], \\ g(u,v) &\le M_2''(u+v), \quad \forall\, u,\ v \ge 0, \quad u+v \in [r_2', \infty). \end{aligned}$$

We consider $M_2 = \max\left\{M_2', M_2'', \max\limits_{r_2 \le u+v \le r_2'} \dfrac{g(u,v)}{u+v}\right\} > 0$. Then, we obtain

$$g(u,v) \le M_2(u+v), \quad \forall\, u,\ v \ge 0.$$

We define $\lambda_0 = \dfrac{1}{2M_1B}$ and $\mu_0 = \dfrac{1}{2M_2D}$, where $B = \displaystyle\int_0^T J_1(s)a(s)\,ds$ and $D = \displaystyle\int_0^T J_2(s)b(s)\,ds$. We shall show that for every $\lambda \in (0,\lambda_0)$ and $\mu \in (0,\mu_0)$, the problem $(S)-(BC)$ has no positive solution.

Let $\lambda \in (0,\lambda_0)$ and $\mu \in (0,\mu_0)$. We suppose that $(S)-(BC)$ has a positive solution $(u(t), v(t))$, $t \in [0,T]$. Then, we have

$$\begin{aligned} u(t) &= Q_1(u,v)(t) = \lambda \int_0^T G_1(t,s)a(s)f(u(s),v(s))\,ds \\ &\le \lambda \int_0^T J_1(s)a(s)f(u(s),v(s))\,ds \le \lambda M_1 \int_0^T J_1(s)a(s)(u(s)+v(s))\,ds \\ &\le \lambda M_1(\|u\|+\|v\|) \int_0^T J_1(s)a(s)\,ds = \lambda M_1 B \|(u,v)\|_Y, \quad \forall\, t \in [0,T]. \end{aligned}$$

Therefore, we conclude

$$\|u\| \le \lambda M_1 B \|(u,v)\|_Y < \lambda_0 M_1 B \|(u,v)\|_Y = \frac{1}{2}\|(u,v)\|_Y.$$

In a similar manner, we have

$$\begin{aligned} v(t) &= Q_2(u,v)(t) = \mu \int_0^T G_2(t,s)b(s)g(u(s),v(s))\,ds \\ &\le \mu \int_0^T J_2(s)b(s)g(u(s),v(s))\,ds \le \mu M_2 \int_0^T J_2(s)b(s)(u(s)+v(s))\,ds \\ &\le \mu M_2(\|u\|+\|v\|) \int_0^T J_2(s)b(s)\,ds = \mu M_2 D \|(u,v)\|_Y, \quad \forall\, t \in [0,T]. \end{aligned}$$

Therefore, we conclude

$$\|v\| \le \mu M_2 D \|(u,v)\|_Y < \mu_0 M_2 D \|(u,v)\|_Y = \frac{1}{2}\|(u,v)\|_Y.$$

Hence, $\|(u,v)\|_Y = \|u\| + \|v\| < \frac{1}{2}\|(u,v)\|_Y + \frac{1}{2}\|(u,v)\|_Y = \|(u,v)\|_Y$, which is a contradiction. So, the boundary value problem $(S)-(BC)$ has no positive solution. ∎

Theorem 3.2. *Assume that $(H1)-(H3)$ hold.*

a) If f_0^i, $f_\infty^i > 0$, then there exists a positive constant $\widetilde{\lambda}_0$ such that for every $\lambda > \widetilde{\lambda}_0$ and $\mu > 0$, the boundary value problem $(S)-(BC)$ has no positive solution.

b) If g_0^i, $g_\infty^i > 0$, then there exists a positive constant $\widetilde{\mu}_0$ such that for every $\mu > \widetilde{\mu}_0$ and $\lambda > 0$, the boundary value problem $(S)-(BC)$ has no positive solution.

c) If f_0^i, f_∞^i, g_0^i, $g_\infty^i > 0$, then there exist positive constants $\widetilde{\widetilde{\lambda}}_0$ and $\widetilde{\widetilde{\mu}}_0$ such that for every $\lambda > \widetilde{\widetilde{\lambda}}_0$ and $\mu > \widetilde{\widetilde{\mu}}_0$, the boundary value problem $(S)-(BC)$ has no positive solution.

Proof. a) Since f_0^i, $f_\infty^i > 0$, we deduce that there exist m_1', m_1'', r_3, $r_3' > 0$, $r_3 < r_3'$ such that

$$f(u,v) \geq m_1'(u+v),\quad \forall\, u,\ v \geq 0,\quad u+v \in [0, r_3],$$
$$f(u,v) \geq m_1''(u+v),\quad \forall\, u,\ v \geq 0,\quad u+v \in [r_3', \infty).$$

We introduce $m_1 = \min\left\{m_1', m_1'', \min\limits_{u+v\in[r_3,r_3']} \dfrac{f(u,v)}{u+v}\right\} > 0$. Then we obtain

$$f(u,v) \geq m_1(u+v),\quad \forall\, u,\ v \geq 0.$$

We define $\widetilde{\lambda}_0 = \dfrac{T^{n-1}}{\gamma c^{n-1} m_1 A} > 0$, where $A = \displaystyle\int_c^{T-c} J_1(s)a(s)\,ds$. We shall show that for every $\lambda > \widetilde{\lambda}_0$ and $\mu > 0$ the problem $(S)-(BC)$ has no positive solution.

Let $\lambda > \widetilde{\lambda}_0$ and $\mu > 0$. We suppose that $(S)-(BC)$ has a positive solution $(u(t), v(t))$, $t \in [0,T]$. Then, we obtain

$$\begin{aligned}
&u(c) = Q_1(u,v)(c) = \lambda \int_0^T G_1(c,s)a(s)f(u(s),v(s))\,ds \\
&\geq \lambda \int_c^{T-c} G_1(c,s)a(s)f(u(s),v(s))\,ds \geq \lambda m_1 \int_c^{T-c} G_1(c,s)a(s)(u(s)+v(s))\,ds \\
&\geq \frac{\lambda m_1 c^{n-1}}{T^{n-1}} \int_c^{T-c} J_1(s)a(s)\gamma(\|u\|+\|v\|)\,ds = \frac{\lambda\gamma m_1 c^{n-1} A}{T^{n-1}}\|(u,v)\|_Y.
\end{aligned}$$

Therefore, we deduce

$$\|u\| \geq u(c) \geq \frac{\lambda\gamma m_1 c^{n-1}A}{T^{n-1}}\|(u,v)\|_Y > \frac{\widetilde{\lambda}_0\gamma m_1 c^{n-1}A}{T^{n-1}}\|(u,v)\|_Y = \|(u,v)\|_Y,$$

and so, $\|(u,v)\|_Y = \|u\| + \|v\| \geq \|u\| > \|(u,v)\|_Y$, which is a contradiction. Therefore, the boundary value problem $(S)-(BC)$ has no positive solution.

b) Since g_0^i, $g_\infty^i > 0$, we deduce that there exist m_2', m_2'', r_4, $r_4' > 0$, $r_4 < r_4'$ such that

$$g(u,v) \geq m_2'(u+v),\quad \forall\, u,\ v \geq 0,\quad u+v \in [0, r_4],$$
$$g(u,v) \geq m_2''(u+v),\quad \forall\, u,\ v \geq 0,\quad u+v \in [r_4', \infty).$$

We introduce $m_2 = \min\left\{m_2', m_2'', \min\limits_{u+v\in[r_4,r_4']} \dfrac{g(u,v)}{u+v}\right\} > 0$. Then we obtain

$$g(u,v) \geq m_2(u+v), \quad \forall\, u,\ v \geq 0.$$

We define $\widetilde{\mu}_0 = \dfrac{T^{m-1}}{\gamma c^{m-1} m_2 C} > 0$, where $C = \displaystyle\int_c^{T-c} J_2(s)b(s)\,ds$. We shall show that for every $\mu > \widetilde{\mu}_0$ and $\lambda > 0$ the problem $(S)-(BC)$ has no positive solution.

Let $\mu > \widetilde{\mu}_0$ and $\lambda > 0$. We suppose that $(S)-(BC)$ has a positive solution $(u(t), v(t))$, $t \in [0,T]$. Then, we obtain

$$\begin{aligned}
&v(c) = Q_2(u,v)(c) = \mu \int_0^T G_2(c,s)b(s)g(u(s),v(s))\,ds \\
&\geq \mu \int_c^{T-c} G_2(c,s)b(s)g(u(s),v(s))\,ds \geq \mu m_2 \int_c^{T-c} G_2(c,s)b(s)(u(s)+v(s))\,ds \\
&\geq \frac{\mu m_2 c^{m-1}}{T^{m-1}} \int_c^{T-c} J_2(s)b(s)\gamma(\|u\|+\|v\|)\,ds = \frac{\mu\gamma m_2 c^{m-1} C}{T^{m-1}} \|(u,v)\|_Y.
\end{aligned}$$

Therefore, we deduce

$$\|v\| \geq v(c) \geq \frac{\mu\gamma m_2 c^{m-1} C}{T^{m-1}} \|(u,v)\|_Y > \frac{\widetilde{\mu}_0 \gamma m_2 c^{m-1} C}{T^{m-1}} \|(u,v)\|_Y = \|(u,v)\|_Y,$$

and so, $\|(u,v)\|_Y = \|u\| + \|v\| \geq \|v\| > \|(u,v)\|_Y$, which is a contradiction. Therefore, the boundary value problem $(S)-(BC)$ has no positive solution.

c) Because $f_0^i,\ f_\infty^i,\ g_0^i,\ g_\infty^i > 0$, we deduce as above, that there exist $m_1,\ m_2 > 0$ such that

$$f(u,v) \geq m_1(u+v), \quad g(u,v) \geq m_2(u+v), \quad \forall\, u,\ v \geq 0.$$

We define $\widetilde{\widetilde{\lambda}}_0 = \dfrac{T^{n-1}}{2\gamma c^{n-1} m_1 A}\left(= \dfrac{\widetilde{\lambda}_0}{2}\right)$ and $\widetilde{\widetilde{\mu}}_0 = \dfrac{T^{m-1}}{2\gamma c^{m-1} m_2 C}\left(= \dfrac{\widetilde{\mu}_0}{2}\right)$. Then for every $\lambda > \widetilde{\widetilde{\lambda}}_0$ and $\mu > \widetilde{\widetilde{\mu}}_0$, the problem $(S)-(BC)$ has no positive solution. Indeed, let $\lambda > \widetilde{\widetilde{\lambda}}_0$ and $\mu > \widetilde{\widetilde{\mu}}_0$. We suppose that $(S)-(BC)$ has a positive solution $(u(t), v(t))$, $t \in [0,T]$. Then in a similar manner as above, we deduce

$$\|u\| \geq \frac{\lambda\gamma m_1 c^{n-1} A}{T^{n-1}} \|(u,v)\|_Y, \quad \|v\| \geq \frac{\mu\gamma m_2 c^{m-1} C}{T^{m-1}} \|(u,v)\|_Y,$$

and so,

$$\begin{aligned}
&\|(u,v)\|_Y = \|u\| + \|v\| \geq \frac{\lambda\gamma m_1 c^{n-1} A}{T^{n-1}} \|(u,v)\|_Y + \frac{\mu\gamma m_2 c^{m-1} C}{T^{m-1}} \|(u,v)\|_Y \\
&> \frac{\widetilde{\widetilde{\lambda}}_0 \gamma m_1 c^{n-1} A}{T^{n-1}} \|(u,v)\|_Y + \frac{\widetilde{\widetilde{\mu}}_0 \gamma m_2 c^{m-1} C}{T^{m-1}} \|(u,v)\|_Y \\
&= \tfrac{1}{2}\|(u,v)\|_Y + \tfrac{1}{2}\|(u,v)\|_Y = \|(u,v)\|_Y,
\end{aligned}$$

which is a contradiction. Therefore, the boundary value problem $(S)-(BC)$ has no positive solution. ∎

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EDELSTEIN-SUZUKI FIXED POINT THEOREM IN METRIC AND CONE METRIC SPACES; AN IMPLICIT RELATION APPROACH

Fridoun Moradlou, Peyman Salimi

Department of Mathematics, Sahand University of Technology, Tabriz, Iran

moradlou@sut.ac.ir, salimipeyman@gmail.com

Abstract In this paper, we prove some fixed point theorems in compact metric and compact cone metric spaces by using implicit relation. The presented theorems extend, generalize and improve many existing results in the literature such as a theorem by D. Dorić et al. [Dragan Dorić, Zoran Kadelburg and Stojan Radenović, Edelstein - Suzuki-type fixed point results in metric and abstract metric spaces, Nonlinear Anal. TMA 75 (2012) 1927 - 1932.]

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1. INTRODUCTION

In 1962, M. Edelstein [6] proved another version of Banach contraction Principle. He assumed a compact metric space (X, d) and a self-mapping T on X such that $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$, and he proved T has a unique fixed point. In 2009, T. Suzuki [19] improved the results of Banach and Edelstein. Suzuki replaced the condition"$d(Tx, Ty) < d(x, y)$" by "$\frac{1}{2}d(x, Tx) < d(x, y) \Rightarrow d(Tx, Ty) < d(x, y)$" for all $x, y \in X$. By this assumption, he established that T has a unique fixed point. Recently D. Dorić et al. in [5] proved the following theorem and extended the results of Edelstein and Suzuki:

Theorem 1.1. *Let (X, d) be a compact metric space and let $T : X \to X$. Assume that*

$$\frac{1}{2}d(x, Tx) < d(x, y) \Rightarrow$$

$$d(Tx, Ty) < Ad(x, y) + Bd(x, Tx) + Cd(y, Ty) + Dd(x, Ty) + Ed(y, Tx)$$

holds for all $x, y \in X$, where the nonnegative constants A, B, C, D, E satisfy

$$A + B + C + 2D = 1 \qquad and \qquad C \neq 1.$$

Then T has a fixed point in X. If $E \leq B + C + D$, then the fixed point of T is unique.

Also, they gave an example which does not satisfy Suzuki's condition but it satisfies condition of Theorem 1.1.

In 2007, Huang and Zhang [8] introduced cone metric spaces and defined some properties of convergence of sequences and completeness in cone metric spaces. They also proved a fixed point theorem of cone metric spaces. A number of authors were attracted by these results of Huang and Zhang and were stimulated to investigate the fixed point theorems in cone metric spaces. During the recent years, cone metric spaces and properties of these spaces have been studied by a number of authors. Also many mathematicians have extensively investigated fixed point theorems in cone metric spaces (see [15], [18], [21]).

Furthermore, many authors considered implicit relation technique to investigation of fixed point theorems in metric spaces (see [2], [11]-[13], [17], [20]).

In this paper, we introduce a new version of implicit relation technique by using two functions. This helps us to extend our results on cone metric spaces.

This paper is organized as follows: In Section 2, we prove the generalization of Theorem 1.1 in compact metric spaces by using implicit relation technique.

In Section 3, we generalize our results on compact cone metric spaces.

2. IMPLICIT RELATION

In this section, we introduce an implicit relation by using two functions. Also, we prove a theorem in compact metric spaces. Our result extends Theorem 3 of [19] and Theorem 3.1 of [6].

Let $\psi : [0,\infty) \longrightarrow [0,\infty)$ and $\varphi : [0,\infty)^5 \longrightarrow [0,\infty)$ be two continuous functions which satisfy the following conditions:

$(M1)$ $\varphi(t_1, t_2, t_3, t_4, t_5)$ is increasing in variable t_3;

$(M2)$ $\psi(u) \leq \varphi(v, v, u+v, u, 0)$ implies $u \leq v$;

$(M3)$ $\psi(u) < \varphi(v, v, u+v, u, 0)$ implies $u < v$ where $u \geq 0$ and $v > 0$;

$(M4)$ $\psi(u) < \varphi(v, 0, v, 0, v)$ implies $u < v$, where $u \geq 0$ and $v > 0$.

Example 2.1. *Let*

(A) $\psi(r) = r$ *and* $\varphi(t_1, t_2, t_3, t_4, t_5) = t_1$*;*

(B) $\psi(r) = 2r$ *and* $\varphi(t_1, t_2, t_3, t_4, t_5) = t_3$*;*

(C) $\psi(r) = 2r$ *and* $\varphi(t_1, t_2, t_3, t_4, t_5) = t_1 + t_4$*;*

(D) $\psi(r) = 5r$ *and* $\varphi(t_1, t_2, t_3, t_4, t_5) = t_1 + t_2 + t_3 + t_4 + t_5$*;*

(E) $\psi(r) = 2r$ *and* $\varphi(t_1, t_2, t_3, t_4, t_5) = \max\{t_1, t_2, t_3, t_4, t_5\}$*;*

(F) $\psi(r) = 2r^2$ *and* $\varphi(t_1, t_2, t_3, t_4, t_5) = t_1^2 + t_4^2$.

It is easy to see that $(M1) - (M4)$ *are satisfied for* ψ *and* φ *in* (A), (B), (C), (D), (E) *and* (F).

(G) *$\psi(r) = r$ and $\varphi(t_1, t_2, t_3, t_4, t_5) = at_1 + bt_2 + ct_3 + dt_4 + et_5$, where a, b, c, d and e are nonnegative numbers, $a + b + 2c + d = 1$, $d \neq 1$ and $e \leq b + c + d$.*

Clearly, $(M1)$ holds. Now, let $\psi(u) - \varphi(v, v, u + v, u, 0) = (1-c-d)u-(a+b+c)v \leq 0$. By the assumption, we conclude $1-c-d = (a+b+c)$. So $1 - c - d \leq 0$ implies $a = b = c = 0$. Therefore, $d = 1$, which is a contradiction. Hence, $1 - c - d > 0$. Thus, $u \leq v$. So $(M2)$ is satisfied. A similar argument shows that $(M3)$ is satisfied. Moreover, if $\psi(u) - \varphi(v, 0, v, 0, v) = u - (a + c + e)v < 0$, then $u < (a + c + e)v$. So, by the hypothesis we can write $u < (a + c + e)v \leq (a + b + 2c + d)v = v$. Therefore, $(M4)$ is satisfied.

(H) *$\psi(r) = r$ and $\varphi(t_1, t_2, t_3, t_4, t_5) = a \min\{t_1, t_2\} + b \min\{t_2, t_3\} + c \min\{t_3, t_4\} + t_5$ where a, b and c are nonnegative numbers, $a + b + c = 1$ and $c \neq 1$.*

Clearly $(M1)$ holds. If $\psi(u) - \varphi(v, v, u + v, u, 0) = (1 - c)u - (a + b)v \leq 0$ then, by using $a + b + c = 1$, we conclude that $u \leq v$. This means $(M2)$ is satisfied. Similarly, we can show that $(M3)$ is satisfied. Now, if $\psi(u) - \varphi(v, 0, v, 0, v) < 0$ then $u < v$, since $\varphi(v, 0, v, 0, v) = v$. Hence, $(M4)$ is satisfied.

Theorem 2.1. *Let (X, d) be a compact metric space and T be a self-mapping on X. Suppose that $\psi : [0, \infty) \longrightarrow [0, \infty)$ and $\varphi : [0, \infty)^5 \longrightarrow [0, \infty)$ are two continuous mappings such that $(M1) - (M3)$ are satisfied. Assume that*

$$\frac{1}{2}d(x, Tx) < d(x, y) \Longrightarrow \tag{1}$$

$$\psi(d(Tx, Ty)) < \varphi\Big(d(x, y), d(x, Tx), d(x, Ty), d(y, Ty), d(y, Tx)\Big),$$

for all $x, y \in X$. Then T has at least one fixed point. Moreover, if ψ and φ satisfy $(M4)$, then T has a unique fixed point.

Proof. Let $\alpha = \inf\{d(x, Tx) : x \in X\}$. There exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} d(x_n, Tx_n) = \alpha$. By compactness of X, there exist $w_1, w_2 \in X$ such that $\lim_{n\to\infty} x_n = w_1$ and $\lim_{n\to\infty} Tx_n = w_2$. Hence

$$\lim_{n\to\infty} d(x_n, w_2) = \lim_{n\to\infty} d(x_n, Tx_n) = d(w_1, w_2) = \alpha.$$

Now, we show that α must be equal to 0.

If $\alpha > 0$, then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\frac{2}{3}\alpha < d(x_n, w_2)$ and $d(x_n, Tx_n) < \frac{4}{3}\alpha$. Therefore, for all $n \geq N$, $\frac{1}{2}d(x_n, Tx_n) < \frac{2}{3}\alpha < d(x_n, w_2)$. Now, by (1), we have

$$\psi(d(Tx_n, Tw_2)) < \varphi\Big(d(x_n, w_2), d(x_n, Tx_n), d(x_n, Tw_2), d(w_2, Tw_2), d(w_2, Tx_n)\Big). \tag{2}$$

By taking the limit as $n \to \infty$ in (2), we get

$$\begin{aligned}
\psi(d(w_2, Tw_2)) &\leq \varphi\Big(\alpha, \alpha, d(w_1, Tw_2), d(w_2, Tw_2), 0\Big) \\
&\leq \varphi\Big(\alpha, \alpha, d(w_1, w_2) + d(w_2, Tw_2), d(w_2, Tw_2), 0\Big) \\
&= \varphi\Big(\alpha, \alpha, \alpha + d(w_2, Tw_2), d(w_2, Tw_2), 0\Big),
\end{aligned}$$

so by $(M2)$, we have $d(w_2, Tw_2) \leq \alpha$. Therefore, $d(w_2, Tw_2) = \alpha > 0$. Hence, $\frac{1}{2}d(w_2, Tw_2) < d(w_2, Tw_2)$. Now by (1), we can obtain

$$\begin{aligned}\psi(d(Tw_2, T^2w_2)) &< \varphi\Big(d(w_2, Tw_2), d(w_2, Tw_2), d(w_2, T^2w_2), d(Tw_2, T^2w_2), 0\Big)\\ &\leq \varphi\Big(d(w_2, Tw_2), d(w_2, Tw_2), d(Tw_2, T^2w_2) + d(w_2, Tw_2), d(Tw_2, T^2w_2), 0\Big).\end{aligned}$$

By $(M3)$, we get $d(Tw_2, T^2w_2) < d(w_2, Tw_2) = \alpha$, which is a contradiction of the definition of α. So $\alpha = 0$, that is, $w_1 = w_2$.

Now, we must show that T has at least one fixed point. Assume towards a contradiction that T does not have a fixed point. Hence $0 < \frac{1}{2}d(x_n, Tx_n) < d(x_n, Tx_n)$. Then by (1), we have

$$\psi(d(Tx_n, T^2x_n)) < \varphi\Big(d(x_n, Tx_n), d(x_n, Tx_n), d(x_n, T^2x_n), d(Tx_n, T^2x_n), d(Tx_n, Tx_n)\Big).$$

By taking the limit as $n \to \infty$ in above inequality, we get

$$\psi(\lim_{n\to\infty} d(w_1, T^2x_n)) \leq \varphi\Big(0, 0, \lim_{n\to\infty} d(w_1, T^2x_n), \lim_{n\to\infty} d(w_1, T^2x_n), 0\Big).$$

It follows from $(M2)$ that $\lim_{n\to\infty} d(w_1, T^2x_n) \leq 0$, so $\lim_{n\to\infty} T^2x_n = w_1$. Furthermore, by using (1) and $(M1)$, we obtain

$$\begin{aligned}\psi(d(Tx_n, T^2x_n)) &< \varphi\Big(d(x_n, Tx_n), d(x_n, Tx_n), d(x_n, T^2x_n), d(Tx_n, T^2x_n), d(Tx_n, Tx_n)\Big)\\ &\leq \varphi\Big(d(x_n, Tx_n), d(x_n, Tx_n), d(Tx_n, T^2x_n) + d(x_n, Tx_n), d(Tx_n, T^2x_n), 0\Big).\end{aligned}$$

Then by $(M3)$ we have, $d(Tx_n, T^2x_n) < d(x_n, Tx_n)$.

Now, suppose that both of the following inequalities hold for some $n \in \mathbb{N}$,

$$\frac{1}{2}d(x_n, Tx_n) \geq d(x_n, w_1) \qquad \text{and} \qquad \frac{1}{2}d(Tx_n, T^2x_n) \geq d(Tx_n, w_1),$$

so, we have

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, w_1) + d(w_1, Tx_n)\\ &\leq \frac{1}{2}d(x_n, Tx_n) + \frac{1}{2}d(Tx_n, T^2x_n)\\ &< \frac{1}{2}d(x_n, Tx_n) + \frac{1}{2}d(x_n, Tx_n) = d(x_n, Tx_n),\end{aligned}$$

which is a contradiction. Thus, for each $n \in \mathbb{N}$, either

$$\frac{1}{2}d(x_n, Tx_n) < d(x_n, w_1),$$

or

$$\frac{1}{2}d(Tx_n, T^2x_n) < d(Tx_n, w_1),$$

holds. So by hypotheses, we conclude that one of the following inequalities holds for all n in an infinite subset of $\mathbb{N}$:

$$\psi(d(Tx_n, Tw_1)) < \varphi\Big(d(x_n, w_1), d(x_n, Tx_n), d(x_n, Tw_1), d(w_1, Tw_1), d(w_1, Tx_n)\Big),$$

or

$$\psi(d(T^2x_n, Tw_1)) < \varphi\Big(d(Tx_n, w_1), d(Tx_n, T^2x_n), d(Tx_n, Tw_1), d(w_1, Tw_1), d(w_1, T^2x_n)\Big).$$

If we take the limit as $n \to \infty$ in each of these inequalities, then we have

$$\psi(d(w_1, Tw_1)) \leq \varphi\Big(0, 0, d(w_1, Tw_1), d(w_1, Tw_1), 0\Big).$$

So $(M2)$ implies that $d(w_1, Tw_1) \leq 0$, i.e., $w_1 = Tw_1$. Hence, we conclude that w_1 is a fixed point of T.

To prove the uniqueness of w_1, suppose that w_0 is another fixed point of T such that $w_1 \neq w_0$. Hence, $0 = \frac{1}{2}d(w_1, Tw_1) < d(w_1, w_0)$. By (1), we have

$$\psi(d(Tw_1, Tw_0)) < \varphi\Big(d(w_1, w_0), d(w_1, Tw_1), d(w_1, Tw_0), d(w_0, Tw_0), d(w_0, Tw_1)\Big).$$

So

$$\psi(d(w_1, w_0)) < \varphi\Big(d(w_1, w_0), 0, d(w_1, w_0), 0, d(w_0, w_1)\Big).$$

Considering $(M4)$, we have $d(w_1, w_0) < d(w_1, w_0)$, which is a contradiction. Therefore $w_1 = w_0$. Then w_1 is the unique fixed point of T. ∎

Theorem 2.2. *Let (X, d) be a metric space and let F and T be two self-mappings on X such that $TX \subseteq FX$ and FX is compact. Suppose that $\psi : [0, \infty) \longrightarrow [0, \infty)$ and $\varphi : [0, \infty)^5 \longrightarrow [0, \infty)$ are two continuous mappings such that $(M1) - (M3)$ are satisfied. Assume that*

$$\begin{aligned} &\frac{1}{2}d(Fx, Tx) < d(Fx, Fy) \Longrightarrow \\ &\qquad \psi(d(Tx, Ty)) < \varphi\Big(d(Fx, Fy), d(Fx, Tx), d(Fx, Ty), d(Fy, Ty), d(Fy, Tx)\Big), \end{aligned} \tag{3}$$

for all $x, y \in X$. Then F and T have at least one point of coincidence. Moreover, if ψ and φ satisfy $(M4)$ and F and T are weakly compatible, then F and T have a unique common fixed point.

Proof. Define $G : FX \longrightarrow FX$ by $G(F(w)) = Tw$. Replacing Tx and Ty by $G(Fx)$ and $G(Fy)$, respectively, in (3), we have

$$\begin{aligned} &\frac{1}{2}d(Fx, G(Fx)) < d(Fx, Fy) \Longrightarrow \\ &\psi(d(G(Fx), G(Fy))) < \\ &\qquad \varphi(d(Fx, Fy), d(Fx, G(Fx)), d(Fx, G(Fy)), d(Fy, G(Fy)), d(Fy, G(Fx))), \end{aligned}$$

for all $Fx, Fy \in FX$. Since FX is compact, by Theorem 2.1, G has a fixed point, i.e., there exists $z \in X$ such that $Fz = G(Fz) = Tz := u$. Moreover, if ψ and φ satisfy $(M4)$ then G has a unique fixed point. So we conclude that z is a unique point of coincidence of F and T. Furthermore, if F and T are weakly compatible mappings, we get $FTz = TFz$, so $Fu = Tu$. Therefore $z = u$ and $Fz = Tz = z$. This yields z as the unique common fixed point of F and T. ∎

Corollary 2.1. *Let (X, d) be a metric space and let F and T be two self-mappings on X such that $TX \subseteq FX$ and FX is compact. Assume that*

$$\begin{aligned}&\frac{1}{2}d(Fx, Tx) < d(Fx, Fy) \Longrightarrow \\ &\quad d(Tx, Ty) < Ad(Fx, Fy) + Bd(Fx, Tx) + Cd(Fx, Ty) + Dd(Fy, Ty) + Ed(Fy, Tx)\end{aligned}$$

for all $x, y \in X$, where $A, B, C, D, E \geq 0$, $A + B + 2C + D = 1$ and $D \neq 1$. Then F and T have at least one point of coincidence. Moreover, if $E \leq B + C + D$ and F and T are weakly compatible, then F and T have a unique common fixed point.

Proof. The proof follows from Theorem 2.2 and part (G) of example 2.1. ∎

Corollary 2.2. *Let (X, d) be a metric space and let F and T be two weakly compatible self-mappings on X such that $TX \subseteq FX$ and FX is compact. Assume that*

$$\frac{1}{2}d(Fx, Tx) < d(Fx, Fy) \Longrightarrow d(Tx, Ty) < d(Fx, Fy),$$

for all $x, y \in X$ with $x \neq y$. Then F and T have a unique common fixed point.

Proof. The proof follows from Theorem 2.2 and part (A) of example 2.1. ∎

Corollary 2.3. *Let (X, d) be a metric space and let F and T be two weakly compatible self-mappings on X such that $TX \subseteq FX$ and FX is compact. Assume that*

$$\begin{aligned}\frac{1}{2}d(Fx, Tx) < d(Fx, Fy) \Longrightarrow d(Tx, Ty) &< a\min\{d(Fx, Fy), d(Fx, Tx)\} \\ &\quad + b\min\{d(Fx, Tx), d(Fx, Ty)\} \\ &\quad + c\min\{d(Fx, Ty), d(Fy, Ty)\} + d(Fy, Tx),\end{aligned}$$

for all $x, y \in X$ where $a + b + c = 1$, $c \neq 1$. Then F and T have a unique common fixed point.

Proof. The proof follows from Theorem 2.2 and part (H) of example 2.1. ∎

Remark 2.1. *We can obtain some new results by using Theorem 2.2 and other examples of ψ and φ.*

3. CONE METRIC SPACES

In this section, we generalize our results on compact cone metric spaces.

Definition 3.1. *[8] Let E be a real Banach space with norm $\|.\|$ and P be a subset of E. P is called a cone if and only if the following conditions are satisfied:*

(i) P is closed, nonempty and $P \neq \{\theta\}$;

(ii) $a, b \geq 0$ and $x \in P$ implies $ax + by \in P$;

(iii) $x \in P$ and $-x \in P$ implies $x = \theta$.

Let $P \subset E$ be a cone, we define a partial ordering $\preceq$ on E with respect to P by $x \preceq y$ if and only if $y - x \in P$. We write $x \prec y$ whenever $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$ (interior of P). The cone $P \subset E$ is called normal if there is a positive real number K such that for all $x, y \in E$, $\theta \preceq x \preceq y \Rightarrow \|x\| \leq K\|y\|$. The least positive number satisfying the last inequality is called the normal constant of P. If $K = 1$, then the cone P is called monotone.

Definition 3.2. *[8] A cone metric space is an ordered pair (X, d), where X is any set and $d : X \times X \longrightarrow E$ is a mapping satisfying:*

(D1) $\theta \preceq d(x, y)$ for all $x, y \in X$; and $d(x, y) = \theta$ if and only if $x = y$;

(D2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(D3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Let (X, d) be a cone metric space, P be a normal cone in X with normal constant K, $x \in X$ and $\{x_n\}$ a sequence in X. The sequence $\{x_n\}$ converges to x if and only if $d(x_n, x) \longrightarrow \theta$. Limit point of every sequence is unique.

It is well known that there exists a norm $\|.\|_1$ on E, equivalent with the given $\|.\|$, such that the cone P is monotone w.r.t. $\|.\|_1$(see [1], [10], [16], [22]). By using this fact, from now on, we assume that the cone P is solid and monotone. In this case, we can define a metric on X by $D(x, y) = \|d(x, y)\|$. Furthermore, it is proved that D and d give the same topology on X (see [14]).

We will use the following lemma in the proof of the next results.

Lemma 3.1. *[7] Let (X,d) be a cone metric space. Then*

$$\theta \preceq x \ll y \Rightarrow \|x\| < \|y\|.$$

Proof. According to ([22], Proposition (2.2), page 20) $[-(y - x), y - x]$ is the neighborhood of θ. Hence, for a sufficiently large n, we have $\frac{1}{n}y \in [-(y - x), y - x]$, i.e., $\frac{y}{n} \preceq y - x$. From this, it follows that $x \preceq \left(1 - \frac{1}{n}\right)y$, that is $\|x\| \leq \left(1 - \frac{1}{n}\right)\|y\| < \|y\|$. ∎

Lemma 3.2. *Let $\psi_p : [0, \infty) \longrightarrow P$ and $\varphi_p : [0, \infty)^5 \longrightarrow P$ be two mappings satisfying the following conditions:*

$(P1)$ *$u \leq v$ implies $\varphi_p(.,.,v,.,.) - \varphi_p(.,.,u,.,.) \in P$;*

$(P2)$ *$\psi_p(u) - \varphi_p(v, v, u+v, u, 0) \notin intP$ implies $u \leq v$;*

$(P3)$ *$\psi_p(u) - \varphi_p(v, v, u+v, u, 0) \notin P$ implies $u < v$, where $u \geq 0$ and $v > 0$;*

$(P4)$ *$\psi_p(u) - \varphi_p(u, 0, v, 0, v) \notin P$ implies $u < v$, where $u \geq 0$ and $v > 0$.*

Define $\psi : [0, \infty) \longrightarrow [0, \infty)$ and $\varphi : [0, \infty)^5 \longrightarrow [0, \infty)$ by

$$\psi(r) = \|\psi_p(r)\| \qquad \text{and} \qquad \varphi(t_1, t_2, t_3, t_4, t_5) = \|\varphi_p(t_1, t_2, t_3, t_4, t_5)\|.$$

Then ψ and φ satisfy $(M1) - (M4)$.

Proof. First, notice that $\|\psi_p(u)\| \leq \|\varphi_p(v, v, u + v, u, 0)\|$ implies $\psi_p(u) - \varphi_p(v, v, u+v, u, 0) \notin intP$. Indeed, if $\psi_p(u) - \varphi_p(v, v, u+v, u, 0) \in intP$, then $\varphi_p(v, v, u+v, u, 0) \ll \psi_p(u)$. Therefore, by Lemma 3.1, we get $\|\varphi_p(v, v, u+v, u, 0)\| < \|\psi_p(u)\|$, which is a contradiction. So, we conclude from $(P2)$ that $u \leq v$. Now, suppose that $\|\psi_p(u)\| < \|\varphi_p(v, v, u+v, u, 0)\|$, then $\psi_p(u) - \varphi_p(v, v, u+v, u, 0) \notin P$. (Arguing by contradiction, if $\psi_p(u) - \varphi_p(v, v, u+v, u, 0) \in P$, then $\|\varphi_p(v, v, u+v, u, 0)\| \leq \|\psi_p(u)\|$.) Hence, $(P3)$ implies $u < v$. By a similar method, it can be shown that $\|\psi(u)\| < \|\varphi(u, 0, v, 0, v)\|$ implies $u < v$. ∎

Example 3.1. *Suppose that $p \in P$. Let*

(A) *$\psi_p(r) = rp$ and $\varphi_p(t_1, t_2, t_3, t_4, t_5) = t_1 p$;*

(B) *$\psi_p(r) = 2rp$ and $\varphi_p(t_1, t_2, t_3, t_4, t_5) = t_3 p$;*

(C) *$\psi_p(r) = 2rp$ and $\varphi_p(t_1, t_2, t_3, t_4, t_5) = (t_1 + t_4)p$;*

(D) *$\psi_p(r) = 5rp$ and $\varphi_p(t_1, t_2, t_3, t_4, t_5) = (t_1 + t_2 + t_3 + t_4 + t_5)p$;*

(E) *$\psi_p(r) = 2rp$ and $\varphi_p(t_1, t_2, t_3, t_4, t_5) = p \max\{t_1, t_2, t_3, t_4, t_5\}$;*

(F) *$\psi_p(r) = 2r^2 p$ and $\varphi_p(t_1, t_2, t_3, t_4, t_5) = (t_1^2 + t_4^2)p$.*
It is easy to show that $(P1) - (P4)$ are satisfied for ψ_p and φ_p in (A), (B), (C), (D), (E) and (F).

(G) *$\psi_p(r) = rp$ and $\varphi_p(t_1, t_2, t_3, t_4, t_5) = (at_1 + bt_2 + ct_3 + dt_4 + et_5)p$, where a, b, c, d and e are nonnegative numbers, $a + b + 2c + d = 1$ and $d \neq 1$. So, $(P1) - (P3)$ are satisfied. Moreover, if $e \leq b + c + d$ then (P_4) is satisfied.*

(H) *$\psi_p(r) = rp$ and $\varphi_p(t_1, t_2, t_3, t_4, t_5) = (a \min\{t_1, t_2\} + b \min\{t_2, t_3\} + c \min\{t_3, t_4\} + t_5)p$, where a, b and c are nonnegative numbers, $a + b + c = 1$ and $c \neq 1$. Then $(P1) - (P4)$ are satisfied.*

Theorem 3.1. *Let (X, d) be a compact cone metric space and T be a self-mapping on X. Suppose that $\psi_p : [0,\infty) \longrightarrow P$ and $\varphi_p : [0,\infty)^5 \longrightarrow P$ are two continuous mappings such that $(P1) - (P3)$ are satisfied. Assume that*

$$\begin{aligned}&\frac{1}{2}d(x,Tx) - d(x,y) \notin intP \Longrightarrow \\ &\qquad \psi_p(D(Tx,Ty)) \ll \varphi_p\Big(D(x,y), D(x,Tx), D(x,Ty), D(y,Ty), D(y,Tx)\Big),\end{aligned} \tag{4}$$

for all $x, y \in X$ where $D(x,y) = \|d(x,y)\|$. Then T has at least one fixed point. Moreover, if ψ_p and φ_p satisfy $(P4)$, then T has a unique fixed point.

Proof. Let $\frac{1}{2}D(x.Tx) < D(x,y)$. So $\frac{1}{2}d(x,Tx) - d(x,y) \notin intP$. Therefore, by (4), we have

$$\psi_p(D(Tx,Ty)) \ll \varphi_p\Big(D(x,y), D(x,Tx), D(x,Ty), D(y,Ty), D(y,Tx)\Big).$$

Thus, by Lemma 3.2, we get

$$\psi := \|\psi_p(D(Tx,Ty))\| < \|\varphi_p\Big(D(x,y), D(x,Tx), D(x,Ty), D(y,Ty), D(y,Tx)\Big)\| := \varphi.$$

It is easy to see that ψ and φ are continuous. Also, it follows from Lemma 3.2 that ψ and φ satisfy $(M1) - (M3)$. Hence, the conditions of Theorem 2.1 are satisfied. Therefore, T has at least one fixed point. Furthermore, ψ and φ satisfy $(M4)$. Then T has a unique fixed point. ∎

Theorem 3.2. *Let (X, d) be a cone metric space and let F and T be two self-mappings on X such that $TX \subseteq FX$ and FX is compact. Suppose that $\psi_p : [0,\infty) \longrightarrow P$ and $\varphi_p : [0,\infty)^5 \longrightarrow P$ are two continuous mappings satisfying $(P1) - (P3)$. Assume that*

$$\begin{aligned}&\frac{1}{2}d(Fx,Tx) - d(Fx,Fy) \notin intP \Longrightarrow \\ &\qquad \psi_p(D(Tx,Ty)) \ll \varphi_p\Big(D(Fx,Fy), D(Fx,Tx), D(Fx,Ty), D(Fy,Ty), D(Fy,Tx)\Big),\end{aligned}$$

for all $x, y \in X$ where $D(x,y) = \|d(x,y)\|$. Then F and T have at least one point of coincidence. Moreover, if ψ and φ satisfy $(P4)$ and F and T are weakly compatible, then F and T have a unique common fixed point.

Now, we obtain the following new results by using Theorem 3.2 and parts (A), (G) and (H) of example 3.1.

Corollary 3.1. *Let (X, d) be a cone metric space and let F and T be two weakly compatible self-mappings on X such that $TX \subseteq FX$ and FX is compact. Assume that*

$$\frac{1}{2}d(Fx,Tx) - d(Fx,Fy) \notin intP \Longrightarrow (D(Fx,Fy) - D(Tx,Ty))intP \subseteq intP,$$

for all $x, y \in X$ *with* $x \neq y$, *where* $D(x, y) = \|d(x, y)\|$. *Then* F *and* T *have a unique common fixed point.*

Corollary 3.2. *Let* (X, d) *be a cone metric space and let* F *and* T *be two self-mappings on* X *such that* $TX \subseteq FX$ *and* FX *is compact. Assume that*

$$\frac{1}{2}d(Fx, Tx) - d(Fx, Fy) \notin intP \Longrightarrow M(x, y)intP \subseteq intP,$$

for all $x, y \in X$, *where*

$$\begin{aligned} M(x, y) = aD(Fx, Fy) + bD(Fx, Tx) + cD(Fx, Ty) + dD(Fy, Ty) + eD(Fy, Tx) \\ - D(Tx, Ty), \end{aligned}$$

and $a, b, c, d, e \geq 0$, $a + b + 2c + d = 1$ *and* $d \neq 1$. *Then* F *and* T *have at least one point of coincidence. Moreover, if* $e \leq b + c + d$ *and* F *and* T *are weakly compatible, then* F *and* T *have a unique common fixed point.*

Corollary 3.3. *Let* (X, d) *be a cone metric space and let* F *and* T *be two weakly compatible self-mappings on* X *such that* $TX \subseteq FX$ *and* FX *is compact. Assume that*

$$\frac{1}{2}d(Fx, Tx) - d(Fx, Fy) \notin intP \Longrightarrow N(x, y)intP \subseteq intP,$$

for all $x, y \in X$, *where*

$$\begin{aligned} N(x, y) = a \min\{D(Fx, Fy), D(Fx, Tx)\} + b \min\{D(Fx, Tx), D(Fx, Ty)\} \\ + c \min\{D(Fx, Ty), D(Fy, Ty)\} + D(Fy, Tx) - D(Tx, Ty), \end{aligned}$$

and $a, b, c \geq 0$, $a + b + c = 1$ *and* $c \neq 1$. *Then* F *and* T *have a unique common fixed point.*

Remark 3.1. *We can obtain some new results by using Theorem 3.2 and other examples of* ψ_p *and* φ_p.

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ON THE NUMERICAL APPROXIMATION OF THE PHASE-FIELD SYSTEM WITH NON-HOMOGENEOUS CAUCHY-NEUMANN BOUNDARY CONDITIONS. CASE 1D

Costică Moroşanu, Ana - Maria Moşneagu
"A. I. Cuza" University; Iaşi, Romania
costica.morosanu@uaic.ro, anamaria.mosneagu@uaic.ro

Abstract A *scheme of fractional steps type*, associated to the nonlinear phase-field transition system in one dimension, is considered in this paper. To approximate the solution of the linear parabolic system introduced by such approximating scheme, we consider three finite differences schemes: **1-IMBDF** (first-order **IM**plicit **B**ackward **D**ifferentiation **F**ormula), **2-IMBDF** (second-order IMBDF) and **2-SBDF** (second-order **S**emi-implicit BDF). A study of stability and the numerical efficiency analysis of this new approach, as well as physical experiments, are performed too.

Keywords: fractional steps method, stability and convergence of numerical methods, computer aspects of numerical algorithms, phase-field transition system, phase changes.
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1. INTRODUCTION

Consider the nonlinear parabolic boundary value problem

$$\begin{cases} \rho c \frac{\partial}{\partial t} u + \frac{\ell}{2} \frac{\partial}{\partial t} \varphi = k \Delta u \\ \tau \frac{\partial}{\partial t} \varphi = \xi^2 \Delta \varphi + \frac{1}{2a}(\varphi - \varphi^3) + 2u \end{cases} \quad \text{in } Q := [0, T] \times \Omega, \tag{1.1}$$

subject to the non-homogeneous Cauchy-Neumann boundary conditions:

$$\begin{cases} \frac{\partial}{\partial \nu} u + hu = w(t, x) \\ \frac{\partial}{\partial \nu} \varphi = 0 \end{cases} \quad \text{on } \Sigma := [0, T] \times \partial\Omega, \tag{1.2}$$

and initial conditions:

$$u(0, x) = u_0(x), \quad \varphi(0, x) = \varphi_0(x) \quad \text{on } \Omega, \tag{1.3}$$

where:

- Ω is a bounded domain in $\boldsymbol{R}$ with smooth boudary $\partial\Omega$,
- $T > 0$ is a given positive number,

- the unknown functions u and φ represent the *reduced temperature distribution* and the *phase function* (used to distinguish between the phases of Ω), respectively,
- $u_0, \varphi_0 : \Omega \to \boldsymbol{R}$ are given functions,
- $w : [0,T] \times \partial\Omega \to \boldsymbol{R}$ also is a given function - *the temperature surrounding at* $\partial\Omega$,
- the positive parameters ρ, c, τ, ξ, ℓ, k, h, a, have the following physical meaning: ρ - is the density, c - is the heat capacity, τ - is the relaxation time, ξ - is the length scale of the interface, ℓ - denotes the latent heat, k - the heat conductivity, h - the heat transfer coeficient and a is an probabilistic measure on the individual atoms (a depends on ξ).

The mathematical model (1.1), introduced by Caginalp [3], has been established in literature as an alternative of the classic two-phase Stefan problem to capture, among others, the effects of *surface tension*, *supercooling*, and *superheating*.

As regards the existence, it is known that under appropiate conditions on u_0, φ_0 and w, the system (1.1)-(1.3) has a unique solution $u, \varphi \in W_p^{2,1}(Q) \cap L^\infty(Q)$, $p > \frac{3}{2}$ (see Morosanu [6]).

Numerical approximation of the phase-field system (1.1) subject to the homogeneous Neumann boundary conditions: $\frac{\partial}{\partial \nu} u + hu = 0$ on Σ, has been analyzed in Morosanu [5]. For other numerical investigation of the phase-field model (subject to various other boundary conditions), see Arnautu & Morosanu [1], Morosanu [4, 6] and references there in.

In order to approximate the above nonlinear problem, a *scheme of fractional steps type* was introduced and analyzed in Benincasa & Morosanu [2], namely, for every $\varepsilon > 0$, it was associated to system (1.1)-(1.3) the following approximating scheme:

$$\begin{cases} \rho c \frac{\partial}{\partial t} u^\varepsilon + \frac{\ell}{2} \frac{\partial}{\partial t} \varphi^\varepsilon = k \Delta u^\varepsilon \\ \tau \frac{\partial}{\partial t} \varphi^\varepsilon = \xi^2 \Delta \varphi^\varepsilon + \frac{1}{2a} \varphi^\varepsilon + 2u^\varepsilon \end{cases} \quad \text{in } Q_i^\varepsilon, \tag{1.4}$$

$$\begin{cases} \frac{\partial}{\partial \nu} u^\varepsilon + h u^\varepsilon = w(t,x) \\ \frac{\partial}{\partial \nu} \varphi^\varepsilon = 0 \end{cases} \quad \text{on } \Sigma_i^\varepsilon, \tag{1.5}$$

$$\begin{cases} u^\varepsilon(0,x) = u_0(x) \\ \varphi_+^\varepsilon(i\varepsilon, x) = z(\varepsilon,\ \varphi_-^\varepsilon(i\varepsilon, x)) \end{cases} \quad \text{on } \Omega \tag{1.6}$$

where $z(\varepsilon, \varphi_-^\varepsilon(i\varepsilon, x))$ is the solution of Cauchy problem:

$$\begin{cases} z'(s) + \frac{1}{2a} z^3(s) = 0 \quad s \in (0, \varepsilon), \\ z(0) = \varphi_-^\varepsilon(i\varepsilon, x) \quad \varphi_-^\varepsilon(0,x) = \varphi_0(x), \end{cases} \tag{1.7}$$

for $i = 0, 1, \cdots, M_\varepsilon - 1$, with $Q_i^\varepsilon = (i\varepsilon, (i+1)\varepsilon) \times \Omega$, $\Sigma_i^\varepsilon = (i\varepsilon, (i+1)\varepsilon) \times \partial\Omega$, $M_\varepsilon = \left[\frac{T}{\varepsilon}\right]$, $Q_{M_\varepsilon - 1}^\varepsilon = [(M_\varepsilon - 1)\varepsilon, T] \times \Omega$ and $\varphi_+^\varepsilon(i\varepsilon, x) = \lim\limits_{t \downarrow i\varepsilon} \varphi^\varepsilon(t, x)$, $\varphi_-^\varepsilon(i\varepsilon, x) = \lim\limits_{t \uparrow i\varepsilon} \varphi^\varepsilon(t, x)$.

In other words, the fractional steps method consists in decoupling the nonlinear system (1.1)-(1.3) in a linear parabolic system and a nonlinear ordinary differential equation containing the nonlinearity φ^3 of $(1.1)_2$, expressed on a partition of the time interval $[0, T]$ which is composed from M_ε subintervals, the first $M_\varepsilon - 1$ having the same length ε.

The following result establishes the relationship between the solution (u, φ) in (1.1)-(1.3) and the solution $(u^\varepsilon, \varphi^\varepsilon)$ in (1.4)-(1.7).

Theorem 1.1. *Assume that* $u_0, \varphi_0 \in W_\infty^1(\Omega)$ *satisfying* $\frac{\partial}{\partial \nu} u_0 + h u_0 = w(0, x)$, $\frac{\partial}{\partial \nu} \varphi_0 = 0$ *and* $w \in W^1([0, T], L^2(\partial\Omega))$. *Furthermore,* $\Omega \subset \boldsymbol{R}^n$ $(n = 1, 2, 3)$ *is a bounded domain with a smooth boundary. Let* $(u^\varepsilon, \varphi^\varepsilon)$ *be the solution of the approximating scheme* (1.4)-(1.7). *Then, for* $\varepsilon \to 0$*, one has*

$$(u^\varepsilon(t), \varphi^\varepsilon(t)) \to (u(t), \varphi(t)) \quad \textit{strongly in } L^2(\Omega) \quad \textit{for any } t \in (0, T], \tag{1.8}$$

where $u, \varphi \in W_p^{2,1}([0, T]; L^2(\Omega)) \cap L^2([0, T]; H^2(\Omega))$ *is the solution of the nonlinear system* (1.1)-(1.3).

Based on the result of convergence given by Theorem 1, we will concerned in this work with the numerical approximatin of the solution $(u^\varepsilon, \varphi^\varepsilon)$ of the linear system (1.4)-(1.7).

The rest of paper is organized as follows: in Section 2, for each type of scheme:*1-IMBDF, 2-IMBDF, 2-SBDF*, we have introduced the discrete equations corresponding to (1.4)-(1.7); consequently, conceptual algorithms have been formulated: **Alg_1-IMBDF**, **Alg_2-IMBDF**, **Alg_2-SBDF**, respectively. A stability result for each new approach is stated and proved too. Some physical experiments are reported in the last Section.

2. NUMERICAL METHODS

In this Section we are concerned with the numerical approximation of the solution $(u^\varepsilon, \varphi^\varepsilon)$ in (1.4)-(1.7). As already stated, we will work in one dimension, i.e. $\Delta u^\varepsilon = u_{xx}^\varepsilon$ and $\Delta \varphi^\varepsilon = \varphi_{xx}^\varepsilon$. To fix the ideas, let $\Omega = [0, b] \subset \boldsymbol{R}_+$ and we introduce over it the grid with N equidistant nodes

$$x_j = (j - 1)dx \qquad j = 1, 2, \ldots, N, \quad dx = b/(N - 1).$$

Given a positive value T and considering $M \equiv M_\varepsilon$ as the number of equidistant nodes in which is divided the time interval $[0, T]$, we set

$$t_i = (i - 1)\varepsilon \qquad i = 1, 2, \ldots, M, \quad \varepsilon = T/(M - 1).$$

Now we denote by (u_j^i, φ_j^i) the approximate values in the point (t_i, x_j) of the unknown functions $(u^\varepsilon, \varphi^\varepsilon)$. More precisely

$$\begin{array}{l} u_j^i = u^\varepsilon(t_i, x_j) \\ \varphi_j^i = \varphi^\varepsilon(t_i, x_j) \end{array} \qquad i = 1, 2, \ldots, M, \quad j = 1, 2, \ldots, N,$$

or, for later use

$$u^i \stackrel{\text{not}}{=} \left(u_1^i, u_2^i, \ldots, u_N^i\right)^T \quad \varphi^i \stackrel{\text{not}}{=} \left(\varphi_1^i, \varphi_2^i, \ldots, \varphi_N^i\right)^T \quad i = 1, 2, \ldots, M. \tag{2.1}$$

We continue by explaining how we treat each term in (1.4)-(1.7). The Laplace operator in (1.4) will be approximated by a second order centred finite differences, which means:

$$\begin{array}{l} u_{xx}^\varepsilon(t_i, x_j) = \Delta_{dx} u_j^i \approx \frac{u_{j-1}^i - 2u_j^i + u_{j+1}^i}{dx^2} \\ \varphi_{xx}^\varepsilon(t_i, x_j) = \Delta_{dx} \varphi_j^i \approx \frac{\varphi_{j-1}^i - 2\varphi_j^i + \varphi_{j+1}^i}{dx^2} \end{array} \quad i = 1, 2, ..., M, \quad j = 1, 2, ..., N, \tag{2.2}$$

(Δ_{dx} is the discrete Laplacian depending on the step-size dx).

From the initial condition $(1.6)_1$, we have

$$u_j^1 = u^\varepsilon(t_1, x_j) = u_0(x_j) \quad j = 1, 2, \ldots, N. \tag{2.3}$$

Involving the separation of variables method to solve the Cauchy problem (1.7) (see Morosanu [4]), we get

$$\begin{cases} z(\varepsilon, \varphi_-^\varepsilon(t_1, x)) = z(\varepsilon, \varphi_0(x)) = \varphi_0(x) \sqrt{\frac{a}{a + \varepsilon \varphi_0(x)}}, \\ z(\varepsilon, \varphi_-^\varepsilon(t_i, x)) = \varphi_-^\varepsilon(t_i, x) \sqrt{\frac{a}{a + \varepsilon \varphi_-^\varepsilon(t_i, x)}} \quad i = 2, ..., M-1. \end{cases} \tag{2.4}$$

Corresponding to Ω, already choosen in one dimension, the boundary $\partial\Omega$ is reduced to the set $\{0, b\}$. Thus the boundary conditions $(1.5)_1$ become

$$\begin{cases} -u_x(0) + h\, u(0) = w(t, 0) \\ u_x(b) + h\, u(b) = w(t, b), \end{cases} \tag{2.5}$$

where the sign in front of $\frac{\partial}{\partial \nu} u = u_x$ is $\mp$ because the normal to $[0, b]$ at 0 (b) point in the negative (positive) direction.

Using in (2.5) a farward (backward) finite differences to approximate $u_x(0)$ $(u_x(b))$, we get

$$\begin{cases} -\dfrac{u_2^i - u_1^i}{dx} + h\, u_1^i = w^i(0) \\ \dfrac{u_N^i - u_{N-1}^i}{dx} + h\, u_N^i = w^i(b) \end{cases} \qquad i = 1, 2, \ldots, M,$$

i.e.

$$\begin{cases} (1 + dx\, h)u_1^i - u_2^i = dx\, w^i(0) \\ -u_{N-1}^i + (1 + dx\, h)u_N^i = dx\, w^i(b) \end{cases} \qquad i = 1, 2, \ldots, M, \tag{2.6}$$

where $w^i(0) = w(t_i, 0)$, $w^i(b) = w(t_i, b)$, $i = 1, 2, ..., M$.

To approximate $\varphi_x(0)$ ($\varphi_x(b)$) we will use a backward (forward) finite differences; this leads to

$$\varphi_0^i = \varphi_1^i, \quad \varphi_{N+1}^i = \varphi_N^i \quad i = 1, 2, \ldots, M, \tag{2.7}$$

where φ_0^i and φ_{N+1}^i are dummy variables.

For approximating the partial derivative with respect to time, we employed a *first-order scheme* and a *second-order scheme*, namely:

$$\tfrac{\partial}{\partial t} u^\varepsilon(t_i, x_j) \approx \tfrac{u_j^i - u_j^{i-1}}{\varepsilon}, \quad \tfrac{\partial}{\partial t} \varphi^\varepsilon(t_i, x_j) \approx \tfrac{\varphi_j^i - \varphi_j^{i-1}}{\varepsilon} \tag{2.8}$$

$i = 2, 3, \ldots, M, \quad j = 1, 2, \ldots, N$, and

$$\tfrac{\partial}{\partial t} u^\varepsilon(t_i, x_j) \approx \tfrac{3u_j^i - 4u_j^{i-1} + u_j^{i-2}}{2\varepsilon}, \quad \tfrac{\partial}{\partial t} \varphi^\varepsilon(t_i, x_j) \approx \tfrac{3\varphi_j^i - 4\varphi_j^{i-1} + \varphi_j^{i-2}}{2\varepsilon} \tag{2.9}$$

$i = 2, 3, \ldots, M, \quad j = 1, 2, \ldots, N$.

Finally we refer to the right hand in (1.4): $\frac{1}{2a}\varphi^\varepsilon(t_i, x_j) + 2u^\varepsilon(t_i, x_j)$. To approximate this quantity (the *reaction term*), will involve two approaches: an implicit and a semi-implicit formula, i.e.:

$$\frac{1}{2a}\varphi^\varepsilon(t_i, x_j) + 2u^\varepsilon(t_i, x_j) \approx \frac{1}{2a}\varphi_j^i + 2u_j^i, \tag{2.10}$$

$i = 1, 2, \ldots, M$, $j = 1, 2, \ldots, N$, and

$$\frac{1}{2a}\varphi^\varepsilon(t_i, x_j) + 2u^\varepsilon(t_i, x_j) \approx 2\left[\frac{1}{2a}\varphi_j^{i-1} + 2u_j^{i-1}\right] - \left[\frac{1}{2a}\varphi_j^{i-2} + 2u_j^{i-2}\right], \tag{2.11}$$

$i = 2, 3, ..., M$, $j = 1, 2, ..., N$ (see Ruuth [7, pp. 156]).

We are now ready to build those three approximation schemes, mentioned at the begining.

A. 1-IMBDF - First-order Implicit Backward Difference Formula. To develop such a scheme, we begin by replacing in (1.4) approximations stated in (2.2), (2.8) and (2.10). We deduce:

$$\begin{cases} \rho c\, \frac{u_j^i - u_j^{i-1}}{\varepsilon} + \frac{\ell}{2}\frac{\varphi_j^i - \varphi_j^{i-1}}{\varepsilon} = k\Delta_{dx} u_j^i \\ \tau\, \frac{\varphi_j^i - \varphi_j^{i-1}}{\varepsilon} = \xi^2 \Delta_{dx}\varphi_j^i + \frac{1}{2a}\varphi_j^i + 2u_j^i, \end{cases} \tag{2.12}$$

for $i = 2, 3, \ldots, M$, $j = 1, 2, \ldots, N$.

Using in (2.12) the equalities from (2.2) and arranging convenient, we conclude that, via 1-IMBDF, the system (1.4) is discretized as follows

$$\begin{cases} -k\frac{\varepsilon}{dx^2}u^i_{j-1} + \left[\rho c + 2k\frac{\varepsilon}{dx^2}\right]u^i_j - k\frac{\varepsilon}{dx^2}u^i_{j+1} + \frac{\ell}{2}\varphi^i_j = \rho c u^{i-1}_j + \frac{\ell}{2}\varphi^{i-1}_j \\ -2\varepsilon\, u^i_j - \xi^2\frac{\varepsilon}{dx^2}\varphi^i_{j-1} + \left[\tau + 2\xi^2\frac{\varepsilon}{dx^2} - \frac{\varepsilon}{2a}\right]\varphi^i_j - \xi^2\frac{\varepsilon}{dx^2}\varphi^i_{j+1} = \tau\varphi^{i-1}_j, \end{cases} \tag{2.13}$$

for $i = 2, 3, \ldots, M,\ j = 1, 2, \ldots, N$.

In order to compute the matrix $\begin{pmatrix} u^i_j \\ \varphi^i_j \end{pmatrix}_{i=\overline{2,M},\ j=\overline{1,N}}$, the linear system (2.13) will be solved ascending with respect to time levels. For the first time level ($i = 1$), the values of u^1_j and φ^1_j are computed by (2.3) and (2.4), respectively. Moreover, let us point out from (2.13) and (2.6)-(2.7) that we have $2N$ unknowns for each time-level i, $i = 2, 3, ..., M$ (see also (2.1)).

If, corresponding to $j = 1$ and $j = N$, in $(2.13)_1$ we take $u^i_0 = u^i_1$ and $u^i_{N+1} = u^i_N$, respectively, and if we set

$$\begin{array}{lll} c_1 = -k\frac{\varepsilon}{dx^2} & c_2 = \rho c - 2c_1 & c_3 = \frac{\ell}{2} \\ c_5 = -\xi^2\frac{\varepsilon}{dx^2} & c_6 = \tau - 2c_5 - \frac{\varepsilon}{2a}, & \end{array}$$

than the system (2.13), coupled with (2.6)-(2.7), can be rewritten in matrix form as

$$A\begin{pmatrix} u^i \\ \varphi^i \end{pmatrix} = B\begin{pmatrix} u^{i-1} \\ \varphi^{i-1} \end{pmatrix} + \begin{pmatrix} d^i_1 \\ d^i_2 \end{pmatrix} \qquad i = 2, 3, ..., M, \tag{2.14}$$

where

$$A = \begin{pmatrix} A_{11} & A_{12} \\ -2A_{21} & A_{22} \end{pmatrix} \qquad B = \begin{pmatrix} A_{13} & A_{12} \\ 0 & A_{23} \end{pmatrix}$$

with $A_{11}, A_{12}, A_{13}, A_{21}, A_{22}, A_{23}$ having the same size $N \times N$, and

$$A_{11} = \begin{pmatrix} a_1 & c_1 - 1 & 0 & \cdots & 0 & 0 & 0 \\ c_1 & c_2 & c_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_1 & c_2 & c_1 \\ 0 & 0 & 0 & \cdots & 0 & c_1 - 1 & a_1 \end{pmatrix}$$

$a_1 = c_1 + c_2 + 1 + dx \cdot h$,

$$A_{22} = \begin{pmatrix} c_5{+}c_6 & c_5 & 0 & \cdots & 0 & 0 & 0 \\ c_5 & c_6 & c_5 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_5 & c_6 & c_5 \\ 0 & 0 & 0 & \cdots & 0 & c_5 & c_5{+}c_6 \end{pmatrix}$$

$$A_{12} = \begin{pmatrix} c_3 & 0 & \cdots & 0 & 0 \\ 0 & c_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & c_3 & 0 \\ 0 & 0 & \cdots & 0 & c_3 \end{pmatrix} \quad A_{21} = \begin{pmatrix} dt & 0 & \cdots & 0 & 0 \\ 0 & dt & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & dt & 0 \\ 0 & 0 & \cdots & 0 & dt \end{pmatrix}$$

$$A_{13} = \begin{pmatrix} \rho c & 0 & \cdots & 0 & 0 \\ 0 & \rho c & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \rho c & 0 \\ 0 & 0 & \cdots & 0 & \rho c \end{pmatrix} \quad A_{23} = \begin{pmatrix} \tau & 0 & \cdots & 0 & 0 \\ 0 & \tau & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \tau & 0 \\ 0 & 0 & \cdots & 0 & \tau \end{pmatrix}$$

$$d_1^i = \begin{pmatrix} dx \cdot w^i(0) \\ 0 \\ \vdots \\ 0 \\ dx \cdot w^i(b) \end{pmatrix} \quad d_2^i = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Therefore, the general design of the algorithm to calculate the approximate solution of nonlinear system (1.1)-(1.3), via *fractional steps method* and *1-IMBDF*, is the following one

```
Begin Alg_1-IMBDF
    Choose T > 0, b > 0;
    Choose M > 0, N > 0 and compute ε, dx;
    Choose u_0, φ_0, w;
    i := 1 → u^1 from the initial conditions (2.3);
    For i = 2 to M do
        Compute φ^{i-1} = z(ε, φ_-^ε(t_{i-1}, ·)) using (1.6)_2 and (2.4);
        Compute u^i, φ^i solving the linear system (2.14);
    End-for;
End.
```

B. 2-IMBDF - Second-order Implicit Backward Difference Formula. To solve the system (1.4) we consider now a *second-order implicit scheme*, i.e.:

$$\begin{cases} \rho c \, \frac{3u_j^i - 4u_j^{i-1} + u_j^{i-2}}{2\varepsilon} + \frac{\ell}{2} \frac{3\varphi_j^i - 4\varphi_j^{i-1} + \varphi_j^{i-2}}{2\varepsilon} = k \Delta_{dx} u_j^i \\ \tau \, \frac{3\varphi_j^i - 4\varphi_j^{i-1} + \varphi_j^{i-2}}{2\varepsilon} = \xi^2 \Delta_{dx} \varphi_j^i + \frac{1}{2a} \varphi_j^i + 2u_j^i, \end{cases} \tag{2.15}$$

for $i = 2, 3 \ldots, M$, $j = 1, 2, \ldots, N$, and u^0, φ^0 considered as dummy variables.

Following the same schedule as above, we conclude that, via *2-IMBDF*, the system (1.4) is discretized as follows:

$$\begin{cases} 2c_1 u_{j-1}^i + \left(3\rho c + 4k\frac{\varepsilon}{dx^2}\right) u_j^i + 2c_1 u_{j+1}^i + 3c_3\varphi_j^i \\ \qquad = \rho c\left(4u_j^{i-1} - u_j^{i-2}\right) + c_3\left(4\varphi_j^{i-1} - \varphi_j^{i-2}\right), \\ -4\varepsilon\, u_j^i + 2c_5\varphi_{j-1}^i + \left(3\tau - 4c_5 - \frac{\varepsilon}{a}\right)\varphi_j^i \\ \qquad +2c_5\varphi_{j+1}^i = \tau\left(4\varphi_j^{i-1} - \varphi_j^{i-2}\right), \end{cases} \tag{2.16}$$

for $i = 2, 3, \ldots, M,\ j = 1, 2, \ldots, N$.

Remembering the same considerations (developed at begining of Section) with respect to: *initial conditions* - relations (2.3)-(2.4), *boundary conditions* - relations (2.6)-(2.7), *unknown vector* for each time-level i - which was denoted by u^i and φ^i, and setting

$$c_7 = 3\rho c + 4k\frac{\varepsilon}{dx^2} \quad c_8 = 3\tau - 4c_5 - \frac{\varepsilon}{a},$$

the system (2.16) can be written as a matrix equation,

$$E\begin{pmatrix} u^i \\ \varphi^i \end{pmatrix} = 4B\begin{pmatrix} u^{i-1} \\ \varphi^{i-1} \end{pmatrix} - B\begin{pmatrix} u^{i-2} \\ \varphi^{i-2} \end{pmatrix} + \begin{pmatrix} d_1^i \\ d_2^i \end{pmatrix} \quad i = 2, 3, ..., M, \tag{2.17}$$

where

$$E = \begin{pmatrix} E_{11} & 3A_{12} \\ -4A_{21} & E_{22} \end{pmatrix}$$

with E_{11}, E_{22} having the same size $N \times N$, and

$$E_{11} = \begin{pmatrix} e_1 & 2c_1 - 1 & 0 & \cdots & 0 & 0 & 0 \\ 2c_1 & c_7 & 2c_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2c_1 & c_7 & 2c_1 \\ 0 & 0 & 0 & \cdots & 0 & 2c_1 - 1 & e_1 \end{pmatrix}$$

$e_1 = 2c_1 + c_7 + 1 + dx \cdot h$,

$$E_{22} = \begin{pmatrix} 2c_5 + c_8 & 2c_5 & 0 & \cdots & 0 & 0 & 0 \\ 2c_5 & c_8 & 2c_5 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2c_5 & c_8 & 2c_5 \\ 0 & 0 & 0 & \cdots & 0 & 2c_5 & 2c_5 + c_8 \end{pmatrix}.$$

Summing up, we can conclude that the general design of the algorithm to calculate the approximate solution of nonlinear system (1.1)-(1.3), via *fractional steps method* and *2-IMBDF*, is the following one

```
Begin Alg_2-IMBDF
    Choose T > 0, b > 0;
    Choose M > 0, N > 0 and compute ε, dx;
    Choose u_0, φ_0, w;
    i := 1 → u^1 from the initial conditions (2.3);
            φ^1 = z(ε, φ^ε_(t_1, ·)) from (2.4)_1;
    i := 0 → u^0 = u^1, φ^0 = φ^1;
    For i = 2 to M do
        Compute φ^{i-1} = z(ε, φ^ε_(t_{i-1}, ·)) using (1.6)_2 and (2.4);
        Compute u^i, φ^i solving the linear system (2.17);
    End-for;
End.
```

C. 2-SBDF - Second-order Semi-implicit Backward Difference Formula. The purpose of this Subsection is to implement a *2-SBDF* method to approximate the solution $(u^\varepsilon, \varphi^\varepsilon)$ in (1.4)-(1.7). The work is based especially on relations (2.9) and (2.11). Consequently, replacing in (1.4) the approximations mentioned above, we deduce the following system of equations:

$$\begin{cases} \rho c\,\frac{3u_j^i-4u_j^{i-1}+u_j^{i-2}}{2\varepsilon} + \frac{\ell}{2}\frac{3\varphi_j^i-4\varphi_j^{i-1}+\varphi_j^{i-2}}{2\varepsilon} = k\Delta_{dx}u_j^i \\ \tau\,\frac{3\varphi_j^i-4\varphi_j^{i-1}+\varphi_j^{i-2}}{2\varepsilon} = \xi^2\Delta_{dx}\varphi_j^i + 2\left[\frac{1}{2a}\varphi_j^{i-1}+2u_j^{i-1}\right] - \left[\frac{1}{2a}\varphi_j^{i-2}+2u_j^{i-2}\right] \end{cases} \tag{2.18}$$

$i = 2, 3, ..., M$, $j = 1, 2, ..., N$, where, following the same strategy as in previous Subsection, we obtain the discrete system (see also (2.16)):

$$\begin{cases} 2c_1u_{j-1}^i + c_7u_j^i + 2c_1u_{j+1}^i + 3c_3\varphi_j^i \\ \quad = \rho c\left(4u_j^{i-1} - u_j^{i-2}\right) + c_3\left(4\varphi_j^{i-1} - \varphi_j^{i-2}\right), \\ 2c_5\varphi_{j-1}^i + (3\tau - 4c_5)\,\varphi_j^i + 2c_5\varphi_{j+1}^i \\ \quad = 8\varepsilon\,u_j^{i-1} + \left(4\tau + \frac{2\varepsilon}{a}\right)\varphi_j^{i-1} - 4\varepsilon\,u_j^{i-2} - \left(\tau + \frac{\varepsilon}{a}\right)\varphi_j^{i-2}, \end{cases} \tag{2.19}$$

$i = 2, 3, \ldots, M$, $j = 1, 2, \ldots, N$.

Setting

$$c_9 = 3\tau - 4c_5 \quad c_{10} = 4\tau + 2\frac{\varepsilon}{a} \quad c_{11} = \tau + \frac{\varepsilon}{a},$$

the system (2.19) can be rewritten in matrix form as

$$X\text{ł}\begin{pmatrix} u^i \\ \varphi^i \end{pmatrix} = Y\begin{pmatrix} u^{i-1} \\ \varphi^{i-1} \end{pmatrix} - Z\begin{pmatrix} u^{i-2} \\ \varphi^{i-2} \end{pmatrix} + \begin{pmatrix} d_1^i \\ d_2^i \end{pmatrix} \quad i = 2, 3, ..., M, \tag{2.20}$$

where

$$X = \begin{pmatrix} E_{11} & 3A_{12} \\ 0 & X_{22} \end{pmatrix} \quad Y = \begin{pmatrix} 4A_{13} & 4A_{12} \\ 8A_{21} & Y_{22} \end{pmatrix} \quad Z = \begin{pmatrix} A_{13} & A_{12} \\ 4A_{21} & Z_{22} \end{pmatrix}$$

with $A_{12}, X_{22}, A_{13}, A_{21}, Y_{22}, Z_{22}$ having the same size $N \times N$, and

$$X_{22} = \begin{pmatrix} 2c_5 + c_9 & 2c_5 & 0 & \cdots & 0 & 0 & 0 \\ 2c_5 & c_9 & 2c_5 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2c_5 & c_9 & 2c_5 \\ 0 & 0 & 0 & \cdots & 0 & 2c_5 & 2c_5 + c_9 \end{pmatrix},$$

$$Y_{22} = \begin{pmatrix} c_{10} & 0 & \cdots & 0 & 0 \\ 0 & c_{10} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & c_{10} & 0 \\ 0 & 0 & \cdots & 0 & c_{10} \end{pmatrix}, \quad Z_{22} = \begin{pmatrix} c_{11} & 0 & \cdots & 0 & 0 \\ 0 & c_{11} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & c_{11} & 0 \\ 0 & 0 & \cdots & 0 & c_{11} \end{pmatrix}.$$

Summing up, we can conclude that the general design of the algorithm to calculate the approximate solution of nonlinear system (1.1)-(1.3) by *fractional steps scheme* via *2-SBDF method* is the following one

```
Begin Alg_2-SBDF
    Choose T > 0, b > 0;
    Choose M > 0, N > 0 and compute ε, dx;
    Choose u_0, φ_0, w;
    i := 1 → u^1 from the initial conditions (2.3);
            φ^1 = z(ε, φ^ε_(t_1, ·)) from (2.4)_1;
    i := 0 → u^0 = u^1, φ^0 = φ^1;
    For i = 2 to M do
        Compute φ^{i-1} = z(ε, φ^ε_(t_{i-1}, ·)) using (1.6)_2 and (2.4);
        Compute u^i, φ^i solving the linear system (2.20);
    End-for;
End.
```

As it is well known, most initial value problems reduce to solving large sparse linear systems of the form (2.14), (2.17) or (2.20). For later use (e.g., numerical implementation of conceptual algorithms), we will proof the following

Lemma 2.1. *If*

$$\tau + \xi^2 \frac{\varepsilon}{dx^2} \neq \frac{\varepsilon}{2a}, \tag{2.21}$$

then the matrix coefficients in linear system (2.14) *can be factored into the product of a lower-triangular matrix and an upper-triangular matrix* (*LU - factorization*).

Proof. Let denote by a_{mn}, $m, n = 1, 2, \cdots, 2N$, the elements of matrix coefficients in linear system (2.14). Analyzing the main diagonal elements of block matrices A_{11} and A_{22} in (2.14), first we finding that $a_1 = c_1 + c_2 + 1 + dx \cdot h = \rho c + k\frac{\varepsilon}{dx^2} + 1 + dx \cdot h \neq 0$

and $c_2 = \rho c - 2c_1 = \rho c + 2k\frac{\varepsilon}{dx^2} \neq 0$. Observing now that $c_5 + c_6 \neq 0$ reflect the assumptions expressed in (2.21), as well as that $c_6 \neq 0$, we find easily that $a_{mm} \neq 0$ $\forall m = 1, 2, \cdots, 2N$. So Gaussian elimination can be performed on the system (2.14) without interchanges; consequently A has an LU factorization. ▮

Remark 2.1. *i. if*

$$\tau + \xi^2 \frac{\varepsilon}{dx^2} \neq \frac{\varepsilon}{2a},$$

then the matrix coefficients E in linear system (2.17) *has a LU factorization;*

ii. always, the matrix coefficients X in linear system (2.20) *has a LU factorization.*

3. STABILITY CONDITIONS

To establish conditions of stability for the linear difference equations (2.14), (2.17) and (2.20) introduced in the previous section, we will use in our analysis the Lax-Richtmyer definition of stability, expressed in terms of norm $\|\cdot\|_\infty$ (see Smith [8], pp. 48). To fixed the ideas, we will focus our atention on equation (2.14). This may be rewritten in a more convenient form as

$$\begin{pmatrix} u^i \\ varphi^i \end{pmatrix} = A^{-1}B\begin{pmatrix} u^{i-1} \\ \varphi^{i-1} \end{pmatrix} + A^{-1}\begin{pmatrix} d_1^i \\ d_2^i \end{pmatrix} \quad i = 2, 3, ..., M \tag{3.1}$$

(the existence of A^{-1} will be proved in the proof of Proposition 3.1 below). In addition, the matrix A can be written in the form

$$A = D(I + D^{-1}G) \tag{3.2}$$

where $D = diag(a_1, c_2, \cdots, c_2, a_1, c_5 + c_6, c_6, \cdots, c_6, c_5 + c_6)$ and $G = A - D$. Thus, noting $a_2 = c_5 + c_6$, we have

$$D^{-1}G = \begin{pmatrix}
0 & \frac{c_1-1}{a_1} & 0 & \cdots & 0 & 0 & \frac{c_3}{a_1} & 0 & 0 & \cdots & 0 & 0 & 0 \\
\frac{c_1}{c_2} & 0 & \frac{c_1}{c_2} & \cdots & 0 & 0 & 0 & \frac{c_3}{c_2} & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & \frac{c_1}{c_2} & 0 & 0 & 0 & \cdots & 0 & \frac{c_3}{c_2} & 0 \\
0 & 0 & 0 & \cdots & \frac{c_1-1}{a_1} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{c_3}{a_1} \\
-\frac{2\varepsilon}{a_2} & 0 & 0 & \cdots & 0 & 0 & 0 & \frac{c_5}{a_2} & 0 & \cdots & 0 & 0 & 0 \\
0 & -\frac{2\varepsilon}{c_6} & 0 & \cdots & 0 & 0 & \frac{c_5}{c_6} & 0 & \frac{c_5}{c_6} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -\frac{2\varepsilon}{c_6} & 0 & 0 & 0 & 0 & \cdots & \frac{c_5}{c_6} & 0 & \frac{c_5}{c_6} \\
0 & 0 & 0 & \cdots & 0 & -\frac{2\varepsilon}{a_2} & 0 & 0 & 0 & \cdots & 0 & \frac{c_5}{a_2} & 0
\end{pmatrix}$$

and a simple analysis of all lines in matrix $D^{-1}G$ allows us to deduce that we only have four distinct lines. The sum of each such line is written in vector v below (recall that $a_1 = c_1 + c_2 + 1 + dx \cdot h$ and $a_2 = c_5 + c_6$)

$$v = \left[\frac{c_1 + c_3 - 1}{a_1}, \frac{2c_1 + c_3}{c_2}, \frac{-2\varepsilon + c_5}{a_2}, \frac{-2\varepsilon + 2c_5}{c_6} \right]. \tag{3.3}$$

Let's denote by

$v_{max} = \max\{|c_1 + c_3 - 1|, |2c_1 + c_3|, |-2\varepsilon + c_5|, |-2\varepsilon + 2c_5|\}$,

and

$v_{min} = \min\{|c_1 + c_2 + 1 + dx \cdot h|, |c_2|, |a_2|, |c_6|\}$.

Now we are able to prove the following result with respect to the stability in matrix equation (3.1).

Proposition 3.1. *Suppose that $v_{min} - v_{max} > 0$. If one of the following conditions is true:*

i) $\rho c + \frac{\ell}{2} > \tau$ & $\dfrac{\rho c + \frac{\ell}{2}}{v_{min} - v_{max}} < 1$

or

ii) $\rho c + \frac{\ell}{2} \le \tau$ & $\dfrac{\tau}{v_{min} - v_{max}} < 1$,

then the equation (3.1) *is stable. Otherwise, it is unstable.*

Proof. The proof is reduced to checking the condition of stability which, based on the Lax-Richtmyer definition mentioned above and taking into account the relation (3.1), it reduces to check the inequality

$$\|A^{-1}B\|_\infty < 1.$$

We begin our analyse by determining an estimate for $\|D^{-1}G\|_\infty$. As we have already noted (see relation (3.3)), this is equivalent with the following equality: $\|D^{-1}G\|_\infty = \max|v|$, wherefrom we easily derive the estimate

$$\|D^{-1}G\|_\infty < \frac{v_{max}}{v_{min}}. \tag{3.4}$$

The estimate (3.4) allows us now to prove the existence of A^{-1}. Indeed, since by hypothesis we have assumed that $v_{max} < v_{min}$ than $\|D^{-1}G\|_\infty < 1$ which guarantees that there exist $(I + D^{-1}G)^{-1}$. Moreover, there exist A^{-1} and $A^{-1} = (I + D^{-1}G)^{-1}D^{-1}$. Using the well known inequality: $\|(I + D^{-1}G)^{-1}\|_\infty \le \frac{1}{1 - \|D^{-1}G\|_\infty}$ and making use of (3.2), it follows that

$$\|A^{-1}\|_\infty \le \|(I + D^{-1}G)^{-1}\|_\infty \|D^{-1}\|_\infty \le \frac{1}{1 - \|D^{-1}G\|_\infty}\|D^{-1}\|_\infty. \tag{3.5}$$

How $\|D^{-1}G\|_\infty \le 1$ imply that $1 - \|D^{-1}G\|_\infty \ge 1 - \frac{v_{max}}{v_{min}} > 0$, we easily deduce from this that

$$0 < \frac{1}{1 - \|D^{-1}G\|_\infty} \le \frac{v_{min}}{v_{min} - v_{max}}.$$

Since $\|D^{-1}\|_\infty \le \frac{1}{v_{min}}$ and involving the above estimate, from (3.5) we finaly obtain

$$\|A^{-1}\|_\infty < \frac{1}{v_{min} - v_{max}}. \tag{3.6}$$

Now we turn our attention to matrix B. Analyzing the matrix B lines, it follows that

$$\|B\|_\infty = \max\left\{\rho c + \frac{\ell}{2}, \tau\right\}. \tag{3.7}$$

Summing up and making use of (3.6)-(3.7) we derive the following estimate

$$\|A^{-1}B\|_\infty \le \|A^{-1}\|_\infty \|B\|_\infty < \frac{1}{v_{min} - v_{max}} \|B\|_\infty,$$

which, in either cases i) or ii) leads us to the estimate $\|A^{-1}B\|_\infty < 1$ as we claimed at begining of proof. ∎

Remark 3.1. *Concerning the stability of the linear system* (2.17) *we can finding easily that the conditions* i), ii) *in* Proposition 3.1 *are kept and,*

$v_{max} = \max\{|2c_1 + 3c_3 - 1|, |2c_1 + 3c_3|, |-4\varepsilon + 2c_5|, |-4\varepsilon + 4c_5|\}$,

$v_{min} = \min\{|2c_1 + c_7 + 1 + dx \cdot h|, |c_7|, |2c_5 + c_8|, |c_8|\}$,

while, for the linear system (2.20) *the parameter* τ *in conditions* i), ii) - Proposition 3.1*, must be replaced with* $2\varepsilon + \tau + \frac{\varepsilon}{2a}$ *and,*

$v_{max} = \max\{|2c_1 + 3c_3 - 1|, |2c_1 + 3c_3|, |2c_5|, |4c_5|\}$,

$v_{min} = \min\{|2c_1 + c_7 + 1 + dx \cdot h|, |c_7|, |2c_5 + c_9|, |c_9|\}$.

4. NUMERICAL EXPERIMENTS

The aim of this Section is to present numerical experiments implementing the conceptual algorithms **Alg_1-IMBDF**, **Alg_2-IMBDF** and **Alg_2-SBDF**. Corresponding to input data T, b, M, N, we have used several different values while, for the model's parameters we have considered industrial values, which are:

- the casting speed (c = 12.5 mm/s),
- physical parameters:
  - the density (ρ = 7.85 kg/m^3),
  - the latent heat ($\ell = 65.28\ kcal/kg$),
  - the termal conductivity ($k = 7.8e - 2$),
  - the length of separating zone ($\xi = .5$),
  - the relaxation time ($\tau = 1.0e + 3 * \xi^2$),

- the coefficients of heat transfer ($h = 32.012$),
- $a = \sqrt{\xi}$;

The initial values $\varphi_0(x_j)$, $j = 1, 2, ..., N$, ploted in Figure 4.1 - left side, were computed via Matlab function `csapi(fi0)` - cubic spline interpolant to the given data:

```
fi0=[-1.4 -1.4 -1.44 -1.42 -1.42 -1.44 -1.43 -1.43 -1.42 -1.42 -1.4 -1.4 -1.25 -1.2 -1.17 -1.15 ...
-1.1 -1.08 -1.0 -.95 -.9 -.85 -.88 -.6 .0 .5 -.92 -.25 .8 -.7 .58 .75 .58 -.63 -.59 .69 -.72 .7 -.59 -.5 ...
.7 -.79 -.87 -.88 .0 .72 -.8 .81 .0 -.89 .0 .7 .55 .68 -.49 .79 .0 -.1 -.8 -.78 -.83 .69 -.8 .68 .5 .7 ...
.59 1. 1.08 1.1 1.15 1.17 1.2 1.25 1.3 1.3 1.25 1.24 1.3 1.31 1.3 1.32 1.3 1.3];
```

The initial values $u_0(x_j)$, $j = 1, 2, ..., N$, ploted in Figure 1 - right side, were computed as solution of the discrete form to the stationary equation $(2a)^{-1}[\varphi_0(x) - \varphi_0^3(x)] + 2u_0(x) = 0$ (see Caginalp [3]), i.e.:

$$(2a)^{-1}[\varphi_0(x_j) - (\varphi_0(x_j)^3)] + 2u_0(x_j) = 0 \quad j = 1, 2, ..., N.$$

Now (see $(2.4)_1$) we are able to calculate the vector $\left(z\left(\varepsilon, \varphi_0(x_j)\right)\right)_{j=\overline{1,N}}$, ploted in Figure 2, and the vectors: $\varphi^1 = \left(\varphi_j^1\right)_{j=\overline{1,N}}$ and $u^1 = \left(u_j^1\right)_{j=\overline{1,N}}$ (see relations (2.3), $(1.6)_2$ and (2.4)). As the schemes 2-IMBDF and 2-SBDF involves three time levels, we consider at the first time level $i := 0$ the values $u^0 = u^1$ and $\varphi^0 = \varphi^1$. Consequently, the right side of the linear systems (2.17) and (2.20), corresponding to the first iteration of the cycle "`for`" in algorithms **Alg_2-IMBDF** and **Alg_2-SBDF** ($i = 2$), become:

$3B\begin{pmatrix} u^1 \\ \varphi^1 \end{pmatrix} + \begin{pmatrix} d_1^2 \\ d_2^2 \end{pmatrix}$ and $\begin{pmatrix} 3A_{13} & 3A_{12} \\ 4A_{21} & Y_{22} - Z_{22} \end{pmatrix}\begin{pmatrix} u^1 \\ \varphi^1 \end{pmatrix} + \begin{pmatrix} d_1^2 \\ d_2^2 \end{pmatrix}$, respectively.

We will continue with the presentation of numerical experiments regarding the *stability* of equation (3.1) (see Proposition 3.1). The shape of the graphs ploted in Figures 3 and 4 shows the stability and accuracy of the numerical results obtained by algorithm **Alg_1-IMBDF**. For this test we have used $T = 2$, $b = 1$, $M = 100$, $N = 40$ and the temperature surrounding at $\partial\Omega = \{0, b\}$ given by: $w(t_i, 0) = -15$, $w(t_i, b) = 7.5$, $i = 1, 2, ..., M$.

Taking now *k=.785*, we can verify that $v_{min} - v_{max}$=*-15.2372* which means that the first hypothesis in Proposition 3.1 in not verified. Consequently the numerical scheme is unstable. Figure 5 shows that it really is. Furthemore, if we keep *k=.785* and take τ=*1.0e+2**ξ^2 (in place of τ=*1.0e+3**ξ^2), we get also $v_{min} - v_{max} < 0$. So, again we are in a unstable case. Moreover, analyzing the graph in Figure 6 we found a more pronounced instability. Let's remark that the instability of the solution occurred following a slight change (modification) of only two physical parameters (k and τ in this case). This highlights the strong dependence of approximation scheme regarding physical parameters.

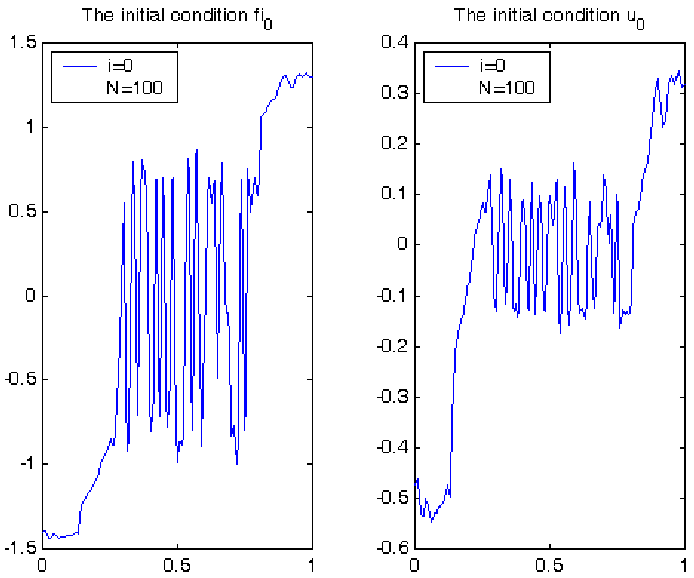

Fig. 1. The initial conditions φ_0 and u_0

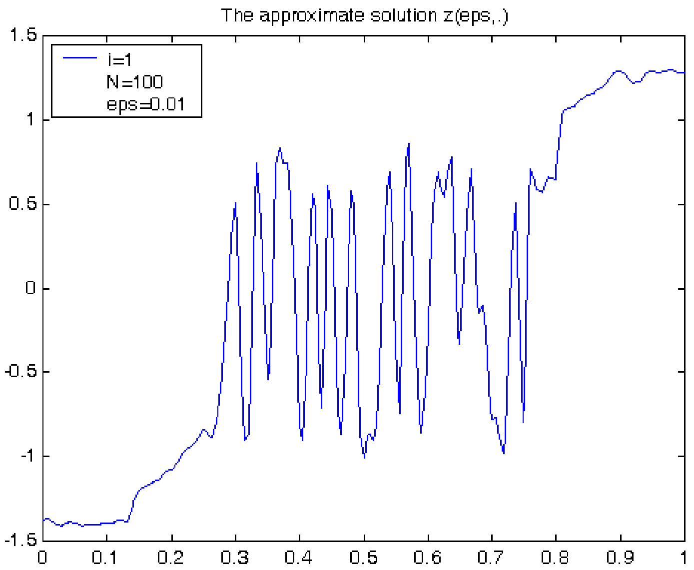

Fig. 2. The approximate solution $z(\varepsilon, \cdot)$ of Cauchy problem (1.7)

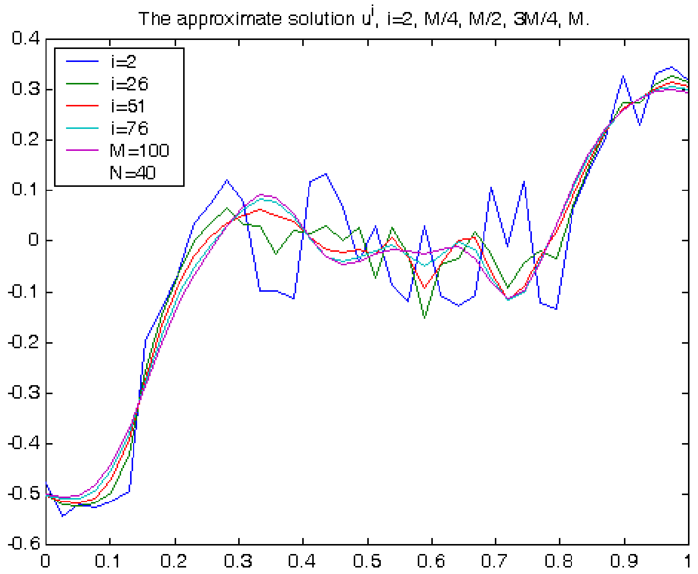

Fig. 3. Example of numerical stability: u^i at different levels of time

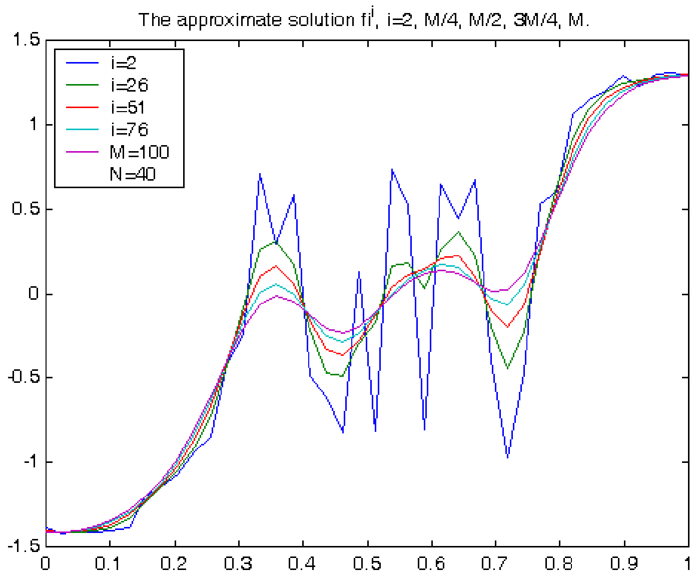

Fig. 4. Example of numerical stability: φ^i at different levels of time

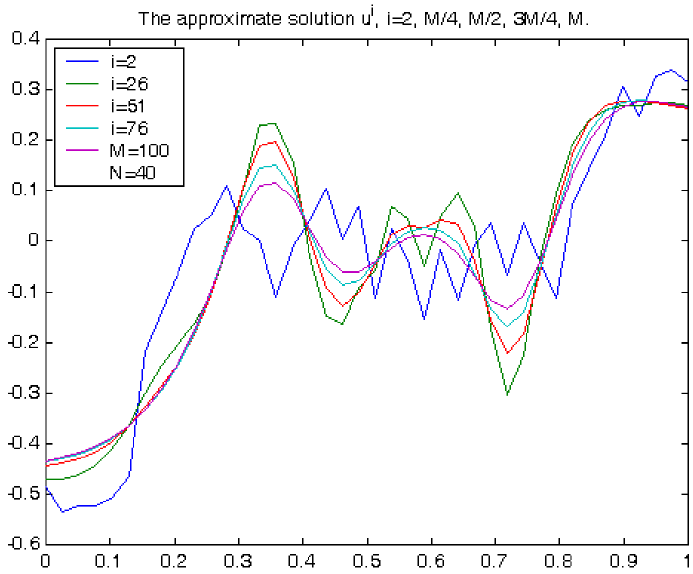

Fig. 5. An example of slight numerical instability

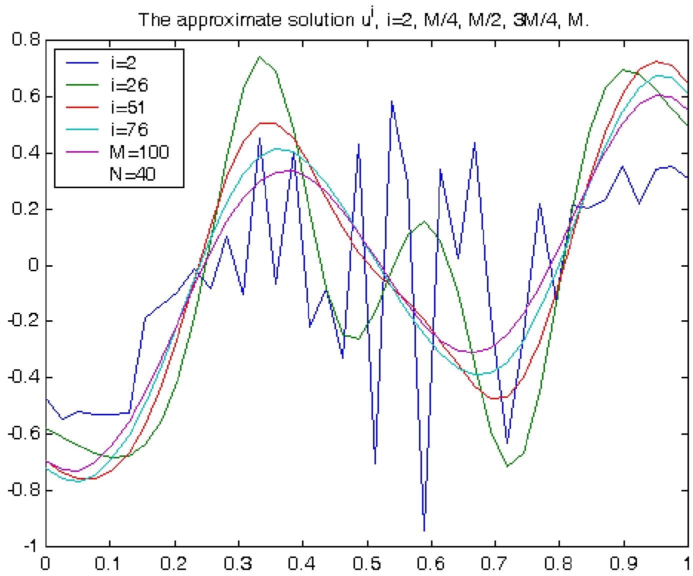

Fig. 6. An example of strong numerical instability

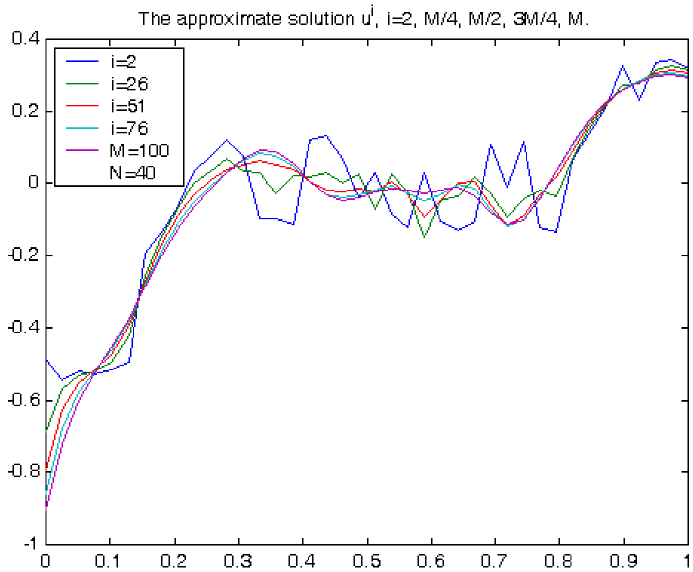

Fig. 7. u^i corresponding to $w^i(0) = -60$, via Alg_1-IMBDF

We turn to numerical stability conditions and we change the temperature surrounding at $0 \in \partial\Omega$ by setting $w(t_i, 0) = -60$, $i = 1, 2, ..., M$. The numerical results, obtained by algorithms **Alg_1-IMBDF** and **Alg_2-SBDF**, were ploted in Figures 7 and Figure 8 below, respectively. Analyzing the approximations near to zero, we observe a instability just for u, due to the nature of boundary conditions that we have considered $(1.2)_1$. In addition we also find a difference in the error of approximation.

On stability, we mention that similar results were also obtained by implementing the algorithms **Alg_2-IMBDF** and **Alg_2-SBDF**. In this sense, we reproduce in Figure 9 the numerical result obtained by Alg_2-IMBDF, executed with the same values as in Alg_1-IMBDF (see Figure3).

5. CONCLUSIONS

As the novelty of this work we notice the use of three finite difference schemes in order to approximate the linear system given by a scheme of fractional steps type. Even if each brings particularities in the implementation (memory space required, the right side), executed in the same conditions, produced essentially the same numerical results (see figures 3 and 9). Not least, let's remark that conditions of stability are sustained by both theory and numerical experiment and that are significantly dependent on the physical parameters.

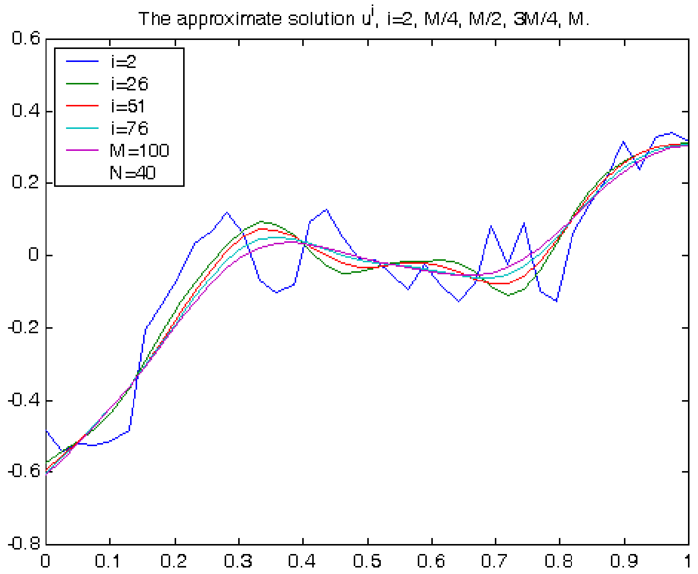

Fig. 8. u^i corresponding to $w^i(0) = -60$, via Alg_2-SBDF

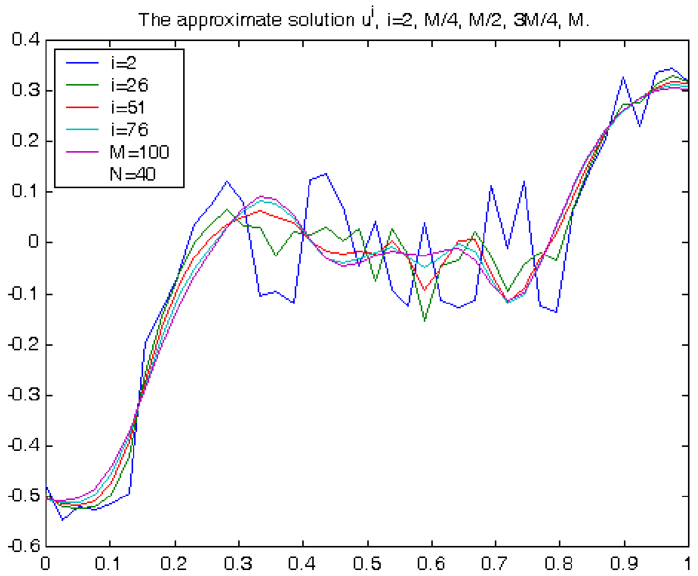

Fig. 9. u^i obtained by Alg_2-IMBDF

Analyzing the numerical results in terms of physical phenomena, we constat that *the temperature distribution* tends to become parabolic and *the phase function* distribution say that the instability of the portion of material will disappear. Moreover, analyzed together (see figures 3 and 4, for example), highlight theoretical meaning assigned to functions u and φ as well as the *zone of separation* between material phases.

The numerical solution obtained by this way can be considered as an admissible one for the corresponding boundary optimal control problem (from this perspective, compare figures 7 and 8). Generally, the numerical method considered here can be used to approximate the solution of a nonlinear parabolic phase-field system containing a general nonlinear part.

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ON THE SYMBOL OF SINGULAR INTEGRAL OPERATORS WITH COMPLEX CONJUGATION

Vasile Neagu

Tiraspol State University, Chişinău, Republic of Moldova

vasileneagu45@gmail.com

Abstract In this paper the symbol of singular integral operator with complex conjugation is constructed. Noetherien conditions and index of operators are expressed by the determinant of its symbol.

Keywords: singular integral operator, Noetherian operators, Riemann boundary value problems, spaces with weights.

2010 MSC: 45E05, 45E10.

1. INTRODUCTION

Let Γ be an orientated, closed and of piecewise Lyapunov type contour, which divides the complex plane in domains F^+ and F^- $(\infty \in F^-)$, $t_1, \ldots, t_n$ angular points of Γ with angles θ_k, formed by lateral tangents to Γ in these points. In the space $L_p(\Gamma, \rho)$ consider the operator

$$\begin{aligned}(A\varphi)(t) = a_1(t)\varphi(t) + a_2(t)\frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - t}d\tau + \\ + a_3(t)\bar{\varphi}(t) + a_4(t)\frac{1}{\pi i}\int_\Gamma \frac{\bar{\varphi}(\tau)}{\tau - t}d\tau,\end{aligned} \tag{1}$$

where $\rho(t) = \prod_{k=1}^{n} |t - t_k|^{\beta_k}$ $(-1 < \beta_k < p - 1)$ and $a_j(t)$ $(j = 1, 2, 3, 4)$ are continuous functions in every point $t \in \Gamma$ excepting points t_k $(k = 1, \ldots, n)$ in which there exist finite limits $(a(t_k \pm 0)$. In what will follow it is comfortable to write operator (1) in other form. With this purpose we do the following notations:

$$\begin{aligned}&a_1(t) + a_2(t) = a(t), \quad a_1(t) - a_2(t) = b(t),\\ &a_3(t) + a_4(t) = c(t), \quad a_3(t) - a_4(t) = d(t),\\ &(V\varphi)(t) = \overline{\varphi}(t), \quad P = (I + S)/2, \quad Q = I - P,\end{aligned}$$

where S is singular integral operator with Cauchy kernel. With these notations operator (1) is written in the form

$$A = aP + bQ + (cP + dQ)V. \tag{2}$$

The operator A becomes linear in the space $L_p(\Gamma, \rho)$ if this space is considered over the field of real numbers. Denote it by $\tilde{L}_p(\Gamma, \rho)$.

In the case of Lyapunov contour operator A was studied in monograph [1] see also the bibliography of this work. In determining Noetherian conditions for operator (2) an important role played the fact that in the case of contour of Lyanupov type the operator $VSV + S$ is compact in the space $L_p(\Gamma, \rho)$. As it is shown in this work, if contour Γ has angular points, then operator $VSV + S$ is not more compact and reasoning from above mentioned works cannot be applied. Moreover, it turns out that the very Noetherian conditions for operator A depend also of measures of angles on the contour Γ.

In this work the symbol of singular integral operator with complex conjugation of form (2) is constructed. It is proved that the symbol is a matrix of variable order: in points t_k $(k = 1, \dots, n)$ of forth order, but in the other pointes this order is equal to two. The symbol depends also of coefficients of the operator, of the space $L_p(\Gamma, \rho)$ and of the measures of angles on the contour of integration. Noetherien conditions and index of operator A are expressed by the determinant of its symbol. We establish certain relations between operators of form (2) and boundary problems of Riemann type [1], [2], [3] for analytic functions.

Similar results are obtained also for operators $\sum_{j=1}^{m} \prod_{k=1}^{r} A_{jk}$, where A_{jk} are operators of form (2).

2. PROPERTIES OF OPERATOR $VSV + S$

Theorem 2.1. *Let Γ be a closed contour of Lyapunov type. Then operator $VSV + S$ is compact in the space $L_p(\Gamma, \rho)$*

Proof. Denote by Γ_0 the unit disc ($\Gamma_0 = \{t : |t| = 1\}$) and by S_0 operator S_{Γ_0}. Then

$$(VS_0V + S_0)\varphi = -\frac{1}{\pi i}\int\limits_{\Gamma_0} \frac{\varphi(\tau) d\bar{\tau}}{\bar{\tau} - \bar{t}} + \frac{1}{\pi i}\int\limits_{\Gamma_0} \frac{\varphi(\tau)}{\tau - t} d\tau = \frac{1}{\pi i}\int\limits_{\Gamma_0} \frac{\varphi(\tau)}{\tau} d\tau.$$

Thus, if Γ is the unit disc, then $VS_0V + S_0$ is compact in $L_p(\Gamma, \rho)$. We shall consider the case in which Γ is any closed Lyapunov contour. Let $\nu : \Gamma_0 \to \Gamma$ be a map which verifies conditions: there exists the derivative $\nu'(t)$ not equal to zero and $\nu'(t)$ verifies Hölder conditions. Denote $B : (L_p(\Gamma, \rho) \to (L_p(\Gamma_0, \rho_0)$, where

$$\rho_0(z) = \prod_{k=1}^{n} |\nu(z) - \nu(z_k)|^{\beta_k} \ (\nu(z_k) = t_k)$$

where operator defined by relation $(B\varphi)(z) = \varphi(\nu(z))$ $(z \in \Gamma_0)$. Then

$$(BSB^{-1} - S_0)\varphi = \frac{1}{\pi i}\int\limits_{\Gamma_0} \Big(\frac{\nu'(\xi)}{\nu(\xi) - \nu(z)} - \frac{1}{\xi - z}\Big)\varphi(\xi) d\xi. \tag{3}$$

Since $\nu'(\xi)$ is not equal to zero and satisfies Hölder conditions, operator $BSB^{-1} - S_0$ has (see [4]) weak singularity on $\Gamma_0 \times \Gamma_0$ and, hence is compact in the space $L_p(\Gamma, \rho)$.

As operators V and B commute, from (3) and from what was already proved we obtain that operator

$$B(VSV + S)B^{-1} - VS_0V - S_0 \tag{4}$$

is compact in $L_p(\Gamma,\rho)$, from which it results that $VSV+S$ is also compact in $L_p(\Gamma,\rho)$. The theorem is proved. ∎

Let us show that assertions of Theorem 2.1 are false if contour Γ has angular points. Suppose for example, that $\Gamma \supset \Gamma_1 \cup \Gamma_2$, where Γ_1 and Γ_2 are segments of straight line which joins point $z = 0$ with $z = 1$ and, respectively $z = 0$ with $z = i$. In point $z = 0 \in \Gamma$ contour forms an angle of measure $\pi/2$. We shall show that in this case operator $VSV + S$ is not compact in $L_2(\Gamma)$. Suppose, by absurd, that $VSV + S \in \mathbf{T}(L_2(\Gamma))$. Let X be characteristic function of Γ_2 and $M = X(VSV + S)$. We will show that $M \notin \mathbf{T}(L_2(\Gamma))$ and as a result we shall obtain a contradiction.

Consider in the space $L_2(\Gamma)$ sequence $\{\varphi_n(t)\}$ of functions defined by relations

$$\varphi_n(t) = \begin{cases} \sqrt{n}, & \text{for } t \in [0, \frac{1}{n}] \\ 0, & \text{for } t \in \Gamma \setminus [0, \frac{1}{n}]. \end{cases}$$

We have $\|\varphi_n\|_{L_2(\Gamma)} = 1$. We will show that from the sequence $\psi_n = M\varphi_n$ is not possible to extract any convergent subsequence. By the definition of operator M we have

$$(M\varphi_n)(t) = X(t)(VSV + S)\varphi_n = \frac{X(t)\sqrt{n}}{\pi i} \int_0^{1/n} \Big(\frac{1}{\tau - t} - \frac{1}{\tau - \bar{t}}\Big) d\tau =$$

$$= \frac{X(t)}{\pi i} \sqrt{n} \int_0^{1/n} \frac{t - \bar{t}}{(\tau - t)(\tau - \bar{t})} d\tau.$$

Therefore,

$$\|M\varphi_n\|^p_{L_p(\Gamma)} = \frac{n^{p/2}}{\pi^p} \int_\Gamma \Big| \frac{t - \bar{t}}{(\tau - t)(\tau - \bar{t})} d\tau \Big|^p |dt| =$$

$$= c_p n^{p/2} \int_0^1 \Big| arctg \frac{1}{nt} \Big|^p dt \le c_p n^{(p-2)/2} \int_0^n \Big(arctg \frac{1}{t} \Big)^p dt \le$$

$$\le c_p n^{(p-2)/2} \int_0^\infty \Big(arctg \frac{1}{t} \Big)^p dt = \tilde{c}_p n^{(p-2)/2},$$

where c_p and $\tilde{c}_p$ are constants depending only of p. Hence it results that

$$\lim_{n\to\infty} \|M\varphi_n\|_{L_p(\Gamma)} = 0, \quad \text{for } \ 1 < p < 2.$$

Thus if the sequence $\psi_n = M\varphi_n (\in L_2(\Gamma))$ would contain a convergent subsequence, then this subsequence necessarily would converge to zero. But

$$\|\psi_n\|^2_{L_2(\Gamma)} = \|M\varphi_n\|^2_{L_2(\Gamma)} \geq \tilde{\tilde{c}}_p \int_0^n arctg^2 \frac{1}{t} dt \geq \tilde{\tilde{c}}_p \int_0^1 arctg^2 \frac{1}{t} dt > 0,$$

from which it results that $\{\psi_n\}$ in the space $L_2(\Gamma)$ does not contain any convergent subsequence. Therefore operator M is not compact in the space $L_2(\Gamma)$.

3. NOETHERIEN CRITERIONS

Conditions in which operator of the form (2) (and more complicated operators) are of Noether type are expressed with the help of symbol. That is why we shall firstly define the symbol of operators aI, P, Q and V. Denote by $a(t,\xi)$, $P(t,\xi)$, $Q(t,\xi)$ and $V(t,\xi)$ $(t \in \Gamma,\ -\infty \leq \xi \leq \infty)$ the symbols of these operators respectively. Put

$$a(t,\xi) = \begin{cases} \left\| \begin{matrix} a(t) & 0 \\ 0 & a(t) \end{matrix} \right\|, & \text{for } t \in \Gamma \backslash \{t_1, \dots, t_n\} \\ \left\| \begin{matrix} a(t_k+0) & 0 & 0 & 0 \\ 0 & \overline{a(t_k+0)} & 0 & 0 \\ 0 & 0 & a(t_k-0) & 0 \\ 0 & 0 & 0 & \overline{a(t_k-0)} \end{matrix} \right\|, & (k = 1, \dots, n); \end{cases} \tag{5}$$

$$P(t,\xi) = \begin{cases} \left\| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right\|, & \text{for } t \in \Gamma \backslash \{t_1, \dots, t_n\}, \\ \frac{1}{z_k^{2\pi}-1} \left\| \begin{matrix} z_k^{2\pi} & 0 & -z_k^{\theta_k} & 0 \\ 0 & -1 & 0 & z_k^{2\pi-\theta_k} \\ z_k^{2\pi-\theta_k} & 0 & -1 & 0 \\ 0 & -z_k^{\theta_k} & 0 & z_k^{2\pi} \end{matrix} \right\|, & (k = 1, \dots, n), \end{cases} \tag{6}$$

where $z_k = \exp(\xi + i\frac{1+\beta_k}{p})$ $(-\infty \leq \xi \leq +\infty)$; $Q(t,\xi) = E(t) - P(t,\xi)$, where

$$E(t) = \begin{cases} \left\| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right\|, & \text{for } t \in \Gamma \backslash \{t_1, \dots, t_n\}, \\ \left\| \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \right\|, & \text{for } t = t_k \ (k = 1, \dots, n). \end{cases} \tag{7}$$

Finally

$$V(t,\xi) = \begin{cases} \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix}, & \text{for } t \in \Gamma \backslash \{t_1, \dots, t_n\}, \\ \begin{Vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{Vmatrix}, & \text{for } \ t = t_k \ (k = 1, \dots, n). \end{cases} \tag{8}$$

If operator A has form (2), then we define its symbol, $A(t,\xi)$, by relation

$$\begin{aligned} A(t,\xi) = a(t,\xi)P(t,\xi) + b(t,\xi)Q(t,\xi) + \\ + (c(t,\xi)P(t,\xi) + d(t,\xi)Q(t,\xi))V(t,\xi). \end{aligned} \tag{9}$$

If $a, b, c, d \in CP_m(\Gamma)$, then the symbol of operator $A \in L(L_p^m(\Gamma,\rho))$ is defined by relation (9), in which $a(t,\xi)$, $b(t,\xi)$, $c(t,\xi)$ and $d(t,\xi)$ are, respectively, matrices of order m, defined by equalities (5).

Theorem 3.1. *The operator*

$$A = aP + bQ + (cP + dQ)V$$

($a, b, c, d \in CP(\Gamma)$) *is Noetherian in the space* $\tilde{L}_p(\Gamma,\rho)$, *if and only if* $\det A(t,\xi) \neq 0 \quad (t \in \Gamma, \ -\infty \leq \xi \leq \infty)$.

Beforehand we will prove two lemmas.

Lemma 3.1. *The operator A is Noetherian in the space $\tilde{L}_p(\Gamma,\rho)$, if and only if in the space $\tilde{L}_p^2(\Gamma,\rho) = \tilde{L}_p(\Gamma,\rho) \times \tilde{L}_p(\Gamma,\rho)$ the operators*

$$\tilde{A} = \begin{Vmatrix} aP + bQ & cP + dQ \\ \bar{c}VPV + \bar{d}VQV & \bar{a}VPV + \bar{b}VQV \end{Vmatrix} \tag{10}$$

is Noetherian. Moreover, $IndA = \frac{1}{2}Ind\tilde{A}$.

Proof. The identity

$$\begin{Vmatrix} X + YW & 0 \\ 0 & X - YW \end{Vmatrix} = \frac{1}{2} \begin{Vmatrix} I & W \\ I & -W \end{Vmatrix} \begin{Vmatrix} X & Y \\ WYW & WXW \end{Vmatrix} \begin{Vmatrix} I & I \\ W & -W \end{Vmatrix}, \tag{11}$$

is true [5], where X, Y, W are any linear and bounded operators which act in Banach space B and $W^2 = I$.

Put in identity (11) $X = aP + bQ, \quad Y = cP + dQ$, then

$$\tilde{A} = H \begin{Vmatrix} A & 0 \\ 0 & A_1 \end{Vmatrix} H^{-1}, \tag{12}$$

where

$$A_1 = aP + bQ - (cP + dQ)V, \quad H = \frac{1}{2}\left\| \begin{matrix} I & I \\ V & -V \end{matrix} \right\|.$$

Let $(M\varphi)(t) = i\varphi(t)$, then one can immediately verify that $MA_1M^{-1} = A$ and assertions of lemma follow from equality (12). ∎

Remark 3.1. *Let* Γ *be of Lyapunov type, then by Theorem 2.1 we have* $VSV = S + T_1$ *and, hence,* $VPV = Q + T_2$, $VQV = P + T_3$, *where* $T_j \in T(\tilde{L}_p(\Gamma, \rho))$ $(j = 1, 2, 3)$. *From this it results that operator* $\tilde{A}$ *differs from operator*

$$\tilde{A}_0 = \left\| \begin{matrix} a & c \\ \bar{d} & \bar{b} \end{matrix} \right\| P + \left\| \begin{matrix} a & c \\ \bar{c} & \bar{a} \end{matrix} \right\| Q$$

by a compact term.

Operator $\tilde{A}_0$ is a singular integral operator with piecewise continuous matrix entries. For this operators conditions under which they are of Noether type are known (see [5]). These conditions consists in the fact that $\det \tilde{A}_0(t, \xi) \neq 0$ for every $(t, \xi) \in \Gamma \times \bar{R}$. One can observe than in this case $\det \tilde{A}_0(t, \xi)$ coincides with $\det A(t, \xi)$, defined by equality (9).

From Lemma 3.1 it follows

Corollary 3.1. *Let* Γ *be of Lyapunov type. For the operator A to be Noetherian it is necessary and sufficient that*

$$\det A(t, \xi) \neq 0 \ \ (t \in \Gamma, \ -\infty \leq \xi \leq \infty).$$

Lemma 3.2. *The operator A is locally Noetherian in point* $t_0 \in \Gamma \backslash \{t_1, ..., t_n\}$, *if and only if* $\det A_0(t, \xi) \neq 0 \ \ (-\infty \leq \xi \leq \infty)$.

Proof. Denote by $u(t_0) \subset \Gamma \backslash \{t_1, \ldots, t_n\}$ a neighborhood of the point t_0. Let $\tilde{\Gamma}$ be a closed Lyaponov contour which contains the neighborhood $u(t_0)$. In the space $\tilde{L}_p(\Gamma)$ consider the operator

$$B = \tilde{a}\tilde{P} + \tilde{b}\tilde{Q} + (\tilde{c}\tilde{P} + \tilde{d}\tilde{Q})V,$$

where $\tilde{P} = (I + S_{\tilde{\Gamma}})/2$, $\tilde{Q} = I - \tilde{P}$ and $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$ are continuous functions on $\tilde{\Gamma}$ restrictions of which on $u(t_0)$ coincide with functions a, b, c, d. Obvious operators A and B are quasi equivalent in point t_0. Hence, both are locally Noetherian in point t_0. By Corollary 3.1, condition $\det B(t_0, \xi) \neq 0$ $(\xi \in \bar{R})$ is necessary and sufficient for B to be locally Noetherian in point t_0. Since $\det B(t, \xi) = \det A(t_0, \xi)$, lemma is proved. ∎

Now we can give the proof of the theorem.

Proof. By Lemma 3.2, it is sufficient to show that condition $\det A(t_k, \xi) \neq 0 \ \ (-\infty \leq \xi \leq \infty)$ is necessary and sufficient for the operator A to be Noetherian in point t_k $(k = 1, \ldots, n)$.

To begin with suppose that $n = 1$. Together with contour Γ consider contour $\Gamma_1(= \Gamma_{\theta_1})$ which also has an unique singular point z_1 of the some measure $\theta = \theta(t_1)$ with the property that if $z \in \Gamma_1$, then also the point $\bar{z} \in \Gamma_1$. Hence, $z_1 = 0$. Then there exists a map $\mu : \Gamma_1 \to \Gamma$, so that $\mu'(t) \neq 0$ $(t \in \Gamma_1)$ and satisfies Hölder conditions. Denote by $B : L_p(\Gamma, \rho) \to L_p(\Gamma_1, \rho_1)(\rho(t) = |t - t_1|^{\beta_1},\ \rho_1(t) = |z|^{\beta_1})$ the operator

$$(B\varphi)(z) = \varphi(\mu(z)).$$

We have $BaB^{-1} = a_1 I$ $(a_1(z) = a(\mu(z)))$, $BVB^{-1} = V$ and $BSB^{-1} = S_1 + T_1$, where $S_1 = S_{\Gamma_1}$ and $T_1 \in T(L_p(\Gamma_1, \rho_1))$. Taking this into account we get

$$\tilde{B}\tilde{A}\tilde{B}^{-1} = \left\| \begin{array}{ll} a_1 P_1 + b_1 Q_1 & c_1 P_1 + d_1 Q_1 \\ \bar{c} V P_1 V + \bar{d}_1 V Q_1 V & \bar{a} V P_1 V + \bar{b}_1 V Q_1 V \end{array} \right\| + T, \tag{13}$$

where

$$\tilde{B} = \left\| \begin{array}{cc} B & 0 \\ 0 & B \end{array} \right\|, \quad P_1 = (I + S_1)/2,\ Q_1 = I - P_1 \text{ and } T \in \mathbf{T}(L^2_p(\Gamma_1, \rho_1)).$$

Denote by W the operator of shift, defined by relation $(W\varphi)(z) = \varphi(\omega(z))$, where $\omega(z) = \bar{z}$ $(z \in \Gamma_1)$. We observe that the derivative $\omega'(z)$ is discontinuous in point $z = 0$, and $\omega'(+0) = \exp(i\theta_1)$, $\omega'(-0) = \exp(-i\theta_1)$. It is easy to verity that

$$VS_1V = WS_1W \tag{14}$$

Substituting (14) in (13) and using Lemma 3.1, we obtain that operator A is locally Noetherian in point t_1, if and only if the operator

$$M_1 = \left\| \begin{array}{ll} a_1 P_1 + b_1 Q_1 & c_1 P_1 + d_1 P_1 \\ \bar{c}_1 W P_1 W + \bar{d}_1 W Q_1 W & \bar{a}_1 W P_1 W + \bar{b}_1 W Q_1 W \end{array} \right\|$$

has this property in point $z = 0$.

The operator M_1 is a singular integral operator with shift W studied in the work [7]. From this work we get that operator M_1 is locally Noetherien in point $z = 0$, if and only if $\det M_1(0, \xi) = 0$ $(\xi \in \bar{R})$. Since $\det A(t_1, \xi) = \det M_1(0, \xi)$ it follows that for $n = 1$ the theorem is proved.

Pass to the general case. Let $u_k = u(t_k)$ $(\subset \Gamma)$ be a neighborhood of point t_k containing no points $t_j \neq t_k$. As before consider contour Γ_k $(= \Gamma_{\theta_k})$ with a single angular point $z = 0$ with condition that together with every point z it contains the point $\bar{z}$ two. Denote by μ_k a map from a neighborhood u_k to a neighborhood $v_k = \mathbf{v}_k(0)$ $(\subset \Gamma_k)$ and, moreover, $\mu_k(t_k) = 0$. Since Γ and Γ_k in point t_k and respectively $z = 0$ form angles of the same measure θ_k, then μ_k may be chosen in such a way that $\mu'_k(t) \neq 0$ $(t \in u(t_k))$ and this derivative to satisfy Hölder conditions. If $f \in CP(\Gamma)$, then we agree to denote by $f_k(z)$ $(z \in v_k(0))$ the function $f(\mu_k^{-1}(z))$, where μ_k^{-1} is the inverse of μ_k. Extend functions $a_k(z)$, $b_k(z)$, $c_k(z)$, $d_k(z)$ by continuity on contour Γ_k and denote then by the same letters.

In the space $L_p^2(\Gamma_k, |z|^{\beta_k})$ consider the operator

$$M_k = \left\| \begin{array}{ll} a_k P_k + b_k Q_k & c_k P_k + d_k Q_k \\ \bar{c}_k W P_k W + \bar{d}_k W Q_k W & \bar{a}_k W P_k W + \bar{b}_k W Q_k W \end{array} \right\|,$$

where $(W\varphi)(z) = \varphi(\bar{z})$ and $S_k = S_{\Gamma_k}$. The operator $\tilde{A}$, defined by relation (12), is quasiequivalent in point t_k to operator M_k in point $z = 0$:

$$T_{\mu_k} P_{u_k} \tilde{A} P_{u_k} T_{\mu_k^{-1}} \overset{o}{\sim} P_{\mathbf{v}_k} M_k P_{\mathbf{v}_k},$$

where

$$(P_F\varphi)(t) = \begin{cases} \varphi(t), & t \in \Gamma, \\ 0, & t \in \Gamma \backslash F, \end{cases} \qquad (T_\varphi f)(t) = \begin{cases} f(\varphi(t)), & t \in u, \\ 0, & t \in \Gamma \backslash u. \end{cases}$$

Therefore, $\tilde{A}$ and M_k are locally Noetherian operators ($\tilde{A}$ in point t_k and M_k in $z = 0$). By Theorem 3.1, the operator M_k has this propriety if and only if $\det M_k(0, \xi) \neq 0$ $(\xi \in \bar{R})$. It remains to convince ourselves that $\det M_k(0, \xi) = \det A(t, \xi)$ and the theorem is proved. ∎

As a consequences one can formulate the following result.

Theorem 3.2. *Let functions a, b, c and d belong to the set $CP_m(\Gamma)$. Operator*

$$A = aP + bQ + (cP + dQ)V$$

is Noetherian in the space $L_p^m(\Gamma, \rho)$, if and only if

$$\det A(t, \xi) \neq 0 \ (t \in \Gamma, \ \xi \in \bar{R}).$$

Let operator A have the form

$$A = \sum_{j=1}^{r} A_{j1} A_{j2} \cdots A_{js},$$

where

$$A_{jk} = a_{jk} P + b_{jk} Q + (c_{jk} P + d_{jk} Q)V \ \ (a_{jk},\ b_{jk},\ c_{jk},\ d_{jk} \in CP_m(\Gamma)).$$

Define the symbol of operator A as follows

$$A(t, \xi) = \sum_{j=1}^{r} A_{j1}(t, \xi) A_{j2}(t, \xi) \cdots A_{js}(t, \xi),$$

where $A_{jk}(t, \xi)$ is the symbol of operator A_{jk}. With the help of Theorem 3.2, repeating reasoning from the proof of Theorem 3.1, it is easy to obtain the following result.

Theorem 3.3. *Operator A is Noetherian in the space $L_p^m(\Gamma, \rho)$, if and only if*

$$\det A(t, \xi) = \det \sum_{j=1}^{r} \prod_{k=1}^{s} A_{jk}(t, \xi) \neq 0, \ \ (t \in \Gamma, \ \xi \in \bar{R}) .$$

4. CONCLUDING REMARKS

In this section it is shown that conditions under which operator

$$A = aP + bQ + (cP + dQ)V$$

is Noetherian depend of the presence of angular points an contour Γ and of measure of these angles.

In Section 2 it was proved that operator $VSV + S$, in general, is not compact in space $\tilde{L}_p(\Gamma, \rho)$. Using Theorem 3.1 it is possible to prove that $VSV + S \in \mathbf{T}(L_p(\Gamma, \rho))$, if and only if contour Γ is of Lyapunov type. Really, the sufficiency is established by Theorem 2.1. Let Γ be a piecewise Lyapunov contour and t_0 an angular point with angle $\theta_0 = \theta(t_0)$ $(0 < \theta_0 < \pi)$. Admit that if operator $VSV + S$ is supposed to be compact in space $L_p(\Gamma, |t - t_0|^{\beta_0})$, then operator $A_\lambda = VSV + S - \lambda I$ must be Noetherian for every $\lambda \in C\backslash\{0\}$.

The symbol of operator A_λ in point t_0 has the form

$$A_\lambda(t_0, \xi) = \left\| \begin{matrix} -\lambda & 0 & \omega(\xi) & 0 \\ 0 & -\lambda & 0 & \omega(\xi) \\ \omega(\xi) & 0 & -\lambda & 0 \\ 0 & \omega(\xi) & 0 & -\lambda \end{matrix} \right\|,$$

where

$$\omega(\xi) = 2 \cdot \frac{\exp\left[(2\pi - \theta_0)(\xi + i(1 + \beta_0)/p)\right] - \exp\left[\theta_0(\xi + i(1 + \beta_0)/p)\right]}{\exp\left[2\pi(\xi + i(1 + \beta_0)/p)\right] - 1}.$$

By Theorem 3.1, operator A_λ is Noetherian if and only if $\det A_\lambda(t, \xi) \neq 0$ $(t \in \Gamma,\ -\infty \leq \xi \leq \infty)$. Particularity, for all values of λ which verify conditions

$$\det A_\lambda(t_0, \xi) = 0 \ (-\infty \leq \xi \leq \infty)$$

operator A_λ is not Noetherian in $\tilde{L}_p(\Gamma, |t - t_0|^{\beta_0})$. That is, for all values $\lambda = \pm\omega(\xi)$ $(-\infty \leq \xi \leq \infty)$ operator A_λ is not Noetherian. Since $\theta_0 \neq \pi$, it result that $\omega(\xi) \not\equiv 0$, a contradiction to hypothesis.

The symbol of operator

$$A = aP + bQ + (cP + dQ)V$$

depends on measures of angles formed by lateral tangents in points of contour Γ. This is seen from the definition of symbol of operators P and Q. It we consider that $c(t) \equiv d(t) \equiv 0$, then operator A has the form $A = aP + bQ$ and, as it is known [3], [4], [8], conditions under which it is do not depend angles $\theta(t_k)$, though the symbol, defined in this work, depends on $\theta(t_k)$ explicitly. In connection with this it appears naturally the question whether the dependence of the symbol of operator A on measures of $\theta(t_k)$ $(k = 1, ..., n)$ is essential. In other words, do conditions under which operator A $(|c(t)| + |d(t)| \not\equiv 0)$ is Noetherian really depend on $\theta_k = \theta(t_k)$? We will know that the answer to this question is affirmative.

Let

$$A = (1+\sqrt{2})P + (1-\sqrt{2})Q + V.$$

If contour Γ is of Lyapunov type, then operator A is Noetherian in all spaces $\tilde{L}_p(\Gamma,\rho)$. Let contour Γ have an angular point t_0 with angle $\theta(t_0) = \frac{\pi}{2}$ and $p = 2$. Then the symbol of this operator in point $(t_0, 0)$ has the form

$$A(t_0,0) = \left\| \begin{matrix} 1 & 1 & 1+i & 0 \\ 1 & 1 & 0 & 1-i \\ 1-i & 0 & 1 & 1 \\ 0 & 1+i & 1 & 1 \end{matrix} \right\|$$

and $\det A(t_0, 0) = 0$. So, operator

$$A = (1+\sqrt{2})P + (1-\sqrt{2})Q + V$$

is not Noetherian in the space $\tilde{L}_2(\Gamma)$. This example shows that the presence of angular points influences essentially Noetherian conditions of operator (2).

Concluding this section we consider the generalized boundary problem of Riemann which consists of the following. Determine two analytical functions $\Phi^+(z)$ and $\Phi^-(z)$ in F^+ and, respectively in F^- with the following properties: can be represented in F^+ and respectively in F^- using Cauchy integral; limit values $\Phi^+(t)$ and $\Phi^-(t)$ on contour Γ belong to the space $L_p(\Gamma,\rho)$; limits $\Phi^+(t)$ and $\Phi^-(t)$ at boundary verify conditions

$$\Phi^+(t) = a(t)\Phi^-(t) + b(t)\overline{\Phi^-(t)} + c(t), \tag{15}$$

where a, b, c are known functions. In the case of Lyapunov contour Noether theorems for problem (15) are proved in works [1], [2] and others. From these works, particularly one can deduce that if $a, b, c \in C(\Gamma)$, then the boundary problem (15) is Noetherian if and only if $|a(t)| \neq 0$ $(\forall t \in \Gamma)$. In the case of piecewise Lyapunov contour the following result is true.

Theorem 4.1. *The Riemann boundary problem (15) is Noetherian in space $\tilde{L}_p(\Gamma,\rho)$ it and only if the conditions are verified:*

(*i*) $|a(t)| > 0, \quad \forall t \in \Gamma$;

(ii) $|a(t_k)|^2 - |b(t_k)|^2\Big(\frac{z_k^{2\pi-\theta_k} - z_k^{\theta_k}}{z_k^{2\pi}-1}\Big) \neq 0$ *for every* $k = 1,\ldots,n$ *and every* $t \in \Gamma$, *where* $z_k = \exp(\xi + i(1+\beta_k)/p), \quad -\infty \le \xi \le \infty$.

The proof is done ordinary. Using Plemelj and Sohotski formulae, the problem (15) can be reduced to a singular integral equation with complex conjugation. We write the symbol of this equation and apply Theorem 3.1.

Remark that in the case of piecewise Lyapunov contour Noetherian condition for problem (15) also depend on measures of angles θ_k and, moreover, they depend also on the coefficient $b(t)$, that is not observed in the case of Lyapunov contour.

The result of this work can be generalized also to the case when contour is formed from a finite number at piecewise Lyapunov curves without points of self-intersection.

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LIE THEOREM ON INTEGRATING FACTOR FOR POLYNOMIAL DIFFERENTIAL SYSTEMS

Victor Orlov

Institute of Mathematics and Computer Science of the Academy of Sciences of Moldova, Chişinău, Republic of Moldova

orlovictor@gmail.com

Abstract A generalization of the classical Lie theorem on integrating factor for autonomous polynomial first-order differential system was obtained. Its application for differential system with cubic nonlinearities on nonsingular invariant manifolds (containing $GL(2,\mathbb{R})$–orbit of maximal dimension 4) and singular invariant manifolds (containing $GL(2,\mathbb{R})$–orbits of dimension < 4) was shown.

Keywords: autonomous polynomial system of first-order differential equations, $GL(2,\mathbb{R})$–orbit, Lie algebras, Lie's integrating factor, invariants and comitants, singular and nonsingular invariant manifolds.
2010 MSC: 34C14, 34C30.

1. INTRODUCTION

Let consider an autonomous system of first-order differential equations with polynomial right-hand sides as follows:

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1}$$

where the coefficients and variables in P and Q take the values from the field of real numbers $\mathbb{R}$.

Systems of the form (1) arise in solving of various problems in engineering [1], medicine [2,3], biology [5], energy security [4], etc.

In [6,7,8,9,10,11] the classifications of orbit dimensions for different polynomial differential systems with respect to the groups of centroaffine transformations $GL(2,\mathbb{R})$ as well as the group of affine transformations $Aff(2,\mathbb{R})$ have been carried out. It has been remarked that differential systems on singular invariant manifolds (containing the orbits having dimensions less than the maximum dimension) can be quite successful studied qualitatively by using invariants and comitants [10,13]. However, the most complex systems belong to nonsingular invariant manifolds, i.e. to the systems on orbits of maximal dimension. Therefore the approach to study of such systems is not always clear and single-valued. From these considerations, it is the necessary to develop an approach that enables to single out some of these systems and the strategy for their studying as well.

One of the method to study of autonomous first-order differential systems is the integrating factor method. However, the classical approach to this issue leads to the solving of a partial differential equation, which is not always successfully done. So, to get around this issue the Lie theorem on integrating factor was applied [14]. However, the classical Lie theorem on integrating factor also has to deal with a partial differential equation associated with problems in its solving. In this paper a generalization of the Lie theorem on integrating factor for polynomial differential systems was obtained, which allows getting away from the solving of partial differential equations to a system of algebraic equations. It turned out that this theorem is also related to the systems on nonsingular invariant manifolds. On an example of the differential system (1), with cubic nonlinearities, it is shown that the generalized Lie theorem on integrating factor can be applied for study of certain classes of polynomial differential systems, belonging to the orbits of dimension 4 and 3 relatively to the centroaffine group $GL(2,\mathbb{R})$.

2. THE CLASSIFICATION OF ORBITS FOR DIFFERENTIAL SYSTEMS WITH CUBIC NONLINEARITIES

Let denote the set of coefficients of the right-hand sides of (1) as a and their Euclidean space as $E^N(a)$. We denote by $a(q)$ the point from $E^N(a)$, corresponding to the system obtained from (1) with coefficients a after the transformation $q \in GL(2,\mathbb{R})$, where

$$q: \ \bar{x} = \alpha x + \beta y, \quad \bar{y} = \gamma x + \delta y, \quad \Delta_q = \alpha\delta - \beta\gamma \neq 0.$$

Definition 2.1. *The set* $O(a) = \{a(q); q \in GL(2,\mathbb{R})\}$ *is called the* $GL(2,\mathbb{R})$*–orbit of point* a *for the system* (1).

Let consider the differential system (1) with cubic nonlinearities, written in tensor form

$$\dot{x}^j = a^j_\alpha x^\alpha + a^j_{\alpha\beta\gamma} x^\alpha x^\beta x^\gamma \ (j,\alpha,\beta,\gamma = 1,2), \tag{2}$$

where the coefficient tensor $a^j_{\alpha\beta\gamma}$ is symmetric in the lower indices, for which the complete convolution is performed here. Following [12], the differential Lie operators corresponding to centroaffine group GL(2, R), which is admitted by the system (2) have the following form:

$$\mathcal{X}_1 = x\frac{\partial}{\partial x} + \mathcal{D}_1, \quad \mathcal{X}_2 = y\frac{\partial}{\partial x} + \mathcal{D}_2, \quad \mathcal{X}_3 = x\frac{\partial}{\partial y} + \mathcal{D}_3, \quad \mathcal{X}_4 = y\frac{\partial}{\partial y} + \mathcal{D}_4,$$

where

$$\begin{aligned}
&\mathcal{D}_1 = -d\frac{\partial}{\partial d} + e\frac{\partial}{\partial e} + 2p\frac{\partial}{\partial p} + q\frac{\partial}{\partial q} - s\frac{\partial}{\partial s} + 3t\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v},\\
&\mathcal{D}_2 = -e\frac{\partial}{\partial c} + (c-f)\frac{\partial}{\partial d} + e\frac{\partial}{\partial f} - t\frac{\partial}{\partial p} + (p-u)\frac{\partial}{\partial q} + (2q-v)\frac{\partial}{\partial r} +\\
&\qquad +(3r-w)\frac{\partial}{\partial s} + t\frac{\partial}{\partial u} + 2u\frac{\partial}{\partial v} + 3v\frac{\partial}{\partial w},\\
&\mathcal{D}_3 = d\frac{\partial}{\partial c} + (f-c)\frac{\partial}{\partial e} - d\frac{\partial}{\partial f} + 3q\frac{\partial}{\partial p} + 2r\frac{\partial}{\partial q} + s\frac{\partial}{\partial r} + (3u-p)\frac{\partial}{\partial t} +\\
&\qquad +(2v-q)\frac{\partial}{\partial u} + (w-r)\frac{\partial}{\partial v} - s\frac{\partial}{\partial w},\\
&\mathcal{D}_4 = d\frac{\partial}{\partial d} - e\frac{\partial}{\partial e} + q\frac{\partial}{\partial q} + 2r\frac{\partial}{\partial r} + 3s\frac{\partial}{\partial s} - t\frac{\partial}{\partial t} + v\frac{\partial}{\partial v} + 2w\frac{\partial}{\partial w},
\end{aligned} \tag{3}$$

and

$$\begin{aligned}
&c = a^1_1, \quad d = a^1_2, \quad p = a^1_{111}, \quad q = a^1_{112}, \quad r = a^1_{122}, \quad s = a^1_{222},\\
&e = a^2_1, \quad f = a^2_2, \quad t = a^2_{111}, \quad u = a^2_{112}, \quad v = a^2_{122}, \quad w = a^2_{222}.
\end{aligned} \tag{4}$$

Operators $\mathcal{X}_1$–$\mathcal{X}_4$ and thereafter (3) generate a reductive Lie algebra L_4.

From [12] it is known that

$$dim_{\mathbb{R}} O(a) = rank M_1, \tag{5}$$

where M_1 is the matrix constructed on the coordinate vectors of Lie algebra L_4 from (3). From (5) we obtain that $rank M_1$ can be equal to $4, 3, 2, 1, 0$, and consequently we have that $dim_{\mathbb{R}} O(a) = 4, 3, 2, 1, 0$, respectively. Following [14], it can be argued that $GL(2,\mathbb{R})$–orbits of maximal dimension 4 generate nonsingular invariant manifolds, and $GL(2,\mathbb{R})$–orbits of dimension < 4 generate singular invariant manifolds of the system (2).

From [15,16] let's give the following centroaffine invariants and comitants of the system (2):

$$P_1 = a^\alpha_{\alpha\beta\gamma} x^\beta x^\gamma, \; P_2 = a^p_{\alpha\beta\gamma} x^\alpha x^\beta x^\gamma x^q \varepsilon_{pq},$$

$$P_3 = a^\alpha_{p\alpha\beta} a^\beta_{q\gamma\delta} x^\gamma x^\delta \varepsilon^{pq}, \; P_4 = a^\alpha_{\alpha\beta\gamma} a^\beta_{\delta\mu\theta} x^\gamma x^\delta x^\mu x^\theta, \; P_5 = a^\alpha_{\beta\gamma\delta} a^\beta_{\alpha\mu\theta} x^\gamma x^\delta x^\mu x^\theta,$$

$$P_6 = a^\alpha_{\alpha pr} a^\beta_{\gamma\delta q} a^\gamma_{\beta\nu s} x^\delta x^\nu \varepsilon^{pq} \varepsilon^{rs}, \; Q_1 = a^p_\alpha a^q_{\beta\gamma\delta} x^\alpha x^\beta x^\gamma x^\delta \varepsilon_{pq}, \; Q_2 = a^\alpha_\beta a^\beta_{\alpha\gamma\delta} x^\gamma x^\delta,$$

$$Q_3 = a^\alpha_\gamma a^\beta_{\alpha\beta\delta} x^\gamma x^\delta, \; Q_7 = a^\alpha_\beta a^\beta_{p\alpha\gamma} a^\gamma_{q\eta\mu} x^\eta x^\mu \varepsilon^{pq}, \; K_2 = a^\alpha_\beta x^\beta x^\gamma \varepsilon_{\alpha\gamma}, \; I_1 = a^\alpha_\alpha,$$

$$I_2 = a^\alpha_\beta a^\beta_\alpha, \; J_1 = a^\alpha_{\alpha pr} a^\beta_{\beta qs} \varepsilon^{pq} \varepsilon^{rs}, \; J_2 = a^\alpha_{\beta pr} a^\beta_{\alpha qs} \varepsilon^{pq} \varepsilon^{rs},$$

$$J_4 = a^\alpha_{pru} a^\beta_{\gamma qs} a^\gamma_{\alpha\beta v} \varepsilon^{pq} \varepsilon^{rs} \varepsilon^{uv}, \tag{6}$$

where ε^{pq} (ε_{pq}) is the unit bivector with coordinates $\varepsilon^{11} = \varepsilon^{22} = 0$, $\varepsilon^{12} = -\varepsilon^{21} = 1$ ($\varepsilon_{11} = \varepsilon_{22} = 0$, $\varepsilon_{12} = -\varepsilon_{21} = 1$). Using the expressions (6), the following comitants are constructed:

$$\mathcal{M}_1 = 3P_1P_3 - 2J_1P_2, \quad \mathcal{N}_1 = P_2^2(I_1^2 - I_2) + 2Q_1^2 + 3K_2^2P_5 + 2I_1P_2Q_1,$$

$$\mathcal{N}_2 = J_2P_5 - J_4P_2 \not\equiv 0, \quad \mathcal{N}_3 = 2P_1P_2(4Q_1 - 3K_2P_1) + 2P_2^2(2Q_3 + I_1P_1),$$

$$\mathcal{N}_4 = K_2^2(2P_1^2 - 6P_4 + 4P_5) + 2P_2K_2(Q_3 - Q_2). \tag{7}$$

Moreover, by using the equation (5), the expressions (6), (7), and Theorem 5.9 from [12] the following theorem is proved:

Theorem 2.1. [9] *The dimension of the $GL(2,\mathbb{R})$-orbit of the system* (2) *is equal to*

4 $\Leftrightarrow$ $K_2P_1P_2(\mathcal{N}_3 + \mathcal{N}_4) \not\equiv 0$, *or* $K_2 \equiv 0$, $P_1P_2\mathcal{M}_1 \not\equiv 0$,
or $K_2 \equiv P_1 \equiv 0$, $P_2\mathcal{N}_2 \not\equiv 0$, *or* $P_1 \equiv 0$, $K_2\mathcal{N}_1 \not\equiv 0$,
or $P_2 \equiv 0$, $K_2P_1Q_7 \not\equiv 0$;

3 $\Leftrightarrow$ $\mathcal{N}_3 + \mathcal{N}_4 \equiv 0$, $K_2P_1P_2 \not\equiv 0$, *or* $K_2 \equiv \mathcal{M}_1 \equiv 0$, $P_1P_2 \not\equiv 0$,
or $K_2 \equiv P_1 \equiv \mathcal{N}_2 \equiv 0$, $P_2P_5 \not\equiv 0$, *or* $P_1 \equiv \mathcal{N}_1 \equiv 0$, $K_2P_2 \not\equiv 0$,
or $P_2 \equiv Q_7 \equiv 0$, $K_2P_1(P_1Q_1 + P_6) \not\equiv 0$, *or* $P_2 \equiv K_2 \equiv 0$, $J_1 \neq 0$;

2 $\Leftrightarrow$ $P_2 \equiv Q_7 \equiv P_1Q_1 + P_6 \equiv 0$, $P_1^2 + K_2^2 \not\equiv 0$, *or* $P_1 \equiv P_2 \equiv 0$, $K_2 \not\equiv 0$,
or $P_1 \equiv P_5 \equiv K_2 \equiv \mathcal{N}_2 \equiv 0$, $P_2 \not\equiv 0$, *or* $J_1 = 0$, $P_2 \equiv K_2 \equiv 0$, $P_1 \not\equiv 0$;

0 $\Leftrightarrow$ $P_1 \equiv P_2 \equiv K_2 \equiv 0$.

3. GENERALIZED LIE THEOREM ON INTEGRATING FACTOR FOR POLYNOMIAL DIFFERENTIAL SYSTEMS

Let consider the polynomial differential system (1), where $PQ \not\equiv 0$, and the corresponding equation

$$y' = \frac{Q(x,y)}{P(x,y)}. \tag{8}$$

From Marius Sophus Lie (1842-1899) it is known the following theorem:

Theorem 3.1. [14] *The differential equation $y' = f(x,y)$ admits an one-parameter continuous group G_1 with the operator $\mathcal{X} = \xi^1(x,y)\dfrac{\partial}{\partial x} + \xi^2(x,y)\dfrac{\partial}{\partial y}$ if and only if the coordinates of the operator satisfy the defining equation $\xi^2_x + f(\xi^2_y - \xi^1_x) - f^2\xi^1_y = \xi^1 f_x + \xi^2 f_y$, where $\xi^i_x, \xi^i_y, (i = 1,2)$ and f_x, f_y there are partial derivatives of the corresponding functions in x and y. Moreover $\mu = \left(\xi^2 - f\xi^1\right)^{-1}$ is an Lie's integrating factor for the equation $dy - f dx = 0$.*

Theorem 3.2. [17] *If the polynomial system* (1) *with $PQ \not\equiv 0$ admits Lie operator $\mathcal{Y} = \xi^1(x,y)\dfrac{\partial}{\partial x} + \xi^2(x,y)\dfrac{\partial}{\partial y} + \mathcal{D}$, where $\mathcal{D} \neq 0$ is an operator of linear representation of a* *one-parameter continuous group G_1 in the space of coefficients of the polynomials P and Q of the system* (1)*, then in order the corresponding equation* (8) *allows the same operator $\mathfrak{Y}$, it is necessary and sufficient the fulfillment of the identity $\mathcal{D}\left(\dfrac{Q(x,y)}{P(x,y)}\right) \equiv 0$, which is equivalent to the identity $P\mathcal{D}(Q) - Q\mathcal{D}(P) \equiv 0$.*

Corollary 3.1. *If the conditions of Theorem 3.2 are satisfied, we have that the equation $y' = \dfrac{Q(x,y)}{P(x,y)}$ allows the operator $\overline{\mathfrak{Y}} = \xi^1(x,y)\dfrac{\partial}{\partial x} + \xi^2(x,y)\dfrac{\partial}{\partial y}$, obtained from the operator $\mathfrak{Y}$ by excluding the operator $\mathcal{D}$.*

From Corollary 3.1 and Theorems 3.1-3.2 we obtain

Theorem 3.3. *(Generalized Lie theorem) If the system* (1) *with $PQ \not\equiv 0$ admits the operator $\mathfrak{Y} = \xi^1(x,y)\dfrac{\partial}{\partial x} + \xi^2(x,y)\dfrac{\partial}{\partial y} + \mathcal{D}$, with $\mathcal{D} \neq 0$ and the identity*

$$P\mathcal{D}(Q) - Q\mathcal{D}(P) \equiv 0, \tag{9}$$

holds then the function $\mu = (\xi^1 Q - \xi^2 P)^{-1}$ is the Lie's integrating factor for this system, and $\xi^1 Q - \xi^2 P = 0$ is its particular integral.

4. LIE'S INTEGRATING FACTORS FOR THE SYSTEM WITH CUBIC NONLINEARITIES

Let write the linear combination of the operators (3) in the form

$$\mathcal{D} = \alpha\mathcal{D}_1 + \beta\mathcal{D}_2 + \gamma\mathcal{D}_3 + \delta\mathcal{D}_4, \tag{10}$$

where $\alpha, \beta, \gamma, \delta$ are undefined real parameters.

Remark 4.1. *Hereafter let consider the general case, i.e. when in* (2)*,* (4) *and* (10) *all coefficients and parameters are different from zero.*

Remark 4.2. *Let call degenerate the system of the form* (2) *with proportional right-hand sides, where the proportionality factor is a number or an expression, depending only on the coefficients of this system.*

Theorem 4.1. *The differential system* (2)*,* (4) *with inverse polynomial Lie's integrating factor μ^{-1} with degree ≤ 4 with respect to phase variables is subdivided into the following classes: 17 (13 degenerate) systems on nonsingular invariant manifolds (which contain $GL(2,\mathbb{R})$-orbits of dimension 4) and 9 (2 degenerate) systems on singular invariant manifolds (which include $GL(2,\mathbb{R})$-orbits of dimension 3), i.e. 4 systems for which μ^{-1} is represented as a product of two polynomials, one being homogeneous of the second order and other – nonhomogeneous of the second order, 1 system, for which μ^{-1} is represented as a product of two polynomials, one being of* *the first order, and other heterogeneous of the third order, 5 systems for which μ^{-1} is represented as a product of three polynomials, two being of the first order and the third – heterogeneous of the second order, 1 system, for which μ^{-1} is represented as a product of two linear homogeneous polynomials.*

Proof explanation of the theorem 4.1. Subject to the expressions of the operators (3) for the differential system (2), (4) we obtain that the identity (9) is split into 15 polynomial equations linear relative to the parameters $\alpha, \beta, \gamma, \delta$ and of the second degree relative to the coefficients of the mentioned system:

$$
\begin{gathered}
u_1 \equiv ce\alpha + e^2\beta - (c^2 + de - cf)\gamma - ce\delta = 0,\\
u_2 \equiv de\alpha + ef\beta - cd\gamma - de\delta = 0,\\
u_3 \equiv df\alpha + (de - cf + f^2)\beta - d^2\gamma - df\delta = 0,\\
u_4 \equiv (ep - 3ct)\alpha - 2et\beta + (2cp - fp + 3eq + dt - 3cu)\gamma + (ep + ct)\delta = 0,\\
u_5 \equiv (fp - 2dt - 3cu)\alpha + (ep - ct - ft - 3eu)\beta +\\
+(dp + 3cq + 3er - 3cv)\gamma + (3eq + dt)\delta = 0,\\
u_6 \equiv pt\alpha + t^2\beta - (p^2 + 3qt - 3pu)\gamma - pt\delta = 0,\\
u_7 \equiv qt\alpha + tu\beta - (pq + rt - pv)\gamma - qt\delta = 0,\\
u_8 \equiv (fq - er - 3du - cv)\alpha + (fp + eq - dt - cu - 2fu - 2ev)\beta +\\
+(2dq + 2cr + fr + es - dv - cw)\gamma + (fq + 3er + du - cv)\delta = 0,\\
u_9 \equiv (3rt + 3qu - pv)\alpha + (qt - pu + 3u^2 + 2tv)\beta - (3q^2 + 2pr +\\
+st + 3ru - 3qv - pw)\gamma - (3rt + 3qu - pv)\delta = 0,\\
u_{10} \equiv sv\alpha + (su - qw + vw)\beta - rs\gamma - sv\delta = 0,\\
u_{11} \equiv (es + 3dv)\alpha - (3fq - 3du - 3fv - ew)\beta -\\
-(3dr + cs + fs - dw)\gamma - (3fr + 2es - cw)\delta = 0,\\
u_{12} \equiv (fs + dw)\alpha - (3fr - es - 3dv + cw - 2fw)\beta -\\
-2ds\gamma - (3fs - dw)\delta = 0,\\
u_{13} \equiv (2st + 9ru - pw)\alpha + (3rt - 3pv + 9uv + tw)\beta -\\
-(9qr + ps + 3su - 3qw)\gamma - (2st + 9ru - pw)\delta = 0,\\
u_{14} \equiv (3su + 3rv - qw)\alpha + (st + 3ru - 3qv + 3v^2 - pw + 2uw)\beta -\\
-(3r^2 + 2qs + sv - rw)\gamma - (3su + 3rv - qw)\delta = 0,\\
u_{15} \equiv sw\alpha + (3sv - 3rw + w^2)\beta - s^2\gamma - sw\delta = 0.
\end{gathered}
\tag{11}
$$

By solving the algebraic system (11) relative to the coefficients of the system (2), (4) and the parameters $\alpha, \beta, \gamma, \delta$, subject to Theorem 2.1 and the operators $\mathcal{X}_1$–$\mathcal{X}_4$ mentioned above, using Theorem 3.3, we obtain the following classes of systems with polynomial inverse Lie's integrating factor μ^{-1} (degenerate systems are not presented):

I. Systems on nonsingular invariant manifolds of dimension 4 relative to the group $GL(2,\mathbb{R})$:

$$1)\quad \begin{aligned} \dot{x} &= cx+dy+\frac{t}{e}x^3+3qx^2y+3rxy^2+\frac{d\Theta}{c^2e}y^3,\\ \dot{y} &= ex+\frac{e\Phi}{ct}y+tx^3+3ux^2y-\frac{3\Omega}{c^2t}xy^2+\frac{\Theta\Phi}{c^3t}y^3. \end{aligned}$$

where $\Omega = 3e^2q^2 - deqt - cert - 3cequ + cdtu$, $\Theta = -3deq + 3cer + d^2t$, $\Phi = -3eq + dt + 3cu$.

Polynomial inverse Lie's integrating factor of this system has the following form: $\mu^{-1} = \dfrac{e\alpha - c\gamma}{c^3e^2t}\mathcal{F}_1\mathcal{F}_2$, where $\mathcal{F}_1 \equiv cetx^2 + (-3e^2q - c^2t + det + 3ceu)xy - cdty^2 = 0$, $\mathcal{F}_2 \equiv c^2e + c^2tx^2 + c(3eq - dt)xy + (-3deq + 3cer + d^2t)y^2 = 0$ are the particular integrals of the mentioned system.

$$2)\quad \dot{x} = \frac{p\gamma x - (p\alpha + 3u\alpha + 3v\gamma)y}{t\gamma^3}\,\Theta,\quad \dot{y} = \frac{t\gamma x + (2t\alpha + 3u\gamma)y}{t\gamma^3}\,\Theta.$$

where $\Theta = e\gamma^2 + t\gamma^2x^2 - 2t\alpha\gamma xy + (4t\alpha^2 + 6u\alpha\gamma + 3v\gamma^2)y^2$.

Polynomial inverse Lie's integrating factor of this system has the following form: $\mu^{-1} = \frac{t\alpha - p\gamma}{t^2\gamma^3}\mathcal{F}_1\mathcal{F}_2$, where $\mathcal{F}_1 \equiv t\gamma x^2 + (2t\alpha - p\gamma + 3u\gamma)xy + (p\alpha + 3u\alpha + 3v\gamma)y^2 = 0$, $\mathcal{F}_2 \equiv \Theta = 0$ are the particular integrals of the mentioned system.

$$3)\quad \begin{aligned} \dot{x} &= \tfrac{1}{c^2e^4}(ex+cy)\left[c^3e^3 - 3e^5rx^2 + ce(3e^3r + c^3t)xy - c^5ty^2\right],\\ \dot{y} &= \tfrac{1}{c^2e^2}(ex+cy)\left[c^2e^2 + c^2etx^2 - (3e^3r + c^3t)xy + 3ce^2ry^2\right]. \end{aligned}$$

Polynomial inverse Lie's integrating factor of this system has the following form: $\mu^{-1} = \frac{\gamma}{c^2e^2}\mathcal{F}_1\mathcal{F}_2\mathcal{F}_3$, where $\mathcal{F}_1 \equiv -ex + cy = 0$, $\mathcal{F}_2 \equiv ex + cy = 0$, $\mathcal{F}_3 \equiv c^3 - 3e^2rx^2 + 3c^2ry^2 = 0$ are the particular integrals of the mentioned system.

$$4)\quad \begin{aligned} \dot{x} &= \tfrac{1}{2ce^4t}\left[2c^2etx + (3e^3r + c^3t)y\right]\left(e^3 - e^2tx^2 + 2cetxy - c^2ty^2\right),\\ \dot{y} &= \tfrac{1}{2c^2e^3t}\left[2c^2etx + (3e^3r + c^3t)y\right]\left(e^3 + e^2tx^2 - 2cetxy + c^2ty^2\right). \end{aligned}$$

Polynomial inverse Lie's integrating factor of this system has the following form: $\mu^{-1} = \frac{\gamma}{2ce^5t}\mathcal{F}_1\mathcal{F}_2\mathcal{F}_3$, where $\mathcal{F}_1 \equiv -ex + cy = 0$, $\mathcal{F}_2 \equiv 2c^2etx + (3e^3r + c^3t)y = 0$, $\mathcal{F}_3 \equiv e^3 - e^2tx^2 + c^2ty^2 = 0$ are the particular integrals of the mentioned system.

II. Systems on singular invariant manifolds of dimension 3 relative to the group $GL(2,\mathbb{R})$:

$$1)\quad \begin{aligned} \dot{x} &= cx+\frac{\Theta}{et}y+\frac{2t(c-f)+3eu}{e}x^3+3qx^2y-\frac{3(ct-ft+eu)\Theta}{e^3t}xy^2-\frac{\Theta^2}{e^4t}y^3,\\ \dot{y} &= ex+fy+tx^3+3ux^2y-\frac{3\Omega}{e^2}xy^2-\frac{(ct-ft+3eu)\Theta}{e^3t}y^3, \end{aligned}$$

where $\Omega = e^2q + c^2t - 2cft + f^2t + 2ceu - 2efu$, Ă $\Theta = 3e^2q + 2c^2t - 4cft + 2f^2t + 3ceu - 3efu$.

Polynomial inverse Lie's integrating factor of this system has the following form: $\mu^{-1} = -\frac{\gamma}{2e^5t^2}\mathcal{F}_1\mathcal{F}_2$, where $\mathcal{F}_1 \equiv -e^2tx^2 + e(c-f)txy + \Theta y^2 = 0$ Ă¨ $\mathcal{F}_2 \equiv -e^3(c+f)t - 3t(ct - ft + 2eu)\left[e^2x^2 - e(c-f)xy - \Theta y^2\right] = 0$ are the particular integrals of the mentioned system.

$$2)\quad \begin{aligned} \dot{x} &= cx - \frac{e\Theta}{4t^2}y + px^3 - \frac{3\Phi}{4t}x^2y + \frac{3(p-u)\Theta}{8t^2}xy^2 - \frac{\Theta^2}{16t^3}y^3, \\ \dot{y} &= ex + \frac{\Psi}{2t}y + tx^3 + 3ux^2y + 3vxy^2 + \frac{(p+3u)\Theta}{8t^2}y^3. \end{aligned}$$

where $\Theta = p^2 - 9u^2 + 12tv$, $\Phi = p^2 - 2pu - 3u^2 + 4tv$, $\Psi = -ep + 2ct + 3eu$.

Polynomial inverse Lie's integrating factor of this system has the following form: $\mu^{-1} = -\frac{\gamma}{64t^4}\mathcal{F}_1\mathcal{F}_2$, where $\mathcal{F}_1 \equiv 4t^2x^2 - 2t(p-3u)xy + (p^2 - 9u^2 + 12tv)y^2 = 0$ and $\mathcal{F}_2 \equiv -4t(ep - 4ct - 3eu) + 12t^2(p+u)x^2 - 6t(p-3u)(p+u)xy + 3(p+u)(p^2 - 9u^2 + 12tv)y^2 = 0$ are the particular integrals of the mentioned system.

$$3)\quad \begin{aligned} \dot{x} &= cx - \frac{eu(2t\alpha + 3u\gamma)}{t^2\gamma}y - ux^3 - \frac{3u^2}{t}x^2y - \frac{3u^3}{t^2}xy^2 - \frac{u^4}{t^3}y^3, \\ \dot{y} &= ex + \frac{2et\alpha + ct\gamma + 4eu\gamma}{t\gamma}y + tx^3 + 3ux^2y + \frac{3u^2}{t}xy^2 + \frac{u^3}{t^2}y^3. \end{aligned}$$

Polynomial inverse Lie's integrating factor of this system has the following form: $\mu^{-1} = \frac{\mathcal{F}_1\mathcal{F}_2}{t^4\gamma}$, where $\mathcal{F}_1 \equiv tx + uy = 0$ and $\mathcal{F}_2 \equiv t^3\gamma(e\alpha - c\gamma)x + t^2(e\alpha - c\gamma)(2t\alpha + 3u\gamma)y + t^3\gamma(t\alpha + u\gamma)x^3 + 3t^2u\gamma(t\alpha + u\gamma)x^2y + 3tu^2\gamma(t\alpha + u\gamma)xy^2 + u^3\gamma(t\alpha + u\gamma)y^3 = 0$ are the particular integrals of the mentioned system.

$$4)\quad \begin{aligned} \dot{x} &= cx + \frac{epu}{t^2}y + px^3 + \frac{3pu}{t}x^2y + \frac{3pu^2}{t^2}xy^2 + \frac{pu^3}{t^3}y^3, \\ \dot{y} &= ex + \frac{ct + eu - ep}{t}y + tx^3 + 3ux^2y + \frac{3u^2}{t}xy^2 + \frac{u^3}{t^2}y^3. \end{aligned}$$

Polynomial inverse Lie's integrating factor of this system has the following form: $\mu^{-1} = -\frac{\gamma}{t^4}\mathcal{F}_1\mathcal{F}_2\mathcal{F}_3$, where $\mathcal{F}_1 \equiv tx - py = 0$, $\mathcal{F}_2 \equiv tx + uy = 0$, and $\mathcal{F}_3 \equiv t(ct + eu) + (p+u)(tx + uy)^2 = 0$ are the particular integrals of the mentioned system.

$$5)\qquad \dot{x} = cx + \frac{e\alpha(3t\alpha - p\gamma + 3u\gamma)}{t\gamma^2}y + px^3 + \frac{3\alpha(t\alpha - p\gamma + u\gamma)}{\gamma^2}x^2y -$$

$$-\frac{3\alpha^2(2t\alpha - p\gamma + 2u\gamma)}{\gamma^3}xy^2 + \frac{\alpha^3(3t\alpha - p\gamma + 3u\gamma)}{\gamma^4}y^3,$$

$$\dot{y} = ex + \frac{2et\alpha - (ep - ct - 3eu)\gamma}{t\gamma} y + tx^3 + 3ux^2y - \\ - \frac{3\alpha(t\alpha + 2u\gamma)}{\gamma^2} xy^2 + \frac{\alpha^2(2t\alpha + 3u\gamma)}{\gamma^3} y^3.$$

Polynomial inverse Lie's integrating factor of this system has the following form: $\mu^{-1} = \frac{\mathcal{F}_1\mathcal{F}_2\mathcal{F}_3}{t\gamma^4}$, where $\mathcal{F}_1 \equiv -\alpha y + \gamma x = 0$, $\mathcal{F}_2 \equiv t\gamma x + (3t\alpha - p\gamma + 3u\gamma)y = 0$, $\mathcal{F}_3 \equiv \gamma^2(e\alpha - c\gamma) + (t\alpha - p\gamma)[\alpha y - \gamma x]^2 = 0$ are the particular integrals of the mentioned system.

$$6) \quad \begin{aligned} \dot{x} &= cx + \frac{ep(p+3u)}{2t^2} y + px^3 + \frac{3pu}{t} x^2 y - \frac{3p\,\Theta}{4t^2} xy^2 - \frac{p^2(p+3u)^2}{4t^3} y^3, \\ \dot{y} &= ex + \frac{2ct + 3eu - ep}{2t} y + tx^3 + 3ux^2y - \frac{3\Theta}{4t} xy^2 - \frac{p(p+3u)^2}{4t^2} y^3, \end{aligned}$$

where $\Theta = (p-u)(p+3u)$.

Polynomial inverse Lie's integrating factor of this system has the following form: $\mu^{-1} = -\frac{\gamma}{16t^4}\mathcal{F}_1\mathcal{F}_2\mathcal{F}_3$, where $\mathcal{F}_1 \equiv tx - py = 0$, $\mathcal{F}_2 \equiv 2tx + (p+3u)y = 0$, $\mathcal{F}_3 \equiv -2t(ep - 4ct - 3eu) + 3(p+u)[2t^2x^2 - t(p-3u)xy - p(p+3u)y^2] = 0$ are the particular integrals of the mentioned system.

$$7) \quad \begin{aligned} \dot{x} &= cx + \frac{ep(2p+3u)}{t^2} y + px^3 + \frac{3pu}{t} x^2 y - \frac{3p^2(p+2u)}{t^2} xy^2 + \frac{p^3(2p+3u)}{t^3} y^3, \\ \dot{y} &= ex + \frac{ep + ct + 3eu}{t} y + tx^3 + 3ux^2y - \frac{3p(p+2u)}{t} xy^2 + \frac{p^2(2p+3u)}{t^2} y^3, \end{aligned}$$

Polynomial inverse Lie's integrating factor of this system has the following form: $\mu^{-1} = \frac{(ep-ct)\gamma}{t^3}\mathcal{F}_1\mathcal{F}_2$, where $\mathcal{F}_1 \equiv tx - py = 0$ and $\mathcal{F}_2 \equiv tx + (2p+3u)y = 0$ are the particular integrals of the mentioned system.

Taking into consideration Remark 4.1, the propositions of Theorem 4.1 are proved.□

Remark 4.3. *It can be verified that the differential systems above obtained have an interesting geometry, for example, for some of them the origin of coordinates may be a center or a focus, a saddle or a node.*

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CUBIC SYSTEMS WITH DEGENERATE INFINITY AND STRAIGHT LINES OF TOTAL PARALLEL MULTIPLICITY SIX

Vadim Repeşco
Tiraspol State University, Chişinău, Republic of Moldova
repescov@gmail.com

Abstract In this paper the cubic systems with the infinite line filled up with singularities (i.e. with the degenerated infinity) and having invariant straight lines of total multiplicity six are classified. It is proved that there are 11 affine classes of such systems. For every class was carried out the qualitative investigation in the Poincaré disc.

Keywords: cubic differential system, invariant straight line, phase portrait.
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1. INTRODUCTION

We consider the real cubic differential system

$$\begin{cases} \frac{dx}{dt} = \sum\limits_{r=0}^{3} P_r(x,y) \equiv P(x,y), \\ \frac{dy}{dt} = \sum\limits_{r=0}^{3} Q_r(x,y) \equiv Q(x,y), \end{cases} \tag{1}$$

where P_r, Q_r are homogeneous polinomials of degree r, $GCD(P,Q) = 1$ and $|P_3(x,y)| + |Q_3(x,y)| \not\equiv 0$.

The curve $f(x,y) = 0,\ f \in \mathbb{C}[x,y]$ is said to be an *invariant algebraic curve* of (1) if there exists a polynomial $K_f \in \mathbb{C}[x,y]$, such that the identity $\frac{\partial f}{\partial x}P(x,y)+\frac{\partial f}{\partial y}Q(x,y) \equiv f(x,y)K_f(x,y)$ holds.We say that an invariant algebraic curve $f(x,y) = 0$ has the *parallel multiplicity* equal to m, if m is the greatest positive integer such that f^{m-1} divides K_f.

The system (1) is called *Darboux integrable* if there exists a non-constant function of the form $f = f_1^{\lambda_1} \cdots f_s^{\lambda_s}$, where f_j is an invariant algebraic curve and $\lambda_j \in \mathbb{C},\ j = \overline{1,s}$, such that either $f = const$ is a first integral or f is an integrating factor for (1). We will be interested in invariant algebraic curves of degree one, that is invariant straight lines $\alpha x + \beta y + \gamma = 0,\ (\alpha,\beta) \neq (0,0)$.

At present, a great number of works are dedicated to the investigation of polynomial differential systems with invariant straight lines.

The problem of estimation the number of invariant straight lines which can have a polynomial differential system was considered in [1]; the problem of coexistence of the invariant straight lines and limit cycles in [4], [5]; the problem of coexistence of the invariant straight lines and singular points of a center type for cubic system in [3], [10]. The classification of all cubic systems with the maximum number of invariant straight lines, taking into account their multiplicities, is given in [6].

In [1] it was proved that the cubic system (1) can have in the finite part of the phase plane at most eight invariant straight lines. The cubic systems with exactly eight invariant straight lines has been studied in [6], [7] and with invariant straight lines with total parallel multiplicity equal to seven in [11], [13]. A qualitative investigation of systems (1) with six real invariant straight lines along two (three) directions is given in [8] ([9]). In [12] were examined some cubic systems with degenerate infinity having invariant straight lines of total parallel multiplicity five or six, three of which are parallel. In this paper we continued the investigation from [8], [9], [12] and a complet qualitative study of cubic systems (1) with degenerated infinity and invariant straight lines (real or complex) of total multiplicity six is given.

Theorem 1.1. *Assume that a cubic system with degenerate infinity possesses invariant straight lines of total parallel multiplicity six. Then via an affine transformation and time rescaling this system can be brought to one of the 11 system 1)-11). Moreover, its phase portrait on the Poincaré disc corresponds up to topological equivalence to one of the portraiths given in Fig. 1 - Fig. 11. In the table below for each one of the systems 1) - 11) the first arrow shows the straight lines and either the first integral or integrating factor that corresponds to each system.*

$$1)\quad \begin{cases} \dot{x} = x(x+1)(x-a),\ a>0, \\ \dot{y} = y(-a+2x-y+x^2); \\ \text{Configuration } (3r, 1r, 1r, 1r) \end{cases} \qquad \to (2) \quad \to \text{Fig. 1;}$$

$$2)\quad \begin{cases} \dot{x} = (x-a)(x^2+1),\ a\in\mathbb{R}, \\ \dot{y} = y(-1-2ax-y+x^2); \\ \text{Configuration } (1r+2c_0, 1r, 1c_1, 1c_1) \end{cases} \qquad \to (3) \quad \to \text{Fig. 2;}$$

$$3)\quad \begin{cases} \dot{x} = x(x-1)(2y-1), \\ \dot{y} = y(y-1)(2x-1); \\ \text{Configuration } (2r, 2r, 1r, 1r) \end{cases} \qquad \to (4) \quad \to \text{Fig. 3;}$$

$$4)\quad \begin{cases} \dot{x} = 2xy(x-1), \\ \dot{y} = (y^2+1)(2x-1); \\ \text{Configuration } (2r, 2c_0, 1c_1, 1c_1) \end{cases} \qquad \to (5) \quad \to \text{Fig. 4;}$$

$$5)\quad \begin{cases} \dot{x} = y(x^2+1), \\ \dot{y} = x(y^2+1); \\ \text{Configuration } (2c_0, 2c_0, 1r, 1r) \end{cases} \qquad \to (6) \quad \to \text{Fig. 5;}$$

$$6)\quad \begin{cases} \dot{x} = x(1-2y+2x^2+2y^2), \\ \dot{y} = (2y-1)(-y+x^2+y^2); \\ \text{Configuration } (2c_1, 2c_1, 1r, 1r) \end{cases} \qquad \to (7) \quad \to \text{Fig. 6;}$$

$$\begin{gathered} l_1 = x+1,\ l_2 = x,\ l_3 = x-a,\ l_4 = y,\ l_5 = (a+1)x-y, \\ l_6 = x-y-a; \quad l_1l_4/(l_3l_5) = const. \end{gathered} \tag{2}$$

$$7)\quad \begin{cases} \dot{x} = x(x-1)(1+ax-2y), \\ \dot{y} = y(-1+2x+y+ax^2-2xy); \\ \text{Configuration } (2r, 1r, 1r, 1r, 1r) \end{cases} \quad \to (8) \quad \to \text{Fig. 7;}$$

$$8)\quad \begin{cases} \dot{x} = y(1+(x-a)^2),\ a \neq 0, \\ \dot{y} = -(1+a^2)x + a(x^2-y^2) + xy^2; \\ \text{Configuration } (2c_0, 1c_1, 1c_1, 1r, 1r) \end{cases} \quad \to (9) \quad \to \text{Fig. 8;}$$

$$9)\quad \begin{cases} \dot{x} = y + x^2 + 2axy - y^2 + ax^3 + \\ \qquad +(a^2+b^2-1)x^2y - axy^2, \\ \dot{y} = -x - ax^2 + 2xy + ay^2 + ax^2y + \\ \qquad +(a^2+b^2-1)xy^2 - ay^3,\ ab \neq 0; \\ \text{Configuration } (1r, 1r, 1c_1, 1c_1, 1c_1, 1c_1) \end{cases} \quad \to (10) \quad \to \text{Fig. 9;}$$

$$10)\quad \begin{cases} \dot{x} = x(1+2ax-2y+(a^2+b^2-b)x^2 - \\ \qquad -2axy+(1-b)y^2), \\ \dot{y} = y + bx^2 + 2axy + (b-2)y^2 + (a^2 + \\ \qquad +b^2-b)x^2y - 2axy^2 + (1-b)y^3, \\ b(b-1)(|a|+|b+1|) \neq 0; \\ \text{Configuration } (1r, 1r, 1c_1, 1c_1, 1c_1, 1c_1) \end{cases} \quad \to (11) \quad \to \text{Fig. 10;}$$

$$11)\quad \begin{cases} \dot{x} = x(1+(a+b)x-2y+abx^2-2axy+ \\ \qquad +(a-b+1)y^2), \\ \dot{y} = y(1+2ax+(b-a-2)y+abx^2 - \\ \qquad -2axy+(a-b+1)y^2), \\ ab(a-1)(b-1)(b-a-1) \neq 0,\ a > b; \\ \text{Configuration } (1r, 1r, 1r, 1r, 1r, 1r). \end{cases} \quad \to (12) \quad \to \text{Fig. 11;}$$

$$\begin{gathered} l_1 = x - i,\ l_2 = x - a,\ l_3 = x + i,\ l_4 = y, \\ l_{5,6} = (a \mp i)x + y + 1 \pm ai; \quad \mu(x,y) = 1/(l_1 l_3 l_5 l_6). \end{gathered} \tag{3}$$

$$\begin{gathered} l_1 = x\ l_2 = x-1,\ l_3 = y,\ l_4 = y-1,\ l_5 = x-y,\ l_6 = x+y-1; \\ l_1 l_2/(l_3 l_4) = const. \end{gathered} \tag{4}$$

$$\begin{gathered} l_1 = x,\ l_2 = x-1,\ l_{3,4} = y \pm i,\ l_{5,6} = y \pm i(2x-1); \\ l_3 l_4/(l_5 l_6) = const. \end{gathered} \tag{5}$$

$$\begin{gathered} l_{1,2} = x \pm i,\ l_{3,4} = y \pm i,\ l_5 = x-y,\ l_6 = x+y; \\ (l_1 l_2)/(l_3 l_4) = const. \end{gathered} \tag{6}$$

$$\begin{gathered} l_{1,2} = y \mp ix,\ l_{3,4} = y \mp ix - 1,\ l_5 = x,\ l_6 = 2y-1; \\ l_1 l_2 l_3 l_4/(l_5 l_6)^2 = const. \end{gathered} \tag{7}$$

$$\begin{gathered} l_1 = x,\ l_2 = x-1,\ l_3 = y,\ l_4 = x+y-1,\ l_5 = (a+1)x-y, \\ l_6 = ax-y+1; \quad l_1 l_2/(l_4 l_5) = const. \end{gathered} \tag{8}$$

$$\begin{gathered} l_{1,2} = x - a \pm i,\ l_{3,4} = y \pm ix,\ l_5 = ax + y - a^2 - 1, \\ l_6 = ax - y - a^2 - 1; \quad l_3 l_4/(l_1 l_2) = const. \end{gathered} \tag{9}$$

$$\begin{gathered} l_{1,2} = y \mp xi,\ l_{3,4} = y - (a \pm bi)x - 1,\ l_{5,6} = 1 + ax - y \pm by; \\ l_1 l_2/(l_3 l_4) = const. \end{gathered} \tag{10}$$

$$l_{1,2} = y \mp xi,\ l_{3,4} = y - (a \pm bi)x - 1, l_5 = 1 + ax - y + by, \\ l_6 = x; \quad l_1 l_2 l_3 l_4/(l_5 l_6)^2 = const. \tag{11}$$

$$l_1 = x,\ l_2 = y,\ l_3 = y - x,\ l_4 = y - ax - 1,\ l_5 = y - bx - 1, \\ l_6 = ax + (b - a - 1)y + 1; \quad l_1 l_6/(l_2 l_5) = const. \tag{12}$$

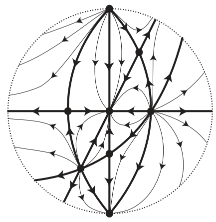

Fig. 1

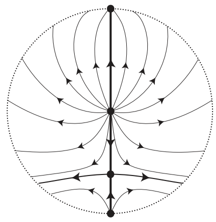

Fig. 2

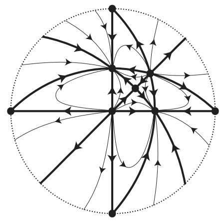

Fig. 3

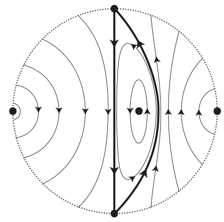

Fig. 4

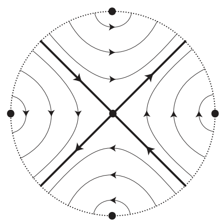

Fig. 5

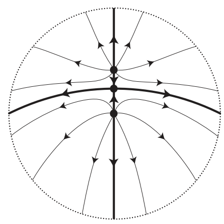

Fig. 6

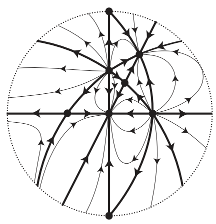

Fig. 7

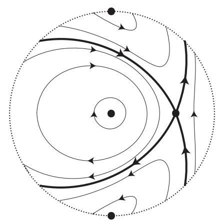

Fig. 8

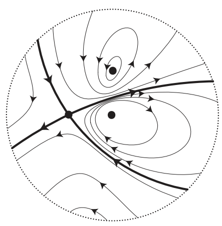

Fig. 9

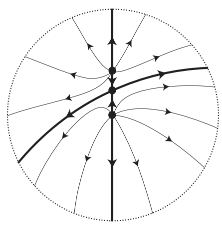

Fig. 10.a)

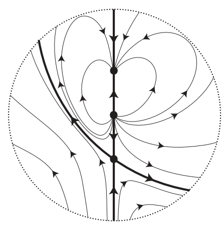

Fig. 10.b)

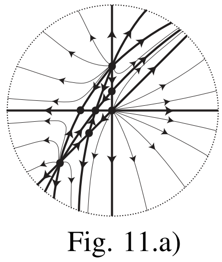

Fig. 11.a)

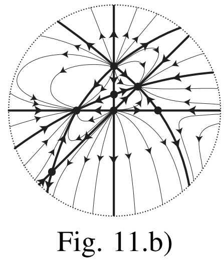

Fig. 11.b)

2. SOME PROPERTIES OF THE CUBIC SYSTEMS WITH STRAIGHT LINES

By *configuration of straight lines* we understand the $\mathbb{R}^2$ plane with a certain number of straight lines.

To each bidimensional differential system (with invariant straight lines) we can associate a configuration consisting of invariant straight lines of this system. It's easy to show that reciprocal affirmation is not always true.

The problem arise to determine such properties for invariant straight lines that allow to construct all realizable configurations of straight lines for (1). Below we shall enumerate such properties. Theirs proofs are not complicated and are not given in this paper.

Proposition 2.1. *In the finite part of the phase plane the system* (1) *has at most nine singular points.*

Proposition 2.2. *In the finite part of the phase plane on any straight line there are at most 3 singular points of the system* (1)

Proposition 2.3. *If system* (1) *has complex invariant straight lines then they occur in complex conjugated pairs (l and $\bar{l}$).*

Proposition 2.4. *The intersection point* (x_0, y_0) *of two invariant straight lines* l_1 *and* l_2 *of the system* (1) *is a singular point for this system. Moreover, if* $l_1, l_2 \in \mathbb{R}[x, y]$ *or* $l_2 \equiv \bar{l}_1$*, then* $x_0, y_0 \in \mathbb{R}$.

Proposition 2.5. *A complex straight line l can pass through at most one point with real coordinates.*

Proposition 2.6. *If a straight line passes through two distinct real points or through two complex conjugated points, then this straight line is real.*

A complex straight line passing through a real point will be called *a relative complex straight line* and a complex straight line not passing through any real point - *a purely imaginary complex straight line*.

Proposition 2.7. *Through one and the same point of a purely imaginary straight line can pass at most one real straight line.*

Proposition 2.8. *A complex invariant straight line of the system* (1) *is purely imaginary iff this straight line is parallel with his conjugate ($l \parallel \bar{l}$).*

Proposition 2.9. *If* l_1 *and* l_2 *are two parallel invariant straight lines of the system* (1)*, then only one of the following properties occurs:*

1. $l_1, l_2 \in \mathbb{R}[x, y]$*,* *2.* l_1 *is real and* l_2 *is purely imaginary,*
3. l_1 *and* l_2 *are purely imaginary,* *4.* l_1 *and* l_2 *are relative complex.*

We say that the cubic system (1) has *degenerate infinity* if the following identity holds

$$yP_3(x, y) - xQ_3(x, y) \equiv 0. \tag{13}$$

If (13) holds, then infinity consists only of singular points.

Proposition 2.10. *The identity* (13) *is invariant under affine transformation of the system* (1).

Proposition 2.11. *The invariant straight lines of the cubic system* (1) *with degenerate infinity passing through the same point* $M_0\,(x_0, y_0)$, $x_0, y_0 \in \mathbb{C}$ *have at most three slopes.*

Proposition 2.12. *Through one and the same point of a complex invariant straight line of the cubic system with degenerate infinity can not pass more than one real straight line.*

Proposition 2.13. *The straight line passing through three distinct singular points of system (1) with degenerate infinity is invariant for (1).*

Proposition 2.14. *The maximum number of the invariant straight lines for a differential cubic system with degenerate infinity is equal to six.*

Proposition 2.15. *Let the cubic system* (1) *has two concurrent invariant straight lines* l_1, l_2. *If* l_1 *has the parallel multiplicity equal to* m, $1 \leq m \leq 3$, *then this system cannot have more than* $3 - m$ *singular points on* $l_2 \setminus l_1$.

We say that three straight lines are in generic position if all lines have different slopes and no more that two lines pass through the same point.

Proposition 2.16. *If the cubic system* (1) *has three invariant straight lines in generic position, then their total parallel multiplicity is at most four.*

Proposition 2.17. *The cubic system* (1) *with degenerate infinity can have at most one triplet of parallel invariant straight lines.*

Proposition 2.18. *The cubic system* (1) *with degenerate infinity can have at most two pair of parallel invariant straight lines.*

3. THE PROOF OF THEOREM 1.1

Using the Propositions 2.17 and 2.18, the family of cubic systems [(1),(13)] with six invariant straight lines can be divided in four classes:

A) Systems with a triplet of parallel invariant straight lines;

B) Systems with two pairs of parallel invariant straight lines;

C) Systems with only a pair of parallel invariant straight lines;

D) Systems with invariant straight lines of different slopes.

The class A) was studied in [8], [12] and is characterized by the systems 1) and 2) of Theorem 1.1.

3.1. CLASS B): TWO PAIRS OF PARALLEL INVARIANT STRAIGHT LINES

For cubic systems from the class B) are possible the next 13 configurations of the straight lines:

B1) $(\mathbf{2r}, \mathbf{2r}, \mathbf{1r}, \mathbf{1r})$	B2) $(2(2)r, 2r, 1r, 1r)$
B3) $(2(2)r, 2(2)r, 1r, 1r)$	B4) $(2r, 2c_0, 1r, 1r)$
B5) $(2(2)r, 2c_0, 1r, 1r)$	**B6)** $(\mathbf{2r}, \mathbf{2c_0}, \mathbf{1c_1}, \mathbf{1c_1})$
B7) $(2(2)r, 2c_0, 1c_1, 1c_1)$	**B8)** $(\mathbf{2c_0}, \mathbf{2c_0}, \mathbf{1r}, \mathbf{1r})$
B9) $(2c_0, 2c_0, 1c_1, 1c_1)$	**B10)** $(\mathbf{2c_1}, \mathbf{2c_1}, \mathbf{1r}, \mathbf{1r})$
B11) $(2(2)c_1, 2(2)c_1, 1r, 1r)$	B12) $(2c_1, 2c_1, 1c_1, 1c_1)$
B13) $(2(2)c_1, 2(2)c_1, 1c_1, 1c_1)$	

By $(2r, 2r, 1r, 1r)$ we denoted the configurations which consists of six distinct real straight lines $l_1, \ldots, l_6 \in \mathbb{R}[x, y]$, of which l_1, l_2 and l_3, l_4 form two pairs of parallel straight lines, i.e. $l_1 \parallel l_2$, $l_3 \parallel l_4$, $l_1 \nparallel l_3$ and $l_j \nparallel l_k$, $j = 1, \ldots, 4$, $k = 5, 6$. In the case of configuration $(2c_0, 2c_0, 1c_1, 1c_1)$ we have six straight lines $l_1, \ldots, l_6$, where l_1, l_2, l_3 and l_4 are purely imaginary, l_5 and l_6 are relative complex, l_1, l_2 and l_3, l_4 form two pairs of parallel straight lines. The configuration $(2(2)r, 2r, 1r, 1r)$ consists of six real straight lines, where $l_1 \equiv l_2$, $l_3 \parallel l_4$, $l_1 \nparallel l_3$, $l_j \nparallel l_k$, $j = 1, \ldots, 4$, $k = 5, 6$ and the straight line l_1 (or l_2) has parallel multiplicity equal to two.

Remark 3.1. *The propositions 2.2, 2.5, 2.12, 2.15 and 2.16 do not allow the realization of the configurations B2) - B5), B7), B9) and B11)-B13) in the class of the cubic systems with degenerate infinity.*

Configuration B1) $(\mathbf{2r}, \mathbf{2r}, \mathbf{1r}, \mathbf{1r})$**.** Via an affine transformation and time rescaling the system [(1),(13)] with two pairs of real invariant straight lines can be written into the following form:

$$\dot{x} = x(x-1)(y+a), \quad \dot{y} = y(y-1)(x+b), \quad a, b \notin \{-1;\ 0\}. \tag{14}$$

The system (14) has the invariant straight lines $l_1 = x$, $l_2 = x - 1$, $l_3 = y$, $l_4 = y - 1$ and the singular points $(0, 0)$, $(1, 0)$, $(0, 1)$, $(1, 1)$, $(-b, -a)$. Therefore, any other invariant straight line of (14) must pass through the singular points $(0, 0)$ and $(1, 1)$ or through the singular points $(1, 0)$ and $(0, 1)$. The straight lines $l_5 = x - y$ and $l_6 = x + y - 1$ passing through these points are invariant for (14) iff $a = b = -1/2$. Replacing in (14) the values $a = -1/2$, $b = -1/2$ and $t = 2\tau$, we get the system 3) from Theorem 1.1.

Configuration B6) $(\mathbf{2r}, \mathbf{2c_0}, \mathbf{1c_1}, \mathbf{1c_1})$. The cubic system with degenerate infinity possessing two pairs of parallel invariant straight lines with the configuration $(2r, 2c_0)$ can be written as:

$$\dot{x} = x(x-1)(y+a), \quad \dot{y} = (y^2+1)(x+b), \quad b \notin \{-1;\ 0\}. \tag{15}$$

The system has the invariant straight lines $l_1 = x$, $l_2 = x - 1$, $l_3 = y + i$, $l_4 = y - i$ and the singular points $(0, -i)$, $(0, i)$, $(1, i)$, $(1, -i)$, $(-b, -a)$. Let l_5 and l_6 are two relative complex straight lines. According to the Proposition 2.3, they must pass through the intersection points of the straight lines $l_1, \ldots, l_4$. Let l_5 passes through the singular points $(0, -i)$, $(1, i)$, and l_6 through $(0, i)$, $(1, -i)$, therefore they are described by the equations $l_5 \equiv i(2x - 1) + y = 0$ and $l_6 \equiv i(1 - 2x) + y = 0$. The straight lines l_5 and l_6 are invariant for (15) if and only if $a = 0$ and $b = -1/2$. So, we obtained the system 4) from Theorem 1.1.

Configuration B8) $(\mathbf{2c_0}, \mathbf{2c_0}, \mathbf{1r}, \mathbf{1r})$ In this case the pairs of parallel invariant straight lines can be brought to form $l_{1,2} = x \pm i$ and $l_{3,4} = y \pm i$. The system [(1),(13)] with these invariant straight lines is:

$$\dot{x} = (x^2 + 1)(y + a), \quad \dot{y} = (y^2 + 1)(x + b). \tag{16}$$

The system (16) has the singular points: $(-i, -i)$, $(-i, i)$, (i, i), $(i, -i)$, $(-b, -a)$. The real straight lines l_5 and l_6 can pass only through the pairs of reciprocaly conjugate singular points $(-i, -i)$, (i, i) or $(-i, i)$, $(i, -i)$, therefore they are described by the equations $l_5 = x - y$ and $l_6 = x + y$. The invariance of these straight lines for the (16) it is conditioned by $a = b = 0$, i.e. we have the system 5) from Theorem 1.1.

Configuration B10) $(\mathbf{2c_1}, \mathbf{2c_1}, \mathbf{1r}, \mathbf{1r})$ Through an affine change of coordinates, the straight lines $l_1, \ldots, l_4$ can be brought to the form $l_{1,2} = y \pm ix$, $l_{3,4} = y \pm ix - 1$. The cubic system [(1),(13)] possessing these invariant straight lines can be written into the following form:

$$\begin{cases} \dot{x} = ax + by + bx^2 - 2axy - by^2 + x^3 + xy^2, \\ \dot{y} = -bx + ay + (a - 1)x^2 + 2bxy - (a + 1)y^2 + x^2y + y^3. \end{cases} \tag{17}$$

This system has the singular points: $(0, 0)$, $(-i/2, 1/2)$, $(0, 1)$, $(i/2, 1/2)$, $(-b, a)$. The real invariant straight lines l_5 and l_6 can pass only through the singular points $(0, 0)$, $(0, 1)$ and $(-i/2, 1/2)$, $(i/2, 1/2)$, therefore they are described by the equations $l_5 = x$ and $l_6 = 2y - 1$. These straight lines are invariant for the system (17) iff $b = 0$ and $a = 1/2$. Thus, was obtained the system 6) from Theorem 1.1.

3.2. CLASS C): ONE PAIR OF PARALLEL INVARIANT STRAIGHT LINES

For cubic systems from the class C) are possible the next 9 configurations of the straight lines:

C1) $(\mathbf{2r}, \mathbf{1r}, \mathbf{1r}, \mathbf{1r}, \mathbf{1r})$	C2) $(2(2)r, 2r, 1r, 1r, 1r)$
C3) $(2r, 1r, 1r, 1c_1, 1c_1)$	C4) $(2(2)r, 1r, 1r, 1c_1, 1c_1)$
C5) $(2r, 1c_1, 1c_1, 1c_1, 1c_1)$	C6) $(2(2)r, 1c_1, 1c_1, 1c_1, 1c_1)$
C7) $(2c_0, 1r, 1r, 1r, 1r)$	**C8)** $(\mathbf{2c_0}, \mathbf{1r}, \mathbf{1r}, \mathbf{1c_1}, \mathbf{1c_1})$
C9) $(2c_0, 1c_1, 1c_1, 1c_1, 1c_1)$	

Remark 3.2. *The propositions 2.2, 2.5, 2.11, 2.15 and 2.16 do not allow the realization of the configurations C2) - C7) and C9) in the class of the cubic systems with degenerate infinity.*

Configuration C1) $(\mathbf{2r}, \mathbf{1r}, \mathbf{1r}, \mathbf{1r}, \mathbf{1r})$. Let the straight lines l_1, l_2, l_3, l_4 with the configuration $(2r, 1r, 1r)$ are invariant for the system [(1),(13)]. These straight lines can be brought to the form $l_1 = x$, $l_2 = x - 1$, $l_3 = y$ and $l_4 = x + y - 1$. Therefore, the system [(1),(13)] has the following form:

$$\begin{cases} \dot{x} = x(x-1)(b_{01} + b_{11} + a_{30}x + a_{21}y), \\ \dot{y} = y(b_{01} + b_{11}x - b_{01}y + a_{30}x^2 + a_{21}xy), \end{cases} \tag{18}$$

The intersection points of the straight lines of the system (18) are $(0,0)$, $(0,1)$ and $(1,0)$. Through the singular point $(1,0)$ pass the invariant straight lines l_2, l_3 and l_4. According to the Proposition 2.11 any other real invariant straight line must pass through $(0,0)$ or $(0,1)$. Let l_5 and l_6 are real invariant straight lines of the system (18), according to the Proposition 2.2 their intersection point belongs to the l_2: $l_5 \cap l_6 = (1, a+1) \in l_2$, where $a \neq 0$, $a \in \mathbb{R}$. Let l_5 passes through $(0,0)$, $(1, a+1)$ and l_6 through $(0,1)$, $(1, a+1)$, i.e. they are described by the equations $l_5 = (a+1)x - y$, $l_6 = ax - y + 1$. These straight lines are invariant for the (18) iff $a_{30} = a$, $a_{21} = -b_{11} = -2$. Using these condition and rescaling the time $t = -1/b_{01}\tau$ in (18), we obtain the system 7) from the Theorem 1.1.

Configuration C8) $(\mathbf{2c_0}, \mathbf{1r}, \mathbf{1r}, \mathbf{1c_1}, \mathbf{1c_1})$. Let the system [(1),(13)] has four invariant straight lines of the configuration $(2c_0, 1c_1, 1c_1)$. The straight lines can be written as $l_{1,2} = x - a \pm i$ and $l_{3,4} = y \pm ix$. The system [(1),(13)] with these invariant straight lines has the form:

$$\begin{cases} \dot{x} = ((x-a)^2 + 1)(a_{30}x + a_{21}y), \\ \dot{y} = (a^2+1)(a_{30}y - a_{21}x) + b_{20}x^2 - 2aa_{30}xy + (b_{20} - 2aa_{21})y^2 + \\ \qquad + a_{30}x^2y + a_{21}xy^2. \end{cases} \tag{19}$$

This system has the singular points $O_1(a - i, 1 + ai)$, $O_2(a + i, 1 - ai)$, $O_3(a + i, -1 + ai)$, $O_4(a - i, -1 - ai)$, $O_5(0,0)$, $O_6(a_{21}(1 + a^2)/b_{20}, -a_{30}(1 + a^2)/b_{20})$, $O_1 = l_1 \cap l4$, $O_2 = l_2 \cap l_3$, $O_3 = l_2 \cap l_4$, $O_4 = l_1 \cap l_3$. Any other real invariant straight line of the system (19) must pass through one of the two pairs of conjugate complex singular points O_1, O_2 and O_3, O_4. Therefore, $l_5 \equiv ax + y - a^2 - 1 = 0$ and $l_6 \equiv ax - y - a^2 - 1 = 0$. This straight lines are invariant for the system (19) iff $a_{30} = 0$ and $b_{20} = a$. Moreover, after rescaling the time $t = 1/a_{21}\tau$ we get the system 8) from the Theorem 1.1

3.3. CLASS D): INVARIANT STRAIGHT LINES WITH DIFFERENT SLOPES

For cubic systems from the class D) are possible the next 4 configurations of the straight lines:

D1) $(1c_1, 1c_1, 1c_1, 1c_1, 1c_1, 1c_1)$ **D2)** $(\mathbf{1c_1}, \mathbf{1c_1}, \mathbf{1c_1}, \mathbf{1c_1}, \mathbf{1r}, \mathbf{1r})$
D3) $(1c_1, 1c_1, 1r, 1r, 1r, 1r)$ **D4)** $(\mathbf{1r}, \mathbf{1r}, \mathbf{1r}, \mathbf{1r}, \mathbf{1r}, \mathbf{1r})$

Remark 3.3. *The propositions 2.2, 2.5 and 2.11 do not allow the realization of the configurations D1) and D3) in the class of the cubic systems with degenerate infinity.*

Configuration D2) $(\mathbf{1c_1}, \mathbf{1c_1}, \mathbf{1c_1}, \mathbf{1c_1}, \mathbf{1r}, \mathbf{1r})$**.** Let the system [(1),(13)] has the invariant straight lines $l_j \in \mathbb{C}[x, y] \setminus \mathbb{R}[x, y]$, $j = \overline{1, 4}$, $l_j = \bar{l}_{j+1}$, $j = 1, 3$, $l_j \nparallel l_k$, $j \neq k$. Via an affine transformation and time rescaling we can write $l_{1,2} \equiv y \pm ix = 0$, $l_{3,4} = y - (a \pm bi)x - 1 = 0$, $a, b \in \mathbb{R}$, $b(|a| + |b \pm 1|) \neq 0$. There are two affine different systems [(1),(13)] with these invariant straight lines:

$$\begin{cases} \dot{x} = y + x^2 + 2axy - y^2 + (2a - b_{02})x^3 + (a^2 + b^2 - 1)x^2y - b_{02}xy^2, \\ \dot{y} = -x + (b_{02} - 2a)x^2 + 2xy + b_{02}y^2 + (2a - b_{02})x^2y + \\ \qquad +(a^2 + b^2 - 1)xy^2 - b_{02}y^3; \end{cases} \tag{20}$$

$$\begin{cases} \dot{x} = x + cy + (2a + c)x^2 + 2(-1 + ac)xy - cy^2 + (-2 + a^2 + b^2 - b_{02} + \\ \qquad +2ac)x^3 + (-2a - c + a^2c + b^2c)x^2y - (1 + b_{02})xy^2, \\ \dot{y} = -cx + y + (2 + b_{02} - 2ac)x^2 + 2(a + c)xy + b_{02}y^2 + (-2 + a^2 + \\ \qquad b^2 - b_{02} + 2ac)x^2y + (-2a - c + a^2c + b^2c)xy^2 - (1 + b_{02})y^3. \end{cases} \tag{21}$$

Let $O_{j,k}$ is the intersection point of the straight lines l_j and l_k, $j \neq k$. Then we have

$$O_{1,2} = (0, 0),\ O_{1,3} = (-1/(-i + a + bi), 1/(1 - b + ai)),$$
$$O_{1,4} = (-1/(-i + a - bi), 1/(1 + b + ai)),\ \ O_{3,4} = (0, 1),\ \ O_{2,3} \equiv \overline{O_{1,4}} \text{ and}$$
$$O_{2,4} \equiv \overline{O_{1,3}}.$$

The straight line passing through the singular points $O_{1,3}$ and $O_{2,4}$ ($O_{1,4}$ and $O_{2,3}$) it is described by the equation $1 + ax - y + by = 0$ ($1 + ax - y - by = 0$). Using only the information provided by the singular points we can state that besides the invariant straight lines $l_{1,2,3,4}$, both systems can have invariant straight lines described by the equations $1 + ax - y + by = 0$, $1 + ax - y - by = 0$ and $x = 0$.

The straight line $x = 0$ can't be invariant for (20), because the coefficients of the monomials y, $-y^2$ from the right side of the first equation of the system (20) are constant. The straight lines $l_5 = 1 + ax - y + by$ and $l_6 = 1 + ax - y - by$ are invariant for (20) iff $b_{02} = a$. Therefore, replacing $b_{02} = a$ in (20) we obtain the system 9) from the Theorem 1.1. The straight line $1 + ax - y + by = 0$ is not invariant for the system (21). Asking for $x = 0$ to be invariant we obtain $c = 0$ and $b_{02} = b - 2$ or $b_{02} = -b - 2$, i.e. the system 10) from the Theorem 1.1 or the system

$$\dot{x} = \varphi(x, y, a, -b, 0),\ \ \dot{y} = \psi(x, y, a, -b, 0),\ \ b(b + 1)(|a| + |b - 1|) \neq 0\ ,$$

where $\varphi(x, y, a, b, c)$ and $\psi(x, y, a, b, c)$ are the right sides of the system (21). The two systems are topologically equivalent.

Configuration D4) (1r, 1r, 1r, 1r, 1r, 1r). Let the system [(1),(13)] has at least five real invariant straight lines with diffefrent slopes $l_j,\ j = \overline{1,5}$. Via an affine transformation we can bring these straight lines to be described by the equations: $x = 0,\ y = 0,\ y = x,\ y = ax + 1,\ y = bx + 1,\ ab(a-1)(b-1) \neq 0,\ a < b$. The cubic system with these invariant straight lines has the form:

$$\begin{cases} \dot{x} = x(1 + (a+b)x - 2y + abx^2 - \alpha xy + cy^2), \\ \dot{y} = y(1 + \alpha x - (c+1)y + abx^2 - \alpha xy + cy^2). \end{cases} \tag{22}$$

Let $O_{j,k} = l_j \cap l_k,\ j \neq k$. Any other invariant straight line l_6 of the cubic system (22) must pass through the singular points $O_{24} = (-1/a, 0)$ and $O_{3,5} = (1/(1-b), 1/(1-b))$ or through the singular points $O_{2,5} = (-1/b, 0)$ and $O_{3,4} = (1/(1-b), 1/(1-b)$, therefore it is described by the equation: $ax + (b-a-1)y + 1 = 0,\ b - a - 1 \neq 0$ or $bx - (b-a+1)y + 1 = 0,\ b - a + 1 \neq 0$. The straight line $ax + (b-a-1)y + 1 = 0$ is invariant for the system (22) iff $c = 1 + a - b$. Replacing $c = 1 + a - b$ in (22) we obtain the system 11) from the Theorem 1.1.

The straight line $bx - (b-a+1)y + 1 = 0$ is invariant for the system (22), iff $c = 1 - a + b$, but this system is affine equivalent with the system 11).

3.4. QUALITATIVE INVESTIGATION OF THE SYSTEMS 3)-11)

In this section, the qualitative study of the systems 3) – 11) from Theorem 1.1 will be done. For this purpose, in order to determine the topological behavior of trajectories, the finite and the infinite singular points will be examined. This information and the information provided by the existence of invariant straight lines, we will be taken into account constructing the phase portraits of systems 3) – 11) on Poincaré disk.

We denoted by SP singular points; λ_1 and λ_2 the characteristic roots of the SP; TSP – type of SP; S – saddle ($\lambda_1\lambda_2 < 0$); N^s – stable node ($\lambda_1, \lambda_2 < 0$); N^i – instable node ($\lambda_1, \lambda_2 > 0$); $DN^{s(i)}$ – "decritic" stable (instable) node ($\lambda_1 = \lambda_2 \neq 0$); C - centre.

In the next tables, the first column will indicate the singular points of the systems; the second column - the eigenvalues corresponding to these singular points and the third column - the types of the singularities. All these points are simple and together with the invariant straight lines, entirely determine the phase portrait of each of the systems 3)-11).

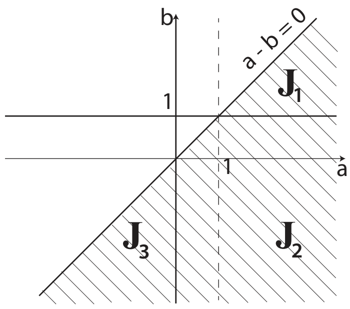

Fig. 12

System 11) from Th. 1.1 was obtained for $ab(a-1)(b-1)(b-a-1) \neq 0,\ a > b$. In the space of parameters we get three sectors as shown in Fig. 12. The sectors J_1 and J_3 provide two topologically equivalent phase portraits.

Tab. 1-3

System 3)					
SP	$\lambda_1;\lambda_2$	TSP	SP	$\lambda_1;\lambda_2$	TSP
$O_1(0,0)$	$-a\,;-a$	DN^s	$O_2(1,0)$	$a;-a$	S
$O_3(0,1)$	$-a\,;-a$	DN^s	$O_4(1,1)$	$a;-a$	S
$O_5(-a,-a)$	$-a\,;-a$	DN^s	$X_\infty(1,0,0)$	$a;-a$	S
$Y_\infty(0,1,0)$	$-1;-1$	DN^s			
System 4)					
$O(\frac{1}{2},0)$	$\pm i$	C	$X_\infty(1,0,0)$	$\pm 2i$	C
$Y_\infty(0,1,0)$	$1\,;-1$	S			
System 5)					
$O(0,0)$	± 1	S	$X_\infty(1,0,0)$	$\pm i$	C
$Y_\infty(0,1,0)$	$\pm i$	C			

SP	$\lambda_1;\lambda_2$	TSP	SP	$\lambda_1;\lambda_2$	TSP
System 6)					
$O_1(0,0)$	$1;1$	DN^i	$O_2(0,1)$	$1;1$	DN^i
$O_3(0,\frac{1}{2})$	$\pm\frac{1}{2}$	S			
System 7)					
$O_1(0,0)$	$-1;-1$	DN^s	$O_2(0,1)$	$1;1$	DN^i
$O_3(1,0)$	$a+1;a+1$	DN^i	$O_4(-\frac{1}{a},0)$	$\pm\frac{a+1}{a}$	S
$O_5(1,a+1)$	$-a-1;-a-1$	N^s	$O_6(\frac{1}{a+2},\frac{a+1}{a+2})$	$\pm\frac{a+1}{a+2}$	S
$I_\infty(0,1,0)$	$1;-1$	S			
System 8)					
$O_1(0,0)$	$\pm i(a^2+1)$	C	$O_2(\frac{a^2+1}{a},0)$	$\pm\frac{a^2+1}{a}$	S
$I_\infty(0,1,0)$	$\pm i$	C			
System 9)					
$O_1(0,0)$	$\pm i$	C	$O_2(0,1)$	$\pm bi$	C
$O_3(-\frac{1}{a},0)$	$\pm\frac{b}{a}$	S			

SP	$\lambda_1; \lambda_2$	TSP	
System 10)		$b<0$	$b>0$
$O_1(0,0)$	$1; 1$	DN^i	
$O_2(0, \frac{1}{1-b})$	$\pm\frac{b}{1-b}$	S	
$O_3(0,1)$	$-b; -b$	DN^i	DN^s
System 11)		J_1 (J_3)	J_2
$O_1(0,0)$	$1; 1$	N^i	
$O_2(0,1)$	$a-b; a-b$	N^i	
$O_3(0, \frac{1}{1+a-b})$	$\pm\frac{a-b}{1+a-b}$	S	
$O_4(-\frac{1}{a}, 0)$	$\frac{b-a}{a}; \frac{b-a}{a}$	$N^s(N^i)$	N^s
$O_5(-\frac{1}{b}, 0)$	$\pm\frac{a-b}{b}$	S	
$O_6(\frac{1}{1-a}, \frac{1}{1-a})$	$\pm\frac{b-a}{a-1}$	S	
$O_7(\frac{1}{1-b}, \frac{1}{1-b})$	$\frac{a-b}{b-1}; \frac{a-b}{b-1}$	$N^i(N^s)$	N^s

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GENERALIZED FRACTIONAL INTEGRATION OF THE PRODUCT OF TWO ℵ-FUNCTIONS ASSOCIATED WITH THE APPELL FUNCTION F_3

Ram Kishore Saxena, Jeta Ram, Dinesh Kumar

Department of Mathematics and Statistics, Jai Narain Vyas University, Jodhpur, India.

ram.saxena@yahoo.com, bishnoi_jr@yahoo.com dinesh_dino03@yahoo.com

Abstract The object of the present article is to study and develop the generalized fractional integral operators introduced by Saigo and Maeda for ℵ-function. The considered generalized fractional integration operators contain the Appell hypergeometric function F_3 as a kernel. We establish two results of the product of two ℵ-functions involving Saigo-Maeda operators which are also believed to be new. On account of the general nature of the Saigo-Maeda operators and also of the ℵ-functions, a large number of new and known results involving Saigo, Riemann-Liouville and Erdélyi-Kober integral operators are special cases of our main results.

The results obtained provide extension of the results given by Ram and Kumar [6] for the generalized fractional integration of the product of two H-functions.

Keywords: Aleph function, generalized fractional integration, fractional calculus, Mellin-Barnes type integrals, Appell function F_3, H-function, I-function.

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1. INTRODUCTION AND PRELIMINARIES

The object of this paper is to study the generalized fractional integration operators associated with the Appell function F_3 [10] as a kernel, introduced by Saigo-Maeda [9].

The Aleph-function is defined by means of a Mellin-Barnes type integral in the following manner [13, 14]:

$$\aleph[z] = \aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[z\left|\begin{matrix}(a_j,A_j)_{1,n},\ldots,[\tau_j(a_j,A_j)]_{n+1,p_i}\\(b_j,B_j)_{1,m},\ldots,[\tau_j(b_j,B_j)]_{m+1,q_i}\end{matrix}\right.\right] := \frac{1}{2\pi i}\int_L \Omega^{m,n}_{p_i,q_i,\tau_i;r}(s)\, z^{-s}\, ds, \quad (1)$$

where $z \neq 0$, $i = \sqrt{-1}$ and

$$\Omega^{m,n}_{p_i,q_i,\tau_i;r}(s) = \frac{\prod_{j=1}^{m}\left\{\Gamma\left(b_j + B_j s\right)\right\}\prod_{j=1}^{n}\left\{\Gamma\left(1 - a_j - A_j s\right)\right\}}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\left\{\Gamma\left(1 - b_{ji} - B_{ji}s\right)\right\}\prod_{j=n+1}^{p_i}\left\{\Gamma\left(a_{ji} + A_{ji}s\right)\right\}}. \quad (2)$$

The integration path $L = L_{i\gamma\infty}$, $(\gamma \in \Re)$ extends from $\gamma - i\infty$ to $\gamma + i\infty$, and is such that the poles of $\Gamma\left(1 - a_j - A_j s\right)$, $j = \left(\overline{1,n}\right)$ (the symbol $\left(\overline{1,n}\right)$ is used for 1,2,...,n) do not coincide with the poles of $\Gamma\left(b_j + B_j s\right),\ j = \left(\overline{1,m}\right)$. The parameters p_i, q_i are non-negative integers satisfying the condition $0 \le n \le p_i,\ 1 \le m \le q_i,\ \tau_i > 0$ for $i = \overline{1,r}$. The parameters $A_j, B_j, A_{ji}, B_{ji} > 0$ and $a_j, b_j, a_{ji}, b_{ji} \in C$. The empty product in (2) is interpreted as unity. The existence conditions for the defining integral (1) are given below:

$$\varphi_l > 0, \quad \left|\arg(z)\right| < \frac{\pi}{2}\varphi_l\,, \quad l = \overline{1,r}\,; \tag{3}$$

$$\varphi_l \ge 0, \quad \left|\arg(z)\right| < \frac{\pi}{2}\varphi_l \ \textit{and}\ \Re\{\zeta_l\} + 1 < 0\,, \tag{4}$$

where

$$\varphi_l = \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j - \tau_l\left(\sum_{j=n+1}^{p_l} A_{jl} + \sum_{j=m+1}^{q_l} B_{jl}\right) \tag{5}$$

$$\zeta_l = \sum_{j=1}^{m} b_j - \sum_{j=1}^{n} a_j + \tau_l\left(\sum_{j=m+1}^{q_l} b_{jl} - \sum_{j=n+1}^{p_l} a_{jl}\right) + \frac{1}{2}\left(p_l - q_l\right), \quad \left(l = \overline{1,r}\right). \tag{6}$$

Remark 1.1. *For $\tau_i = 1,\ i = \overline{1,r}$ in (1), we get the I-function due to V.P. Saxena [16], defined in the following manner:*

$$I^{m,n}_{p_i,q_i;r}[z] = \aleph^{m,n}_{p_i,q_i,1;r}[z] = \aleph^{m,n}_{p_i,q_i,1;r}\left[z\left|\begin{matrix}(a_j,A_j)_{1,n},\ldots,(a_j,A_j)_{n+1,p_i}\\(b_j,B_j)_{1,m},\ldots,(b_j,B_j)_{m+1,q_i}\end{matrix}\right.\right] :=$$

$$:= \frac{1}{2\pi i}\int_L \Omega^{m,n}_{p_i,q_i,1;r}(s)\, z^{-s}\, ds, \tag{7}$$

where the kernel $\Omega^{m,n}_{p_i,q_i,1;r}(s)$ is given in (2). The existence conditions for the integral in (1.1) are the same as given in (3) - (6) with $\tau_i = 1,\ i = \overline{1,r}$.

Remark 1.2. *If we further set $r = 1$, then (1.1) reduces to the familiar H- function [3, 4]:*

$$H^{m,n}_{p,q}[z] = \aleph^{m,n}_{p_i,q_i,1;1}[z] = \aleph^{m,n}_{p_i,q_i,1;1}\left[z\left|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right.\right] := \frac{1}{2\pi i}\int_L \Omega^{m,n}_{p_i,q_i,1;1}(s)\, z^{-s}\, ds, \tag{8}$$

where the kernel $\Omega^{m,n}_{p_i,q_i,1;1}(s)$ can be obtained from (2).

Remark 1.3. *Fractional integration of Aleph function is discussed by Saxena and Pogány [14]. A detailed account of $\aleph$- function is given in the papers by Saxena et al. [13, 14].*

The $\aleph$- function of two variables occurring in the present paper will be defined and represented in the following manner, which is also believed to be new and given first time by authors as following:

$$\aleph\,[x,y]$$
$$= \aleph^{0,n:\,m_1,n_1;m_2,n_2}_{p,q:p_i,q_i,\tau_i;p'_i,q'_i,\tau'_i;r}\left[\begin{array}{c|c} x & \left(a_j;\alpha_j,A_j\right)_{1,p}:\left(c_j,C_j\right)_{1,n_1},...,\left[\tau_j\left(c_j,C_j\right)\right]_{n_1+1,p_i};\left(e_j,E_j\right)_{1,n_2},...,\left[\tau'_j\left(e_j,E_j\right)\right]_{n_2+1,p'_i} \\ y & \left(b_j;\beta_j,B_j\right)_{1,q}:\left(d_j,D_j\right)_{1,m_1},...,\left[\tau_j\left(d_j,D_j\right)\right]_{m_1+1,q_i};\left(f_j,F_j\right)_{1,m_2},...,\left[\tau'_j\left(f_j,F_j\right)\right]_{m_2+1,q'_i}\end{array}\right]$$
$$= \frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}\phi\,(s,\xi)\;\theta_1\,(s)\;\theta_2\,(\xi)\;x^{-s}y^{-\xi}\,ds\,d\xi, \tag{9}$$

where

$$\phi\,(s,\xi)\;=\;\frac{\prod_{j=1}^{n}\left\{\Gamma\left(1-a_j-\alpha_j s-A_j\xi\right)\right\}}{\prod_{j=n+1}^{p}\left\{\Gamma\left(a_j+\alpha_j s+A_j\xi\right)\right\}\prod_{j=1}^{q}\left\{\Gamma\left(1-b_j-\beta_j s-B_j\xi\right)\right\}}, \tag{10}$$

$$\theta_1(s)\;=\;\Omega^{m_1,n_1}_{p_i,q_i,\tau_i;\,r}\,(s)\;=\;\frac{\prod_{j=1}^{m_1}\left\{\Gamma\left(d_j+D_js\right)\right\}\prod_{j=1}^{n_1}\left\{\Gamma\left(1-c_j-C_js\right)\right\}}{\sum_{i=1}^{r}\tau_i\,\prod_{j=m_1+1}^{q_i}\left\{\Gamma\left(1-d_{ji}-D_{ji}s\right)\right\}\prod_{j=n_1+1}^{p_i}\left\{\Gamma\left(c_{ji}+C_{ji}s\right)\right\}}, \tag{11}$$

$$\theta_2(\xi)\;=\;\Omega^{m_2,n_2}_{p'_i,q'_i,\tau'_i;\,r}\,(\xi)\;=\;\frac{\prod_{j=1}^{m_2}\left\{\Gamma\left(f_j+F_j\xi\right)\right\}\prod_{j=1}^{n_2}\left\{\Gamma\left(1-e_j-E_j\xi\right)\right\}}{\sum_{i=1}^{r}\tau'_i\,\prod_{j=m_2+1}^{q'_i}\left\{\Gamma\left(1-f_{ji}-F_{ji}\xi\right)\right\}\prod_{j=n_2+1}^{p'_i}\left\{\Gamma\left(e_{ji}+E_{ji}\xi\right)\right\}}, \tag{12}$$

2. GENERALIZED FRACTIONAL CALCULUS OPERATORS

Let $\alpha,\alpha',\beta,\beta',\gamma\,\in\,C$, $Re\,(\gamma)\,>\,0$ and $x\,\in\,R_+$, then the generalized fractional integration operators involving Appell function F_3 as a kernel are defined by Saigo and Maeda [9] as following:

$$\left(I^{\alpha,\alpha',\beta,\beta',\gamma}_{0,x}f\right)(x)=\frac{x^{-\alpha}}{\Gamma\,(\gamma)}\int_0^x(x-t)^{\gamma-1}\,t^{-\alpha'}F_3\,(\alpha,\alpha',\beta,\beta';\gamma;1-t/x,1-x/t)\,f\,(t)\;dt, \tag{13}$$

and

$$\left(I^{\alpha,\alpha',\beta,\beta',\gamma}_{x,\infty}f\right)(x)=\frac{x^{-\alpha'}}{\Gamma\,(\gamma)}\int_x^\infty(t-x)^{\gamma-1}\,t^{-\alpha}F_3\,(\alpha,\alpha',\beta,\beta';\gamma;1-x/t,1-t/x)\,f\,(t)\;dt, \tag{14}$$

here, $Re\,(\gamma)$ denotes the real part of γ, and $F_3\,(\alpha,\alpha',\beta,\beta';\gamma;z,\xi)$ is the familiar Appell hypergeometric function of two variables is defined by:

$$F_3\,(\alpha,\alpha',\beta,\beta';\gamma;z,\xi)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_m\,(\alpha')_n\,(\beta)_m\,(\beta')_n}{(\gamma)_{m+n}}\frac{z^m}{m!}\frac{\xi^n}{n!}\quad(|z|<1,\;|\xi|<1)\,. \tag{15}$$

Lemma 2.1 ([9], p. 394, eqns. (4.18) and (4.19)). *Let $\alpha, \alpha', \beta, \beta', \gamma \in C$, then there holds the following power function formulae:*
(i) If $Re(\gamma) > 0$, $Re(\rho) > \max[0, Re(\alpha + \alpha' + \beta - \gamma), Re(\alpha' - \beta')]$, then

$$I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1} = x^{\rho-\alpha-\alpha'+\gamma-1} \frac{\Gamma(\rho)\,\Gamma(\rho+\gamma-\alpha-\alpha'-\beta)\,\Gamma(\rho+\beta'-\alpha')}{\Gamma(\rho+\gamma-\alpha-\alpha')\,\Gamma(\rho+\gamma-\alpha'-\beta)\,\Gamma(\rho+\beta')}, \quad (16)$$

(ii) If $Re(\gamma) > 0$, $Re(\rho) < 1 + \min[Re(-\beta), Re(\alpha + \alpha' - \gamma), Re(\alpha + \beta' - \gamma)]$, then

$$I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1} = x^{\rho-\alpha-\alpha'+\gamma-1} \frac{\Gamma(1+\alpha+\alpha'-\gamma-\rho)\,\Gamma(1+\alpha+\beta'-\gamma-\rho)\,\Gamma(1-\beta-\rho)}{\Gamma(1-\rho)\,\Gamma(1+\alpha+\alpha'+\beta'-\gamma-\rho)\,\Gamma(1+\alpha-\beta-\rho)}. \quad (17)$$

Remark 2.1. *Generalized fractional integration formulas for the product of special functions are discussed by Ram and Kumar [6], Gupta et al. [2] and Saigo et al. [10].*

3. LEFT-SIDED GENERALIZED FRACTIONAL INTEGRATION OF THE PRODUCT OF TWO ℵ-FUNCTIONS

In this section, we study the left-sided generalized fractional integration $I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}$ defined in (13).

Theorem 3.1. *Let $\alpha, \alpha', \beta, \beta', \gamma, \sigma, \lambda, \omega \in C$, $Re(\gamma) > 0$, $(\mu, \upsilon > 0)$, and $Re(\sigma) + \mu \min_{1\le j\le m_1} Re\left(\frac{b_j}{B_j}\right) + \upsilon \min_{1\le j\le m_2} Re\left(\frac{d_j}{D_j}\right) > \max[0, Re(\alpha' - \beta'), Re(\alpha + \alpha' + \beta' - \gamma)]$. Further, let the constants $a_j, b_j, a_{ji}, b_{ji} \in C$, $A_j, B_j, A_{ji}, B_{ji} \in R_+$ $(i = 1, ..., p_i;\ j = 1, ..., q_i)$; $c_j, d_j, c_{ji}, d_{ji} \in C$, $C_j, D_j, C_{ji}, D_{ji} \in R_+$ $\left(i = 1, ..., p_i';\ j = 1, ..., q_i'\right)$, $\tau_i, \tau_i' > 0$ for $i = \overline{1, r}$ also satisfy the conditions are given (3) - (6). Then the left-sided generalized fractional integration $I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}$ of the product of two ℵ-functions exists and the following relation holds:*

$$\left\{ I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} \left(t^{\sigma-1} \aleph_{p_i,q_i,\tau_i;r}^{m_1,n_1} \left[\lambda t^{\mu} \,\middle|\, \begin{matrix} (a_j, A_j)_{1,n_1}, ..., [\tau_j(a_j, A_j)]_{n_1+1,p_i} \\ (b_j, B_j)_{1,m_1}, ..., [\tau_j(b_j, B_j)]_{m_1+1,q_i} \end{matrix} \right] \right.\right.$$
$$\left.\left. .\,\aleph_{p_i',q_i',\tau_i';r}^{m_2,n_2} \left[\omega t^{\upsilon} \,\middle|\, \begin{matrix} (c_j, C_j)_{1,n_2}, ..., [\tau_j'(c_j, C_j)]_{n_2+1,p_i'} \\ (d_j, D_j)_{1,m_2}, ..., [\tau_j'(d_j, D_j)]_{m_2+1,q_i'} \end{matrix} \right] \right) \right\}(x) = x^{\sigma-\alpha-\alpha'+\gamma-1}$$
$$\times\, \aleph_{3,3:p_i,q_i,\tau_i;p_i',q_i',\tau_i';r}^{0,3:\,m_1,n_1;m_2,n_2} \left[\begin{matrix} \lambda x^{\mu} \\ \omega x^{\upsilon} \end{matrix} \,\middle|\, \begin{matrix} (1-\sigma;\mu,\upsilon),\ (1-\sigma-\gamma+\alpha+\alpha'+\beta;\mu,\upsilon), \\ (1-\sigma-\gamma+\alpha+\alpha';\mu,\upsilon),\ (1-\sigma-\beta';\mu,\upsilon), \end{matrix} \right.$$
$$\left. \begin{matrix} (1-\sigma-\beta'+\alpha';\mu,\upsilon):(a_j,A_j)_{1,n_1},...,[\tau_j(a_j,A_j)]_{n_1+1,p_i};(c_j,C_j)_{1,n_2},...,[\tau_j'(c_j,C_j)]_{n_2+1,p_i'} \\ (1-\sigma-\gamma+\alpha'+\beta;\mu,\upsilon):(b_j,B_j)_{1,m_1},...,[\tau_j(b_j,B_j)]_{m_1+1,q_i};\ (d_j,D_j)_{1,m_2},...,[\tau_j'(d_j,D_j)]_{m_2+1,q_i'} \end{matrix} \right]. \quad (18)$$

Proof. In order to prove (18), we first express the product of two Aleph functions occurring on the left hand side of (18) in terms of Mellin-Barnes contour integral with the help of equation (1) and interchanging the order of integration, we obtain (say I):

$$I = \frac{1}{(2\pi i)^2}\int_{L_1}\Omega^{m_1,n_1}_{p_i,q_i,\tau_i;r}(s)\,\lambda^{-s}ds\int_{L_2}\Omega^{m_2,n_2}_{p'_i,q'_i,\tau'_i;r}(\xi)\,\omega^{-\xi}\,d\xi\left(I^{\alpha,\alpha',\beta,\beta',\gamma}_{0+}\,t^{\sigma-\mu s-\upsilon\xi-1}\right)(x)$$

$$= \frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}\Omega^{m_1,n_1}_{p_i,q_i,\tau_1;r}(s)\,\Omega^{m_2,n_2}_{p'_i,q'_i,\tau'_i;r}(\xi)\,\lambda^{-s}\omega^{-\xi}\left(I^{\alpha,\alpha',\beta,\beta',\gamma}_{0+}\,t^{\sigma-\mu s-\upsilon\xi-1}\right)(x)\,ds\,d\xi,$$

from (16), we arrive at

$$I = \frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}\frac{\Gamma(\sigma-\mu s-\upsilon\xi)\,\Gamma(\sigma-\mu s-\upsilon\xi+\gamma-\alpha-\alpha'-\beta)}{\Gamma(\sigma-\mu s-\upsilon\xi+\gamma-\alpha-\alpha')\,\Gamma(\sigma-\mu s-\upsilon\xi+\gamma-\alpha'-\beta)}$$

$$\times\frac{\Gamma(\sigma-\mu s-\upsilon\xi+\beta'-\alpha)}{\Gamma(\sigma-\mu s-\upsilon\xi+\beta')}\;\frac{\prod_{j=1}^{m_1}\left\{\Gamma\left(b_j+B_js\right)\right\}\prod_{j=1}^{n_1}\left\{\Gamma\left(1-a_j-A_js\right)\right\}}{\sum_{i=1}^{r}\tau_i\prod_{j=m_1+1}^{q_i}\left\{\Gamma\left(1-b_{ji}-B_{ji}s\right)\right\}\prod_{j=n_1+1}^{p_i}\left\{\Gamma\left(a_{ji}+A_{ji}s\right)\right\}}$$

$$\times\frac{\prod_{j=1}^{m_2}\left\{\Gamma\left(d_j+D_j\xi\right)\right\}\prod_{j=1}^{n_2}\left\{\Gamma\left(1-c_j-C_j\xi\right)\right\}}{\sum_{i=1}^{r}\tau'_i\prod_{j=m_2+1}^{q'_i}\left\{\Gamma\left(1-d_{ji}-D_{ji}\xi\right)\right\}\prod_{j=n_2+1}^{p'_i}\left\{\Gamma\left(c_{ji}+C_{ji}\xi\right)\right\}}$$

$$\times\, x^{\sigma-\mu s-\upsilon\xi-\alpha-\alpha'+\gamma-1}\,\lambda^{-s}\omega^{-\xi}\,ds\,d\xi\,. \tag{19}$$

By interpreting the Mellin-Barnes counter integral thus obtained in terms of the ℵ-function of two variables as given in (9), we obtain the result (18). This completes the proof of Theorem 1. ∎

Special Cases of Theorem 1:

If we put $\tau_i = 1,\ \tau'_i = 1\ (i = 1, 2, ..., r)$ in (18) and take (1.1) into account, then we arrive at the following result in the term of I-function [16].

Corollary 3.1.

$$\left\{I^{\alpha,\alpha',\beta,\beta',\gamma}_{0+}\left(t^{\sigma-1}\,I^{m_1,n_1}_{p_i,q_i;r}\left[\lambda\,t^{\mu}\left|\begin{matrix}\left(a_j,A_j\right)_{1,n_1},...,\left(a_j,A_j\right)_{n_1+1,p_i}\\\left(b_j,B_j\right)_{1,m_1},...,\left(b_j,B_j\right)_{m_1+1,q_i}\end{matrix}\right.\right]\right.\right.$$

$$\left.\left..\,I^{m_2,n_2}_{p'_i,q'_i;r}\left[\omega\,t^{\upsilon}\left|\begin{matrix}\left(c_j,C_j\right)_{1,n_2},...,\left(c_j,C_j\right)_{n_2+1,p'_i}\\\left(d_j,D_j\right)_{1,m_2},...,\left(d_j,D_j\right)_{m_2+1,q'_i}\end{matrix}\right.\right]\right)\right\}(x) = x^{\sigma-\alpha-\alpha'+\gamma-1}$$

$$\times\, I^{0,3:\,m_1,n_1;m_2,n_2}_{3,3:p_i,q_i;p'_i,q'_i;r}\left[\begin{matrix}\lambda x^{\mu}\\\omega x^{\upsilon}\end{matrix}\left|\begin{matrix}(1-\sigma;\mu,\upsilon),\,(1-\sigma-\gamma+\alpha+\alpha'+\beta;\mu,\upsilon),\\(1-\sigma-\gamma+\alpha+\alpha';\mu,\upsilon),\ (1-\sigma-\beta';\mu,\upsilon),\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(1-\sigma-\beta'+\alpha';\mu,\upsilon):\left(a_j,A_j\right)_{1,n_1},...,\left(a_j,A_j\right)_{n_1+1,p_i};\left(c_j,C_j\right)_{1,n_2},...,\left(c_j,C_j\right)_{n_2+1,p'_i}\\(1-\sigma-\gamma+\alpha'+\beta;\mu,\upsilon):\left(b_j,B_j\right)_{1,m_1},...,\left(b_j,B_j\right)_{m_1+1,q_i};\left(d_j,D_j\right)_{1,m_2},...,\left(d_j,D_j\right)_{m_2+1,q'_i}\end{matrix}\right] \tag{20}$$

The existence conditions for (20) are the same as given in Theorem 1.

If we put $\tau_i = 1, \tau'_i = 1 \left(i = \overline{1, r}\right)$ and set $r = 1$ in (18) and take (8) into account, then we arrive at the following result in the term of product of two H-functions given by Ram and Kumar [[6], Eqn. (17), p. 36].

Corollary 3.2.

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\sigma-1}\ H_{p,q}^{m_1,n_1}\left[\lambda\, t^{\mu}\ \middle|\ \begin{matrix}\left(a_j, A_j\right)_{1,p}\\ \left(b_j, B_j\right)_{1,q}\end{matrix}\right] H_{p',q'}^{m_2,n_2}\left[\omega\, t^{\upsilon}\ \middle|\ \begin{matrix}\left(c_j, C_j\right)_{1,p'}\\ \left(d_j, D_j\right)_{1,q'}\end{matrix}\right]\right)\right\}(x)$$
$$= x^{\sigma-\alpha-\alpha'+\gamma-1} H_{3,3:p,q;p',q'}^{0,3:\,m_1,n_1;m_2,n_2}\left[\begin{matrix}\lambda x^{\mu}\\ \omega x^{\upsilon}\end{matrix}\ \middle|\ \begin{matrix}(1-\sigma;\,\mu,\upsilon),\,(1-\sigma-\gamma+\alpha+\alpha'+\beta;\,\mu,\upsilon),\\ (1-\sigma-\gamma+\alpha+\alpha';\mu,\upsilon),\,(1-\sigma-\beta';\,\mu,\upsilon),\end{matrix}\right.$$
$$\left.\begin{matrix}(1-\sigma-\beta'+\alpha';\mu,\upsilon):\left(a_j, A_j\right)_{1,p};\left(c_j, C_j\right)_{1,p'}\\ (1-\sigma-\gamma+\alpha'+\beta;\mu,\upsilon):\left(b_j, B_j\right)_{1,q};\left(d_j, D_j\right)_{1,q'}\end{matrix}\right]. \tag{21}$$

The existence conditions for (21) are the same as given in Theorem 1.

Now, if we follow Theorem 1 in respective case $\alpha' = \beta' = 0, \beta = -\eta, \alpha = \alpha+\beta, \gamma = \alpha$. Then we arrive at the following corollary concerning left-sided Saigo fractional integration operator [7].

Corollary 3.3. *Let $\alpha, \beta, \eta, \sigma, \lambda, \omega \in C$, $Re(\alpha) > 0$, $\mu, \upsilon > 0$ and let the constants $a_j, b_j, a_{ji}, b_{ji} \in C$, $A_j, B_j, A_{ji}, B_{ji} \in R_+\ (i = 1, ..., p_i;\ j = 1, ..., q_i)$; $c_j, d_j, c_{ji}, d_{ji} \in C$, $C_j, D_j, C_{ji}, D_{ji} \in R_+ \left(i = 1, ..., p'_i;\ j = 1, ..., q'_i\right)$, $\tau_i, \tau'_i > 0$ for $i = \overline{1, r}$. Further, satisfy the condition $Re(\sigma) + \mu \min\limits_{1\le j\le m_1} Re\left(\frac{b_j}{B_j}\right) + \upsilon \min\limits_{1\le j\le m_2} Re\left(\frac{d_j}{D_j}\right) > \max\,[0, Re(\beta-\eta)]$. Then the left-sided Saigo fractional integral $I_{0+}^{\alpha,\beta,\eta}$ of the product of two $\aleph$-functions exists and the following relation holds:*

$$\left\{I_{0+}^{\alpha,\beta,\eta}\left(t^{\sigma-1}\,\aleph_{p_i,q_i,\tau_i;r}^{m_1,n_1}\left[\lambda\, t^{\mu}\ \middle|\ \begin{matrix}\left(a_j, A_j\right)_{1,n_1}, ..., \left[\tau_j\left(a_j, A_j\right)\right]_{n_1+1,p_i}\\ \left(b_j, B_j\right)_{1,m_1}, ..., \left[\tau_j\left(b_j, B_j\right)\right]_{m_1+1,q_i}\end{matrix}\right]\right.\right.$$
$$\left.\left. .\,\aleph_{p'_i,q'_i,\tau'_i;r}^{m_2,n_2}\left[\omega\, t^{\upsilon}\ \middle|\ \begin{matrix}\left(c_j, C_j\right)_{1,n_2}, ..., \left[\tau'_j\left(c_j, C_j\right)\right]_{n_2+1,p'_i}\\ \left(d_j, D_j\right)_{1,m_2}, ..., \left[\tau'_j\left(d_j, D_j\right)\right]_{m_2+1,q'_i}\end{matrix}\right]\right)\right\}(x)$$
$$= x^{\sigma-\beta-1}\,\aleph_{2,2:p_i,q_i,\tau_i;p'_i,q'_i,\tau'_i;r}^{0,2:\,m_1,n_1;m_2,n_2}\left[\begin{matrix}\lambda x^{\mu}\\ \omega x^{\upsilon}\end{matrix}\ \middle|\ \begin{matrix}(1-\sigma;\,\mu,\upsilon),\,(1-\sigma-\eta+\beta;\,\mu,\upsilon):\\ (1-\sigma+\beta;\mu,\upsilon),\,(1-\sigma-\alpha-\eta;\,\mu,\upsilon):\end{matrix}\right.$$
$$\left.\begin{matrix}\left(a_j, A_j\right)_{1,n_1}, ..., \left[\tau_j\left(a_j, A_j\right)\right]_{n_1+1,p_i};\left(c_j, C_j\right)_{1,n_2}, ..., \left[\tau'_j\left(c_j, C_j\right)\right]_{n_2+1,p'_i}\\ \left(b_j, B_j\right)_{1,m_1}, ..., \left[\tau_j\left(b_j, B_j\right)\right]_{m_1+1,q_i};\left(d_j, D_j\right)_{1,m_2}, ..., \left[\tau'_j\left(d_j, D_j\right)\right]_{m_2+1,q'_i}\end{matrix}\right]. \tag{22}$$

For $\beta = -\alpha$ in Corollary 1.3, the Saigo operator reduces to Riemann-Liouville operator [17] and we obtain the following result:

Corollary 3.4.

$$\left\{ I_{0+}^{\alpha} \left(t^{\sigma-1} \aleph_{p_i,q_i,\tau_i;r}^{m_1,n_1} \left[\lambda t^{\mu} \middle| \begin{matrix} (a_j, A_j)_{1,n_1}, ..., [\tau_j (a_j, A_j)]_{n_1+1,p_i} \\ (b_j, B_j)_{1,m_1}, ..., [\tau_j (b_j, B_j)]_{m_1+1,q_i} \end{matrix} \right] \right.\right.$$
$$\left.\left. . \aleph_{p_i',q_i',\tau_i';r}^{m_2,n_2} \left[\omega t^{\upsilon} \middle| \begin{matrix} (c_j, C_j)_{1,n_2}, ..., [\tau_j' (c_j, C_j)]_{n_2+1,p_i'} \\ (d_j, D_j)_{1,m_2}, ..., [\tau_j' (d_j, D_j)]_{m_2+1,q_i'} \end{matrix} \right] \right) \right\} (x)$$
$$= x^{\sigma+\alpha-1} \aleph_{1,1:p_i,q_i,\tau_i;p_i',q_i',\tau_i';r}^{0,1:m_1,n_1;m_2,n_2} \left[\begin{matrix} \lambda x^{\mu} \\ \omega x^{\upsilon} \end{matrix} \middle| \begin{matrix} (1-\sigma;\mu,\upsilon) : (a_j, A_j)_{1,n_1}, ..., [\tau_j (a_j, A_j)]_{n_1+1,p_i}; \\ (1-\sigma-\alpha;\mu,\upsilon) : (b_j, B_j)_{1,m_1}, ..., [\tau_j (b_j, B_j)]_{m_1+1,q_i}; \end{matrix} \right.$$
$$\left. \begin{matrix} (c_j, C_j)_{1,n_2}, ..., [\tau_j' (c_j, C_j)]_{n_2+1,p_i'} \\ (d_j, D_j)_{1,m_2}, ..., [\tau_j' (d_j, D_j)]_{m_2+1,q_i'} \end{matrix} \right]. \tag{23}$$

Now, if we set $\beta = 0$ in Corollary 1.4, the Riemann-Liouville operator reduces to Erdélyi-Kober operator [17] and we obtain the following result:

Corollary 3.5.

$$\left\{ I_{\eta,\alpha}^{+} \left(t^{\sigma-1} \aleph_{p_i,q_i,\tau_i;r}^{m_1,n_1} \left[\lambda t^{\mu} \middle| \begin{matrix} (a_j, A_j)_{1,n_1}, ..., [\tau_j (a_j, A_j)]_{n_1+1,p_i} \\ (b_j, B_j)_{1,m_1}, ..., [\tau_j (b_j, B_j)]_{m_1+1,q_i} \end{matrix} \right] \right.\right.$$
$$\left.\left. . \aleph_{p_i',q_i',\tau_i';r}^{m_2,n_2} \left[\omega t^{\upsilon} \middle| \begin{matrix} (c_j, C_j)_{1,n_2}, ..., [\tau_j' (c_j, C_j)]_{n_2+1,p_i'} \\ (d_j, D_j)_{1,m_2}, ..., [\tau_j' (d_j, D_j)]_{m_2+1,q_i'} \end{matrix} \right] \right) \right\} (x)$$
$$= x^{\sigma-1} \aleph_{1,1:p_i,q_i,\tau_i;p_i',q_i',\tau_i';r}^{0,1:m_1,n_1;m_2,n_2} \left[\begin{matrix} \lambda x^{\mu} \\ \omega x^{\upsilon} \end{matrix} \middle| \begin{matrix} (1-\sigma-\eta;\mu,\upsilon) : (a_j, A_j)_{1,n_1}, ..., \\ (1-\sigma-\alpha-\eta;\mu,\upsilon) : (b_j, B_j)_{1,m_1}, ..., \end{matrix} \right.$$
$$\left. \begin{matrix} [\tau_j (a_j, A_j)]_{n_1+1,p_i}; (c_j, C_j)_{1,n_2}, ..., [\tau_j' (c_j, C_j)]_{n_2+1,p_i'} \\ [\tau_j (b_j, B_j)]_{m_1+1,q_i}; (d_j, D_j)_{1,m_2}, ..., [\tau_j' (d_j, D_j)]_{m_2+1,q_i'} \end{matrix} \right]. \tag{24}$$

We can also obtain results of I-function and H-function for the corollaries 1.3, 1.4 and 1.5 by following the same method as done in corollaries 1.1 and 1.2.

4. RIGHT-SIDED GENERALIZED FRACTIONAL INTEGRATION OF THE PRODUCT OF TWO ℵ-FUNCTIONS

In this section, we study the right-sided generalized fractional integration $I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}$ defined in (14).

Theorem 4.1. *Let* $\alpha,\alpha',\beta,\beta',\gamma,\sigma,\lambda,\omega\in C$, $Re(\gamma)>0,(\mu,\upsilon>0)$*;* $Re(\sigma)-\mu\min\limits_{1\le j\le m_1}Re\left(\frac{b_j}{B_j}\right)$ $-\upsilon\min\limits_{1\le j\le m_2}Re\left(\frac{d_j}{D_j}\right)<1+\min\left[Re(-\beta),Re(\alpha+\alpha'-\gamma),Re(\alpha+\beta'-\gamma)\right].$
Further, let the constants $a_j,b_j,a_{ji},b_{ji}\in C$, $A_j,B_j,A_{ji},B_{ji}\in R_+$ $(i=1,...,p_i;\ j=1,...,q_i)$; $c_j,d_j,c_{ji},d_{ji}\in C$, $C_j,D_j,C_{ji},D_{ji}\in R_+$ $\left(i=1,...,p_i';\ j=1,...,q_i'\right)$, $\tau_i,\tau_i'>0$ *for* $i=\overline{1,r}$ *also satisfy the conditions as given (3) - (6). Then the right-sided generalized fractional integration* $I_-^{\alpha,\alpha',\beta,\beta',\gamma}$ *of the product of two* $\aleph$*-functions exists and the following relation holds:*

$$\left\{I_-^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\sigma-1}\,\aleph^{m_1,n_1}_{p_i,q_i,\tau_i;r}\left[\lambda\,t^{-\mu}\left|\begin{matrix}(a_j,A_j)_{1,n_1},...,\left[\tau_j(a_j,A_j)\right]_{n_1+1,p_i}\\(b_j,B_j)_{1,m_1},...,\left[\tau_j(b_j,B_j)\right]_{m_1+1,q_i}\end{matrix}\right.\right]\right.\right.$$
$$\left.\left..\,\aleph^{m_2,n_2}_{p_i',q_i',\tau_i';r}\left[\omega\,t^{-\upsilon}\left|\begin{matrix}(c_j,C_j)_{1,n_2},...,\left[\tau_j'(c_j,C_j)\right]_{n_2+1,p_i'}\\(d_j,D_j)_{1,m_2},...,\left[\tau_j'(d_j,D_j)\right]_{m_2+1,q_i'}\end{matrix}\right.\right]\right)\right\}(x)=x^{\sigma-\alpha-\alpha'+\gamma-1}$$
$$\times\,\aleph^{0,3:m_1,n_1;m_2,n_2}_{3,3:p_i,q_i,\tau_i;p_i',q_i',\tau_i';r}\left[\begin{matrix}\lambda x^{-\mu}\\\omega x^{-\upsilon}\end{matrix}\left|\begin{matrix}(\sigma+\gamma-\alpha-\alpha';\mu,\upsilon),(\sigma+\gamma-\alpha-\beta';\mu,\upsilon),(\sigma+\beta;\mu,\upsilon):\\(\sigma;\mu,\upsilon),(\sigma+\gamma-\alpha-\alpha'-\beta';\mu,\upsilon),(\sigma-\alpha+\beta;\mu,\upsilon):\end{matrix}\right.\right.$$
$$\left.\begin{matrix}(a_j,A_j)_{1,n_1},...,\left[\tau_j(a_j,A_j)\right]_{n_1+1,p_i};(c_j,C_j)_{1,n_2},...,\left[\tau_j'(c_j,C_j)\right]_{n_2+1,p_i'}\\(b_j,B_j)_{1,m_1},...,\left[\tau_j(b_j,B_j)\right]_{m_1+1,q_i};\ (d_j,D_j)_{1,m_2},...,\left[\tau_j'(d_j,D_j)\right]_{m_2+1,q_i'}\end{matrix}\right].\quad(25)$$

Proof. In order to prove (25), we first express the product of two Aleph functions occurring on the left hand side of (25) in terms of Mellin-Barnes contour integral with the help of equation (1) and interchanging the order of integration, we obtain (say I):

$$I=\frac{1}{(2\pi i)^2}\int_{L_1}\Omega^{m_1,n_1}_{p_i,q_i,\tau_i;r}(s)\,\lambda^{-s}ds\int_{L_2}\Omega^{m_2,n_2}_{p_i',q_i',\tau_i';r}(\xi)\,\omega^{-\xi}\,d\xi\left(I_-^{\alpha,\alpha',\beta,\beta',\gamma}\,t^{\sigma+\mu s+\upsilon\xi-1}\right)(x)$$
$$=\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}\Omega^{m_1,n_1}_{p_i,q_i,\tau_1;r}(s)\,\Omega^{m_2,n_2}_{p_i',q_i',\tau_i';r}(\xi)\,\lambda^{-s}\omega^{-\xi}\left(I_-^{\alpha,\alpha',\beta,\beta',\gamma}\,t^{\sigma+\mu s+\upsilon\xi-1}\right)(x)\,ds\,d\xi,$$

from (17), we arrive at

$$I=\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}\frac{\Gamma(1+\alpha+\alpha'-\gamma-\sigma-\mu s-\upsilon\xi)\,\Gamma(1+\alpha+\beta'-\gamma-\sigma-\mu s-\upsilon\xi)}{\Gamma(1-\sigma-\mu s-\upsilon\xi)\,\Gamma(1+\alpha+\alpha'+\beta'-\gamma-\sigma-\mu s-\upsilon\xi)}$$
$$\times\frac{\Gamma(1-\beta-\sigma-\mu s-\upsilon\xi)}{\Gamma(1+\alpha-\beta-\sigma-\mu s-\upsilon\xi)}\,\frac{\prod_{j=1}^{m_1}\left\{\Gamma\left(b_j+B_js\right)\right\}\prod_{j=1}^{n_1}\left\{\Gamma\left(1-a_j-A_js\right)\right\}}{\sum_{i=1}^{r}\tau_i\,\prod_{j=m_1+1}^{q_i}\left\{\Gamma\left(1-b_{ji}-B_{ji}s\right)\right\}\prod_{j=n_1+1}^{p_i}\left\{\Gamma\left(a_{ji}+A_{ji}s\right)\right\}}$$
$$\times\frac{\prod_{j=1}^{m_2}\left\{\Gamma\left(d_j+D_j\xi\right)\right\}\prod_{j=1}^{n_2}\left\{\Gamma\left(1-c_j-C_j\xi\right)\right\}}{\sum_{i=1}^{r}\tau_i'\,\prod_{j=m_2+1}^{q_i'}\left\{\Gamma\left(1-d_{ji}-D_{ji}\xi\right)\right\}\prod_{j=n_2+1}^{p_i'}\left\{\Gamma\left(c_{ji}+C_{ji}\xi\right)\right\}}$$
$$\times\,x^{\sigma+\mu s+\upsilon\xi-\alpha-\alpha'+\gamma-1}\,\lambda^{-s}\omega^{-\xi}\,ds\,d\xi\,.\quad(26)$$

By interpreting the Mellin-Barnes counter integral thus obtained in terms of the ℵ-function of two variables as given in (9), we obtain the result (25). This completes the proof of Theorem 2. ∎

Special Cases of Theorem 2:

If we put $\tau_i = 1$, $\tau'_i = 1$ $(i = 1, 2, ..., r)$ in (25) and take (1.1) into account, then we arrive at the following result in the term of I-function [16].

Corollary 4.1.

$$\Bigg\{I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\Bigg(t^{\sigma-1}\, I_{p_i,q_i;r}^{m_1,n_1}\left[\lambda\, t^{-\mu}\,\middle|\,\begin{matrix}(a_j,A_j)_{1,n_1}, ..., (a_j,A_j)_{n_1+1,p_i}\\(b_j,B_j)_{1,m_1}, ..., (b_j,B_j)_{m_1+1,q_i}\end{matrix}\right]$$

$$.\, I_{p'_i,q'_i;r}^{m_2,n_2}\left[\omega\, t^{-\upsilon}\,\middle|\,\begin{matrix}(c_j,C_j)_{1,n_2}, ..., (c_j,C_j)_{n_2+1,p'_i}\\(d_j,D_j)_{1,m_2}, ..., (d_j,D_j)_{m_2+1,q'_i}\end{matrix}\right]\Bigg)\Bigg\}(x) = x^{\sigma-\alpha-\alpha'+\gamma-1}$$

$$\times\, I_{3,3:p_i,q_i;p'_i,q'_i;r}^{0,3:\,m_1,n_1;m_2,n_2}\left[\begin{matrix}\lambda x^{-\mu}\\ \omega x^{-\upsilon}\end{matrix}\,\middle|\,\begin{matrix}(\sigma+\gamma-\alpha-\alpha';\,\mu,\upsilon),\,(\sigma+\gamma-\alpha-\beta';\,\mu,\upsilon),\\ (\sigma;\mu,\upsilon),\,(\sigma+\gamma-\alpha-\alpha'-\beta';\,\mu,\upsilon),\end{matrix}\right.$$

$$\left.\begin{matrix}(\sigma+\beta;\mu,\upsilon) : (a_j,A_j)_{1,n_1}, ..., (a_j,A_j)_{n_1+1,p_i}; (c_j,C_j)_{1,n_2}, ..., (c_j,C_j)_{n_2+1,p'_i}\\ (\sigma-\alpha+\beta;\mu,\upsilon) : (b_j,B_j)_{1,m_1}, ..., (b_j,B_j)_{m_1+1,q_i}; (d_j,D_j)_{1,m_2}, ..., (d_j,D_j)_{m_2+1,q'_i}\end{matrix}\right]. \tag{27}$$

The existence conditions for (27) are the same as given in Theorem 2.

If we put $\tau_i = 1$, $\tau'_i = 1$ $\left(i = \overline{1,r}\right)$ and set $r = 1$ in (25) and take (8) into account, then we arrive at the following result in the term of product of two H-functions given by Ram and Kumar [[6], Eqn. (20), p. 39].

Corollary 4.2.

$$\Bigg\{I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\Bigg(t^{\sigma-1}\, H_{p,q}^{m_1,n_1}\left[\lambda\, t^{-\mu}\,\middle|\,\begin{matrix}(a_j,A_j)_{1,p}\\(b_j,B_j)_{1,q}\end{matrix}\right]\; H_{p',q'}^{m_2,n_2}\left[\omega\, t^{-\upsilon}\,\middle|\,\begin{matrix}(c_j,C_j)_{1,p'}\\(d_j,D_j)_{1,q'}\end{matrix}\right]\Bigg)\Bigg\}(x)$$

$$= x^{\sigma-\alpha-\alpha'+\gamma-1} H_{3,3:p,q;p',q'}^{0,3:m_1,n_1;m_2,n_2}\left[\begin{matrix}\lambda x^{-\mu}\\ \omega x^{-\upsilon}\end{matrix}\,\middle|\,\begin{matrix}(\sigma+\gamma-\alpha-\alpha';\,\mu,\upsilon),(\sigma+\beta;\mu,\upsilon),\\ (\sigma;\mu,\upsilon),(\sigma+\gamma-\alpha-\alpha'-\beta';\,\mu,\upsilon),\end{matrix}\right.$$

$$\left.\begin{matrix}(\sigma+\gamma-\alpha-\beta';\,\mu,\upsilon) : (a_j,A_j)_{1,p}; (c_j,C_j)_{1,p'}\\ (\sigma+\beta-\alpha;\mu,\upsilon) : (b_j,B_j)_{1,q}; (d_j,D_j)_{1,q'}\end{matrix}\right]. \tag{28}$$

The existence conditions for (28) are the same as given in Theorem 2.

Now, if we follow Theorem 2 in respective case $\alpha' = \beta' = 0, \beta = -\eta, \alpha = \alpha+\beta, \gamma = \alpha$. Then we arrive at the following corollary concerning right-sided Saigo fractional integration operator [7].

Corollary 4.3. *Let $\alpha, \beta, \eta, \sigma, \lambda, \omega \in C$, $Re(\alpha) > 0$, $\mu, \upsilon > 0$ and let the constants $a_j, b_j, a_{ji}, b_{ji} \in C$, $A_j, B_j, A_{ji}, B_{ji} \in R_+$ $(i = 1, ..., p_i;\ j = 1, ..., q_i)$; $c_j, d_j, c_{ji}, d_{ji} \in C$, $C_j, D_j, C_{ji}, D_{ji} \in R_+ \left(i = 1, ..., p_i';\ j = 1, ..., q_i'\right)$, $\tau_i, \tau_i' > 0$ for $i = \overline{1, r}$. Further, satisfy the condition $Re(\sigma) - \mu \min\limits_{1\le j\le m_1} Re\left(\frac{b_j}{B_j}\right) - \upsilon \min\limits_{1\le j\le m_2} Re\left(\frac{d_j}{D_j}\right) < 1 + \min\left[Re(\beta), Re(\eta)\right]$. Then the right-sided Saigo fractional integral $I_-^{\alpha,\beta,\eta}$ of the product of two $\aleph$-functions exists and the following relation holds:*

$$\begin{aligned}&\left\{I_-^{\alpha,\beta,\eta}\left(t^{\sigma-1}\,\aleph_{p_i,q_i,\tau_i;r}^{m_1,n_1}\left[\lambda\, t^{-\mu}\left|\begin{matrix}\left(a_j,A_j\right)_{1,n_1},...,\left[\tau_j\left(a_j,A_j\right)\right]_{n_1+1,p_i}\\ \left(b_j,B_j\right)_{1,m_1},...,\left[\tau_j\left(b_j,B_j\right)\right]_{m_1+1,q_i}\end{matrix}\right.\right]\right.\right.\\
&\qquad\left.\left..\,\aleph_{p_i',q_i',\tau_i';r}^{m_2,n_2}\left[\omega\, t^{-\upsilon}\left|\begin{matrix}\left(c_j,C_j\right)_{1,n_2},...,\left[\tau_j'\left(c_j,C_j\right)\right]_{n_2+1,p_i'}\\ \left(d_j,D_j\right)_{1,m_2},...,\left[\tau_j'\left(d_j,D_j\right)\right]_{m_2+1,q_i'}\end{matrix}\right.\right]\right)\right\}(x)\\
&= x^{\sigma-\beta-1}\,\aleph_{2,2:p_i,q_i,\tau_i;p_i',q_i',\tau_i';r}^{0,2:\,m_1,n_1;m_2,n_2}\left[\begin{matrix}\lambda x^{-\mu}\\ \omega x^{-\upsilon}\end{matrix}\left|\begin{matrix}(\sigma-\beta;\,\mu,\upsilon),\,(\sigma-\eta;\,\mu,\upsilon):\left(a_j,A_j\right)_{1,n_1},...,\\ (\sigma;\mu,\upsilon),\,(\sigma-\alpha-\beta-\eta;\,\mu,\upsilon):\left(b_j,B_j\right)_{1,m_1},...,\end{matrix}\right.\right.\\
&\qquad\left.\begin{matrix}\left[\tau_j\left(a_j,A_j\right)\right]_{n_1+1,p_i};\left(c_j,C_j\right)_{1,n_2},...,\left[\tau_j'\left(c_j,C_j\right)\right]_{n_2+1,p_i'}\\ \left[\tau_j\left(b_j,B_j\right)\right]_{m_1+1,q_i};\left(d_j,D_j\right)_{1,m_2},...,\left[\tau_j'\left(d_j,D_j\right)\right]_{m_2+1,q_i'}\end{matrix}\right].\end{aligned}\tag{29}$$

For $\beta = -\alpha$ in Corollary 2.3, the Saigo operator reduces to Riemann-Liouville operator [17] and we obtain the following result:

Corollary 4.4.

$$\begin{aligned}&\left\{I_-^{\alpha}\left(t^{\sigma-1}\,\aleph_{p_i,q_i,\tau_i;r}^{m_1,n_1}\left[\lambda\, t^{-\mu}\left|\begin{matrix}\left(a_j,A_j\right)_{1,n_1},...,\left[\tau_j\left(a_j,A_j\right)\right]_{n_1+1,p_i}\\ \left(b_j,B_j\right)_{1,m_1},...,\left[\tau_j\left(b_j,B_j\right)\right]_{m_1+1,q_i}\end{matrix}\right.\right]\right.\right.\\
&\qquad\left.\left..\,\aleph_{p_i',q_i',\tau_i';r}^{m_2,n_2}\left[\omega\, t^{-\upsilon}\left|\begin{matrix}\left(c_j,C_j\right)_{1,n_2},...,\left[\tau_j'\left(c_j,C_j\right)\right]_{n_2+1,p_i'}\\ \left(d_j,D_j\right)_{1,m_2},...,\left[\tau_j'\left(d_j,D_j\right)\right]_{m_2+1,q_i'}\end{matrix}\right.\right]\right)\right\}(x)\\
&= x^{\sigma+\alpha-1}\,\aleph_{1,1:p_i,q_i,\tau_i;p_i',q_i',\tau_i';r}^{0,1:m_1,n_1;m_2,n_2}\left[\begin{matrix}\lambda x^{-\mu}\\ \omega x^{-\upsilon}\end{matrix}\left|\begin{matrix}(\sigma+\alpha;\,\mu,\upsilon):\left(a_j,A_j\right)_{1,n_1},...,\left[\tau_j\left(a_j,A_j\right)\right]_{n_1+1,p_i};\\ (\sigma;\mu,\upsilon):\left(b_j,B_j\right)_{1,m_1},...,\left[\tau_j\left(b_j,B_j\right)\right]_{m_1+1,q_i};\end{matrix}\right.\right.\\
&\qquad\left.\begin{matrix}\left(c_j,C_j\right)_{1,n_2},...,\left[\tau_j'\left(c_j,C_j\right)\right]_{n_2+1,p_i'}\\ \left(d_j,D_j\right)_{1,m_2},...,\left[\tau_j'\left(d_j,D_j\right)\right]_{m_2+1,q_i'}\end{matrix}\right].\end{aligned}\tag{30}$$

Now, if we set $\beta = 0$ in Corollary 2.4, the Riemann-Liouville operator reduces to Erdélyi-Kober operator [17] and we obtain the following result:

Corollary 4.5.

$$\left\{I^{-}_{\eta,\alpha}\left(t^{\sigma-1}\,\aleph^{m_1,n_1}_{p_i,q_i,\tau_i;r}\left[\lambda\,t^{-\mu}\left|\begin{matrix}\left(a_j,A_j\right)_{1,n_1},...,\left[\tau_j\left(a_j,A_j\right)\right]_{n_1+1,p_i}\\ \left(b_j,B_j\right)_{1,m_1},...,\left[\tau_j\left(b_j,B_j\right)\right]_{m_1+1,q_i}\end{matrix}\right.\right]\right.\right.$$

$$\left.\left..\,\aleph^{m_2,n_2}_{p'_i,q'_i,\tau'_i;r}\left[\omega\,t^{-\upsilon}\left|\begin{matrix}\left(c_j,C_j\right)_{1,n_2},...,\left[\tau'_j\left(c_j,C_j\right)\right]_{n_2+1,p'_i}\\ \left(d_j,D_j\right)_{1,m_2},...,\left[\tau'_j\left(d_j,D_j\right)\right]_{m_2+1,q'_i}\end{matrix}\right.\right]\right)\right\}(x)$$

$$= x^{\sigma-1}\,\aleph^{0,1:m_1,n_1;m_2,n_2}_{1,1:p_i,q_i,\tau_i;p'_i,q'_i,\tau'_i;r}\left[\begin{matrix}\lambda x^{-\mu}\\ \omega x^{-\upsilon}\end{matrix}\left|\begin{matrix}(\sigma-\eta;\,\mu,\upsilon):\left(a_j,A_j\right)_{1,n_1},...,\\ (\sigma-\alpha-\eta;\mu,\upsilon):\left(b_j,B_j\right)_{1,m_1},...,\end{matrix}\right.\right.$$

$$\left.\begin{matrix}\left[\tau_j\left(a_j,A_j\right)\right]_{n_1+1,p_i};\left(c_j,C_j\right)_{1,n_2},...,\left[\tau'_j\left(c_j,C_j\right)\right]_{n_2+1,p'_i}\\ \left[\tau_j\left(b_j,B_j\right)\right]_{m_1+1,q_i};\left(d_j,D_j\right)_{1,m_2},...,\left[\tau'_j\left(d_j,D_j\right)\right]_{m_2+1,q'_i}\end{matrix}\right]. \tag{31}$$

We can also obtain results of I-function and H-function for the corollaries 2.3, 2.4 and 2.5 by following the same method as done in corollaries 2.1 and 2.2.

Remark 4.1. (i). *If we specialize the first H-function in Corollary 1.2 and 2.2 to the exponential function by taking $\mu = 1$, then we obtain the result given by Ram and Kumar [[6], Eqn. (21), p.41].*
(ii). *If we further set $\omega = 0$, then we obtain the result given by Ram and Kumar [[6], Eqn. (22), p.41].*
(iii). *If we reduce the H-function to the generalized wright hypergeometric function [18], we have the results given by Ram and Kumar [[6], Eqn. (23), p.41].*
(iv). *A number of several special cases as Mittag-Leffler function, Whittaker function and Bessel function of the first kind can be developed for Corollary 1.2 and 2.2, but we do not mention them here on account of lack space.*

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AN ILL-POSED ELLIPTIC PROBLEM OF RECONSTRUCTING THE TEMPERATURE FROM INTERIOR DATA

Huy Nguyen Tuan[1] , Binh Thanh Tran[2], Hieu Van Nguyen[3], Le Duc Thang[4]

[1] *Division of Applied Mathematics, Ton Duc Thang University, Ho Chi Minh City, Vietnam.*

[2] *Department of Mathematics and Applications, Sai Gon University, Ho Chi Minh City, VietNam.*

[3] *Department of Physics, Faculty of Science, Ho Chi Minh City University of Agriculture and Forestry, VietNam*

[4] *Faculty of Basic Science, Ho Chi Minh City Industry and Trade College, VietNam*

tuanhuy_bs@yahoo.com

Abstract We consider the problem of reconstructing, from the interior data $u(x, 1)$ and $u_y(x, 1)$, a function u satisfying a linear elliptic equation

$$\Delta u = 0, \quad x \in \mathbb{R}, \ 0 < y < 1.$$

The problem is ill-posed. Using the method of Green functions, the method of Fourier transforms, and the quasi-boundary value method, we shall regularize the problem. Error estimate is given.

Keywords: Fourier transform; linearly ill-posed problem; quasi-boundary value methods.
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1. INTRODUCTION

In this paper, we consider the problem of reconstructing the temperature of a body from interior measurements. In fact, in many engineering contexts (see, e.g., [1]), we cannot attach a temperature sensor at the surface of a body (e.g., the skin of a missile). Hence, to get the distribution of temperature on the surface, we have to use the measured temperature inside the body.

Precisely, we consider the problem of finding the temperature $u(x, y)$, $x \in \mathbb{R}$, $0 < y < 1$ satisfying

$$\Delta u = 0, \quad x \in \mathbb{R}, \ 0 < y < 1 \tag{1}$$

subject to the conditions

$$u(x, 1) = \varphi(x), \tag{2}$$

$$u_y(x, 1) = \psi(x), \tag{3}$$

where $\varphi(x)$, $\psi(x)$ are given. The problem is called the Cauchy problem for linear homogeneous elliptic equation. Using the method of Green functions and the method of Fourier transforms, we can rewrite the above system in the following form (see [8])

$$\widehat{u}(p,y) = \frac{1}{2}\widehat{\varphi}(p)\left[e^{(1-y)|p|} + e^{(y-1)|p|}\right] + \frac{1}{2|p|}\widehat{\psi}(p)\left[e^{(y-1)|p|} - e^{(1-y)|p|}\right]. \tag{4}$$

The homogeneous problem was studied, by various methods in many papers. Using the mollification method, the homogeneous sideways parabolic problems were considered in [2, 3, 4, 5-7] and the references therein. Similarly, many methods have been investigated to solve the Cauchy problem for linear homogeneous elliptic equation such as the quasi-reversibility method [9], fourth order modified method [10], Meyer wavelets [11], etc. Moreover, in [11,12,13], the error estimate was not given.

Especially, in [10], the authors considered the same form of the system (1)-(3) as follows

$$\Delta u = 0, \quad 0 < x \le 1, \ y \in \mathbb{R}$$
$$u(0,y) = \varphi(y),$$
$$u_x(0,y) = 0,$$

and in the case $x = 1$, they showed that the error between the aprroximate problem and the exact solution is

$$\|u(1,\cdot) - v(1,\cdot)\| \le \frac{E}{\left(\ln \frac{E}{\delta}\right)^{2p}} + \varepsilon$$

where $\|\cdot\|$ is the norm on $L^2(\mathbb{R})$ and

$$\begin{aligned} \varepsilon &= \max\left\{\mu^p, \frac{1}{2}\mu^{p-1}, \frac{1}{2}\mu^2\right\} E, \\ \mu &= \frac{1}{\ln\left(\frac{E}{\delta}\left(\ln\frac{E}{\delta}\right)^{-2p}\right)}, \end{aligned}$$
$$\|u(1,\cdot)\|_p \le E, \quad p \ge 0,$$

$\|\cdot\|_p$ denotes the norm in $H^p(\mathbb{R})$ defined by

$$\|u(1,\cdot)\|_p = \left(\int_{-\infty}^{+\infty} \left(1+\xi^2\right)^p |\widehat{u}(1,\cdot)| d\xi\right)^{1/2},$$

It is easy to see that the error above is not near to zero if $p = 0$. In the current paper, we shall prove that

$$\|w_\epsilon(\cdot,0) - u_{ex}(\cdot,0)\|_2 \le C\left(\ln\left(\frac{1}{\epsilon}\right)\right)^{-1/4}$$

where $\|\cdot\|_2$ is the norm on $L^2(\mathbb{R})$, C is a positive constant. It is easy to see that the convergence of the approximate solution is also proved.

In the present paper, we shall regularize (1)-(3) using the method of integral equation. We approximate problem (4) by the following problem

$$\begin{aligned} \widehat{u^\epsilon}(p,y) &= \tfrac{1}{2}\widehat{\varphi}(p)\left[\tfrac{e^{-y|p|}}{\epsilon+e^{-|p|}} + e^{(y-1)|p|}\right] \\ &\quad + \tfrac{1}{2}\left[\tfrac{1-e^{2(1-y)|p|}}{|p|e^{2(1-y)|p|}}\right]\left[\tfrac{e^{-2y|p|}}{\epsilon+e^{-2|p|}}\right] e^{(y-1)|p|}\widehat{\psi}(p) \end{aligned}$$

or

$$\begin{aligned} u^\epsilon(x,y) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{1}{2}\widehat{\varphi}(p)\left[\frac{e^{-y|p|}}{\epsilon+e^{-|p|}} + e^{(y-1)|p|}\right]e^{ipx}dp \\ &\quad + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{1}{2}\left[\frac{1-e^{2(1-y)|p|}}{|p|e^{2(1-y)|p|}}\right]\left[\frac{e^{-2y|p|}}{\epsilon+e^{-2|p|}}\right]e^{(y-1)|p|}\widehat{\psi}(p)e^{ipx}dp. \end{aligned} \tag{5}$$

The rest of the article is divided into two sections. In Section 2, we shall study the ill-posedness and the regularization of problem (1)-(3). In Section 3, we shall give a numerical experiment.

2. MAIN RESULTS

2.1. THE ILL-POSEDNESS OF PROBLEM (1)-(3)

The Cauchy problem for linear homogeneous elliptic equation is severely ill-posed. We shall prove solutions do not depend continuously on the given data. Indeed, we choose

$$\widehat{\varphi}_n(p) = \begin{cases} \left[\dfrac{1}{e^{|p|}+e^{-|p|}}\right]\dfrac{n}{|p|^{3/2}} & \text{if } |p| \geq n \\ 0 & \text{if } |p| < n \end{cases} \tag{6}$$

and

$$\widehat{\psi}_n(p) = 0 \tag{7}$$

where $p \in \mathbb{R}$, $n \in \mathbb{N}$.

Then, we have

$$\begin{aligned} \left\|\widehat{\varphi}_n\right\|^2_{L^2(\mathbb{R})} &= \int_{|p|\geq n}\left|\frac{1}{e^{|p|}+e^{-|p|}}\right|^2\frac{n^2}{|p|^3}dp \leq \int_{|p|\geq n} e^{-2|p|}\frac{n^2}{|p|^3}dp \\ &\leq \frac{1}{n}\int_{|p|\geq n} e^{-2|p|}dp \leq \frac{1}{n}\int_{\mathbb{R}} e^{-2|p|}dp = \frac{1}{n}. \end{aligned}$$

From (4) and by choosing $\widehat{\varphi}_n, \widehat{\psi}_n$ in (7)-(8), we have

$$\left\|\widehat{u_n}(\cdot,0)\right\|^2_{L^2(\mathbb{R})} = \frac{n^2}{4}\int_{|p|\geq n} |p|^{-3}\,dp = \frac{1}{4}. \tag{8}$$

Letting $n \to \infty$ in (9) and (10), we have $\left\|\widehat{\varphi_n}\right\|^2_{L^2(\mathbb{R})} \to 0$ while $\left\|\widehat{u_n}(\cdot,0)\right\|^2_{L^2(\mathbb{R})} \to \frac{1}{4}$. So, the problem is ill-posed.

2.2. REGULARIZATION OF PROBLEM (1)-(3)

Assume that u_{ex} is the exact solution of (1)-(3), v_{ex} is the solution of problem (6) corresponding to the exact data φ_{ex}, ψ_{ex} and v_ϵ is the solution of problem (6) corresponding to the measured data φ_ϵ, ψ_ϵ, where φ_{ex}, ψ_{ex}, φ_ϵ, ψ_ϵ are in the right-hand side of (6) such that $\left\|\varphi_\epsilon - \varphi_{ex}\right\|_2 \leq \epsilon$, $\left\|\psi_\epsilon - \psi_{ex}\right\|_2 \leq \epsilon$ where $\|\cdot\|_2$ is the norm on $L^2(\mathbb{R})$. Then, we have

$$\widehat{u_{ex}}(p,y) = \frac{1}{2}\widehat{\varphi_{ex}}(p)\left[e^{(1-y)|p|} + e^{(y-1)|p|}\right] + \frac{1}{2|p|}\widehat{\psi_{ex}}(p)\left[e^{(y-1)|p|} - e^{(1-y)|p|}\right], \tag{9}$$

$$\begin{aligned}
\widehat{v_{ex}}(p,y) &= \tfrac{1}{2}\widehat{\varphi_{ex}}(p)\left[\tfrac{e^{-y|p|}}{\epsilon+e^{-|p|}} + e^{(y-1)|p|}\right] \\
&\quad + \tfrac{1}{2}\left[\tfrac{1-e^{2(1-y)|p|}}{|p|e^{2(1-y)|p|}}\right]\left[\tfrac{e^{-2y|p|}}{\epsilon+e^{-2|p|}}\right]e^{(y-1)|p|}\widehat{\psi_{ex}}(p), \\
\widehat{v_\epsilon}(p,y) &= \tfrac{1}{2}\widehat{\varphi_\epsilon}(p)\left[\tfrac{e^{-y|p|}}{\epsilon+e^{-|p|}} + e^{(y-1)|p|}\right] \\
&\quad + \tfrac{1}{2}\left[\tfrac{1-e^{2(1-y)|p|}}{|p|e^{2(1-y)|p|}}\right]\left[\tfrac{e^{-2y|p|}}{\epsilon+e^{-2|p|}}\right]e^{(y-1)|p|}\widehat{\psi_\epsilon}(p).
\end{aligned}$$

We have the estimate

$$\|v_\epsilon - u_{ex}\|_2 = \left\|\widehat{v_\epsilon} - \widehat{u_{ex}}\right\|_2 \leq \left\|\widehat{v_\epsilon} - \widehat{v_{ex}}\right\|_2 + \left\|\widehat{v_{ex}} - \widehat{u_{ex}}\right\|_2. \tag{10}$$

We first have the following lemma

Lemma 2.1 (The stability of a solution of problem (5))**.** *Suppose that φ_{ex}, ψ_{ex}, φ_ϵ, $\psi_\epsilon \in L^2(\mathbb{R})$ and $\left\|\varphi_\epsilon - \varphi_{ex}\right\|_2 \leq \epsilon$, $\left\|\psi_\epsilon - \psi_{ex}\right\|_2 \leq \epsilon$. Then we have*

$$\left\|\widehat{v_\epsilon}(\cdot,y) - \widehat{v_{ex}}(\cdot,y)\right\|_2 \leq \frac{3}{\sqrt{2}}(\epsilon^y + \epsilon)$$

for all $y \in (0,1)$.

Proof. First, from (12) and (13), we have

$$\widehat{v_\epsilon}(p,y) - \widehat{v_{ex}}(p,y) = \frac{1}{2}\left[\frac{e^{-y|p|}}{\epsilon + e^{-|p|}} + e^{(y-1)|p|}\right]\left[\widehat{\varphi_\epsilon}(p) - \widehat{\varphi_{ex}}(p)\right]$$

$$+\frac{1}{2}\left[\frac{1-e^{2(1-y)|p|}}{|p|e^{2(1-y)|p|}}\right]\left[\frac{e^{-2y|p|}}{\epsilon+e^{-2|p|}}\right]e^{(y-1)|p|}\left[\widehat{\psi}_\epsilon(p)-\widehat{\psi}_{ex}(p)\right]. \tag{11}$$

Using the inequality $\dfrac{e^{|x|}-1}{|x|}\le e^{|x|}$ for every $x\neq 0$, we have

$$\left|\frac{1-e^{2(1-y)|p|}}{|p|e^{2(1-y)|p|}}\right|\le 2, \text{ for } 0<y<1. \tag{12}$$

Moreover, one has, for $s>y>0$ and $\alpha>0$,

$$\frac{e^{-y|p|}}{\alpha+e^{-s|p|}}=\frac{1}{(\alpha e^{s|p|}+1)^{\frac{y}{s}}(\alpha+e^{-s|p|})^{1-\frac{y}{s}}}\le \alpha^{\frac{y}{s}-1}.$$

Letting $\alpha=\epsilon$, $s=1$, we get

$$\frac{e^{-y|p|}}{\epsilon+e^{-|p|}}\le \epsilon^{y-1}. \tag{13}$$

From (15), (16), (17) and take note that $e^{(y-1)|p|}\le 1$ for $0<y<1$, we obtain

$$\begin{aligned}\left|\widehat{v}_\epsilon(p,y)-\widehat{v}_{ex}(p,y)\right| &\le \frac{1}{2}\left[\frac{e^{-y|p|}}{\epsilon+e^{-|p|}}+e^{(y-1)|p|}\right]\left|\widehat{\varphi}_\epsilon(p)-\widehat{\varphi}_{ex}(p)\right|\\ &+\frac{1}{2}e^{(y-1)|p|}\left|\frac{1-e^{2(1-y)|p|}}{|p|e^{2(1-y)|p|}}\right|\left[\frac{e^{-2y|p|}}{\epsilon+e^{-2|p|}}\right]\left|\widehat{\psi}_\epsilon(p)-\widehat{\psi}_{ex}(p)\right|\\ &\le \frac{1}{2}\left(\epsilon^{y-1}+1\right)\left|\widehat{\varphi}_\epsilon(p)-\widehat{\varphi}_{ex}(p)\right|+\epsilon^{y-1}\left|\widehat{\psi}_\epsilon(p)-\widehat{\psi}_{ex}(p)\right|.\end{aligned} \tag{14}$$

Applying the inequality $(a+b)^2\le 2(a^2+b^2)$, we get

$$\begin{aligned}\left|\widehat{v}_\epsilon(p,y)-\widehat{v}_{ex}(p,y)\right|^2 &\le \tfrac{1}{2}\left(\epsilon^{y-1}+1\right)^2\left|\widehat{\varphi}_\epsilon(p)-\widehat{\varphi}_{ex}(p)\right|^2\\ &\quad +2\left(\epsilon^{y-1}\right)^2\left|\widehat{\psi}_\epsilon(p)-\widehat{\psi}_{ex}(p)\right|^2.\end{aligned}$$

In addition, since $(\widehat{\varphi}_\epsilon-\widehat{\varphi}_{ex})\in L^2(\mathbb{R})$ and $\left(\widehat{\psi}_\epsilon-\widehat{\psi}_{ex}\right)\in L^2(\mathbb{R})$, we have

$$(\widehat{v}_\epsilon-\widehat{v}_{ex})\in L^2(\mathbb{R}). \tag{15}$$

From (19), (20) and take note of the inequality $\sqrt{a^2+b^2}\le a+b$ for $a,b\ge 0$, we have

$$\begin{aligned}\left\|\widehat{v}_\epsilon(\cdot,y)-\widehat{v}_{ex}(\cdot,y)\right\|_2 &\le \tfrac{1}{\sqrt{2}}\left(\epsilon^{y-1}+1\right)\left\|\widehat{\varphi}_\epsilon-\widehat{\varphi}_{ex}\right\|_2+\sqrt{2}\epsilon^{y-1}\left\|\widehat{\psi}_\epsilon-\widehat{\psi}_{ex}\right\|_2\\ &\le \tfrac{1}{\sqrt{2}}\left(\epsilon^{y-1}+1\right)\epsilon+\sqrt{2}\epsilon^{y-1}\epsilon=\left(\tfrac{1}{\sqrt{2}}+\sqrt{2}\right)\epsilon^y+\tfrac{1}{\sqrt{2}}\epsilon\\ &\le \tfrac{3}{\sqrt{2}}(\epsilon^y+\epsilon).\end{aligned}$$

This completes the proof of Lemma 2.1. ∎

Theorem 2.1. *Let φ, ψ be as in Lemma 2.1. Assume that $\widehat{\varphi}_{ex}(p)e^{|p|} \in L^2(\mathbb{R})$, $\widehat{\psi}_{ex}(p)e^{2|p|} \in L^2(\mathbb{R})$, then for every $0 < y < 1$ we have*

$$\|v_\epsilon(\cdot, y) - u_{ex}(\cdot, y)\|_2 \le M(\epsilon^y + \epsilon)$$

where

$$M = \frac{3}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(\left\|\widehat{\varphi}_{ex}(p)e^{|p|}\right\|_2 + 2\left\|\widehat{\psi}_{ex}(p)e^{2|p|}\right\|_2\right). \tag{16}$$

Proof. First, from (11) and (12), we have $\widehat{v}_{ex}(p, y) - \widehat{u}_{ex}(p, y) = \frac{1}{2}\widehat{\varphi}_{ex}(p)\left[\frac{e^{-y|p|}}{\epsilon + e^{-|p|}} - e^{(1-y)|p|}\right]$

$$+\frac{1}{2}\left[\frac{1 - e^{2(1-y)|p|}}{|p|e^{2(1-y)|p|}}\right]\left[\frac{e^{-2y|p|}}{\epsilon + e^{-2|p|}} - e^{2(1-y)|p|}\right]e^{(y-1)|p|}\widehat{\psi}_{ex}(p). \tag{17}$$

Moreover, one has, for $1 > y > 0$,

$$e^{(1-y)|p|} - \frac{e^{-y|p|}}{\epsilon + e^{-|p|}} = \frac{\epsilon e^{-y|p|}}{e^{-|p|}(\epsilon + e^{-|p|})}, \tag{18}$$

and take note that (17), we get

$$\frac{\epsilon e^{-y|p|}}{e^{-|p|}(\epsilon + e^{-|p|})} \le \epsilon^y e^{|p|}. \tag{19}$$

From (24) and (25), we obtain

$$\left|\frac{e^{-y|p|}}{\epsilon + e^{-|p|}} - e^{(1-y)|p|}\right| \le \epsilon^y e^{|p|}. \tag{20}$$

Similarly, we also have

$$\left|\frac{e^{-2y|p|}}{\epsilon + e^{-2|p|}} - e^{2(1-y)|p|}\right| \le \epsilon^y e^{2|p|}. \tag{21}$$

From (23), (26), (27), (16) and take note that $e^{(y-1)|p|} \le 1$ for $0 < y < 1$, we obtain

$$\begin{aligned}
\left|\widehat{v}_{ex}(p, y) - \widehat{u}_{ex}(p, y)\right| &\le \frac{1}{2}\left|\widehat{\varphi}_{ex}(p)\right|\left|\frac{e^{-y|p|}}{\epsilon + e^{-|p|}} - e^{(1-y)|p|}\right| \\
&\quad+\frac{1}{2}\left|\frac{1 - e^{2(1-y)|p|}}{|p|e^{2(1-y)|p|}}\right|\left|\frac{e^{-2y|p|}}{\epsilon + e^{-2|p|}} - e^{2(1-y)|p|}\right|\left|e^{(y-1)|p|}\right|\left|\widehat{\psi}_{ex}(p)\right| \\
&\le \frac{1}{2}\epsilon^y\left|\widehat{\varphi}_{ex}(p)e^{|p|}\right| + \epsilon^y\left|\widehat{\psi}_{ex}(p)e^{2|p|}\right|.
\end{aligned}$$

Applying the inequality $(a+b)^2 \leq 2(a^2+b^2)$, we get

$$\left|\widehat{v}_{ex}(p,y) - \widehat{u}_{ex}(p,y)\right|^2 \leq \frac{1}{2}(\epsilon^y)^2 \left|\widehat{\varphi}_{ex}(p)e^{|p|}\right|^2 + 2(\epsilon^y)^2 \left|\widehat{\psi}_{ex}(p)e^{2|p|}\right|^2.$$

Therefore

$$\left\|\widehat{v}_{ex}(\cdot,y) - \widehat{u}_{ex}(\cdot,y)\right\|_2^2 \leq \frac{1}{2}(\epsilon^y)^2 \left\|\widehat{\varphi}_{ex}(p)e^{|p|}\right\|_2^2 + 2(\epsilon^y)^2 \left\|\widehat{\psi}_{ex}(p)e^{2|p|}\right\|_2^2.$$

Since $\widehat{\varphi}_{ex}(p)e^{|p|}$, $\widehat{\psi}_{ex}(p)e^{2|p|} \in L^2(\mathbb{R})$ and the inequality $\sqrt{a^2+b^2} \leq a+b$ for $a, b \geq 0$, we have

$$\begin{aligned}\left\|\widehat{v}_{ex}(\cdot,y) - \widehat{u}_{ex}(\cdot,y)\right\|_2 &\leq \frac{1}{\sqrt{2}}\epsilon^y \left\|\widehat{\varphi}_{ex}(p)e^{|p|}\right\|_2 + \sqrt{2}\epsilon^y \left\|\widehat{\psi}_{ex}(p)e^{2|p|}\right\|_2 \\ &= \frac{1}{\sqrt{2}}\left(\left\|\widehat{\varphi}_{ex}(p)e^{|p|}\right\|_2 + 2\left\|\widehat{\psi}_{ex}(p)e^{2|p|}\right\|_2\right)\epsilon^y.\end{aligned} \tag{22}$$

From (14), using Lemma 2.1 and (29) we get

$$\begin{aligned}\|v_\epsilon(\cdot,y) - u_{ex}(\cdot,y)\|_2 &\leq \frac{3}{\sqrt{2}}(\epsilon^y+\epsilon) + \frac{1}{\sqrt{2}}\left(\left\|\widehat{\varphi}_{ex}(p)e^{|p|}\right\|_2 + 2\left\|\widehat{\psi}_{ex}(p)e^{2|p|}\right\|_2\right)\epsilon^y \\ &\leq M(\epsilon^y+\epsilon)\end{aligned}$$

where

$$M = \frac{3}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(\left\|\widehat{\varphi}_{ex}(p)e^{|p|}\right\|_2 + 2\left\|\widehat{\psi}_{ex}(p)e^{2|p|}\right\|_2\right).$$

This completes the proof of Theorem 2.1.

Theorem 2.1 gives a good approximation for the case $0 < y < 1$.

To get an approximation result for the case $y = 0$, we shall use the result of the following Lemma.

Lemma 2.2. *Let φ, ψ be as in Lemma 2.1. Assume that $\widehat{\varphi}_{ex}(p)e^{|p|} \in L^2(\mathbb{R})$, $\widehat{\psi}_{ex}(p)e^{2|p|} \in L^2(\mathbb{R})$ and that problem (4) has a solution satisfying $u_y \in L^2((0,1);L^2(\mathbb{R}))$. Then for all $\epsilon \in (0,1)$ there exists a $y_\epsilon > 0$ such that*

$$\|v_{ex}(\cdot,y_\epsilon) - u_{ex}(\cdot,0)\|_2 \leq \sqrt[4]{8}C_1\left(\ln\left(\frac{1}{\epsilon}\right)\right)^{-1/4}$$

where

$$N = \sqrt{\int_0^1 \left\|u_y(\cdot,s)\right\|_2^2 ds} \tag{23}$$

and

$$C_1 = \max\left\{N, \frac{1}{\sqrt{2}}\left(\left\|\widehat{\varphi}_{ex}(p)e^{|p|}\right\|_2 + 2\left\|\widehat{\psi}_{ex}(p)e^{2|p|}\right\|_2\right)\right\}. \tag{24}$$

Proof. We have

$$u_{ex}(x,y) - u_{ex}(x,0) = \int_0^y u_y(x,s)ds.$$

It follows that

$$\|u_{ex}(\cdot,y) - u_{ex}(\cdot,0)\|_2^2 \le y\int_0^1 \|u_y(\cdot,s)\|_2^2 ds = N^2 y.$$

Using (29) and (30) and (31), we have

$$\begin{aligned}\|v_{ex}(\cdot,y) - u_{ex}(\cdot,0)\|_2 &\le \|v_{ex}(\cdot,y) - u_{ex}(\cdot,y)\|_2 + \|u_{ex}(\cdot,y) - u_{ex}(\cdot,0)\|_2 \\ &\le C_1(\sqrt{y} + \epsilon^y).\end{aligned}$$

For every $\epsilon \in (0,1)$, there exists uniquely a positive number y_ϵ such that $\sqrt{y_\epsilon} = \epsilon^{y_\epsilon}$, i.e.,

$$\frac{\ln y_\epsilon}{y_\epsilon} = 2\ln\epsilon.$$

Using inequality $\ln y > -(1/y)$ for every $y > 0$, we get

$$\|v_{ex}(\cdot,y_\epsilon) - u_{ex}(\cdot,0)\|_2 \le \sqrt[4]{8}C_1\left(\ln\left(\frac{1}{\epsilon}\right)\right)^{-1/4}.$$

This completes the proof of Lemma 2.2. ∎

In the case of non-exact data, one has

Theorem 2.2. *Let φ, ψ be as in Lemma 2.1. Assume that $\widehat{\varphi}_{ex}(p)e^{|p|} \in L^2(\mathbb{R})$, $\widehat{\psi}_{ex}(p)e^{2|p|} \in L^2(\mathbb{R})$ and that problem (4) has a solution satisfying $u_y \in L^2((0,1);L^2(\mathbb{R}))$. Let $\epsilon \in (0,1)$ such that $\left\|\varphi_\epsilon - \varphi_{ex}\right\|_2 \le \epsilon$, $\left\|\psi_\epsilon - \psi_{ex}\right\|_2 \le \epsilon$. Then from φ_ϵ, ψ_ϵ we can construct a function w_ϵ satisfying*

$$\|w_\epsilon(\cdot,y) - u_{ex}(\cdot,y)\|_2 \le M(\epsilon^y + \epsilon)$$

for every $y \in (0,1)$ and

$$\|w_\epsilon(\cdot,0) - u_{ex}(\cdot,0)\|_2 \le C\left(\ln\left(\frac{1}{\epsilon}\right)\right)^{-1/4}$$

where

$$M = \frac{3}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(\left\|\widehat{\varphi}_{ex}(p)e^{|p|}\right\|_2 + 2\left\|\widehat{\psi}_{ex}(p)e^{2|p|}\right\|_2\right),$$

$$C_1 = \max\left\{\sqrt{\int_0^1 \left\|u_y(\cdot,s)\right\|_2^2 ds}, \frac{1}{\sqrt{2}}\left(\left\|\widehat{\varphi}_{ex}(p)e^{|p|}\right\|_2 + 2\left\|\widehat{\psi}_{ex}(p)e^{2|p|}\right\|_2\right)\right\}$$

and

$$C = \frac{3}{\sqrt[4]{8}} + \frac{3}{\sqrt{2}} + \sqrt[4]{8}C_1.$$

Proof. Let y_ϵ be the unique solution of

$$\sqrt{y_\epsilon} = \epsilon^{y_\epsilon}. \tag{25}$$

We define a function w_ϵ as follow

$$w_\epsilon(\cdot,y) = \begin{cases} v_\epsilon(\cdot,y), & 0 < y < 1 \\ v_\epsilon(\cdot,y_\epsilon), & y = 0. \end{cases}$$

From Theorem 2.1, we have

$$\|w_\epsilon(\cdot,y) - u_{ex}(\cdot,y)\|_2 = \|v_\epsilon(\cdot,y) - u_{ex}(\cdot,y)\|_2 \le M(\epsilon^y + \epsilon) \tag{26}$$

for every $y \in (0,1)$.

From Lemma 2.2, we have

$$\|v_{ex}(\cdot,y_\epsilon) - u_{ex}(\cdot,0)\|_2 \le \sqrt[4]{8}C_1\left(\ln\left(\frac{1}{\epsilon}\right)\right)^{-1/4}. \tag{27}$$

Using Lemma 2.1 and (32), (34), we get

$$\begin{aligned}
\|w_\epsilon(\cdot,0) - u_{ex}(\cdot,0)\|_2 &= \|v_\epsilon(\cdot,y_\epsilon) - u_{ex}(\cdot,0)\|_2 \\
&\le \|v_\epsilon(\cdot,y_\epsilon) - v_{ex}(\cdot,y_\epsilon)\|_2 + \|v_{ex}(\cdot,y_\epsilon) - u_{ex}(\cdot,0)\|_2 \\
&\le \frac{3}{\sqrt{2}}\epsilon^{y_\epsilon} + \frac{3}{\sqrt{2}}\epsilon + \sqrt[4]{8}C_1\left(\ln\left(\frac{1}{\epsilon}\right)\right)^{-1/4} \\
&\le C\left(\ln\left(\frac{1}{\epsilon}\right)\right)^{-1/4}
\end{aligned}$$

where

$$C = \frac{3}{\sqrt[4]{8}} + \frac{3}{\sqrt{2}} + \sqrt[4]{8}C_1.$$

This completes the proof of Theorem 2.2. ∎

Remark 2.1. *The condition $u_y \in L^2((0,1); L^2(\mathbb{R}))$ is difficult to check. We can present some conditions of φ, ψ . Since (4), we have*

$$\widehat{u}(p,y) = \frac{1}{2}\widehat{\varphi}(p)\left[e^{(1-y)|p|} + e^{(y-1)|p|}\right] + \frac{1}{2|p|}\widehat{\psi}(p)\left[e^{(y-1)|p|} - e^{(1-y)|p|}\right]. \qquad (28)$$

Therefore

$$\begin{aligned} \frac{\partial}{\partial y}\widehat{u}(p,y) &= \frac{1}{2}|p|\widehat{\varphi}(p)\left[-e^{(1-y)|p|} + e^{(y-1)|p|}\right] + \frac{1}{2|p|}|p|\widehat{\psi}(p)\left[e^{(y-1)|p|} + e^{(1-y)|p|}\right] \\ &= \frac{1}{2}|p|e^{|p|}\widehat{\varphi}(p)\left[-e^{-y|p|} + e^{(y-2)|p|}\right] + \frac{1}{2|p|}|p|e^{|p|}\widehat{\psi}(p)\left[e^{(y-2)|p|} + e^{-y|p|}\right]. \end{aligned}$$

If $|p|e^{|p|}\widehat{\varphi}(p) \in L^2(\mathbb{R})$, $e^{|p|}\widehat{\psi}(p) \in L^2(\mathbb{R})$ then $u_y \in L^2((0,1); L^2(\mathbb{R}))$.

3. A NUMERICAL EXPERIMENT

Consider the linear homogeneous elliptic equation

$$\Delta u = 0, \quad x \in \mathbb{R}, \ 0 < y < 1$$

where u satisfies

$$u(x,1) = \varphi(x),$$
$$u_y(x,1) = \psi(x).$$

Consider the exact data $\varphi_{ex}(x) = \dfrac{4}{x^2+4}$, $\psi_{ex}(x) = 0$ then

$$\widehat{\varphi}_{ex}(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \frac{4}{x^2+4} e^{-ipx} dx = \sqrt{2\pi} e^{-2|p|} \qquad (29)$$

and

$$\widehat{\psi}_{ex}(p) = 0. \qquad (30)$$

From (35), (36) and (11), we have

$$\widehat{u}_{ex}(p,y) = \sqrt{\frac{\pi}{2}}\left[e^{(1-y)|p|} + e^{(y-1)|p|}\right]e^{-2|p|}.$$

Consider the measured data $\varphi_\epsilon(x) = \left(\dfrac{\epsilon}{\sqrt{\pi}} + 1\right)\varphi_{ex}(x)$, we have

$$\left\|\varphi_\epsilon - \varphi_{ex}\right\|_2 = \left\|\widehat{\varphi}_\epsilon - \widehat{\varphi}_{ex}\right\|_2 = \left(\int_{-\infty}^{+\infty} 2\epsilon^2 e^{-4|p|} dp\right)^{1/2} = \epsilon.$$

From (35), (36) and (13), we have the regularized solution

$$\widehat{v_\epsilon}(p, y) = \sqrt{\frac{\pi}{2}}\left(\frac{\epsilon}{\sqrt{\pi}} + 1\right)\left[\frac{e^{-y|p|}}{\epsilon + e^{-|p|}} + e^{(y-1)|p|}\right]e^{-2|p|}.$$

Let ϵ be $\epsilon_1 = 10^{-1}$, $\epsilon_2 = 10^{-5}$, $\epsilon_3 = 10^{-10}$ respectively. If we put

$$y = \{0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1\}$$

we get the following tables for the case $0 < y < 1$

$\epsilon_1 = 10^{-1}$	
y	$\left\|\widehat{v_\epsilon} - \widehat{u_{ex}}\right\|_2$
0.2	0.1119
0.3	0.0957
0.4	0.0822
0.5	0.0707
0.6	0.0607
0.7	0.0519
0.8	0.0441
0.9	0.0370
1	0.0305

$\epsilon_2 = 10^{-5}$	
y	$\left\|\widehat{v_\epsilon} - \widehat{u_{ex}}\right\|_2$
0.2	2.0428×10^{-5}
0.3	1.6272×10^{-5}
0.4	1.3241×10^{-5}
0.5	1.0952×10^{-5}
0.6	9.1612×10^{-6}
0.7	7.7117×10^{-6}
0.8	6.5021×10^{-6}
0.9	5.4650×10^{-6}
1	4.5545×10^{-6}

$\epsilon_3 = 10^{-10}$	
y	$\left\|\widehat{v_\epsilon} - \widehat{u_{ex}}\right\|_2$
0.2	2.0433×10^{-10}
0.3	1.6275×10^{-10}
0.4	1.3243×10^{-10}
0.5	1.0953×10^{-10}
0.6	9.1619×10^{-11}
0.7	7.7122×10^{-11}
0.8	6.5025×10^{-11}
0.9	5.4653×10^{-11}
1	4.5547×10^{-11}

and we have the graphic is displayed in Figure 2, Figure 3, Figure 4 on the interval $[-5, 5] \times [0.2, 1]$

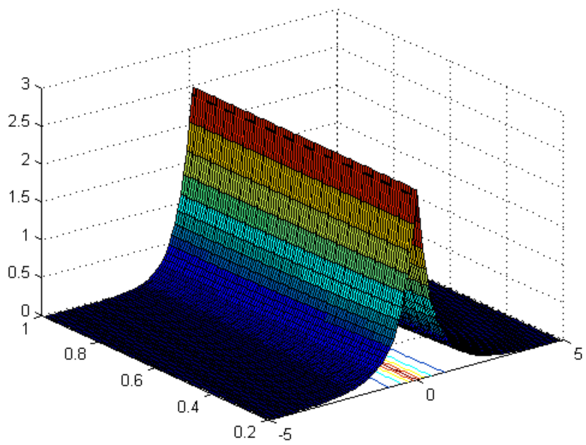

FIGURE 1. The Fourier transform of the exact solution in the case $0 < y < 1$.

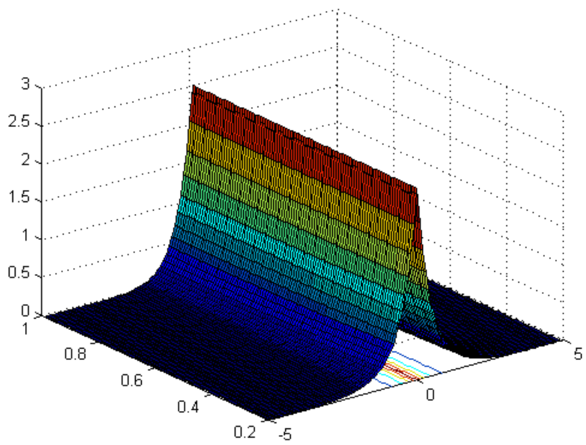

FIGURE 2. The Fourier transform of the regularized solution with $\epsilon_1 = 10^{-1}$.

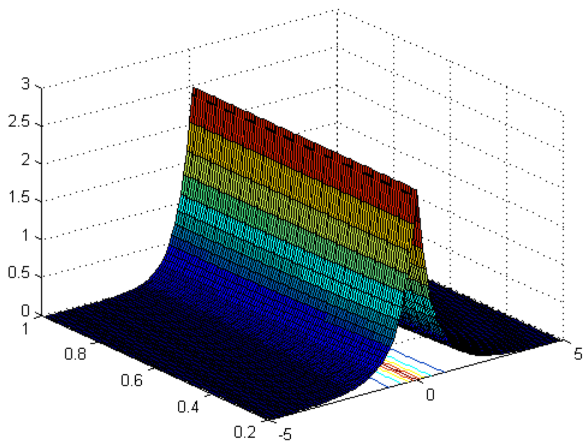

FIGURE 3. The Fourier transform of the regularized solution with $\epsilon_2 = 10^{-5}$.

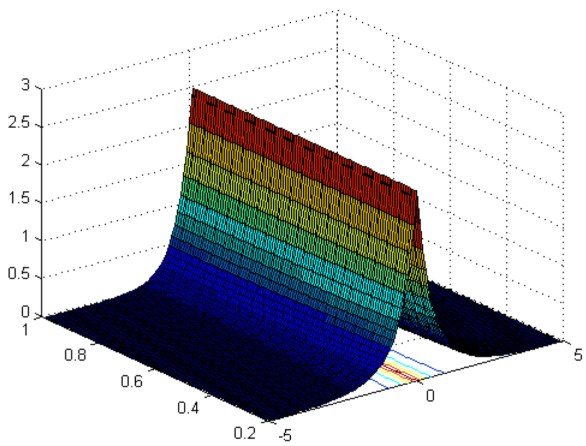

FIGURE 4. The Fourier transform of the regularized solution with $\epsilon_3 = 10^{-10}$.

In the case $y = 0$, from (32) and using inequality $\ln y > -(1/y)$ for every $y > 0$, we get

$$y_\epsilon < \frac{1}{\sqrt{2\ln\left(\frac{1}{\epsilon}\right)}}.$$

Therefore, we will choose $y_{\epsilon_1} = 0.4$, $y_{\epsilon_2} = 0.2$, $y_{\epsilon_3} = 0.01$, with $\epsilon_1 = 10^{-1}$, $\epsilon_2 = 10^{-5}$, $\epsilon_3 = 10^{-10}$ respectively, numerical results are given as follows

		$\left\|\widehat{v}_\epsilon(\cdot, y_\epsilon) - \widehat{u}_{ex}(\cdot, 0)\right\|_2$
$\epsilon_1 = 10^{-1}$	$y_{\epsilon_1} = 0.4$	0.3020
$\epsilon_2 = 10^{-5}$	$y_{\epsilon_2} = 0.2$	0.1311
$\epsilon_3 = 10^{-10}$	$y_{\epsilon_3} = 0.01$	0.0077

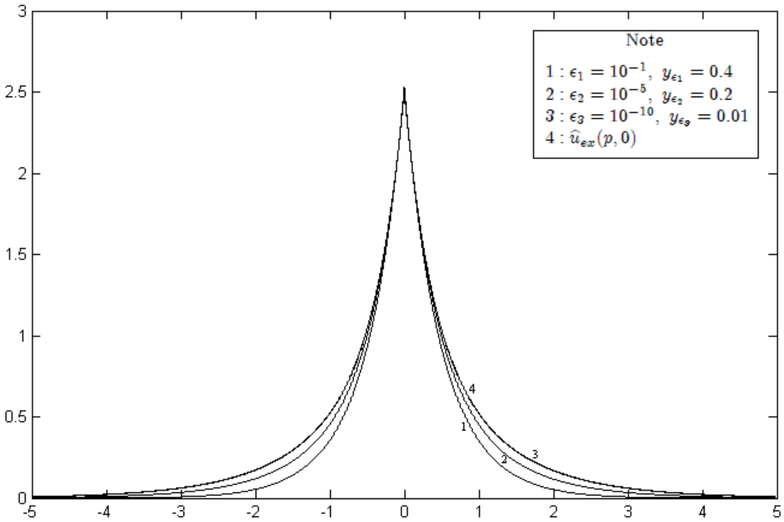

FIGURE 5. The Fourier transform of the exact solution and the Fourier transform of the regularized solution in the case $y = 0$.

Notice that, in Figure 5, the 3rd curve expresses the Fourier transform of the regularized solution corresponding $\epsilon_3 = 10^{-10}, y_{\epsilon_3} = 0.01$ coincides with the 4th curve expresses the Fourier transform of the exact solution.

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WEAKLY CONTRACTIVE MAPS IN ALTERING METRIC SPACES

Mihai Turinici

"A. Myller" Mathematical Seminar; "A. I. Cuza" University; Iaşi, Romania

mturi@uaic.ro

Abstract The weakly contractive metric type fixed point result in Berinde [Nonlin. Anal. Forum, 9 (2004), 45-53] is "almost" covered by the related altering metric one due to Khan et al [Bull. Austral. Math. Soc., 30 (1984), 1-9]. Further extensions of both these results are then provided.

Keywords: complete metric space, contraction, fixed point, altering metric, subunitary and right Boyd-Wong function.

2010 MSC: 47H17 (Primary), 54H25 (Secondary).

1. INTRODUCTION

Let (X, d) be a complete metric space; and $T \in \mathcal{F}(\mathcal{X})$ be a selfmap of X. [Here, for each couple A, B of nonempty sets, $\mathcal{F}(\mathcal{A}, \mathcal{B})$ stands for the class of all functions from A to B; when $A = B$, we simply denote $\mathcal{F}(\mathcal{A}, \mathcal{A})$ as $\mathcal{F}(\mathcal{A})$]. Put $\text{Fix}(T) = \{z \in X; z = Tz\}$; each element of this set is called *fixed* under T. In the metrical fixed point theory, such points are to be determined by a limit process as follows. Let us say that $x \in X$ is a *Picard point* (modulo (d, T)) when **i)** $(T^n x; n \geq 0)$ is d-convergent, **ii)** $\lim_n (T^n x)$ belongs to $\text{Fix}(T)$. If this happens for each $x \in X$, then T is called a *Picard operator* (modulo d); and, if in addition, **iii)** $\text{Fix}(T)$ is a *singleton* ($z_1, z_2 \in \text{Fix}(T)$ implies $z_1 = z_2$), then T is referred to as a *strong Picard operator* (modulo d); cf. Rus [13, Ch 2, Sect 2.2]. In this perspective, a basic result to the question we deal with is the 1922 one due to Banach [2]: it states that, whenever T is *α-contractive* (modulo d), i.e.,

(a01) $d(Tx, Ty) \leq \alpha d(x, y), \ \forall x, y \in X,$

for some $\alpha \in [0, 1[$, then T is a strong Picard operator (modulo d). This result found a multitude of applications in operator equations theory; so, it was the subject of many extensions. For example, a natural way of doing this is by considering "functional" contractive conditions of the form

(a02) $d(Tx, Ty) \leq F(d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)), \quad \forall x, y \in X;$

where $F : R_+^5 \to R_+$ is an appropriate function. For more details about the possible choices of F we refer to the 1977 paper by Rhoades [12]; see also Turinici [15]. Here, we shall be concerned with a 2004 contribution in the area due to Berinde [4]. Given $\alpha, \lambda \geq 0$, let us say that T is a *weak* (α, λ)*-contraction* (modulo d) provided

(a03) $d(Tx, Ty) \leq \alpha d(x, y) + \lambda d(Tx, y)$, for all $x, y \in X$.

Theorem 1.1. *Suppose that T is a weak (α, λ)-contraction (modulo d), where $\alpha \in [0, 1[$. Then, T is a Picard operator (modulo d).*

In a subsequent paper devoted to the same question, Berinde [3] claims that this class of contractions introduced by him is for the first time considered in the literature. Unfortunately, his assertion is not true: conclusions of Theorem 1.1 are "almost" covered by a related 1984 statement due to Khan et al [9], in the context of altering distances. This, among others, motivated us to propose an appropriate extension of the quoted statement; details are given in Section 3. The preliminary material for our device is listed in Section 2. Finally, in Section 4, a "functional" extension of Berinde's result is established. Further aspects will be delineated elsewhere.

2. PRELIMINARIES

Let (X, d) be a metric space. Let us say that the sequence (x_n) in X, *d-converges* to $x \in X$ (and write: $x_n \xrightarrow{d} x$) iff $d(x_n, x) \to 0$; that is

(b01) $\forall \varepsilon > 0, \exists p = p(\varepsilon)$: $n \geq p \Longrightarrow d(x_n, x) \leq \varepsilon$.

Denote $\lim_n(x_n) = \{x \in X; x_n \xrightarrow{d} x\}$; when this set is nonempty, (x_n) is called *d-convergent*. Note that, in this case, $\lim_n(x_n)$ is a singleton, $\{z\}$; as usually, we write $\lim_n(x_n) = z$. Further, let us say that (x_n) is *d-Cauchy* provided $d(x_m, x_n) \to 0$ as $m, n \to \infty$, $m < n$; that is

(b02) $\forall \varepsilon > 0, \exists q = q(\varepsilon)$: $q \leq m < n \Longrightarrow d(x_m, x_n) \leq \varepsilon$.

Clearly, any d-convergent sequence is d-Cauchy too; when the reciprocal holds too, (X, d) is called *complete*. Concerning this aspect, note that any d-Cauchy sequence $(x_n; n \geq 0)$ is *d-semi-Cauchy*; i.e.,

(b03) $\rho_n := d(x_n, x_{n+1}) \to 0$ (hence, $d(x_n, x_{n+i}) \to 0, \forall i \geq 1$), as $n \to \infty$.

The following result about such sequences is useful in the sequel. For each sequence $(z_n; n \geq 0)$ in R and each $z \in R$, put $z_n \downarrow z$ iff $[z_n > z, \forall n]$ and $z_n \to z$.

Proposition 2.1. *Suppose that $(x_n; n \geq 0)$ is d-semi-Cauchy, but not d-Cauchy. There exists then $\eta > 0$, $j(\eta) \in N$ and a couple of rank sequences $(m(j); j \geq 0)$, $(n(j); j \geq 0)$, in such a way that*

$$j \leq m(j) < n(j), \quad \alpha(j) := d(x_{m(j)}, x_{n(j)}) > \eta, \ \forall j \geq 0 \tag{1}$$

$$n(j) - m(j) \geq 2, \ \beta(j) := d(x_{m(j)}, x_{n(j)-1}) \leq \eta, \ \forall j \geq j(\eta) \tag{2}$$

$$\alpha(j) \downarrow \eta \text{ (hence, } \alpha(j) \to \eta) \text{ as } j \to \infty \tag{3}$$

$$\alpha_{p,q}(j) := d(x_{m(j)+p}, x_{n(j)+q}) \to \eta, \text{ as } j \to \infty, \ \forall p, q \in \{0, 1\}. \tag{4}$$

A proof of this may be found in Khan et al [9]. For completeness reasons, we supply an argument which differs, in part, from the original one.

Proof. **(Proposition 2.1)** As (b02) does not hold, there exists $\eta > 0$ with

$$A(j) := \{(m, n) \in N \times N; j \leq m < n, d(x_m, x_n) > \eta\} \neq \emptyset, \ \forall j \geq 0.$$

Having this precise, denote, for each $j \geq 0$,

$$m(j) = \min \mathrm{Dom}(A(j)), \ n(j) = \min A(m(j)).$$

As a consequence, the couple of rank-sequences $(m(j); j \geq 0)$, $(n(j); j \geq 0)$ fulfills (1). On the other hand, letting the index $j(\eta) \geq 0$ be such that

$$d(x_k, x_{k+1}) < \eta, \ \forall k \geq j(\eta), \tag{5}$$

it is clear that (2) holds too. Finally, by the triangular property,

$$\eta < \alpha(j) \leq \beta(j) + \rho_{n(j)-1} \leq \eta + \rho_{n(j)-1}, \ \forall j \geq j(\eta);$$

and this yields (3); hence, the case $(p = 0, q = 0)$ of (4). Combining with

$$\alpha(j) - \rho_{n(j)} \leq d(x_{m(j)}, x_{n(j)+1}) \leq \alpha(j) + \rho_{n(j)}, \ \forall j \geq j(\eta)$$

establishes the case $(p = 0, q = 1)$ of the same. The remaining situations are deductible in a similar way. ∎

3. MAIN RESULT

Let X be a nonempty set; and $d(.,.)$ be a metric over it [in the usual sense]. Further, let $\varphi \in \mathcal{F}(\mathcal{R}_+)$ be an *altering function*; i.e.

(c01) φ is continuous, increasing, and reflexive-sufficient [$\varphi(t) = 0$ iff $t = 0$].

The associated map (from $X \times X$ to R_+)

(c02) $e(x, y) = \varphi(d(x, y)), \ x, y \in X$

has the immediate properties

$$e(x, y) = e(y, x), \ \forall x, y \in X \ (e \text{ is symmetric}) \tag{6}$$

$$e(x, y) = 0 \Longleftrightarrow x = y \ (e \text{ is reflexive-sufficient}). \tag{7}$$

So, it is a (reflexive sufficient) *symmetric*, under Hicks' terminology [8]. In general, $e(.,.)$ is not endowed with the triangular property; but, in compensation to this, one has (as φ is increasing and continuous)

$$e(x, y) > e(u, v) \Longrightarrow d(x, y) > d(u, v) \tag{8}$$

$$x_n \xrightarrow{d} x,\ y_n \xrightarrow{d} y \ \text{ implies } \ e(x_n, y_n) \to e(x, y). \tag{9}$$

Suppose in the following that

(c03) (X, d) is complete (each d-Cauchy sequence is d-convergent).

Let $T \in \mathcal{F}(\mathcal{X})$ be a selfmap of X. The formulation of the problem involving $\mathrm{Fix}(T) = \{x \in X; x = Tx\}$ is the already sketched one. In the following, we are trying to solve it in the precise context. Denote, for $x, y \in X$,

(c04) $M_1(x, y) = e(x, y)$, $M_2(x, y) = (1/2)[e(x, Tx) + e(y, Ty)]$,
$M_3(x, y) = \min\{e(x, Ty), e(Tx, y)\}$,
$M(x, y) = \max\{M_1(x, y), M_2(x, y), M_3(x, y)\}$.

Further, given $\psi \in \mathcal{F}(\mathcal{R}_+)$, we say that T is *$(d, e; M, \psi)$-contractive*, provided

(c05) $e(Tx, Ty) \le \psi(d(x, y))M(x, y)$, $\forall x, y \in X$, $x \ne y$.

The properties of ψ to be used here write

(c06) ψ is strictly subunitary on $R^0_+ :=]0, \infty[$: $\psi(s) < 1$, $\forall s \in R^0_+$

(c07) ψ is right Boyd-Wong on R^0_+: $\limsup_{t \to s+} \psi(t) < 1$, $\forall s \in R^0_+$.

This is related to the developments in Boyd and Wong [6]; we do not give details.
The main result of this exposition is

Theorem 3.1. *Suppose that T is $(d, e; M, \psi)$-contractive, where $\psi \in \mathcal{F}(\mathcal{R}_+)$ is strictly subunitary and right Boyd-Wong on R^0_+. Then, T is a strong Picard operator (modulo d).*

Proof. First, let us check the singleton property for $\mathrm{Fix}(T)$. Let $z_1, z_2 \in \mathrm{Fix}(T)$ be such that $z_1 \ne z_2$; hence $\delta := d(z_1, z_2) > 0$, $\varepsilon := e(z_1, z_2) > 0$. By definition,

$$M_1(z_1, z_2) = \varepsilon,\ M_2(z_2, z_2) = 0,\ M_3(x, y) = \varepsilon;\ \text{hence } M(x, y) = \varepsilon.$$

By the contractive condition (written at (z_1, z_2))

$$\varepsilon = e(z_1, z_2) = e(Tz_1, Tz_2) \le \psi(\delta)M(z_1, z_2) = \psi(\delta)\varepsilon;$$

hence, $1 \le \psi(\delta) < 1$; contradiction. This established the singleton property. It remains now to verify the Picard property. Fix some $x_0 \in X$; and put $x_n = T^n x_0$, $n \ge 0$. If $x_n = x_{n+1}$ for some $n \ge 0$, we are done; so, without loss, one may assume

(c08) $\rho_n := d(x_n, x_{n+1}) > 0$ (hence, $\sigma_n := e(x_n, x_{n+1}) > 0$), for all n.

There are several steps to be passed.

I) For the arbitrary fixed $n \geq 0$, we have

$$
\begin{aligned}
&M_1(x_n, x_{n+1}) = \sigma_n, \\
&M_2(x_n, x_{n+1}) = (1/2)[\sigma_n + \sigma_{n+1}] \leq \max\{\sigma_n, \sigma_{n+1}\}, \\
&M_3(x_n, x_{n+1}) = 0;\ \text{hence } M(x_n, x_{n+1}) \leq \max\{\sigma_n, \sigma_{n+1}\}.
\end{aligned}
$$

By the contractive condition (written at (x_n, x_{n+1})),

$$\sigma_{n+1} \leq \psi(\rho_n) \max\{\sigma_n, \sigma_{n+1}\},\ \forall n.$$

This, along with (c08), yields (as ψ is strictly subunitary on R_+^0)

$$\sigma_{n+1}/\sigma_n \leq \psi(\rho_n) < 1,\ \forall n. \tag{10}$$

As a direct consequence,

$$\sigma_n > \sigma_{n+1} \text{ (hence, } \rho_n > \rho_{n+1}\text{), for all } n.$$

The sequence $(\rho_n; n \geq 0)$ is therefore strictly descending in R_+; hence, $\rho := \lim_n(\rho_n)$ exist in R_+ and $\rho_n > \rho$, $\forall n$. Likewise, the sequence $(\sigma_n = \varphi(\rho_n); n \geq 0)$ is strictly descending in R_+; hence, $\sigma := \lim_n(\sigma_n)$ exists; with, in addition, $\sigma = \varphi(\rho)$. We claim that $\rho = 0$. Assume by contradiction that $\rho > 0$; hence $\sigma > 0$. Passing to lim sup as $n \to \infty$ in (10) yields

$$1 \leq \limsup_n \psi(\rho_n) \leq \limsup_{t \to \rho+} \psi(t) < 1;$$

contradiction. Hence, $\rho = 0$; i.e.,

$$\rho_n := d(x_n, x_{n+1}) \to 0,\ \text{as } n \to \infty. \tag{11}$$

II) We now show that $(x_n; n \geq 0)$ is d-Cauchy. Suppose that this is not true. By Proposition 2.1, there exist $\eta > 0$, $j(\eta) \in N$ and a couple of rank sequences $(m(j); j \geq 0)$, $(n(j); j \geq 0)$, in such a way that (1)-(4) hold. Denote for simplicity $\zeta = \varphi(\eta)$; hence, $\zeta > 0$. By the notations used there, we may write as $j \to \infty$

$$\lambda_j := e(x_{m(j)+1}, x_{n(j)+1}) = \varphi(\alpha_{1,1}(j)) \to \zeta.$$

In addition, we have (again under $j \to \infty$)

$$
\begin{aligned}
&M_1(x_{m(j)}, x_{n(j)}) = \varphi(\alpha(j)) \to \zeta \\
&M_2(x_{m(j)}, x_{n(j)}) = (1/2)[\varphi(\rho_{m(j)}) + \varphi(\rho_{n(j)})] \to 0 \\
&M_3(x_{m(j)}, x_{n(j)}) = \min\{\varphi(\alpha_{0,1}(j)), \varphi(\alpha_{1,0}(j))\} \to \zeta;
\end{aligned}
$$

and this, by definition, yields

$$\mu_j := M(x_{m(j)}, x_{n(j)}) \to \zeta \text{ as } j \to \infty.$$

From the contractive condition (written at $(x_{m(j)}, x_{n(j)})$)

$$\lambda_j/\mu_j \leq \psi(\alpha(j)), \ \forall j \geq j(\eta);$$

so that, passing to lim sup as $j \to \infty$

$$1 \leq \limsup_j \psi(\alpha(j)) \leq \limsup_{t \to \eta+} \psi(t) < 1;$$

contradiction. Hence, $(x_n; n \geq 0)$ is d-Cauchy, as claimed.

III) As (X, d) is complete, there exists a (uniquely determined) $z \in X$ with $x_n \xrightarrow{d} z$; hence $\gamma_n := d(x_n, z) \to 0$ as $n \to \infty$.

Two assumptions are open before us:

i) For each $h \in N$, there exists $k > h$ with $x_k = z$. In this case, there exists a sequence of ranks $(m(i); i \geq 0)$ with $m(i) \to \infty$ as $i \to \infty$ such that $x_{m(i)} = z$, $\forall i$; hence, $x_{m(i)+1} = Tz$, $\forall i$. Letting i tends to infinity and using the fact that $(y_i := x_{m(i)+1}; i \geq 0)$ is a subsequence of $(x_i; i \geq 0)$, we get $z = Tz$.

ii) There exists $h \in N$ such that $n \geq h \Longrightarrow x_n \neq z$. Suppose that $z \neq Tz$; i.e., $\theta := d(z, Tz) > 0$; hence, $\omega := e(z, Tz) > 0$. Note that, in such a case, $\delta_n := d(x_n, Tz) \to \theta$. From our previous notations, we have (as $n \to \infty$)

$$\lambda_n := e(x_{n+1}, Tz) = \varphi(\delta_{n+1}) \to \varphi(\theta) = \omega.$$

In addition (again under $n \to \infty$),

$$M_1(x_n, z) = \varphi(\gamma_n) \to 0, \ M_2(x_n, z) = (1/2)[\sigma_n + \omega] \to \omega/2$$
$$M_3(x_n, z) = \min\{\varphi(\delta_n), \varphi(\gamma_{n+1})\} \to 0;$$

wherefrom,

$$\mu_n := M(x_n, z) \to \omega/2, \text{ as } n \to \infty.$$

By the contractive condition (written at (x_n, z))

$$\lambda_n \leq \psi(\gamma_n)\mu_n < \mu_n, \ \forall n \geq h$$

we then have (passing to limit as $n \to \infty$), $\omega \leq \omega/2$; hence $\omega = 0$. This yields $\theta = 0$; contradiction. Hence, z is fixed under T and the proof is complete. ∎

In particular, the right Boyd-Wong on R^0_+ property of ψ is assured when this function fulfills (c06) and is decreasing on R^0_+. As a consequence, the following particular version of our main result may be stated.

Theorem 3.2. *Suppose that T is $(d, e; M, \psi)$-contractive, where $\psi \in \mathcal{F}(\mathcal{R}_+)$ is strictly subunitary and decreasing on R^0_+. Then, T is a strong Picard operator (modulo d).*

Let $a, b, c \in \mathcal{F}(\mathcal{R}_+)$ be a triple of functions. We say that the selfmap T of X is *$(d, e; a, b, c)$-contractive* if

(c09) $e(Tx, Ty) \leq a(d(x, y))e(x, y) + b(d(x, y))[e(x, Tx) + e(y, Ty)] +$
$c(d(x, y)) \min\{e(x, Ty), e(Tx, y)\}, \ \forall x, y \in X, \ x \neq y.$

Denote for simplicity $\psi = a + 2b + c$; it is clear that, under such a condition, T is $(d, e; M; \psi)$-contractive. Consequently, the following statement is a particular case of Theorem 1.1 above:

Theorem 3.3. *Suppose that T is $(d, e; a, b, c)$-contractive, where the triple of functions $a, b, c \in \mathcal{F}(\mathcal{R}_+)$ is such that their associated function $\psi = a + 2b + c$ is strictly subunitary and right Boyd-Wong on R^0_+. Then, conclusions of Theorem 1.1 hold.*

In particular, when a, b, c are all decreasing on R^0_+, the right Boyd-Wong property on R^0_+ holds; note that, in this case, Theorem 3.3 is also reducible to Theorem 3.2. This is just the 1984 fixed point result in Khan et al [9].

Finally, it is worth mentioning that the nice contributions of these authors was the starting point for a series of results involving altering contractions, like the one in Dutta and Choudhury [7] or Nashine et al [10]. Some other aspects may be found in Akkouchi [1]; see also Pathak and Shahzad [11].

4. FURTHER ASPECTS

Let again (X, d) be a complete metric space and $T \in \mathcal{F}(\mathcal{X})$ be a selfmap of X. A basic particular case of Theorem 3.3 corresponds to the choices φ=identity and $[a, b, c$=constants$]$. The corresponding form of Theorem 3.3 is comparable with Theorem 1.1. However, the inclusion between these is not complete. This raises the question of determining proper extensions of Theorem 1.1, close enough to Theorem 3.3. A direct answer to this is provided by

Theorem 4.1. *Let the numbers $a, b \in R_+$ and the function $K \in \mathcal{F}(\mathcal{R}_+)$ be such that*

(d01) $d(Tx, Ty) \leq ad(x, y) + b[d(x, Tx) + d(y, Ty)] + K(d(Tx, y)), \ \forall x, y \in X$

(d02) $a + 2b < 1$ *and* $K(t) \to 0$ *as* $t \to 0$.

Then, T is a Picard map (modulo d).

Proof. Take an arbitrary fixed $u \in X$. By the very contractive condition (written at $(T^n u, T^{n+1}u)$), we have the evaluation

$$d(T^{n+1}u, T^{n+2}u) \leq \lambda d(T^n u, T^{n+1}u), \quad \forall n \geq 0. \tag{12}$$

where $\lambda := (a + b)/(1 - b) < 1$. This yields

$$d(T^n u, T^{n+1}u) \leq \lambda^n d(u, Tu), \quad \forall n \geq 0. \tag{13}$$

Consequently, $(T^n u; n \geq 0)$ is d-Cauchy; whence (by completeness)

$$T^n u \xrightarrow{d} z := T^\infty u, \text{ for some } z \in X.$$

From the contractive condition (written at $(T^n u, z)$),

$$d(T^{n+1}u, Tz) \leq ad(T^n u, z) + b[d(T^n u, T^{n+1}u) + d(z, Tz)] + K(d(T^{n+1}u, z)),\ \forall n.$$

Passing to limit as $n \to \infty$ gives (via (d02)) $d(z, Tz) \leq bd(z, Tz)$; so that, if $z \neq Tz$, one gets $1 \leq b \leq 1/2$, contradiction. Hence $z = Tz$; and the proof is complete. ∎

In particular, when $b = 0$ and $K(.)$ is linear ($K(t) = \lambda t$, $t \in R_+$, for some $\lambda \geq 0$), this result is just Theorem 1.1. Note that, from (13), one has for these "limit" fixed points, the error approximation formula (which – under the accepted conditions for our data – is available as well in case of Theorem 3.3)

$$d(T^n u, T^\infty u) \leq [\lambda^n/(1-\lambda)]d(u, Tu),\quad \forall n \in N. \tag{14}$$

However, the non-singleton property of Fix(T) makes this "local" evaluation to be without practical effect in Theorem 4.1, by the highly unstable character of the map $u \mapsto T^\infty u$: even if the distance $d(u, v)$ between two initial approximations would decrease, the distance $d(T^\infty u, T^\infty v)$ between the associated fixed points may not decrease.

Finally, another interesting particular case to consider is that of φ being an arbitrary altering function and [a, b, c=constants]; we do not give details. Further aspects may be found in Bhaumik et al [5] see also Sastry and Babu [14];

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LINEAR DISCRETE-TIME SET-VALUED PARETO-NASH-STACKELBERG CONTROL PROCESSES AND THEIR PRINCIPLES

Valeriu Ungureanu, Victoria Lozan

Department of Applied Mathematics, State University of Moldova, Chişinău, Republic of Moldova

v.a.ungureanu@gmail.com, victoria.lozan@gmail.com

Abstract Mathematical models of linear discrete-time set-valued Pareto-Nash-Stackelberg control processes are examined as extension of mono-valued Pareto-Nash-Stackelberg control models proposed by V. Ungureanu in [10]. A straightforward principle is applied to solve Pareto-Nash-Stackelberg control problems. Models and results are presented in natural order by beginning with the simplest case and, by sequential considering of more general cases, the results for the highlighted Pareto-Nash-Stackelberg set-valued control are presented. The maximum principle of Pontryagin is extended for considered control processes, too.

Keywords: linear discrete-time set-valued control problem, non-cooperative game, multi-criteria strategic game, Pareto-Nash-Stackelberg set-valued control.

2010 MSC: 49N05, 62C25, 91A06, 91A10, 91A20, 91A44, 91A50, 91A65, 93C05, 93C55.

1. INTRODUCTION

Pareto-Nash-Stackelberg control processes, examined in [10] as extension and integration of optimal control processes [6, 1] with simultaneous and sequential games [9, 8, 4, 5, 2], are generalized by considering the set-valued multi-criteria control processes of a system with discrete-time dynamics described by a system of set-valued linear equations. The Pareto-Nash-Stackelberg set-valued control problems of linear discrete-time system are solved by applying a straightforward principle [10]. The characteristics and properties of Set-Valued Algebra [7] together with Interval Analysis [3] serve as foundation for obtained results.

Exposure starts with the simplest case of linear discrete-time set-valued optimal control problem and, by sequential considering of more general cases, finalizes with the Pareto-Nash-Stackelberg set-valued control problem. Maximum principle of Pontryagin [6, 10] is extended, formulated and proved for all the considered problems, too. Its equivalence with the straightforward direct principle is established.

2. LINEAR DISCRETE-TIME SET-VALUED OPTIMAL CONTROL PROBLEM

The system may be imagined as an n-dimension dynamic body the state of which is described by the set of points in every time moment. So, the initial state is described by the initial set $X^0 \subset R^n$. The optimal control problem naturally arises:

$$\begin{array}{rcl} F(X,U) & = & \sum_{t=1}^{T}(c^t X^t + b^t U^t) \to \max, \\ X^t & = & A^{t-1}X^{t-1} + B^t X^t,\ t = 1,...,T, \\ D^t U^t & \leq & d^t,\ t = 1,...,T, \end{array} \tag{1}$$

where $X^0, X^t \subset R^n$, $c^t \in R^n$, $U^t \subset R^m$, $b^t \in R^m$, $A^{t-1} \in R^{n\times n}$, $B^t \in R^{n\times m}$, $d^t \in R^k$, $D^t \in R^{k\times n}$, $c^t X^t = \langle c^t, X^t\rangle$, $b^t U^t = \langle b^t, U^t\rangle$, $t = 1,...,T$, $U = (U^1, U^2, ..., U^T)$, $X = (X^0, X^1, ..., X^T)$. Set operations in (1) are defined obviously [7]: $AX = \{Ax : x \in X\}$, $\forall X \subset R^n$, $\forall A \subset R^{n\times n}$.

Remark, the objective set-valued map $F : X \times U \multimap R$, $F(X,U) \subset R$, represents a summation of intervals. So, the applying of interval arithmetic [3] is intrinsic.

By performing direct substitutions in (1):

$$\begin{array}{ll} X^1 & = A^0X^0 + B^1U^1, \\ X^2 & = A^1X^1 + B^2U^2 = A^1(A^0X^0 + B^1U^1) + B^2U^2 = \\ & = A^1A^0X^0 + A^1B^1U^1 + B^2U^2, \\ X^3 & = A^2X^2 + B^3U^3 = A^2(A^1A^0X^0 + A^1B^1U^1 + B^2U^2) + B^3U^3 = \\ & = A^2A^1A^0X^0 + A^2A^1B^1U^1 + A^2B^2U^2 + B^3U^3, \\ & \dots \\ X^T & = A^{T-1}X^{T-1} + B^TU^T = \\ & = \prod_{t=0}^{T-1} A^tX^t + \prod_{t=1}^{T-1} A^tB^1U^1 + \prod_{t=2}^{T-1} A^tB^2U^2 + ...+ \\ & \quad + A^{T-1}B^{T-1}U^{T-1} + B^TU^T, \end{array}$$

and by subsequent substitution of the resulting relations in the objective map:

$$\begin{array}{l} F(X,U) = \\ = c^1(A^0X^0 + B^1U^1) + c^2(A^1A^0X^0 + A^1B^1U^1 + B^2U^2)+ \\ \quad + c^3(A^2A^1A^0X^0 + A^2A^1B^1U^1 + A^2B^2U^2 + B^3U^3)+ \\ \quad + ... + c^T(\prod_{t=0}^{T-1} A^tX^t + \prod_{t=1}^{T-1} A^tB^1U^1 + \prod_{t=2}^{T-1} A^tB^2U^2+ \\ \quad + ... + A^{T-1}B^{T-1}U^{T-1} + B^TU^T)+ \\ \quad + b^1U^1 + b^2U^2 + ... + b^TU^T = \end{array}$$

$$\begin{aligned}
&= (c^1 + c^2A^1 + c^3A^2A^1 + ... + c^TA^{T-1}A^{T-2}...A^1)A^0X^0+\\
&\quad + (c^1B^1 + c^2A^1B^1 + c^3A^2A^1B^1 + ...+\\
&\quad + c^TA^{T-1}A^{T-2}...A^1B^1 + b^1)U^1+\\
&\quad + (c^2B^2 + c^3A^2B^2 + c^4A^3A^2B^2 + ...+\\
&\quad + c^TA^{T-1}A^{T-2}...A^2B^2 + b^2)U^2 + ...+\\
&\quad + (c^TB^T + b^T)U^T,
\end{aligned}$$

the problem (1) is transformed into:

$$\begin{aligned}
F(U) \quad &= (c^1 + c^2A^1 + c^3A^2A^1 + ... + c^TA^{T-1}A^{T-2}...A^1)A^0X^0+\\
&\quad + (c^1B^1 + c^2A^1B^1 + c^3A^2A^1B^1 + ...+\\
&\quad + c^TA^{T-1}A^{T-2}...A^1B^1 + b^1)U^1+\\
&\quad + (c^2B^2 + c^3A^2B^2 + c^4A^3A^2B^2 + ...+\\
&\quad + c^TA^{T-1}A^{T-2}...A^2B^2 + b^2)U^2 + ...+\\
&\quad + (c^TB^T + b^T)U^T \to \max,\\
D^tU^t \quad &\le d^t,\ t = 1, ..., T.
\end{aligned} \tag{2}$$

Obviously, (1) and (2) are equivalent.

The form of the objective map (2) establishes that the optimal control doesn't depends on initial state X^0.

By applying the specific interval arithmetic properties of linear set-valued programming problems, we can conclude that the solution set of problem (2) is equivalent with the solution set of traditional point-valued linear programming problem, that is we can consider that, in general, the cardinality of every control set $U^1, U^2, ..., U^T$ is equal to 1. So, the solution of the problem (2) may be obtained as a sequence of solutions of T linear programming problems. Apparently, we constructed polynomial method of solving (1). In fact, the method has a pseudo-polynomial complexity because of possible exponential value of T on n.

Theorem 2.1. *Let (1) be solvable. The control $\bar{u}^1, \bar{u}^2, ..., \bar{u}^T$, is optimal if and only if $\bar{u}^t$ is the solution of linear programming problem*

$$\begin{aligned}
(c^tB^t + c^{t+1}A^tB^t + \cdots + c^TA^{T-1}A^{T-2}\ldots A^tB^t + b^t)u^t \quad &\to \quad \max,\\
D^tu^t \quad &\le \quad d^t,
\end{aligned}$$

for $t = 1, ..., T$.

The following theorem is an important particular corollary of the precedent theorem.

Theorem 2.2. *If $A^0 = A^1 = ... = A^{T-1} = A$, $B^1 = B^2 = ... = B^T = B$ and (1) is solvable, then the sequence $\bar{u}^1, \bar{u}^2, ..., \bar{u}^T$, forms an optimal control if and only if $\bar{u}^t$ is the solution of linear programming problem*

$$\begin{aligned}
(c^tB + c^{t+1}AB + c^{t+2}(A)^2B + \cdots + c^T(A)^{T-t}B + b^t)u^t \quad &\to \quad \max,\\
D^tu^t \quad &\le \quad d^t,
\end{aligned}$$

for $t = 1, ..., T$.

Theorem 1.1 establishes a principle for solving (1). The maximum principle of Pontryagin may be applied for solving (1), too. Because the cardinality of every control set $U^1, U^2, ..., U^T$ is equal to 1, let us consider the following recurrent relations:

$$\begin{array}{rcl} p^T & = & c^T \\ p^t & = & p^{t+1}A^t + c^t,\ t = T-1, ..., 1. \end{array} \tag{3}$$

Hamiltonian functions are defined on (3) as

$$H_t(u^t) = \langle p^t B^t + b^t, u^t \rangle, t = T, ..., 1.$$

Theorem 2.3. *Let (1) be solvable. The control* $\bar{u}^1, \bar{u}^2, ..., \bar{u}^T$, *is optimal if and only if*

$$H_t(\bar{u}^t) = \max_{u^t : D^t u^t \le d^t} H_t(u^t),\ t = T, ..., 1.$$

It's obvious that theorems 2.1 and 2.3 are equivalent.

3. LINEAR DISCRETE-TIME SET-VALUED STACKELBERG CONTROL PROBLEM

Let us modify the problem (1) by considering the control of Stackelberg type, that is Stackelberg game with T players [8, 2, 9, 10]. In such game, at each stage t ($t = 1, ..., T$) the player t selects his strategy and communicates his and all precedent selected strategies to the following $t+1$ player. After all stage strategy selections, all the players compute their gains on the resulting profile. Let us name such type of system control as Stackelberg control, and the corresponding problem - linear discrete-time set-valued Stackelberg control problem. Described decision process may be formalized in a following manner:

$$\begin{array}{rcl} F_1(X, U) & = & \sum\limits_{t=1}^{T} (c^{1t} X^t + b^{1t} U^t) \underset{U^1}{\longrightarrow} \max, \\ F_2(X, U) & = & \sum\limits_{t=1}^{T} (c^{2t} X^t + b^{2t} U^t) \underset{U^2}{\longrightarrow} \max, \\ \dots & & \\ F_T(X, U) & = & \sum\limits_{t=1}^{T} (c^{Tt} X^t + b^{Tt} U^t) \underset{U^T}{\longrightarrow} \max, \\ X^t & = & X^{t-1} A^{t-1} + B^t X^t, t = 1, ..., T, \\ D^t U^t & \le & d^t, t = 1, ..., T, \end{array} \tag{4}$$

where $X^0, X^t \subset R^n$, $c^{\pi t} \in R^n$, $U^t \subset R^m$, $b^{\pi t} \in R^m$, $A^{t-1} \in R^{n \times n}$, $B^t \in R^{n \times m}$, $d^t \in R^k$, $D^t \in R^{k \times n}$, $c^t X^t = \langle c^t, X^t \rangle$, $b^t U^t = \langle b^t, U^t \rangle$, $t, \pi = 1, ..., T$.

The set of strategies of player π, ($\pi = 1, 2, ..., T$), is determined only by admissible solutions of the problem:

$$\begin{array}{rcl} F_\pi(X, U^\pi \| U^{-\pi}) & = & \sum\limits_{t=1}^{T} (c^{\pi t} X^t + b^{\pi t} U^t) \underset{U^\pi}{\longrightarrow} \max, \\ X^\pi & = & X^{\pi-1} A^{\pi-1} + B^\pi X^\pi, \\ D^\pi U^\pi & \leq & d^\pi. \end{array}$$

Player's π, ($\pi = 1, 2, ..., T$), decision problem is defined by the precedent linear set-valued programming problem. Since, the controlled system is one for all players, by performing the direct substitutions as above, (4) is transformed into

$$\begin{array}{rcl} F_\pi(U^\pi \| U^{-\pi}) & = & (c^{\pi 1} + c^{\pi 2} A^1 + c^{\pi 3} A^2 A^1 + ...+ \\ & & + c^{\pi T} A^{T-1} A^{T-2} ... A^1) A^0 X^0 + \\ & & + (c^{\pi 1} B^1 + c^{\pi 2} A^1 B^1 + c^{\pi 3} A^2 A^1 B^1 + ...+ \\ & & + c^{\pi T} A^{T-1} A^{T-2} ... A^1 B^1 + b^{\pi 1}) U^1 + \\ & & + (c^{\pi 2} B^2 + c^{\pi 3} A^2 B^2 + c^{\pi 4} A^3 A^2 B^2 + ...+ \\ & & + c^{\pi T} A^{T-1} A^{T-2} ... A^2 B^2 + b^{\pi 2}) U^2 + ...+ \\ & & + (c^{\pi T} B^T + b^{\pi T}) U^T \underset{U^\pi}{\longrightarrow} \max, \pi = 1, ..., T, \\ D^t U^t & \leq & d^t, t = 1, ..., T. \end{array} \qquad (5)$$

As in precedent case of optimal control, the cardinality of every Stackelberg control set $U^1, U^2, ..., U^T$ may be reduced to the solution set of the traditional linear programming problem. From equivalence of (4) and (5) the proof of theorem 3.1 follows.

Theorem 3.1. *Let (4) be solvable. The sequence $\bar{u}^1, \bar{u}^2, ..., \bar{u}^T$, forms a Stackelberg equilibrium control if and only if $\bar{u}^\pi$ is optimal optimal solution of*

$$\begin{array}{l} (c^{\pi\pi} B^\pi + c^{\pi\pi+1} A^\pi B^\pi + \cdots + c^{\pi T} A^{T-1} A^{T-2} \ldots A^\pi B^\pi + b^{\pi\pi}) u^\pi \underset{u^\pi}{\longrightarrow} \max, \\ D^\pi u^\pi \leq d^\pi, \end{array}$$

for every $\pi = 1, ..., T$.

The following theorem is an important particular case of theorem 3.1.

Theorem 3.2. *If $A^0 = A^1 = ... = A^{T-1} = A$, $B^1 = B^2 = ... = B^T = B$ and (4) is solvable, then the sequence $\bar{u}^1, \bar{u}^2, ..., \bar{u}^T$, forms a Stackelberg equilibrium control if and only if $\bar{u}^\pi$ is the solution of linear programming problem*

$$\begin{array}{l} (c^{\pi\pi} B + c^{\pi\pi+1} AB + c^{\pi\pi+2} (A)^2 B + ... + c^{\pi T} (A)^{T-\pi} B + b^{\pi\pi}) u^\pi \underset{u^\pi}{\longrightarrow} \max, \\ D^\pi u^\pi \leq d^\pi, \end{array}$$

for $\pi = 1, ..., T$.

Theorem 3.1 establishes a principle for solving (4). The maximum principle of Pontryagin may be applied for solving (4), too. Let us consider the following recurrent relations

$$\begin{array}{lll} p^{\pi T} & = & c^{\pi T}, \\ p^{\pi t} & = & p^{\pi t+1}A^t + c^{\pi t}, t = T-1, ..., 1, \end{array} \tag{6}$$

where $\pi = 1, ..., T$. Hamiltonian functions are defined on (6) as

$$H_{\pi t}(u^t) = \langle p^{\pi t}B^t + b^{\pi t}, u^t \rangle, t = T, ..., 1, \pi = 1, ..., T.$$

Theorem 3.3. *Let (4) be solvable. The sequence of controls $\bar{u}^1, \bar{u}^2, ..., \bar{u}^T$, forms a Stackelberg equilibrium control if and only if*

$$H_{\pi\pi}(\bar{u}^\pi) = \max_{u^\pi : D^\pi u^\pi \leq d^\pi} H_{\pi\pi}(u^\pi),$$

for $\pi = 1, ..., T$.

The proof of theorem 3.3 may be provided by direct substitution of relations (6) in Hamiltonian functions and by comparing the final results with linear programming problems from theorem 3.1. Obviously, theorems 3.1 and 3.3 are equivalent.

From computational point of view, method for solving problem (4) established by theorem 3.1 looks more preferable than the method established by theorem 3.3.

4. LINEAR DISCRETE-TIME SET-VALUED PARETO-STACKELBERG CONTROL PROBLEM

Let us modify the problem (4) by considering control of Pareto-Stackelberg type. At each stage a single player makes decision. Every player selects his strategy (control) on his stage by considering his criteria and communicates his choice and precedent players choices to the following player. At last stage, after all stage strategy selections, the players compute their gains. Such type of control is named Pareto-Stackelberg control, and the corresponding problem is named linear discrete-time set-valued Pareto-Stackelberg control problem.

The decision process is formalized as follows:

$$\begin{array}{rcl} F_1(X,U) & = & \sum\limits_{t=1}^{T}(c^{1t}X^t + b^{1t}U^t) \underset{U^1}{\longrightarrow} \mathrm{ef}\max, \\ F_2(X,U) & = & \sum\limits_{t=1}^{T}(c^{2t}X^t + b^{2t}U^t) \underset{U^2}{\longrightarrow} \mathrm{ef}\max, \\ & \dots & \\ F_T(X,U) & = & \sum\limits_{t=1}^{T}(c^{Tt}X^t + b^{Tt}U^t) \underset{U^T}{\longrightarrow} \mathrm{ef}\max, \\ X^t & = & A^{t-1}X^{t-1} + B^tX^t,\ t=1,...,T, \\ D^tU^t & \le & d^t,\ t=1,...,T, \end{array} \tag{7}$$

where $X^0, X^t \subset R^n$, $c^{\pi t} \in R^{k_\pi \times n}$, $U^t \subset R^m$, $b^{\pi t} \in R^{k_\pi \times m}$, $A^{t-1} \in R^{n\times n}$, $B^t \in R^{n\times m}$, $d^t \in R^k$, $D^t \in R^{k\times n}$, $t, \pi = 1,...,T$. Notation ef max means multi-criteria maximization.

The set of strategies of player π, ($\pi = 1,2,...,T$), is determined formally by the problem:

$$\begin{array}{c} F_\pi(X, U^\pi \| U^{-\pi}) = \sum\limits_{t=1}^{T}(c^{\pi t}X^t + b^{\pi t}U^t) \underset{U^\pi}{\longrightarrow} \mathrm{ef}\max, \\ X^\pi = X^{\pi-1}A^{\pi-1} + B^\pi X^\pi, \\ D^\pi U^\pi \le d^\pi. \end{array}$$

By performing the direct transformations as above, (7) is transformed into

$$\begin{array}{rcl} F_\pi(U^\pi \| U^{-\pi}) & = & (c^{\pi 1} + c^{\pi 2}A^1 + c^{\pi 3}A^2A^1 + ...+ \\ & & + c^{\pi T}A^{T-1}A^{T-2}...A^1)A^0X^0 + \\ & & + (c^{\pi 1}B^1 + c^{\pi 2}A^1B^1 + c^{\pi 3}A^2A^1B^1 + ...+ \\ & & + c^{\pi T}A^{T-1}A^{T-2}...A^1B^1 + b^{\pi 1})U^1 + \\ & & + (c^{\pi 2}B^2 + c^{\pi 3}A^2B^2 + c^{\pi 4}A^3A^2B^2 + ...+ \\ & & + c^{\pi T}A^{T-1}A^{T-2}...A^2B^2 + b^{\pi 2})U^2 + ...+ \\ & & + (c^{\pi T}B^T + b^{\pi T})U^T \underset{U^\pi}{\longrightarrow} \mathrm{ef}\max, \pi = 1,...,T, \\ D^tU^t & \le & d^t, t = 1,...,T. \end{array} \tag{8}$$

By the properties of interval arithmetic relations, we can conclude that (8) is equivalent with simple multi-criteria linear programming problem. Additionally, from equivalence of (7) and (8) the theorem 4.1 follows.

Theorem 4.1. *Let (7) be solvable. The sequence $\bar{u}^1, \bar{u}^2, ..., \bar{u}^T$, forms a Pareto-Stackelberg equilibrium control if and only if $\bar{u}^\pi$ is efficient solution of multi-criteria linear programming problem*

$$\begin{array}{l} (c^{\pi\pi}B^\pi + c^{\pi\pi+1}A^\pi B^\pi + \dots + c^{\pi T}A^{T-1}A^{T-2}\dots A^\pi B^\pi + b^{\pi\pi})u^\pi \underset{u^\pi}{\longrightarrow} \mathrm{ef}\max, \\ D^\pi u^\pi \le d^\pi, \end{array}$$

for $\pi = 1, ..., T$.

As above, a particular cases of (7) is examined.

Theorem 4.2. *If* $A^0 = A^1 = ... = A^{T-1} = A$, $B^1 = B^2 = ... = B^T = B$ *and (7) is solvable, then the sequence* $\bar{u}^1, \bar{u}^2, ..., \bar{u}^T$*, forms a Pareto-Stackelberg equilibrium control if and only if* $\bar{u}^\pi$ *is efficient solution of multi-criteria linear programming problem*

$$\begin{aligned}
&(c^{\pi\pi}B + c^{\pi\pi+1}AB + c^{\pi\pi+2}(A)^2B + ... + c^{\pi T}(A)^{T-\pi}B + b^{\pi\pi})u^\pi \underset{u^\pi}{\longrightarrow} \text{ef max},\\
&D^\pi u^\pi \le d^\pi,
\end{aligned}$$

for $\pi = 1, ..., T$.

Let us extend the Pontryagin maximum principle for (7). By considering the recurrent relations

$$\begin{array}{rcl}
p^{\pi T} & = & c^{\pi T},\\
p^{\pi t} & = & p^{\pi t+1}A^t + c^{\pi t}, t = T-1, ..., 1,
\end{array} \tag{9}$$

where $\pi = 1, ..., T$, the Hamiltonian vector-functions may be defined on (7) and (9) as

$$H_{\pi t}(u^t) = \langle p^{\pi t}B^t + b^{\pi t}, u^t\rangle, t = T, ..., 1, \pi = 1, ..., T.$$

Theorem 4.3. *Let (7) be solvable. The sequence of controls* $\bar{u}^1, \bar{u}^2, ..., \bar{u}^T$*, forms a Pareto-Stackelberg equilibrium control if and only if*

$$\bar{u}^\pi \in \underset{u^\pi : D^\pi u^\pi \le d^\pi}{\text{Arg ef max}} H_{\pi\pi}(u^\pi),$$

for $\pi = 1, ..., T$.

By direct substitution of (9) in Hamiltonian functions and by comparing the final results with multi-criteria linear programming problems from theorem 4.1 the truth of theorem 4.3 arises. Theorems 4.1 and 4.3 are equivalent.

It can be remarked especially that the method of Pareto-Stackelberg control determining, established by theorem 4.1 — 4.3, needs the solutions of multi-criteria linear programming problems.

5. LINEAR DISCRETE-TIME SET-VALUED NASH-STACKELBERG CONTROL PROBLEM

Let us modify the problem (4) by considering the control of Nash-Stackelberg type with T stages and $\nu_1 + \nu_2 + ... + \nu_T$ players, where $\nu_1, \nu_2, ..., \nu_T$ are the numbers of players at stages $1, 2, ..., T$. Every player is identified by two numbers (indices) (τ, π), where τ is the number of stage on which player selects his strategy and $\pi \in \{1, 2, ..., \nu_\tau\}$ is his number at stage τ. In such game, at each stage τ the players

$1, 2, ..., \nu_\tau$ play a Nash game by selecting simultaneously their strategies and by communicating his and all precedent players selected strategies to the following $\tau + 1$ stage players. After all stage strategy selections, on the resulting profile all the players compute their gains. Such type of control is named Nash-Stackelberg control, and the corresponding problem — linear discrete-time set-valued Nash-Stackelberg control problem.

The decision process may be modelled as

$$\begin{array}{rcl} F_{\tau\pi}(X, U^{\tau\pi} \| U^{-\tau\pi}) & = & \sum_{t=1}^{T} (c^{\tau\pi t} X^t + \sum_{\mu=1}^{\nu_t} b^{\tau\pi t\mu} U^{t\mu}) \xrightarrow[U^{\tau\pi}]{} \max, \\ \tau = 1, ..., T, \pi = 1, ..., \nu_\tau, & & \\ X^t & = & X^{t-1} A^{t-1} + \sum_{\pi=1}^{\nu_t} B^{t\pi} U^{t\pi}, t = 1, ..., T, \\ D^{t\pi} U^{t\pi} & \leq & d^{t\pi}, t = 1, ..., T, \pi = 1, ..., \nu_t, \end{array} \tag{10}$$

where $X^0, X^t \subset R^n$, $c^{\tau\pi t} \in R^n$, $U^{\tau\pi} \subset R^m$, $b^{\tau\pi t\mu} \in R^m$, $A^{t-1} \in R^{n\times n}$, $B^{\tau\pi} \in R^{n\times m}$, $d^{\tau\pi} \in R^k$, $D^{\tau\pi} \in R^{k\times n}$, $t, \tau = 1, ..., T$, $\pi = 1, ..., \nu_\tau$, $\mu = 1, ..., \nu_t$.

By performing direct transformations

$$\begin{array}{rcl} X^1 & = & A^0 X^0 + \sum_{\pi=1}^{\nu_1} B^{1\pi} U^{1\pi}, \\ X^2 & = & A^1 X^1 + \sum_{\pi=1}^{\nu_2} B^{2\pi} U^{2\pi} = \\ & = & A^1 \left(A^0 X^0 + \sum_{\pi=1}^{\nu_1} B^{1\pi} U^{1\pi} \right) + \sum_{\pi=1}^{\nu_2} B^{2\pi} U^{2\pi} = \\ & = & A^1 A^0 X^0 + A^1 \sum_{\pi=1}^{\nu_1} B^{1\pi} U^{1\pi} + \sum_{\pi=1}^{\nu_2} B^{2\pi} U^{2\pi}, \end{array}$$

$$\begin{array}{rcl} X^3 & = & A^2 X^2 + \sum_{\pi=1}^{\nu_3} B^{3\pi} U^{3\pi} = \\ & = & A^2 \left(A^1 A^0 X^0 + A^1 \sum_{\pi=1}^{\nu_1} B^{1\pi} U^{1\pi} + \sum_{\pi=1}^{\nu_2} B^{2\pi} U^{2\pi} \right) + \\ & & + \sum_{\pi=1}^{\nu_3} B^{3\pi} U^{3\pi} = \\ & = & A^2 A^1 A^0 X^0 + A^2 A^1 \sum_{\pi=1}^{\nu_1} B^{1\pi} U^{1\pi} + A^2 \sum_{\pi=1}^{\nu_2} B^{2\pi} U^{2\pi} + \\ & & + \sum_{\pi=1}^{\nu_3} B^{3\pi} U^{3\pi}, \\ \ldots & & \end{array}$$

$$\begin{aligned} rllX^T &= A^{T-1}X^{T-1} + \sum_{\pi=1}^{\nu_T} B^{T\pi}U^{T\pi} = \\ &= \prod_{t=0}^{T-1} A^t X^t + \prod_{t=1}^{T-1} A^t \sum_{\pi=1}^{\nu_1} B^{1\pi}U^{1\pi} + \prod_{t=2}^{T-1} A^t \sum_{\pi=1}^{\nu_2} B^{2\pi}U^{2\pi} + ...+ \\ & +A^{T-1}\sum_{\pi=1}^{\nu_{T-1}} B^{T-1\pi}U^{T-1\pi} + \sum_{\pi=1}^{\nu_T} B^{T\pi}U^{T\pi}, \end{aligned}$$

and by subsequent substitution in the objective/cost functions, the problem (10) is reduced to

$$\begin{aligned} F_{\tau\pi}(U^{\tau\pi}\|U^{-\tau\pi}) = \ & (c^{\tau\pi 1} + c^{\tau\pi 2}A^1 + c^{\tau\pi 3}A^2A^1 + ...+ \\ & +c^{\tau\pi T}A^{T-1}A^{T-2}...A^1)A^0X^0 + \\ & +(c^{\tau\pi 1}B^{11} + c^{\tau\pi 2}A^1B^{11} + c^{\tau\pi 3}A^2A^1B^{11} + ...+ \\ & +c^{\tau\pi T}A^{T-1}A^{T-2}...A^1B^{11} + b^{\tau\pi 11})U^{11} + \\ & +(c^{\tau\pi 1}B^{12} + c^{\tau\pi 2}A^1B^{12} + c^{\tau\pi 3}A^2A^1B^{12} + ...+ \\ & +c^{\tau\pi T}A^{T-1}A^{T-2}...A^1B^{12} + b^{\tau\pi 12})U^{12} + \\ & +...+ \\ & +(c^{\tau\pi 1}B^{1\nu_1} + c^{\tau\pi 2}A^1B^{1\nu_1} + c^{\tau\pi 3}A^2A^1B^{1\nu_1} + ...+ \\ & +c^{\tau\pi T}A^{T-1}A^{T-2}...A^1B^{1\nu_1} + b^{\tau\pi 1\nu_1})U^{1\nu_1} + \\ & +(c^{\tau\pi 2}B^{21} + c^{\tau\pi 3}A^2B^{21} + c^{\tau\pi 4}A^3A^2B^{21} + ...+ \\ & +c^{\tau\pi T}A^{T-1}A^{T-2}...A^2B^{21} + b^{\tau\pi 21})U^{21} + ...+ \\ & +(c^{\tau\pi 2}B^{22} + c^{\tau\pi 3}A^2B^{22} + c^{\tau\pi 4}A^3A^2B^{22} + ...+ \\ & +c^{\tau\pi T}A^{T-1}A^{T-2}...A^2B^{22} + b^{\tau\pi 22})U^{22} + ...+ \\ & +...+ \\ & +(c^{\tau\pi 2}B^{2\nu_2} + c^{\tau\pi 3}A^2B^{2\nu_2} + c^{\tau\pi 4}A^3A^2B^{2\nu_2} + ...+ \\ & +c^{\tau\pi T}A^{T-1}A^{T-2}...A^2B^{2\nu_2} + b^{\tau\pi 2\nu_2})U^{2\nu_2} + ...+ \\ & +...+ \\ & +(c^{\tau\pi T}B^{T\nu_T} + b^{\tau\pi T\nu_T})U^{T\nu_T} \underset{U^{\tau\pi}}{\longrightarrow} \max, \\ & \tau = 1,...,T, \pi = 1,...,\nu_\tau, \\ D^{\tau\pi}U^{\tau\pi} \leq \ & d^{\tau\pi}, \tau = 1,...,T, \pi = 1,...,\nu_\tau. \end{aligned} \tag{11}$$

The problem (11) is equivalent to the point-valued problem. The control sets $U^1, U^2, ..., U^T$ may be identified with sets of cardinality 1. Evidently, (11) defines a strategic games for which Nash-Stackelberg equilibrium is also Nash equilibrium and it is simply computed as a sequence of solutions of

$$\begin{aligned} f_{\tau\pi}(u^{\tau\pi}\|u^{-\tau\pi}) &= (c^{\tau\pi\tau}B^{\tau\pi} + c^{\tau\pi\tau+1}A^\tau B^{\tau\pi} + \\ & +c^{\tau\pi\tau+2}A^{\tau+1}A^\tau B^{\tau\pi} + \cdots + \\ & +c^{\tau\pi T}A^{T-1}A^{T-2}\ldots A^\tau B^{\tau\pi} + \\ & +b^{\tau\pi\tau\pi})u^{\tau\pi} \underset{u^{\tau\pi}}{\longrightarrow} \max, \\ D^{\tau\pi}u^{\tau\pi} &\leq d^{\tau\pi}, \tau = 1,...,T, \pi = 1,...,\nu_\tau. \end{aligned} \tag{12}$$

Equivalence of (10) and (12) proves the following theorem 5.1.

Theorem 5.1. *Let (10) be solvable. The sequence $\bar{u}^{11}, \bar{u}^{12}, ..., \bar{u}^{T\nu_T}$, forms a Nash-Stackelberg equilibrium control if and only if $\bar{u}^{\tau\pi}$ is the optimal solution of linear programming problem (12) for $\tau = 1, ..., T$, $\pi = 1, ..., \nu_\tau$.*

An important particular cases of (10) is evident.

Theorem 5.2. *If $A^0 = A^1 = ... = A^{T-1} = A$, $B^{11} = B^{12} = ... = B^{T\nu_T} = B$ and (10) is solvable, then the sequence $\bar{u}^{11}, \bar{u}^{12}, ..., \bar{u}^{T\nu_T}$, forms a Nash-Stackelberg equilibrium control if and only if $\bar{u}^{\tau\pi}$ is optimal in linear programming problem*

$$\begin{aligned} f_{\tau\pi}(u^{\tau\pi} \| u^{-\tau\pi}) &= (c^{\tau\pi\tau}B + c^{\tau\pi\tau+1}AB + c^{\tau\pi\tau+2}(A)^2B + ...+ \\ &+ c^{\tau\pi T}(A)^{T-\tau}B + b^{\tau\pi\tau\pi})u^{\tau\pi} \underset{u^{\tau\pi}}{\longrightarrow} \max, \\ D^{\tau\pi}u^{\tau\pi} &\leq d^{\tau\pi}, \end{aligned}$$

for $\tau = 1, ..., T$, $\pi = 1, ..., \nu_\tau$.

Pontryagin maximum principle is extended for (10). Let us consider the following recurrent relations

$$\begin{array}{lcl} p^{\tau\pi T} &=& c^{\tau\pi T}, \\ p^{\tau\pi t} &=& p^{\tau\pi t+1}A^t + c^{\tau\pi t}, t = T-1, ..., 1, \end{array} \tag{13}$$

where $\tau = 1, ..., T$, $\pi = 1, ..., \nu_\tau$. Hamiltonian functions are defined as

$$H_{\tau\pi t}(u^{\tau\pi}) = \langle p^{\tau\pi t}B^{\tau\pi} + b^{\tau\pi\tau\pi}, u^{\tau\pi} \rangle, t = T, ..., 1,$$

where $\tau = 1, ..., T$, $\pi = 1, ..., \nu_\tau$, and $p^{\tau\pi t}$, $t = T, ..., 1$, $\tau = 1, ..., T$, $\pi = 1, ..., \nu_\tau$, are defined by (13).

Theorem 5.3. *Let (10) be solvable. The sequence of controls $\bar{u}^{11}, \bar{u}^{12}, ..., \bar{u}^{T\nu_T}$, forms a Nash-Stackelberg equilibrium control if and only if*

$$H_{\tau\pi t}(\bar{u}^{\tau\pi}) = \max_{u^{\tau\pi}: D^{\tau\pi}u^{\tau\pi} \leq d^{\tau\pi}} H_{\tau\pi t}(u^{\tau\pi}),$$

for $t = T, ..., 1$, $\tau = 1, ..., T$, $\pi = 1, ..., \nu_\tau$.

Theorems 5.1 and 5.3 are equivalent.

6. LINEAR DISCRETE-TIME SET-VALUED PARETO-NASH-STACKELBERG CONTROL PROBLEM

Let us unify (7) and (10) by considering the control of Pareto-Nash-Stackelberg type with T stages and $\nu_1 + \nu_2 + ... + \nu_T$ players, where $\nu_1, \nu_2, ..., \nu_T$ are the correspondent numbers of players on stages $1, 2, ..., T$. Every player is identified by two numbers as above in Nash-Stackelberg control: τ is stage on which player selects his strategy and π player number at stage τ. In such game, at each stage τ the players $1, 2, ..., \nu_\tau$ play a Pareto-Nash game by selecting simultaneously their strategies accordingly their criteria ($k_{\tau 1}, k_{\tau 2}, ..., k_{\tau \nu_\tau}$ are the numbers of criteria of respective players) and by communicating his and all precedent selected strategies to the following $\tau + 1$ stage players. After all stage strategy selections, all the players compute their gains on the resulting profile. Such type of control is named Pareto-Nash-Stackelberg control, and the corresponding problem linear discrete-time set-valued Pareto-Nash-Stackelberg control problem.

The mathematical model of decision control process may be established as

$$\begin{aligned} F_{\tau\pi}(X, U^{\tau\pi} \| U^{-\tau\pi}) &= \sum_{t=1}^{T} (c^{\tau\pi t} X^t + \sum_{\mu=1}^{\nu_t} b^{\tau\pi t\mu} U^{t\mu}) \underset{U^{\tau\pi}}{\longrightarrow} \text{ef max}, \\ & \quad \tau = 1, ..., T, \pi = 1, ..., \nu_\tau, \\ X^t &= A^{t-1} X^{t-1} + \sum_{\pi=1}^{\nu_t} B^{t\pi} U^{t\pi}, t = 1, ..., T, \\ D^{t\pi} U^{t\pi} &\leq d^{t\pi}, t = 1, ..., T, \pi = 1, ..., \nu_t, \end{aligned} \tag{14}$$

where $X^0, X^t \subset R^n$, $c^{\tau\pi t} \in R^{k_{tp} \times n}$, $U^{\tau\pi} \subset R^m$, $b^{\tau\pi t\mu} \in R^{k_{tp} \times m}$, $A^{t-1} \in R^{n \times n}$, $B^{\tau\pi} \in R^{n \times m}$, $d^{\tau\pi} \in R^k$, $D^{\tau\pi} \in R^{k \times n}$, $t, \tau = 1, ..., T$, $\pi = 1, ..., \nu_\tau$, $\mu = 1, ..., \nu_t$.

By performing similar transformation as above, (14) is reduced to a sequence of multi-criteria linear programming problems

$$\begin{aligned} f_{\tau\pi}(u^{\tau\pi} \| u^{-\tau\pi}) &= (c^{\tau\pi\tau} B^{\tau\pi} + c^{\tau\pi\tau+1} A^\tau B^{\tau\pi} + \\ & \quad + c^{\tau\pi\tau+2} A^{\tau+1} A^\tau B^{\tau\pi} + \cdots + \\ & \quad + c^{\tau\pi T} A^{T-1} A^{T-2} \ldots A^\tau B^{\tau\pi} + \\ & \quad + b^{\tau\pi\tau\pi}) u^{\tau\pi} \underset{u^{\tau\pi}}{\longrightarrow} \text{ef max}, \\ D^{\tau\pi} u^{\tau\pi} &\leq d^{\tau\pi}, \tau = 1, \ldots, T, \pi = 1, \ldots, \nu_\tau. \end{aligned} \tag{15}$$

Equivalence of (14) and (15) proves the following theorem 6.1.

Theorem 6.1. *Let (14) be solvable. The sequence $\bar{u}^{11}, \bar{u}^{12}, ..., \bar{u}^{T\nu_T}$, forms a Pareto-Nash-Stackelberg equilibrium control in (14) if and only if $\bar{u}^{\tau\pi}$ is an efficient solution of multi-criteria linear programming problem (15), for $\tau = 1, ..., T$, $\pi = 1, ..., \nu_\tau$.*

As a corollary follows theorem 6.2.

Theorem 6.2. *If $A^0 = A^1 = ... = A^{T-1} = A$, $B^{11} = B^{12} = ... = B^{T\nu_T} = B$ and (10) is solvable, then the sequence $\bar{u}^{11}, \bar{u}^{12}, ..., \bar{u}^{T\nu_T}$, forms a Pareto-Nash-Stackelberg equilibrium control if and only if $\bar{u}^{\tau\pi}$ is an efficient solution of multi-criteria linear programming problem*

$$\begin{aligned} f_{\tau\pi}(u^{\tau\pi} \| u^{-\tau\pi}) &= (c^{\tau\pi\tau} B + c^{\tau\pi\tau+1} AB + c^{\tau\pi\tau+2} (A)^2 B + ... + \\ & \quad + c^{\tau\pi T} (A)^{T-\tau} B + b^{\tau\pi\tau\pi}) u^{\tau\pi} \underset{u^{\tau\pi}}{\longrightarrow} \text{ef max}, \\ D^{\tau\pi} u^{\tau\pi} &\leq d^{\tau\pi}, \end{aligned}$$

for $\tau = 1, ..., T$, $\pi = 1, ..., \nu_\tau$.

Pontryagin maximum principle may be generalized for (14), too. By considering recurrent relations

$$\begin{aligned} p^{\tau\pi T} &= c^{\tau\pi T}, \\ p^{\tau\pi t} &= p^{\tau\pi t+1}A^t + c^{\tau\pi t}, t = T-1, ..., 1, \end{aligned} \tag{16}$$

where $\tau = 1, ..., T$, $\pi = 1, ..., \nu_\tau$. Hamiltonian vector-functions are defined on (16) as

$$H_{\tau\pi t}(u^{\tau\pi}) = \langle p^{\tau\pi t}B^{\tau\pi} + b^{\tau\pi\tau\pi}, u^{\tau\pi}\rangle, t = T, ..., 1.$$

Remark, the vector nature of (16) via (13).

Theorem 6.3. *Let (14) be solvable. The sequence of controls* $\bar{u}^{11}, \bar{u}^{12}, ..., \bar{u}^{T\nu_T}$*, forms a Pareto-Nash-Stackelberg equilibrium control if and only if*

$$\bar{u}^{\tau\pi} \in \underset{u^{\tau\pi}: D^{\tau\pi}u^{\tau\pi} \leq d^{\tau\pi}}{\mathrm{Arg\,ef\,max}}\; H_{\tau\pi t}(u^{\tau\pi}),$$

for $t = T, ..., 1$, $\tau = 1, ..., T$, $\pi = 1, ..., \nu_\tau$.

Theorems 6.1 and 6.3 are equivalent.

7. CONCLUDING REMARKS

Different types of control processes may be observed in real life: optimal control, Stackelberg control, Pareto-Stackelberg control, Nash-Stackelberg control, Pareto-Nash-Stackelberg control, etc. Traditionally the single valued control is studied. But, really the control may have a set valued nature, too. For such type of control processes the mathematical models and solving principles are established.

The direct-straightforward and classical Pontryagin principle is applied for determining the desired control of set-valued dynamic processes. These principles are the bases for pseudo-polynomial methods, which are exposed as a consequence of theorems for set-valued linear discrete-time Pareto-Nash-Stackelberg control problems.

The results obtained for different types of set-valued non-linear control processes with discrete and continuous time will be exposed in a future paper.

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ON THE INTEGRABILITY OF A STOKES-DIRAC STRUCTURE

Vlad A. Vulcu

"A. I. Cuza" University, Iaşi, Romania

vulcuvlad@gmail.com

Abstract In this paper we recall the notion of Stokes-Dirac structure and we construct several examples of such structures. Then we discuss the integrability of some Stokes-Dirac structures by introducing the convenient Courant brackets. Our theory has potential applications in the control theory and the electromagnetism.

Keywords: semispays, Lie algebroids, Dirac structures, Stokes-Dirac structures.
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1. INTRODUCTION

When studying a complex physical system one can rely on various methods. Two of these are the network modelling and port-based network modelling, which basically mean that the complex physical system is first decomposed into simpler physical subsystems which can be studied separately, and secondly, study the interactions between the subsystems previously determined. In so doing one studies the complex physical system in a hierarchical and controlled manner.

On of the tools used to study the interactions, i.e. the power transfer, between the subsystems, is *the Dirac structure*, as defined by *Courant* and *Weinstein*, in [3]. In the same paper they also define *the integrable Dirac structure* by means of a bilinear skew-symmetric map, which later came to be known as *the Courant bracket*. *The Dirac (integrable) structure* mainly bridges the Poisson manifolds and the presymplectic structures, and has many extensions, see [10]. It also provides conditions for the existence of solutions for important classes of mixed algebraic and differential equations. For more on this subject see [4], [5], [9], [8] or [6].

In 2002, *Schaft* and *Maschke* define in [1], a new type of Dirac structure, called *the Stokes-Dirac structure*. In this case the main ingredients are *the Poincaré duality theorem* and *the Stokes formula*. There, they show that the equations of electromangetism, as given in [11], and other important PDE's can be derived from such structures. *The Hodge-Dirac* and *Laplace-Beltrami-Dirac structures* are later defined in [12]. Some properties of the Stokes-Dirac like structures can be found in [13].

The main goal of this paper is to define *the integrable Stokes-Dirac structures*.

It is structured as follows.

In the first section we define *the Dirac structure* and *the integrable Dirac structure*, using *the Courant bracket*, as defined by *Courant* and *Weinstein*, in [3]. Then we construct several examples of such structures and then give several equivalent conditions for the integrability of a Dirac structure.

In the second section, following [1], we define *the Stokes-Dirac structures*.

In the third section we define *the integrable Stokes-Dirac structures*, using two Courant like brackets, previously defined.

2. DIRAC STRUCTURES

In this section we define the Dirac and integrable Dirac structures. The latter depends on the Courant bracket. Then we give some equivalent conditions to the integrability of a Dirac structure.

Let E and F be linear spaces of dimensions m and n respectively, endowed with a bilinear non-degenerate pairing $(,) : E \times F \to \mathbb{R}$, and consider the total space $(F \times E, \langle,\rangle_+)$.

As an example of such linear spaces and pairing $(,)$, let E be a linear space (of dimension m), $F = E^\star$ and let $(,)$ be the duality pairing of E and $E^\star$. Another non-trivial example is obtained as follows. Let M be a smooth oriented (compact) m-manifold, $F = \Lambda^k(M)$, i. e. the space of all k-forms, on M, and $E = \Lambda^{m-k}(M)$. Now consider the nondegenerate bilinear pairing $(,) : \Lambda^k(M) \times \Lambda^{m-k}(M) \to \mathbb{R}$, given by:

$$(\alpha, \beta) = \int_M (\beta \wedge \alpha), \tag{2.1}$$

for any $\alpha \in \Lambda^k(M)$ and $\beta \in \Lambda^{m-k}(M)$. It is obvious that by defining $(,)$ in this way, by the Poincaré duality theorem we effectively identify the dual of F with E.

The next step is to associate to $(,)$, the non-degenerate symmetric, bilinear pairing $\langle,\rangle_+$, given by:

$$\left\langle \left(f^1, e^1\right), \left(f^2, e^2\right) \right\rangle_+ = \frac{1}{2}\left[\left(f^1, e^2\right) + \left(f^2, e^1\right)\right] \tag{2.2}$$

for any $\left(f^1, e^1\right), \left(f^2, e^2\right) \in F \times E$.

Definition 2.1. *Let F and E be linear spaces, and let $(,) : F \times E \to \mathbb{R}$ be a non-degenerate bilinear pairing and consider a subspace $\mathcal{D} \subset (F \times E, \langle,\rangle_+)$. The orthogonal complement of $\mathcal{D}$, denoted by $\mathcal{D}^\perp$, with respect to $\langle,\rangle_+$, is given by:*

$$\mathcal{D}^\perp = \left\{ \left(\bar{f}, \bar{e}\right) \in F \times E \mid \left\langle (f, e), \left(\bar{f}, \bar{e}\right) \right\rangle_+ = 0, \ \forall (s, \alpha) \in \mathcal{D} \right\}. \tag{2.3}$$

Definition 2.2. *Let F and E be linear spaces (of finite dimensions), endowed with a bilinear nondegenerate pairing $(,)$ and consider the total space $(F \times E, \langle,\rangle_+)$. The linear subspace $\mathcal{D} \subset (F \times E, \langle,\rangle_+)$ is a Dirac structure if*

$$\mathcal{D} = \mathcal{D}^\perp. \tag{2.4}$$

Example 2.1. *Let E be a linear space of dimension m, and let $E^\star$ be the algebraic dual of E, and consider the linear maps $A : E \to E^\star$ and $B : E^\star \to E$ respectively. The maps A and B are skew-symmetric maps if and only if their graphs, are Dirac structures.*

In order to define a Dirac structure with respect to a smooth (m-)manifold M, we consider the big tangent bundle of M, i.e. $\mathbb{T}^{big}M = TM \oplus T^\star M$, where TM is tangent bundle of M and $T^\star M$ is the cotangent bundle of M. The map $(,)$ is defined as the duality pairing between TM and $T^\star M$, respectively. In this case the symmetric bilinear pairing $\langle , \rangle_+$ is given by:

$$\langle (X, \alpha), (Y, \beta) \rangle_+ = \tfrac{1}{2} (i_Y \alpha + i_X \beta), \tag{2.5}$$

for any $(X, \alpha), (Y, \beta) \in \mathbb{T}^{big}M$. Let $\langle , \rangle_-$ be the skew-symmetric and bilinear pairing given by:

$$\langle (X, \alpha), (Y, \beta) \rangle_+ = \tfrac{1}{2} (i_Y \alpha - i_X \beta), \tag{2.6}$$

for any $(X, \alpha), (Y, \beta) \in \mathbb{T}^{big}M$.

The orthogonal complement of a subbundle $\mathcal{D} \subset \left(\mathbb{T}^{big}M, \langle , \rangle_+\right)$, denoted by $\mathcal{D}^\perp$, and it is given by:

$$\mathcal{D}^\perp = \left\{(Y, \beta) \in \mathbb{T}^{big}M \mid \langle (X, \alpha), (Y, \beta) \rangle_+ = 0, \text{ for all } (X, \alpha) \in \mathcal{D}\right\}. \tag{2.7}$$

Definition 2.3. *Let M be a smooth m-manifold and let $\left(\mathbb{T}^{big}M, \langle , \rangle_+\right)$ be the big tangent bundle of M. The subbundle $\mathcal{D} \subset \left(\mathbb{T}^{big}M, \langle , \rangle_+\right)$ is a Dirac structure if*

$$\mathcal{D} = \mathcal{D}^\perp. \tag{2.8}$$

Example 2.2. *Let ω be a 2-form on the smooth manifold M. Then the subbundle*

$$\mathcal{D}_\omega = \left\{(X, \alpha) \in \mathbb{T}^{big}M \mid \alpha = i_X \omega\right\}, \tag{2.8$'$}$$

is a Dirac structure. One can easily check that converse is also true.

Example 2.3. *Let $B : \Lambda^1(M) \to \chi(M)$ be a skew-symmetric map. Then the subbundle*

$$\mathcal{D}_B = \left\{(X, \alpha) \in \mathbb{T}^{big}M \mid X = B(\alpha)\right\} \tag{2.8$''$}$$

is a Dirac structure. This map extends the one previously defined in Example 3. Similarly, the map A from Example 3 is extended to a linear skew-symmetric map, also denoted by A, that give rise to a Dirac structure.

For the definition of the integrable Dirac structure we use the Courant bracket $[,]_C$, which is given by:

$$[(X, \alpha), (Y, \beta)]_C = \left([X, Y]_E, L_X \beta - L_Y \alpha + \tfrac{1}{2} d (i_Y \alpha - i_X \beta)\right), \tag{2.9}$$

for any (X,α), $(Y,\beta) \in \Gamma\left(\mathbb{T}^{big}M\right)$. It is easy to check that $[,]_C$ is bilinear and skew-symmetric.

When restricted to the sections of a Dirac structure $\mathcal{D} \subset \mathbb{T}^{big}M$, the Courant bracket $[,]_C$ is given by:

$$\left[(X,\alpha),(Y,\beta)\right]_C = \left([X,Y],\ L_X\beta - L_Y\alpha + d(\alpha(Y))\right), \tag{2.10}$$

for any (X,α), $(Y,\beta) \in \Gamma\left(\mathbb{T}^{big}M\right)$.

One can easily check that the first component of the Jacobiator of the Courant bracket $[,]_C$ always vanishes, while the second component of the Jacobiator of $[,]_C$ does not, since

$$\begin{aligned} &J_2\left((X,\alpha),(Y,\beta),(Z,\gamma)\right) = \\ &= \tfrac{1}{2}d_E\left(L_s\left(\beta(z)\right) + L_v\left(\gamma(s)\right) + L_z\left(\alpha(v)\right)\right) + \\ &+ \tfrac{1}{2}d_E\left(\gamma\left([s,v]_E\right) + \alpha\left([v,z]_E\right) + \beta\left([z,s]_E\right)\right), \end{aligned} \tag{2.11}$$

for any(X,α), (Y,β), $(Z,\gamma) \in \Gamma(\mathcal{D})$, where $J = (J_1, J_2)$ is the Jacobiator of $[,]_C$. For a detailed computation of J_2 we refer the reader to [2] or [3].

Example 2.4. *Let ω be a 2-form on M. Then the subbundle $\mathcal{D}_\omega \subset \mathbb{T}^{big}M$ is integrable if and only if $\omega|_{\Gamma(\mathcal{D}\cap TM)}$ is closed, i.e. $d\omega|_{\Gamma(\mathcal{D}\cap TM)} = 0$.*

Let T be a map given by

$$\begin{aligned} T\left(X,\alpha\right),\left(Y,\beta\right),\left(Z,\gamma\right) &= \\ = \left(L_X\beta\right)(Z) + \left(L_Y\gamma\right)(X) + \left(L_Z\alpha\right)(Y)&, \end{aligned} \tag{2.12}$$

for any $(X,\alpha),(Y,\beta),(Z,\gamma) \in \Gamma\left(TM \oplus T^\star M\right)$.

The following statements holds good.

Theorem 2.1. *Let $\mathcal{D} \subset \mathbb{T}^{big}M$ be a Dirac structure and consider the map T, given by* (2.12)*. Then $\mathcal{D}$ is an integrable Dirac structure iff $T|_{\Gamma(\mathcal{D})}$ vanishes on the sections of $\mathcal{D}$.*

Theorem 2.2. *Let $\mathcal{D} \subset \mathbb{T}^{big}M$ be a Dirac structure and let $\rho : \mathcal{D}\to TM$, given by $\rho(X,\alpha) = X$. Then $\mathcal{D}$ is an integrable Dirac structure iff the triple $(\mathcal{D}, [,]_C, \rho)$ is a Lie algebroid.*

These theorems provide equivalent conditions to the integrability of a Dirac structure, of which the second one is the most used when solving mixed algebraic and differential equations. We will not go down this path but instead present some interesting extensions of the Dirac structure, called Stokes-Dirac and Hodge-Dirac structures.

3. STOKES-DIRAC STRUCTURES

In this section we define the Stokes-Dirac and Hodge-Dirac structures. Let M be a smooth oriented m-manifold, with smooth boundary ∂M. Let $T^{\star,q}(M)$ be the vector bundle of alternating multilinear forms, of degree q, on M. The fiber at each point is the space $T_p^{\star,q}(M)$ consisting of all q-multilinear alternating continuous functions on the fiber $T_p^{\star,q}(M)$, for each $p \in M$. The sections of $\Lambda^q(M) := \Gamma(T^{\star,q}(M))$ are called q-forms. The set $\Lambda^q(M)$ is an $\mathcal{F}(M)$-module, where $\Lambda^0(M) = \mathcal{F}(M)$ denotes the space of differentiable functions defined on M.

We denote by $\Lambda_c^q(M)$ the space of q-forms with compact support on M and by $\Lambda_c^l(M)$ the space of l-forms with compact support on ∂M. We recall that there is a non-degenerate bilinear pairing $(,)_M : \Lambda_c^q(M) \times \Lambda_c^{m-q}(M) \to \mathbb{R}$, given by $(\alpha,\beta)_M = \int_M \beta \wedge \alpha$, so the dual of $\Lambda_c^q(M)$ is identified with $\Lambda_c^{m-q}(M)$, for each $q \leq m$. Also, there is a non-degenarate bilinear pairing $(,)_{\partial M} : \Lambda_c^l(\partial M) \times \Lambda_c^{m-1-l}(\partial M) \to \mathbb{R}$, given by $(\alpha,\beta)_{\partial M} = \int_{\partial M} \beta \wedge \alpha$, so that the dual of $\Lambda_c^l(\partial M)$ is identified with $\Lambda_c^{m-1-l}(\partial M)$, for each $l \leq m-1$.

Now, we consider the $\mathcal{F}(M)-$modules $\Lambda^p(M)$ and $\Lambda^q(M)$, respectively, such that $p+q = m+1$, and define the linear spaces $\mathcal{F}_{p,q}$ and $\mathcal{E}_{p,q}$, by:

$$\mathcal{F}_{p,q} = \Lambda_c^p(M) \times \Lambda_c^q(M) \times \Lambda_c^{m-p}(\partial M), \tag{3.1}$$

and

$$\mathcal{E}_{p,q} = \Lambda_c^{m-p}(M) \times \Lambda_c^{m-q}(M) \times \Lambda_c^{m-q}(\partial M). \tag{3.2}$$

Now, consider the total space $\mathcal{F}_{p,q} \times \mathcal{E}_{p,q}$. It is obvious that the maps $(,)_M$ and $(,)_{\partial M}$, previously defined, yield a non-degenerate pairing $(,)$, on $\mathcal{F}_{p,q} \times \mathcal{E}_{p,q}$, given by:

$$\left(\left(f_p, f_q, f_b\right), \left(e_p, e_q, e_b\right)\right) = \int_M \left[e_p \wedge f_p + e_q \wedge f_q\right] + \int_{\partial M} e_b \wedge f_b, \tag{3.3}$$

for any $\left(f_p, f_q, f_b\right) \in \mathcal{F}_{p,q}$ and $\left(e_p, e_q, e_b\right) \in \mathcal{E}_{p,q}$, which by symmetrization yields a non-degenerate bilinear pairing

$$\langle,\rangle_+ : \left(\mathcal{F}_{p,q} \times \mathcal{E}_{p,q}\right) \times \left(\mathcal{F}_{p,q} \times \mathcal{E}_{p,q}\right) \to \mathbb{R},$$

given by:

$$\begin{aligned} &\left\langle \left(f_p^1, f_q^1, f_b^1, e_p^1, e_q^1, e_b^1\right), \left(f_p^2, f_q^2, f_b^2, e_p^2, e_q^2, e_b^2\right)\right\rangle_+ = \\ &= \left(\left(f_p^1, f_q^1, f_b^1\right), \left(e_p^2, e_q^2, e_b^2\right)\right) + \left(\left(f_p^2, f_q^2, f_b^2\right), \left(e_p^1, e_q^1, e_b^1\right)\right), \end{aligned} \tag{3.4}$$

for any $\left(f_p^i, f_q^i, f_b^i, e_p^i, e_q^i, e_b^i\right) \in \mathcal{F}_{p,q} \times \mathcal{E}_{p,q}$, and $i = 1, 2$.

Definition 3.1. *Let M be a smooth oriented m-manifold with smooth boundary ∂M, and consider the total space $\left(\mathcal{F}_{p,q} \times \mathcal{E}_{p,q}, \langle,\rangle_+\right)$ given by* (3.1), (3.2) *and* (3.4). *Let*

$\mathcal{D} \subset \mathcal{F}_{p,q} \times \mathcal{E}_{p,q}$ be a subbundle and denote by $\mathcal{D}^{\perp}$ its orthogonal complement with respect to $\langle , \rangle_{+}$. We say $\mathcal{D}$ is a Dirac structure, on M, if

$$\mathcal{D} = \mathcal{D}^{\perp}. \tag{3.5}$$

Example 3.1. *([1]) Let M be a smooth oriented m-manifold with smooth boundary ∂M, and consider the total space $\left(\mathcal{F}_{p,q} \times \mathcal{E}_{p,q}, \langle , \rangle_{+}\right)$. The subbundle $\mathcal{D} \subset \mathcal{F}_{p,q} \times \mathcal{E}_{p,q}$, given by:*

$$\begin{gathered}\mathcal{D} = \{\left(f_p, f_q, f_b, e_p, e_q, e_b\right) \in \mathcal{F}_{p,q} \times \mathcal{E}_{p,q} \mid f_p = (-1)^r \, de_q, \\ f_q = de_p, f_b = e_p|_{\partial M}, e_b = (-1)^{m-q+1} \, e_q|_{\partial M}\},\end{gathered} \tag{3.6}$$

where $r = pq + 1$, is a Dirac structure, i.e. $\mathcal{D} = \mathcal{D}^{\perp}$. This type of Dirac structures are called Stokes-Dirac structures.

Example 3.2. *([1]) Furthermore, let N is a smooth oriented n-manifold (with smooth boundary ∂N), and let $\Lambda_c^d(N)$ denote the space of d-forms, on N, $d \leq n$. Assume that there is a map $G : \Lambda_c^d(N) \to \Lambda_c^p(M) \times \Lambda_c^q(M)$, such that its dual, $G^{\star} : \Lambda_c^{m-p}(M) \times \Lambda_c^{m-q}(M) \to \Lambda_c^{n-d}(N)$, satisfies:*

$$\int_M \left[e_p \wedge G_p\left(f_p\right) + e_q \wedge G_q\left(f_d\right)\right] = \int_N \left[G_p^{\star}\left(e_p\right) + G_q^{\star}\left(e_q\right)\right] \wedge f_d, \tag{3.7}$$

for any $e_p \in \Lambda_c^{m-p}(M)$, $e_q \in \Lambda_c^{m-q}(M)$ and $f_d \in \Lambda_c^d(N)$. In order to define the Stokes-Dirac structure with respect to both M and N, we extend $\mathcal{F}_{p,q} \times \mathcal{E}_{p,q}$ to the total augmented space $\mathcal{F}_{p,q}^a \times \mathcal{E}_{p,q}^a$, defined by:

$$\mathcal{F}_{p,q}^a = \mathcal{F}_{p,q} \times \Lambda_c^d(N) \text{ and } \mathcal{E}_{p,q}^a = \mathcal{E}_{p,q} \times \Lambda_c^{n-d}(N), \tag{3.8}$$

The space $\mathcal{F}_{p,q}^a \times \mathcal{E}_{p,q}^a$ is endowed with the bilinear pairing $\langle , \rangle_{+}^a$, given by

$$\begin{gathered}\left\langle\left(f_p^1, f_q^1, f_b^1 f_d^1, e_p^1, e_q^1, e_b^1, e_d^1\right), \left(f_p^2, f_q^2, f_b^2, f_d^2, e_p^2, e_q^2, e_b^2, e_d^2\right)\right\rangle_{+}^a = \\ = \left\langle\left(f_p^1, f_q^1, f_b^1, e_p^1, e_q^1, e_b^1\right), \left(f_p^2, f_q^2, f_b^2, e_p^2, e_q^2, e_b^2\right)\right\rangle_{+} + \\ + \left(f_d^2, e_d^1\right)_N + \left(f_d^1, e_d^2\right)_N,\end{gathered} \tag{3.9}$$

for any $\left(f_p^i, f_q^i, f_b^i, f_d^i, e_p^i, e_q^i, e_b^i, e_d^i\right) \in \mathcal{F}_{p,q}^a \times \mathcal{E}_{p,q}^a$, and $i = 1, 2$, and thus the definition follows.
The subbundle $\mathcal{D}^a \subset \mathcal{F}_{p,q}^a \times \mathcal{E}_{p,q}^a$, given by:

$$\begin{gathered}\mathcal{D}^a = \{\left(f_p, f_q, f_b, f_d, e_p, e_q, e_b, e_d\right) \in \mathcal{F}_{p,q}^a \times \mathcal{E}_{p,q}^a \mid \\ f_p = (-1)^r \, de_q + G_p\left(f_d\right), f_q = de_p + G_q\left(e_q\right), \\ f_b = e_p|_{\partial M}, e_b = (-1)^{m-q+1} \, e_q|_{\partial M}, e_d = -G_p^{\star}\left(e_p\right) - G_q^{\star}\left(e_q\right)\},\end{gathered} \tag{3.10}$$

is a Stokes-Dirac structure, that is $\mathcal{D}^a = (\mathcal{D}^a)^\perp$.

Example 3.3. *([12]) Suppose that* $m = 2l + 1$ *and let* M *be a smooth oriented m-manifold. Let* $p = q = l + 1$ *and consider the total space* $(\mathcal{F}_{l,l} \times \mathcal{E}_{l,l}, \langle , \rangle_+)$. *The subbundle* $\mathcal{D} \subset \mathcal{F}_{l,l} \times \mathcal{E}_{l,l}$, *given by*

$$\mathcal{D} = \{\left(f_p, f_q, e_p, e_q\right) \in \mathcal{F}_{p,q} \times \mathcal{E}_{p,q} \mid f_p = -\star e_q, f_q = \star e_p\}, \tag{3.11}$$

is a Dirac structure, called the Hodge-Dirac structure.

Now, we define the distributed port-Hamiltonian system as follows.

Let M be a smooth oriented m-manifold and let $\mathcal{D}$ be the Stokes-Dirac strucure, given by (3.6), and consider a smooth Hamiltonian $H : \Lambda_c^p(M) \times \Lambda_c^q(M) \to \mathbb{R}$, given by:

$$H\left(\alpha_p, \alpha_q\right) = \int_M \mathcal{H}\left(\alpha_p, \alpha_q, z\right), \tag{3.12}$$

where $\mathcal{H} : \Lambda_c^p(M) \times \Lambda_c^q(M) \times M \to \Lambda_c^m(M)$ is a smooth density. By computing the time derivative of H, along a trajectory $t \in \mathbb{R} \to \left(\alpha_p(t), \alpha_q(t)\right) \in \Lambda_c^p(M) \times \Lambda_c^q(M)$ one gets:

$$\frac{dH}{dt}\left(\alpha_p(t), \alpha_q(t)\right) = \int_M [\delta_p H \wedge \frac{\partial \alpha_p}{\partial t} + \delta_q H \wedge \frac{\partial \alpha_q}{\partial t}]. \tag{3.13}$$

Definition 3.2. *Let* M *is a smooth oriented m-manifold with smooth boundary* ∂M *and let* $\mathcal{D}$ *be the Stokes-Dirac structure given by* (3.6), *and let* H *be a smooth Hamiltonian, as in* (3.12). *The triple* $(M, \mathcal{D}, H)$ *is a distributed port-Hamiltonian system if there exist trajectories* $t \in I \subset \mathbb{R} \to \left(\alpha_p(t), \alpha_q(t)\right) \in \Lambda_c^p(M) \times \Lambda_c^q(M)$ *such that:*

$$\left(-\frac{\partial \alpha_p}{\partial t}, -\frac{\partial \alpha_q}{\partial t}, \delta_p H, \delta_q H\right) \in \mathcal{D}. \tag{3.14}$$

In practice, the spaces $\Lambda_c^p(M)$ and $\Lambda_c^q(M)$ denote the spaces of energy variables of two different physical energy domains which interact with each other, while the spaces $\Lambda_c^{m-p}(\partial M)$ and respectively $\Lambda_c^{m-q}(\partial M)$ denote the boundary variables, whose "$\wedge$"- product represents the boundary energy flow.

Let $m = 3$, $p = 2$, and $q = 2$. In this case the Stokes-Dirac structure $\mathcal{D}$ is given by:

$$\begin{aligned} &\mathcal{D} = \{\left(f_p, f_q, f_b, e_p, e_q, e_b\right) \in \mathcal{F}_{2,2} \times \mathcal{E}_{2,2} \\ &f_1 = -de_2, \;\; f_2 = de_1, \;\; f_b = e_1|_{\partial M}, \;\; e_b = e_2|_{\partial M}\}. \end{aligned} \tag{3.15}$$

Let M be a 3-dimensional space domain with smooth boundary ∂M, and denote by $\mathcal{B} = B_{ij}(t,x) dx^i \wedge dx^j \in \Lambda_c^2(M)$, and $\mathcal{D} = D_{ij}(t,x) dx^i \wedge dx^j \in \Lambda_c^2(M)$ the magnetic field induction 2-form and the electric field induction 2-form respectively. Let $\mathcal{E} = E_i(t,x) dx^i \in \Lambda_c^1(M)$ and $\mathcal{H} = H_i(t,x) dx^i \in \Lambda_c^1(M)$ denote the electric field intensity and magnetic field intensity.

The constitutive equations of M are $\star\mathcal{D} = \varepsilon\mathcal{E}$ and $\star B = \mu\mathcal{H}$, where $\star$ denotes the Hodge map, ε is the electric permittivity of M, and μ is the magnetic permittivity of M.

Now, consider the triple $(M, \mathcal{D}, H)$, where H is a smooth Hamiltonian, given by:

$$H = \frac{1}{2}\int_M [\mathcal{E} \wedge \mathcal{D} + \mathcal{H} \wedge \mathcal{B}]. \tag{3.16}$$

The triple $(M, \mathcal{D}, H)$ is a distributed port-Hamiltonian system since the implicit Hamiltonian equations for the electromagnetism are given by:

$$\begin{aligned} &\tfrac{\partial\mathcal{D}}{\partial t} = d(\delta_D H) = d\mathcal{H},\ -\tfrac{\partial\mathcal{B}}{\partial t} = d(\delta_B H) = d\mathcal{E}, \\ &f_b = \delta_D H|_{\partial M}, e_b = \delta_B H|_{\partial M}, \end{aligned} \tag{3.17}$$

in the case of a zero density electric current, otherwise $\frac{\partial\mathcal{D}}{\partial t} = d\mathcal{H}+f_d$, where $f_d \in \Lambda^2_c(N)$ and $M = N$.

4. INTEGRABILITY

In this section we define the integrable Stokes-Dirac structure, by means of a Courant like bracket $[,]_{p,q,C}$, and then consider the case of a smooth oriented 3-manifold M, where we explicitely define the Courant like bracket $[,]_{2,2,C}$.

Let M a smooth, oriented m-manifold and let p, $q \in \mathbb{N}$ be such that $p + q = m + 1$ and consider a subbundle $\mathcal{D} \subset \mathcal{F}_{p,q} \times \mathcal{E}_{p,q}$. Also, consider the subbundles $\Lambda^p_c(M) \times \Lambda^{m-p}_c(M) \times \Lambda^{m-d}_c(\partial M)$ and $\Lambda^q_c(M) \times \Lambda^{m-q}_c(M) \times \Lambda^{m-q}_c(\partial M)$, respectively, and define the canonical projections

$$\pi_p : \mathcal{F}_{p,q} \times \mathcal{E}_{p,q} \rightarrow \left(\Lambda^p_c(M) \times \Lambda^{m-p}_c(M) \times \Lambda^{m-p}_c(\partial M)\right), \tag{4.1}$$

and

$$\pi_q : \mathcal{F}_{p,q} \times \mathcal{E}_{p,q} \rightarrow \left(\Lambda^q_c(M) \times \Lambda^{m-q}_c(M) \times \Lambda^{m-q}_c(\partial M)\right), \tag{4.2}$$

which are given by:

$$\pi_p\left(f_p, f_q, f_b, e_p, e_q, e_b\right) = \left(f_p, f_b, e_p\right), \tag{4.3}$$

$$\pi_q\left(f_p, f_q, f_b, e_p, e_q, e_b\right) = \left(f_q, e_q, e_d\right), \tag{4.4}$$

for any $\left(f_p, f_q, f_b, e_p, e_q, e_d\right) \in \mathcal{F}_{p,q} \times \mathcal{E}_{p,q}$.

The subbundles $\Lambda^p_c(M) \times \Lambda^{m-p}_c(M) \times \Lambda^{m-p}_c(\partial M)$ and $\Lambda^q_c(M) \times \Lambda^{m-q}_c(M) \times \Lambda^{m-q}_c(\partial M)$ are endowed with the (non-degenerate) bilinear pairings $\langle,\rangle_{p,+}$ and $\langle,\rangle_{q,+}$, which are given by:

$$\left\langle \left(f^1_p, f^1_b, e^1_p\right), \left(f^2_p, f^2_b, e^2_p\right)\right\rangle_{p,+} = \left\langle \left(f^1_p, 0, f^1_b, e^1_p, 0, 0\right), \left(f^2_p, 0, f^2_b, e^2_p, 0, 0\right)\right\rangle, \tag{4.5}$$

and

$$\left\langle \left(f_q^1, e_q^1, e_b^1\right), \left(f_q^2, e_q^2, e_b^2\right)\right\rangle_{q,+} = \left\langle \left(0, f_q^1, 0, 0, e_q^1, e_b^1\right), \left(0, f_q^2, 0, 0, e_q^2, e_b^2\right)\right\rangle, \tag{4.6}$$

for any $\left(f_p^i, f_q^i, f_b^i, e_p^i, e_q^i, e_d^i\right) \in \mathcal{F}_{p,q} \times \mathcal{E}_{p,q}$, $i = \overline{1,2}$.

Definition 4.1. *Let M be a smooth oriented m-manifold with smooth boudary ∂M, and let $\mathcal{D} \subset \mathcal{F}_{p,q} \times \mathcal{E}_{p,q}$ be a subbundle. We say $\mathcal{D}$ is a pseudo-Dirac structure if the subbundles $\pi_p(\mathcal{D}) \subset \left(\Lambda_c^p(M) \times \Lambda_c^{m-p}(M) \times \Lambda_c^{m-p}(\partial M), \langle,\rangle_{p,+}\right)$ and $\pi_q(\mathcal{D}) \subset \left(\Lambda_c^q(M) \times \Lambda_c^{m-q}(M) \times \Lambda_c^{m-q}(\partial M), \langle,\rangle_{q,+}\right)$ are Dirac structures.*

Definition 4.2. *Let $\mathcal{D} \subset \mathcal{F}_{p,q} \times \mathcal{E}_{p,q}$ be a pseudo-Dirac structure. We say $\mathcal{D}$ is an integrable pseudo-Dirac structure if there exist two maps $[,]_{p,0,C} : \left(\Gamma\left(\pi_p(\mathcal{D})\right)\right)^2 \to \Gamma\left(\pi_p(\mathcal{D})\right)$ and $[,]_{0,q,C} : \left(\Gamma\left(\pi_q(\mathcal{D})\right)\right)^2 \to \Gamma\left(\pi_q(\mathcal{D})\right)$, bilinear and skew-symmetric, such that*

1

$$\left[\Gamma\left(\pi_p(\mathcal{D})\right), \Gamma\left(\pi_p(\mathcal{D})\right)\right]_{p,0,C} \subseteq \Gamma\left(\pi_p(\mathcal{D})\right), \tag{4.7}$$

and respectively,

2

$$\left[\Gamma\left(\pi_q(\mathcal{D})\right), \Gamma\left(\pi_q(\mathcal{D})\right)\right]_{0,q,C} \subseteq \Gamma\left(\pi_q(\mathcal{D})\right). \tag{4.8}$$

Let $m = 3$, $p = q = 2$ and let g be a Riemannian metric on M.

Definition 4.3. *Let M be a smooth oriented 3-manifold with smooth boundary ∂M, and let $\mathcal{D} \subset \mathcal{F}_{2,2} \times \mathcal{E}_{2,2}$ be a pseudo-Dirac structure on M.*
Let $[,]_{2,0,C} : \Gamma\left(\pi_p(\mathcal{F}_{2,2} \times \mathcal{E}_{2,2})\right)^2 \to \Gamma\left(\pi_p(\mathcal{F}_{2,2} \times \mathcal{E}_{2,2})\right)$ and
$[,]_{0,2,C} : \Gamma\left(\pi_q(\mathcal{F}_{2,2} \times \mathcal{E}_{2,2})\right)^2 \to \Gamma\left(\pi_q(\mathcal{F}_{2,2} \times \mathcal{E}_{2,2})\right)$ be given by:

$$\begin{gathered}\left[\left(f_1^1, f_b^1, e_1^1\right), \left(f_1^2, f_b^2, e_1^2\right)\right]_{2,0,C} = \\ = (L_{\flat e_1^1} f_1^2 - L_{\flat e_1^2} f_1^1 + \tfrac{1}{2} d(i_{\flat e_1^2} f_1^1 - i_{\flat e_1^1} f_1^2), 0, \sharp\left[\flat e_1^1, \flat e_1^2\right]) \\ \left[\left(f_2^1, e_2^1, e_b^1\right), \left(f_2^2, e_2^2, e_b^2\right)\right]_{0,2,C} = \\ = (L_{\flat e_2^1} f_2^2 - L_{\flat e_2^2} f_2^1 + \tfrac{1}{2} d(i_{\flat e_2^2} f_2^1 - i_{\flat e_2^1} f_2^2), \sharp\left[\flat e_2^1, \flat e_2^2\right], 0),\end{gathered} \tag{4.9}$$

for any f_1^1, f_1^2, f_2^1, f_2^2, $\in \Lambda_c^2(M)$, f_b^1, $f_b^2 \in \Lambda_c^1(\partial M)$, e_1^1, e_1^2, e_2^1, $e_2^2 \in \Lambda_c^1(M)$,and $e_b^1, e_b^2 \in \Lambda_c^1(M)$, where the maps $\flat : T^\star M \to TM$ and $\sharp : TM \to T^\star M$ are the canonical isomorphisms of the metric g.

The following hold good.

Lemma 4.1. *The maps $[,]_{2,0,C}$ and $[,]_{0,2,C}$ are bilinear and skew-symmetric.*

This proof follows from the properties of the Courant bracket $[,]_C$, the Lie bracket $[,]_{TM}$ and that of the isomorphisms $\flat$ and $\sharp$.

Lemma 4.2. *Let M be a smooth oriented 3-manifold with smooth boundary ∂M, and let $p = q = 2$ and consider the pseudo-Dirac structure $\mathcal{D} \subset \mathcal{F}_{2,2} \times \mathcal{E}_{2,2}$, i.e. $\pi_p(\mathcal{D})$ and $\pi_q(\mathcal{D})$ are Dirac structures. The integrability conditions* (4.7) *and* (4.8) *are equivalent to:*

$$\int_M de_2^3 \wedge \left(\sharp\left[\flat e_1^1, e_1^2\right]\right) = -\int_M (L_{\flat e_1^1} f_1^2 - L_{\flat e_1^2} f_1^1 + \frac{1}{2} d(i_{\flat e_1^2} f_1^1 - i_{\flat e_1^1} f_1^2)) \wedge e_1^3, \quad (4.10)$$

and,

$$\int_M de_1^3 \wedge \left(\sharp\left[\flat e_2^1, \flat e_2^2\right]\right) = -\int_M (L_{\flat e_2^1} f_2^2 - L_{\flat e_2^2} f_2^1 + \frac{1}{2} d(i_{\flat e_2^2} f_2^1 - i_{\flat e_2^1} f_2^2)) \wedge e_2^3, \quad (4.11)$$

for any f_1^1, f_1^2, f_2^1, f_2^2, $\in \Lambda_c^2(M)$ and e_1^1, e_1^2, e_2^1, $e_2^2 \in \Lambda_0^1(M)$.

Now we define the *anchor* maps $\rho_p : (\mathcal{F}_{2,2} \times \mathcal{E}_{2,2}) \rightarrow TM$ and $\rho_q : (\mathcal{F}_{2,2} \times \mathcal{E}_{2,2}) \rightarrow TM$, respectively, by:

$$\rho_p\left(f_1^1, f_1^2, f_b^1, e_1^1, e_1^2, e_b^1\right) = \flat\left(e_1^1\right), \quad (4.12)$$

and,

$$\rho_q\left(f_1^1, f_1^2, f_b^1, e_1^1, e_1^2, e_b^1\right) = \flat\left(e_1^2\right), \quad (4.13)$$

for any f_1^1, f_1^2, f_2^1, f_2^2, $\in \Lambda_c^2(M)$, $f_b^1 \in \Lambda_c^1(\partial M)$, $e_b^1 \in \Lambda_c^1(\partial M)$ and e_1^1, e_1^2, e_2^1, $e_2^2 \in \Lambda_c^1(M)$.

Lemma 4.3. *The following hold:*

1

$$\begin{aligned}\rho_p\left(\left[\left(f_1^1, f_1^2, f_b^1, e_1^1, e_1^2, e_b^1\right), \left(f_2^1, f_2^2, f_b^2, e_2^1, e_2^2, e_b^2\right)\right]_{2,0,C}\right) = \\ = \left[\rho_p\left(f_1^1, f_1^2, f_b^1, e_1^1, e_1^2, e_b^1\right), \rho_p\left(f_2^1, f_2^2, f_b^2, e_2^1, e_2^2, e_b^2\right)\right]_{TM},\end{aligned} \quad (4.14)$$

and, respectively,

2

$$\begin{aligned}\rho_q\left(\left[\left(f_1^1, f_1^2, f_b^1, e_1^1, e_1^2, e_b^1\right), \left(f_2^1, f_2^2, f_b^2, e_2^1, e_2^2, e_b^2\right)\right]_{0,2,C}\right) = \\ = \left[\rho_q\left(f_1^1, f_1^2, f_b^1, e_1^1, e_1^2, e_b^1\right), \rho_q\left(f_2^1, f_2^2, f_b^2, e_2^1, e_2^2, e_b^2\right)\right]_{TM},\end{aligned} \quad (4.15)$$

for any f_1^1, f_1^2, f_2^1, $f_2^2 \in \Lambda_c^2(M)$, $f_b^1, f_b^2 \in \Lambda_c^1(\partial M)$, $e_b^1, e_b^1 \in \Lambda_c^1(\partial M)$ and e_1^1, e_1^2, e_2^1, $e_2^2 \in \Lambda_c^1(M)$.

The proof of this lemma follows from the definition ρ_p and ρ_q, and the properties of $\flat$.

Now we compute the Jacobiators of $[,]_{2,0,C}$ and $[,]_{0,2,C}$. By a straightforward computation we obtain:

$$
\begin{aligned}
&[\left[\left(f_1^1, f_b^1, e_1^1\right), \left(f_1^2, f_b^2, e_1^2\right)\right]_{2,0,C}, \left(f_1^3, f_b^3, e_1^3\right)]_{2,0,C} \qquad (4.16)\\
&= [(L_{\flat e_1^1} f_1^2 - L_{\flat e_1^2} f_1^1 + \frac{1}{2}(d i_{\flat e_1^2} f_1^1 - i_{\flat e_1^1} f_1^2), 0, \sharp\left[\flat e_1^1, \flat e_1^2\right]), \left(f_1^3, f_b^3, e_1^3\right)]_{p,C}\\
&= (L_{\left[\flat e_1^1, \flat e_1^2\right]} f_1^3 - L_{\flat e_1^3} L_{\flat e_1^1} f_1^2 + L_{\flat e_1^3} L_{\flat e_1^2} f_1^1 + \frac{1}{2} d L_{\flat e_1^3}(i_{\flat e_1^2} f_1^1 - i_{\flat e_1^1} f_1^2)\\
&+\frac{1}{2} d(i_{\flat e_1^3} L_{\flat e_1^1} f_1^2 - i_{\flat e_1^3} L_{\flat e_1^2} f_1^1 + \frac{1}{2} i_{\flat e_1^3}(d(i_{\flat e_1^2} f_1^1 - i_{\flat e_1^1} f_1^2) - i_{\left[\flat e_1^1, \flat e_1^2\right]} f_1^3),\\
&0, \sharp\left[\left[\flat e_1^1, \flat e_1^2\right], \flat e_1^3\right])
\end{aligned}
$$

and,

$$
\begin{aligned}
&[\left[\left(f_2^1, e_2^1, e_b^1\right), \left(f_2^2, e_2^2, e_b^2\right)\right]_{0,2,C}, \left(f_2^3, e_2^3, e_b^3\right)]_{0,2,C} = \qquad (4.17)\\
&= [(L_{\flat e_2^1} f_2^2 - L_{\flat e_2^2} f_2^1 + \frac{1}{2} d(i_{\flat e_2^2} f_2^1 - i_{\flat e_2^1} f_2^2), \sharp\left[\flat e_2^1, \flat e_2^2\right], 0), \left(f_2^3, e_2^3, e_b^3\right)]_{q,C}\\
&= (L_{\left[\flat e_2^1, \flat e_2^2\right]} f_2^3 - L_{\flat e_2^3} L_{\flat e_2^1} f_2^2 + L_{\flat e_2^3} L_{\flat e_2^2} f_2^1 + \frac{1}{2} d L_{\flat e_2^3}(i_{\flat e_2^2} f_2^1 - i_{\flat e_2^1} f_2^2)\\
&+\frac{1}{2}(d(i_{\flat e_2^3} L_{\flat e_2^1} f_2^2 - i_{\flat e_2^3} L_{\flat e_2^2} f_2^1 + \frac{1}{2} i_{\flat e_2^3} d(i_{\flat e_2^2} f_2^1 - i_{\flat e_2^1} f_2^2) - i_{\left[\flat e_2^1, \flat e_2^2\right]} f_2^3)),\\
&\sharp\left[\left[\flat e_2^1, \flat e_2^2\right], \flat e_2^3\right], 0),
\end{aligned}
$$

for any $f_1^1, f_1^2, f_1^3, f_2^1, f_2^2, {}_2^3 \in \Lambda_c^2(M)$, $f_b^1, f_b^2, f_b^3 \in \Lambda_c^1(\partial M)$, $e_b^1, e_b^2, e_b^3 \in \Lambda_c^1(\partial M)$ and $e_1^1, e_1^2, e_1^3, e_2^1, e_2^2, e_2^3 \in \Lambda_c^1(M)$. From the previous formulae, follows

Lemma 4.4. *Let* $\left(f_i^1, f_i^2, f_b^i, e_i^1, e_i^2, e_b^i\right) \in \Gamma(\mathcal{D}_{2,2})$, $i = \overline{1,3}$. *The Jacobiators of* $[,]_{2,0,C}$ *and* $[,]_{0,2,C}$ *are given by*

$$
\begin{aligned}
&J_{2,0,C}\left(\left(f_1^1, f_b^1, e_1^1\right), \left(f_1^2, f_b^2, e_1^2\right), \left(f_1^3, f_b^3, e_1^3\right)\right) = \qquad (4.18)\\
&= (\{i_{\flat e_1^2} L_{\flat e_1^3} f_1^1 - i_{\flat e_1^1} L_{\flat e_1^3} f_1^2 + i_{\flat e_1^3} L_{\flat e_1^1} e f_1^2 -\\
&- i_{\flat e_1^2} L_{\flat e_1^1} f_1^3 + i_{\flat e_1^1} L_{\flat e_1^2} f_1^3 - i_{\flat e_1^3} L_{\flat e_1^2} f_1^1\}\\
&+\frac{1}{4} d\{i_{\flat e_1^2} d i_{\flat e_1^1} f_1^3 - i_{\flat e_1^2} d i_{\flat e_1^3} f_1^1 + i_{\flat e_1^1} d i_{\flat e_1^3} f_1^2 -\\
&- i_{\flat e_1^1} d i_{\flat e_1^2} f_1^3 + i_{\flat e_1^2} d i_{\flat e_1^1} f_1^3 - i_{\flat e_1^2} d i_{\flat e_1^3} f_1^1\} -\\
&-\frac{1}{2}\{i_{\left[\flat e_1^2, \flat e_1^3\right]} f_1^1 + i_{\left[\flat e_1^3, \flat e_1^1\right]} f_1^2 + i_{\left[\flat e_1^1, \flat e_1^2\right]} f_1^3\}, 0, 0),
\end{aligned}
$$

and

$$J_{0,2,C}\left(\left(f_2^1,e_2^1,e_b^1\right),\left(f_2^2,e_2^2,e_b^2\right),\left(f_2^3,e_2^3,e_b^3\right)\right)= \tag{4.19}$$
$$=(\{i_{\flat e_2^2}L_{\flat e_2^3}f_2^1-i_{\flat e_2^1}L_{\flat e_2^3}f_2^2+i_{\flat e_2^3}L_{\flat e_2^1}f_2^2-$$
$$-i_{\flat e_2^2}L_{\flat e_2^1}f_2^3+i_{\flat e_2^1}L_{\flat e_2^2}f_2^3-i_{\flat e_2^3}L_{\flat e_2^2}f_2^1\}$$
$$+\frac{1}{4}d\{i_{\flat e_2^2}di_{\flat e_2^1}f_2^3-i_{\flat e_2^2}di_{\flat e_2^3}f_2^1+i_{\flat e_2^1}di_{\flat e_2^3}f_2^2-$$
$$-i_{\flat e_2^1}di_{\flat e_2^2}f_2^3+i_{\flat e_2^2}di_{\flat e_2^1}f_2^3-i_{\flat e_2^2}di_{\flat e_2^3}f_2^1\}+$$
$$+\frac{1}{2}\{i_{\left[\flat e_2^2,\flat e_2^3\right]}f_2^1+i_{\left[\flat e_2^3,\flat e_2^1\right]}f_2^2+i_{\left[\flat e_2^1,\flat e_2^2\right]}f_2^3\},0,0).$$

Theorem 4.1. *Let $\mathcal{D}$ be a pseudo-Dirac structure. Then the Jacobiators $J_{2,0,C}$ and $J_{0,2,C}$ vanish if and only if both $\left(\pi_p(\mathcal{D}),[,]_{2,0,C}|_{\Gamma(\pi_p(\mathcal{D}))}\right)$ and $\left(\pi_q(\mathcal{D}),[,]_{0,2,C}|_{\Gamma(\pi_q(\mathcal{D}))}\right)$ are Lie algebras.*

The if part follows from the vahishing of the Jacobiator of both $[,]_{2,0,C}$ and $[,]_{0,2,C}$, and the anchor properties of ρ_p and ρ_q. The only if part is a consequence of the Lie algebra structure that both $\left(\pi_1(\mathcal{D}),[,]_{2,0,C},\rho_p\right)$ and $\left(\pi_2(\mathcal{D}),[,]_{0,2,C},\rho_q\right)$ respectively are endowed with.

Theorem 4.2. *Let $\mathcal{D}$ be an integrable pseudo-Dirac structure. Then $\left(\pi_p(\mathcal{D}),[,]_{2,0,C}|_{\Gamma(\pi_p(\mathcal{D}))},\rho_p\right)$ and $\left(\pi_q(\mathcal{D}),[,]_{0,2,C}|_{\Gamma(\pi_q(\mathcal{D}))},\rho_q\right)$ are Lie algebroids.*

Definition 4.4. *Let M be a smooth oriented 3-manifold and consider the total space $(\mathcal{F}_{2,2}\times\mathcal{E}_{2,2},\langle,\rangle_+)$. On the sections of $\mathcal{F}_{2,2}\times\mathcal{E}_{2,2}$ we define the Courant bracket:*

$$\left[\left(f_1^1,f_2^1,f_b^1,e_1^1,e_2^1,e_b^1\right),\left(f_1^2,f_2^2,f_b^2,e_1^2,e_2^2,e_b^1\right)\right]_{2,2,C}= \tag{(4.20)}$$
$$(\left[\left(f_1^1,e_b^1,e_1^1\right),\left(f_1^2,f_b^2,e_1^2\right)\right]_{2,0,C},\left[\left(f_2^1,e_2^1,e_b^1\right),\left(f_2^2,e_2^2,e_b^2\right)\right]_{0,2,C}),$$

for any $\left(f_i^1,f_i^2,f_b^i,e_i^1,e_i^2,e_b^i\right)\in\Gamma(\mathcal{F}_{2,2}\times\mathcal{E}_{2,2})$.

Lemma 4.5. *The Courant bracket $[,]_{2,2,C}$ is skew-symmetric and linear. The maps ρ_p and ρ_q are anchor maps.*

It is obvious that $[,]_{2,2,C}$ is not always a Lie bracket

Definition 4.5. *Let M be a smooth oriented 3-manifold with smooth boundary ∂M, and consider the Dirac structure $\mathcal{D}\subset\mathcal{F}_{2,2}\times\mathcal{E}_{2,2}$. We say $\mathcal{D}$ is an integrable Dirac structure if is closed under the Courant bracket $[,]_{2,2,C}$.*

Corollary 4.1. *The Dirac structure $\mathcal{D}\subset\mathcal{F}_{2,2}\times\mathcal{E}_{2,2}$ is integrable if:*

$$\int_M\left[e_3^1\wedge C_1\left(f_1^1,f_2^1\right)+e_3^2\wedge C_2\left(f_1^2,f_2^2\right)\right]= \tag{(4.21)}$$
$$=\int_M\left[\sharp\left[\flat e_2^1,\flat e_1^1\right]\wedge f_1^3+\sharp\left[\flat e_2^2,\flat e_1^2\right]\wedge f_2^3\right],$$

for any $\left(f_i^1, f_i^2, f_b^i, e_i^1, e_i^2, e_b^i\right) \in \Gamma\left(\mathcal{D}_{2,2}\right)$, $i = \overline{1,3}$, *where*

$$C_1\left(f_1^1, f_2^1\right) = L_{\flat e_1^1} f_2^1 - L_{\flat e_2^1} f_1^1 + \tfrac{1}{2} d(i_{\flat e_2^1} f_1^1 - i_{\flat e_1^1} f_2^1), \tag{4.22}$$

and

$$C_2\left(f_2^1, f_2^2\right) = L_{\flat e_1^2} f_2^2 - L_{\flat e_2^2} f_1^2 + \tfrac{1}{2} d(i_{\flat e_2^2} f_1^2 - i_{\flat e_1^2} f_2^2). \tag{4.23}$$

Lemma 4.6. *Let M be a smooth oriented 3-manifold, and let $\mathcal{D}_{2,2} \subset \mathcal{F}_{2,2} \times \mathcal{E}_{2,2}$ be the Stokes-Dirac structure given by* (3.6). *The integrability condition* (4.21) *is equivalent to*

$$\begin{aligned}
&-\int_M e_3^1 \wedge d[L_{\flat e_1^1} e_1^2 - L_{\flat e_2^1} e_2^2 + \tfrac{1}{2}(i_{\flat e_2^1} de_2^2 - i_{\flat e_1^1} de_1^2)] \\
&+\int_M e_3^2 \wedge d[L_{\flat e_1^2} e_2^1 - L_{\flat e_2^2} e_1^1 + \tfrac{1}{2}(i_{\flat e_1^2} de_2^1 - i_{\flat e_2^2} de_1^1)] = \\
&= \int_M [-\left[\sharp\left[\flat e_2^1, \flat e_1^1\right] \wedge de_2^3 + \sharp\left[\flat e_2^2, \flat e_1^2\right] \wedge de_1^3\right],
\end{aligned} \tag{4.24}$$

for any $\left(f_i^1, f_i^2, f_b^i, e_i^1, e_i^2, e_b^i\right) \in \Gamma\left(\mathcal{D}_{2,2}\right)$, $i = \overline{1,3}$.

Corollary 4.2. *Let $\mathcal{D} \subset \mathcal{F}_{2,2} \times \mathcal{E}_{2,2}$ be the Hodge-Dirac structure given by* (3.11). *Then the Dirac structure $\mathcal{D}$ is integrable if:*

$$\begin{aligned}
&-\int_M e_3^1 \wedge [L_{\flat e_1^1}\left(\star e_1^2\right) - L_{\flat e_2^1}\left(\star e_2^2\right) + \tfrac{1}{2} d(i_{\flat e_2^1}\left(\star e_2^2\right) - i_{\flat e_1^1}\left(\star e_1^2\right))] \\
&+\int_M e_3^2 \wedge [L_{\flat e_1^2}\left(\star e_2^1\right) - L_{\flat e_2^2}\left(\star e_1^1\right) + \tfrac{1}{2} d(i_{\flat e_2^2}\left(\star e_1^1\right) - i_{\flat e_1^2}\left(\star e_2^1\right))] = \\
&= \int_M \left[-\sharp\left[\flat e_2^1, \flat e_1^1\right] \wedge \star e_3^2 + \sharp\left[\flat e_2^2, \flat e_1^2\right] \wedge \star e_3^1\right].
\end{aligned} \tag{4.25}$$

for any $\left(f_i^1, f_i^2, f_b^i, e_i^1, e_i^2, e_b^i\right) \in \Gamma\left(\mathcal{D}_{2,2}\right)$, $i = \overline{1,3}$.

Corollary 4.3. *The map* $[,]_{p,q,C}$ *is a Lie bracket iff* $\left(\mathcal{D}_{2,2}, [,]_{2,2,C}\right)$ *is a Lie algebra.*

Theorem 4.3. *The triple* $\left(D_{2,2,} [,]_{2,2,C}, \rho_p\right)$ *is a Lie algebroid if* $[,]_{2,2,C}$ *is a Lie bracket.*

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