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# SELECTIONS OF BOUNDED VARIATION

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**Abstract.** The paper presents recent results concerning the problem of the existence of those selections, which preserve the properties of a given set-valued mapping of one real variable taking on compact values from a metric space. The properties considered are the boundedness of Jordan, essential or generalized variation, Lipschitz or absolute continuity. Selection theorems are obtained by virtue of a single compactness argument, which is the exact generalization of the Helly selection principle. For set-valued mappings with the above properties we obtain a Castaing-type representation and prove the existence of multivalued selections and selections which pass through the boundaries of the images of the set-valued mapping and which are nearest in variation to a given mapping. Multivalued Lipschitzian superposition operators acting on mappings of bounded generalized variation are characterized, and solutions of bounded generalized variation to functional inclusions and embeddings, including variable set-valued operators in the right hand side, are obtained.

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## Introduction

The problem of the existence of selections is, for a given set-valued mapping $F$ from a nonempty set $T$ into a nonempty set $X$ (in symbols, $F : T \rightrightarrows X$), to find a single-valued mapping $f : T \to X$ satisfying the condition: $f(t) \in F(t)$ for all $t \in T$. Here the set-valued mapping (or multifunction) is a rule $F$ which assigns to each point $t \in T$ a nonempty subset $F(t) \subset X$, called the image of $t$ under $F$ or the value of $F$ at $t$, and the mapping $f$ with the above property is called a selection (selector, section, branch) of $F$. By the Axiom of Choice, any set-valued mapping with nonempty images admits at least one selection. So, the problem reduces to finding selections inheriting some (or all) properties of the set-valued mapping. Usually these properties are connected with measurability, continuity, differentiability, etc., which is motivated by the specific problem in question. In the present work the properties under consideration are boundedness of (generalized) variations of the set-valued mapping with respect to the Hausdorff metric in the target space of images.

Let us briefly comment on the existing literature on selections (the references chosen are representative but by no means tend to be exhausting on the subject).

Fundamental results on the existence of measurable selections (for measurable multifunctions) are contained in the works of Castaing [11], Castaing and Valadier [12] and Kuratowski and Ryll-Nardzewski [65]. In [65] the main theorem says that a measurable set-valued mapping $F$ from a measurable space $T$ into a complete separable metric space $X$, having closed images, admits a measurable selection. Castaing [11] showed that $F : T \rightrightarrows X$ with

$T$ and $X$ as above is measurable if and only if it has a countable number of its measurable selections which are pointwise dense in the images of $F$ (the Castaing representation). A survey on measurable selections and a complete bibliography on the subject (up to 1977) is the work of Wagner [109]. Selections with Baire property were obtained by Choban [35, 36]. The existence of measurable selections for maps whose values are compact subsets of a regular Hausdorff space which need not be metrizable or satisfy any restriction on its weight was proved by Graf [51].

The most known results on the existence of continuous selections are due to Michael [79, 80]. One of his theorems claims that a lower semi-continuous set-valued mapping on a paracompact space $T$ with closed convex images from a Banach space $X$ admits a continuous selection. A detailed information on the theory of continuous selections, its development and applications, is contained in the works of Repoš and Semenov [94, 95]. The influence of nonconvexity of images for a set-valued mapping to have continuous selections was studied by Bogatyrev [8], Hasumi [54], Moiseev [81] and Semenov [101]. A universal approach to the existence of measurable and continuous selections was found by Mägerl [67]. Selections of mappings with decomposable values were treated by Bressan and Colombo [9] and Tolstonogov [108].

The existence of Lipschitz continuous selections for set-valued mappings with convex images was recently obtained by Aubin and Cellina [3], Dommisch [41], Polovinkin [89], Przesławski and Yost [90, 91] and Shvartsman [103], and differentiable selections — by Dencheva [40], Gautier and Morchadi [47] and Rockafellar [98]. The basic facts about the way selections preserve the properties of measurability, Lipschitz continuity, etc., are contained in the monograph of Aubin and Frankowska [4].

Continuous and Lipschitz continuous selections exist, as a rule, for set-valued mappings with convex images. If the images are not convex, then in the general case one cannot expect more than measurable selections ([65]) or selections with the Baire property ([36]). In fact, many examples are known to show that a continuous set-valued mapping on an interval $T = [a, b]$ of the real line $\mathbb{R}$ with compact values from a ball in $\mathbb{R}^2$ or a Lipschitz continuous mapping from $\mathbb{R}^3$ into compact subsets of a ball in $X = \mathbb{R}^3$ need not have a continuous selection (Aubin and Cellina [3], Chistyakov and Galkin [31], Hermes [56], Kupka [64], Michael [80, Part I]). In this paper we will show that set-valued mappings $F$ of bounded variation from a nonempty subset $T \subset \mathbb{R}$ into nonempty compact subsets of a metric space $X$ always admit selections of bounded variation passing through a given point in the graph $\mathrm{Gr}(F)$ of $F$, which is defined as usual by $\mathrm{Gr}(F) = \{(t, x) \in T \times X \mid x \in F(t)\}$.

Let $\emptyset \neq T \subset \mathbb{R}$ and $(X, d)$ be a metric space with metric $d$. A mapping $f : T \to X$ is said to be *Lipschitzian* (in symbols, $f \in \mathrm{Lip}(T; X)$), if its (least) Lipschitz constant is finite:

$$L_d(f, T) = \sup\bigl\{\, d(f(t), f(s))/|t - s|\,;\, t,\, s \in T,\, t \neq s\bigr\}.$$

A mapping $f : T \to X$ is called *absolutely continuous* (written $f \in \mathrm{AC}(T; X)$) if there exists a function $\delta : (0, \infty) \to (0, \infty)$ such that for any $\varepsilon > 0$, any $n \in \mathbb{N}$ and any finite collection of points $\{a_i, b_i\}_{i=1}^n \subset T$ such that $a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_n < b_n$ the condition $\sum_{i=1}^n (b_i - a_i) \leq \delta(\varepsilon)$ implies $\sum_{i=1}^n d\bigl(f(b_i), f(a_i)\bigr) \leq \varepsilon$. More precisely, such $f$ will be called $\delta(\cdot)$-*absolutely continuous* and since, in general, the function $\delta(\cdot)$ depends on $f$, we will also write $\delta(\cdot) = \delta_f(\cdot)$.

A mapping $f : T \to X$ is said to be *of bounded* (or *finite*) *variation* (in symbols, $f \in \mathrm{BV}(T; X)$) if its total Jordan variation $V(f, T)$ is finite:

$$V(f, T) = V_d(f, T) = \sup_{\xi} \sum_{i=1}^{m} d\bigl(f(t_i), f(t_{i-1})\bigr)$$

where the supremum is taken over all partitions $\xi = \{t_i\}_{i=0}^m$ of the set $T$, i. e., $m \in \mathbb{N}$, $\{t_0, t_1, \ldots, t_m\} \subset T$ and $t_{i-1} < t_i$, $i = 1, \ldots, m$ (Jordan [58], Schwartz [100, Chapter 4, Section 9]). Single-valued functions and mappings of bounded variation on arbitrary set $T$ have already been treated in various contexts (e.g., [5], [10], [13]–[15], [44], [99], [107]), which is quite natural since the notion of (Jordan) variation depends only on the order relation on $T$ and the distance function(s) in the target space.

The *Hausdorff distance* $D = D_d$ between two nonempty subsets $A$ and $B$ of the metric space $X$ is given by

$$D(A, B) = \max\bigl\{\mathrm{e}(A, B), \mathrm{e}(B, A)\bigr\},$$

where $\mathrm{e}(A, B) = \sup_{x \in A} \mathrm{dist}(x, B)$ and $\mathrm{dist}(x, B) = \inf_{y \in B} d(x, y)$. It is well known (e.g., [12, Chapter II]) that $D$ is a metric on the set $\mathrm{c}(X)$ of all nonempty compact subsets of $X$, called the *Hausdorff metric* (*generated by* $d$).

In [57] Hermes proved that if $T = [a, b]$ and $X = \mathbb{R}^n$, then any set-valued mapping $F \in \mathrm{Lip}(T; \mathrm{c}(X))$ admits a selection $f \in \mathrm{Lip}(T; X)$ such that $L_d(f, T) \leq L_D(F, T)$, and, moreover, that a continuous mapping $F : T \to \mathrm{c}(X)$ of bounded variation admits a *continuous* selection. Similar results for Lipschitzian and absolutely continuous mappings with convex and nonconvex compact values were obtained by Guričan and Kostyrko [53], Kikuchi and Tomita [60] and Qiji [92]. The results of Hermes were generalized by Mordukhovich [82, Section Supplement 1] for a Banach space $X$ and a mapping $F$ with compact graph and by Ślęzak [104] to the general case when $X$ is an arbitrary metric space. Basing on a generalized Helly selection (compactness) principle for metric space valued mappings of bounded variation the author [14] proved that a set-valued mapping $F \in \mathrm{BV}(T; \mathrm{c}(X))$ admits a selection $f \in \mathrm{BV}(T; X)$ such that $V_d(f, T) \le V_D(F, T)$. This result was extended onto mappings of bounded generalized variation in the sense of Riesz-Orlicz and some other classes of mappings in [15]–[23]. By revising the selection principle, the author [24]–[28] in collaboration with Belov and Rychlewicz [6, 7, 33] showed that the assumption that $X$ is a Banach space and the graph $\mathrm{Gr}(F)$ is compact, which was used in the earlier works of the author, is superfluous: that $X$ is a metric space suffices for most of the results (note that Hermes and his successors made use of the Arzelà-Ascoli compactness theorem).

In this work we present the most general results on the existence of selections of bounded generalized variation and their development for solutions to functional inclusions and embeddings. This is done under the assumption that set-valued mappings $F : T \rightrightarrows X$ are defined on a nonempty set $T \subset \mathbb{R}$ and assume compact values from a metric space $(X, d)$. That the domain $T$ has no particular structure (except the linear order) is crucial for the existence of selections of essentially bounded variation (Sections 2 and 5).

The paper is divided into three parts. In the first part (Sections 1–4) we develop the theory of mappings of bounded generalized variation with values in a metric space which is needed for set-valued mappings. The second part (Sections 5 through 10) contains existence theorems for selections of bounded (generalized) variation. And the third part (Sections 11–14) is devoted to the existence of solutions to functional inclusions and embeddings including variable set-valued operators in the right hand side.

## 1. Generalized Helly's selection principle

In what follows, unless otherwise stated, $T \subset \mathbb{R}$ is a nonempty set, $X$ is a metric space with metric $d$ and $X^T$ is the set of all mappings from $T$ into $X$.

Let us recall the main properties of the variation $V(f, T)$ of $f \in \mathrm{BV}(T; X)$ needed below. Setting $f(T) = \{f(t) \mid t \in T\}$ (the image of $T$ under $f$) and $\mathrm{osc}(f, T) = \sup\{d(f(t), f(s)) \mid t, s \in T\}$ (the diameter of $f(T)$), we have (cf. [13], [14] or [15]): 1) if $t \in T$, then $V(f, T) = V(f, T \cap (-\infty, t]) + V(f, T \cap [t, \infty))$ (additivity); 2) $\mathrm{osc}(f, T) \le V(f, T)$; 3) if a sequence of mappings $\{f_n\}_{n=1}^\infty \subset X^T$ converges pointwise on $T$ in metric $d$ to a mapping $f \in X^T$ (i.e., $d(f_n(t), f(t)) \to 0$ as $n \to \infty$ for all $t \in T$), then $V(f, T) \le \liminf_{n\to\infty} V(f_n, T)$ (lower semi-continuity); 4a) if $s = \sup T \in \mathbb{R} \cup \{\infty\}$ and $s \notin T$, then $V(f, T) = \lim_{T \ni t \to s} V(f, T \cap (-\infty, t])$; 4b) if $i = \inf T \in \mathbb{R} \cup \{-\infty\}$ and $i \notin T$, then $V(f, T) = \lim_{T \ni t \to i} V(f, T \cap [t, \infty))$; 4c) if $s \notin T$

and $i \notin T$, then, in addition to 4a) and 4b), the value $V(f,T)$ is also equal to

$$\lim_{T^2 \ni (a,b) \to (i,s)} V(f, T \cap [a,b]) = \lim_{T \ni b \to s} \lim_{T \ni a \to i} V(f, T \cap [a,b]) = \lim_{T \ni a \to i} \lim_{T \ni b \to s} V(f, T \cap [a,b]).$$

**Lemma 1.1** ([15, Theorem 4.3])**.**

(a) *The mapping* $f \in \mathrm{BV}(T;X)$ *is continuous from the right at the point* $t_0 \in T \setminus \{\sup T\}$ *or from the left at* $t_0 \in T \setminus \{\inf T\}$ *if and only if the function* $\varphi(t) = V(f, T \cap (-\infty, t])$, $t \in T$, *has this property at* $t_0$.

(b) $f \in \mathrm{BV}(T;X)$ *is continuous on* $T$ *apart, possibly, at most countable subset of* $T$.

We say that $g : T \to X$ is *natural* ([14], [15]) if $V(g, T \cap [a,b]) = b - a$ for all $a$, $b \in T$, $a \le b$. Clearly, $g \in \mathrm{Lip}(T;X)$ and $L_d(g,T) = 1$. Note also that $\mathrm{Lip}(T;X) \subset \mathrm{AC}(T;X)$ (e. g., with $\delta(\varepsilon) = \varepsilon / \max\{1, L_d(f,T)\}$, $\varepsilon > 0$), $\mathrm{Lip}(T;X) \subset \mathrm{BV}(T;X)$ if $T$ is bounded, and $\mathrm{AC}(T;X) \subset \mathrm{BV}(T;X)$ if $T$ is compact.

Given two mappings $\varphi : T \to J$ and $g : J \to X$, the composite mapping $g \circ \varphi : T \to X$ is given as usual by $(g \circ \varphi)(t) = g(\varphi(t))$ for all $t \in T$.

The following *structural* theorem (Lemma 1.2 below) provides a close relation between mappings of bounded variation and Lipschitzian mappings (with additional assumptions, such as the continuity of mappings or connectedness of their domain, this theorem was employed in [1, II.1.3], [13, 3.19], [42, 2.5.16], [83, Section 5] and [100, IV.9]):

**Lemma 1.2** ([15, Theorem 3.1])**.** *Given* $f : T \to X$, *we have:* $f \in \mathrm{BV}(T;X)$ *if and only if there exist a nondecreasing bounded function* $\varphi : T \to \mathbb{R}$ *and a natural mapping* $g : J = \varphi(T) \to X$ *such that* $f = g \circ \varphi$ *on* $T$. *In the necessity part* $\varphi$ *can be defined by* $\varphi(t) = V(f, T \cap (-\infty, t])$, $t \in T$, *so that* $V(\varphi, T) = V(g, J) = V(f, T)$; *moreover, if* $T$ *is bounded and* $f \in \mathrm{Lip}(T;X)$, *then* $\varphi \in \mathrm{Lip}(T;\mathbb{R})$ *and* $L(\varphi, T) = L_d(f,T)$, *and if* $T$ *is compact and* $f \in \mathrm{AC}(T;X)$, *then* $\varphi \in \mathrm{AC}(T;\mathbb{R})$ *and one can set* $\delta_\varphi(\cdot) = \delta_f(\cdot)$.

The main tool providing the compactness of families of real functions of bounded variation is the well known (pointwise) *Helly selection principle* [55]. In various contexts it was generalized in [84], [86, VIII.4.3], [102, II.4.5], [110] (and others) for real valued functions and in [1, II.1.4], [6], [14], [15], [27], [26], [31], [32], [45] and [46] for families of mappings.

Recall that a family of real functions on $T$ is said to be *bounded* if there exists a constant $C \ge 0$ such that $|\varphi(t)| \le C$ for all $t \in T$ and all functions $\varphi$ from this family. A family $\mathcal{F} \subset X^T$ is called *pointwise precompact* if, for any $t \in T$, the set $\mathcal{F}(t) = \{f(t) \mid f \in \mathcal{F}\}$ is precompact in $X$ (i.e., the closure $\overline{\mathcal{F}(t)}$ of $\mathcal{F}(t)$ in $X$ is compact).

Let $\Phi : \mathbb{R}^+ = [0, \infty) \to \mathbb{R}^+$ be a continuous nondecreasing unbounded function such that $\Phi(\rho) = 0$ if and only if $\rho = 0$. Given $f : T \to X$, we set

$$\mathrm{Var}_\Phi(f, T) = \sup_\xi \sum_{i=1}^m \Phi\Big(d\big(f(t_i), f(t_{i-1})\big)\Big),$$

where the supremum is over all partitions $\xi = \{t_i\}_{i=0}^m$ $(m \in \mathbb{N})$ of $T$. The value $\mathrm{Var}_\Phi(f, T)$ is called the *total $\Phi$-th variation* of $f$ on $T$ (in the sense of Wiener [111] and Young [112]). It is clear that if $\Phi(\rho) = \rho$, $\rho \in \mathbb{R}^+$, then $\mathrm{Var}_\Phi(f, T) = V(f, T)$.

The following properties are known to hold for $\mathrm{Var}_\Phi(f, T)$ (cf. [32] and [84]): $1_\Phi$) if $t \in T$, then $\mathrm{Var}_\Phi(f, T \cap (-\infty, t]) + \mathrm{Var}_\Phi(f, T \cap [t, \infty)) \le \mathrm{Var}_\Phi(f, T)$ (semi-additivity); $2_\Phi$) if $t, s \in T$, then $\Phi\big(d(f(t), f(s))\big) \le \mathrm{Var}_\Phi(f, T)$; $3_\Phi$) under the conditions of 3) above we have: $\mathrm{Var}_\Phi(f, T) \le \liminf_{n\to\infty} \mathrm{Var}_\Phi(f_n, T)$ (lower semi-continuity).

**Theorem 1.3** (generalized Helly's selection principle)**.** *An infinite pointwise precompact family of mappings $\mathcal{F} \subset X^T$ satisfying* $\sup_{f \in \mathcal{F}} \mathrm{Var}_\Phi(f, T) < \infty$ *contains a pointwise convergent* (*in metric $d$*) *sequence whose pointwise limit $f$ is such that* $\mathrm{Var}_\Phi(f, T) < \infty$.

**Proof.** Proof of Theorem 1.3 will be divided into three steps.

1. Let us extend the classical Helly theorem [55] from an interval in $\mathbb{R}$ to an arbitrary set $T \subset \mathbb{R}$: *an infinite bounded family of nondecreasing functions from $T$ into $\mathbb{R}$ contains a sequence which converges pointwise on $T$ to a nondecreasing bounded function.*

First, let $T = \mathbb{R}$. We set $I_k = [-k, k]$, $k \in \mathbb{N}$, and make use of the standard Cantor diagonal process. By Helly's theorem (e. g., [86, VIII.4.2]), applied to the restriction of our family to the interval $I_1$, choose a sequence $\{\varphi_n^1\}_{n=1}^\infty$ in the family which converges pointwise on $I_1$ to a nondecreasing bounded function. Similarly, denote by $\{\varphi_n^2\}_{n=1}^\infty$ a subsequence of $\{\varphi_n^1\}_{n=1}^\infty$ pointwise convergent on the interval $I_2$ to a nondecreasing bounded function, and, inductively, for $k \in \mathbb{N}$, $k \ge 2$, pick a subsequence $\{\varphi_n^k\}_{n=1}^\infty$ of $\{\varphi_n^{k-1}\}_{n=1}^\infty$ which converges pointwise on $I_k$. Then the diagonal sequence $\{\varphi_n^n\}_{n=1}^\infty$ converges pointwise on $\mathbb{R}$ to a nondecreasing bounded function from $\mathbb{R}$ to $\mathbb{R}$.

If $T$ is arbitrary, we extend each function $\varphi$ from our family according to Saks' idea (cf. [99, Chapter 7, Section 4, Lemma (4.1)]) as follows: if $t \in \mathbb{R}$, we set:

$$\widetilde{\varphi}(t) \;=\; \begin{cases} \sup\{\varphi(s) \mid s \in T \cap (-\infty, t]\} & \text{if } T \cap (-\infty, t] \ne \emptyset, \\ \inf\{\varphi(s) \mid s \in T\} & \text{if } T \cap (-\infty, t] = \emptyset. \end{cases} \tag{1.1}$$

Clearly, $\widetilde{\varphi}:\mathbb{R}\to\mathbb{R}$ extends $\varphi$, is nondecreasing and bounded, $\operatorname{osc}(\widetilde{\varphi},\mathbb{R})=\operatorname{osc}(\varphi,T)$ and $\widetilde{\varphi}(\mathbb{R})\subset\overline{\varphi(T)}$ (i.e., the image $\widetilde{\varphi}(\mathbb{R})$ is contained in the closure of $\varphi(T)$). It follows that the family of functions $\{\widetilde{\varphi}\}$ is bounded on $\mathbb{R}$, and so, by the above, it contains a sequence $\{\widetilde{\varphi}_n\}_{n=1}^\infty$, which converges pointwise on $\mathbb{R}$ to a nondecreasing bounded function $\widetilde{\varphi}:\mathbb{R}\to\mathbb{R}$. Since the restriction $\widetilde{\varphi}_n|_T$ of $\widetilde{\varphi}_n$ to $T$ coincides with $\varphi_n$, the sequence $\{\varphi_n\}_{n=1}^\infty$ from the original family converges pointwise on $T$ to the function $\varphi=\widetilde{\varphi}|_T$.

2. Let us show that *if a family of mappings $\mathcal{F}\subset X^T$ is infinite and the set $\mathcal{F}(t)$ is precompact in $X$ for all $t\in T$, then for each countable set $J\subset T$ there exists a sequence in $\mathcal{F}$, which converges in $X$ pointwise on $J$.*

We again employ the diagonal method. In order to be specific, let $J=\{t_k\}_{k=1}^\infty$. Since the family $\{f(t_1)\mid f\in\mathcal{F}\}$ is precompact in $X$, it contains a sequence denoted by $\{f_n^1(t_1)\}_{n=1}^\infty$, which converges in $X$. In a similar manner, let $\{f_n^2(t_2)\}_{n=1}^\infty$ be a convergent subsequence of $\{f_n^1(t_2)\}_{n=1}^\infty$, and, by induction, given $k\in\mathbb{N}$, $k\geq 2$, let $\{f_n^k(t_k)\}_{n=1}^\infty$ be a convergent subsequence of $\{f_n^{k-1}(t_k)\}_{n=1}^\infty$. The diagonal sequence $\{f_n^n\}_{n=1}^\infty\subset\mathcal{F}$ converges in $X$ pointwise on the set $J$.

3. To prove the theorem, we set $\varphi_f(t)=\operatorname{Var}_\Phi(f,T\cap(-\infty,t])$, $f\in\mathcal{F}$, $t\in T$. The family $\{\varphi_f:T\to\mathbb{R}^+\mid f\in\mathcal{F}\}$ of nondecreasing functions is infinite and bounded, since $\varphi_f(t)\leq\operatorname{Var}_\Phi(f,T)$, $t\in T$. By step 1, there exist a sequence $\{f_n\}_{n=1}^\infty\subset\mathcal{F}$ and a nondecreasing bounded function $\varphi:T\to\mathbb{R}^+$ such that $\lim_{n\to\infty}\varphi_{f_n}(t)=\varphi(t)$ for all $t\in T$. Denote by $S$ at most countable dense subset of $T$, so that $S\subset T\subset\overline{S}$ (generally speaking, a separable set need not have separable subsets as is shown, e.g., in [48, 12.8], but in the usual topology of $\mathbb{R}$ this is correct: if $k\in\mathbb{Z}$, i.e. $k$ is integer, and the set $T_k=T\cap[k,k+1]$ is nonempty, then it is totally bounded, and hence, separable, and so, there exists at most countable subset $S_k\subset T_k$ such that $T_k\subset\overline{S_k}$, and it remains to set $S=\bigcup_k S_k$ and note that $T=\bigcup_k T_k$, where the union $\bigcup_k$ is over those $k\in\mathbb{Z}$ for which $T_k\neq\emptyset$). Note that any point $t\in T$, isolated for $T$, belongs to $S$: in fact, $T\cap(\alpha,\beta)=\{t\}$ for some interval $(\alpha,\beta)$, so that $S\cap(\alpha,\beta)\subset T\cap(\alpha,\beta)=\{t\}$ and $t\in S$; for, otherwise, if $t\notin S$, then $S\cap(\alpha,\beta)=\emptyset$ or $S\subset\mathbb{R}\setminus(\alpha,\beta)$, whence $t\in T\subset\overline{S}\subset\mathbb{R}\setminus(\alpha,\beta)$, that is, $t\notin(\alpha,\beta)$, which contradicts the definition of $(\alpha,\beta)$. As $\varphi$ is monotone, the set of its discontinuity points is at most countable, and since the set $\{f_n(t)\}_{n=1}^\infty$ is precompact in $X$ for all $t\in T$, by virtue of step 2 we may assume without loss of generality (passing to a subsequence if necessary) that $f_n(s)$ converges in $X$ at all points $s\in S$ and at all points $s\in T$ of discontinuity of $\varphi$. If $T$ is exhausted by these points $s$, the proof is complete.

It remains to show that $f_n(t)$ converges in $X$ at any point $t\in T\setminus S$, which is a limit point for $T$ and a point of continuity of $\varphi$. The proof of this part is close to the one in [84, Theorem 1.3]. Given $\varepsilon>0$, by the density of $S$ in $T$ and the continuity of $\varphi$ at $t$, choose $s\in S$ such that

$|\varphi(t)-\varphi(s)| \le (1/3)\Phi(\varepsilon)$. By virtue of the pointwise convergence of $\varphi_{f_n}$ to $\varphi$, choose a number $N_0(\varepsilon) \in \mathbb{N}$ such that

$$\max\{|\varphi_{f_n}(t)-\varphi(t)|, |\varphi_{f_n}(s)-\varphi(s)|\} < (1/3)\Phi(\varepsilon), \qquad n \ge N_0(\varepsilon).$$

Properties $2_\Phi$) and $1_\Phi$) imply

$$\Phi\Big(d(f_n(t), f_n(s))\Big) \le \mathrm{Var}_\Phi(f_n, T\cap[s,t]) \le |\varphi_{f_n}(t)-\varphi_{f_n}(s)|$$
$$\le |\varphi_{f_n}(t)-\varphi(t)| + |\varphi(t)-\varphi(s)| + |\varphi(s)-\varphi_{f_n}(s)|,$$

whence $d(f_n(t), f_n(s)) \le \varepsilon$ for all $n \ge N_0(\varepsilon)$. Since $\{f_n(s)\}_{n=1}^\infty$ is convergent, it is Cauchy, and so, there exists $N_1(\varepsilon) \in \mathbb{N}$ such that $d(f_n(s), f_m(s)) \le \varepsilon$ for all $n$, $m \ge N_1(\varepsilon)$. Then for all $n$, $m \ge \max\{N_0(\varepsilon), N_1(\varepsilon)\}$ we have:

$$d(f_n(t), f_m(t)) \le d(f_n(t), f_n(s)) + d(f_n(s), f_m(s)) + d(f_m(s), f_m(t)) \le 3\varepsilon,$$

i. e., the sequence $\{f_n(t)\}_{n=1}^\infty$ is Cauchy in $X$; moreover, since it is precompact in $X$, it follows that it is convergent in $X$.

Setting $f(t) = \lim_{n\to\infty} f_n(t)$ in $X$, $t \in T$, by property $3_\Phi$) we conclude that

$$\mathrm{Var}_\Phi(f, T) \le \liminf_{n\to\infty} \mathrm{Var}_\Phi(f_n, T) \le \sup_{g\in\mathcal{F}} \mathrm{Var}_\Phi(g, T) < \infty.$$

□

**Example 1.4.** In Theorem 1.3 the precompactness of sets $\mathcal{F}(t)$ at all points $t \in T$ cannot be replaced by closedness and boundedness even at a single point. To see this, let $T = [0,1]$ and $X = \ell^1(\mathbb{N})$ be the Banach space of all summable sequences $x = \{x_i\}_{i=1}^\infty \in \mathbb{R}^\mathbb{N}$ equipped with the norm $\|x\| = \sum_{i=1}^\infty |x_i|$. For $n \in \mathbb{N}$ define $f_n : [0,1] \to \ell^1(\mathbb{N})$ by $f_n(t) = 0$ if $0 \le t < 1$ and $f_n(1) = e_n$, where $e_n = \{x_i\}_{i=1}^\infty$ with $x_i = 0$ if $i \ne n$ and $x_n = 1$. Now, if $\mathcal{F} = \{f_n\}_{n=1}^\infty$, we have: $\mathcal{F}(t) = \{0\}$ is compact in $\ell^1(\mathbb{N})$ if $0 \le t < 1$, $\mathcal{F}(1) = \{e_n\}_{n=1}^\infty$ is closed and bounded, $\mathrm{Var}_\Phi(f_n, [0,1]) = \Phi(1)$ for all $n \in \mathbb{N}$, and no subsequence of $\mathcal{F}(1) = \{f_n(1)\}_{n=1}^\infty$ converges in $\ell^1(\mathbb{N})$. Other examples see in [6] and [15].

## 2. Mappings of finite essential variation

The *essential variation* of a mapping $f : T \to X$ is the quantity

$$V_{\mathrm{ess}}(f, T) = \inf\{V(g, T) \mid g \in \mathrm{BV}(T; X) \text{ and } g = f \text{ a.e. on } T\}; \qquad (2.1)$$

here we use the convention that $\inf \emptyset = \infty$, and the term *almost everywhere* (abbreviated *a. e.*) refers to the Lebesgue measure on $\mathbb{R}$. If $V_{\mathrm{ess}}(f, T) < \infty$, we say that $f$ is a mapping of *finite* (or bounded) *essential variation* and write $f \in \mathrm{BV}_{\mathrm{ess}}(T; X)$.

Let $T \subset \mathbb{R}$ be measurable and its Lebesgue measure $\mathrm{meas}(T)$ be positive. Recall (e. g., [86, IX.6]) that the *density of the set* $T$ *at a point* $t \in \mathbb{R}$ is given by

$$\mathrm{dens}(T,t) = \lim_{r \to +0} \mathrm{meas}(T \cap [t-r, t+r])/2r$$

(if the limit exists). A point $t \in \mathbb{R}$ is said to be *a point of density of* $T$ if $\mathrm{dens}(T,t) = 1$; note that such $t$ is a limit point from the left and from the right for $T$. A measurable set $T$ is said to be *density-open* if each point of $T$ is a point of density of $T$; if $t = \inf T \in T$, we assume that the *right density* defined by $2\mathrm{dens}(T \cap [t,\infty), t)$ should be equal to one, and if $t = \sup T \in T$, it holds for the *left density*: $2\mathrm{dens}(T \cap (-\infty, t], t) = 1$.

Throughout this section $T$ is density-open and $X$ is complete.

**Theorem 2.1.** *If* $f \in \mathrm{BV}_{\mathrm{ess}}(T;X)$*, then*

$$V_{\mathrm{ess}}(f,T) = \inf\{V(f, T \setminus E) \mid E \subset T \ \textit{ and } \ \mathrm{meas}(E) = 0\}. \tag{2.2}$$

**Proof.** Let us denote the right hand side of (2.2) by $v$. By definition (2.1), for any number $\alpha > V_{\mathrm{ess}}(f,T)$ we find a mapping $g \in \mathrm{BV}(T;X)$ such that $g = f$ a. e. on $T$ and $V(g,T) \le \alpha$. Since the Lebesgue measure of the set $E = \{t \in T \mid f(t) \ne g(t)\}$ is equal to zero and $f = g$ on $T \setminus E$, we have:

$$V(f, T \setminus E) = V(g, T \setminus E) \le V(g,T) \le \alpha.$$

It follows that $v \le V(f, T \setminus E) \le \alpha$, and so, as $\alpha \to V_{\mathrm{ess}}(f,T)$, we obtain

$$v \le V_{\mathrm{ess}}(f,T). \tag{2.3}$$

Let us establish the reverse inequality. Let $\varepsilon > 0$. By virtue of (2.3), $v$ is finite, so there exists a set $E \subset T$, depending on $\varepsilon$, such that $\mathrm{meas}(E) = 0$ and

$$V(f, T \setminus E) \le v + (\varepsilon/2). \tag{2.4}$$

Hence, $f|_{T_1} \in \mathrm{BV}(T_1;X)$, where $T_1 = T \setminus E$. Let us extend $f$ from $T_1$ to the whole real line. We set $\varphi(t) = V(f, T_1 \cap (-\infty, t])$, $t \in T_1$, and $J = \varphi(T_1)$. By Lemma 1.2, there exists a natural mapping $g : J \to X$ such that $f = g \circ \varphi$ on $T_1$. We extend $\varphi$ to a nondecreasing bounded function $\widetilde{\varphi} : \mathbb{R} \to \mathbb{R}$ according to (1.1) (with $T$ there replaced by $T_1$), so that $\widetilde{\varphi} = \varphi$ on $T_1$ and $\widetilde{\varphi}(\mathbb{R}) \subset \overline{J}$. Since $g$ is uniformly continuous on $J$ and $X$ is complete, there exists a unique extension $\widetilde{g} \in \mathrm{Lip}(\overline{J};X)$ of $g$ such that $L_d(\widetilde{g}, \overline{J}) = L_d(g,J) \le 1$: indeed, if $t \in \overline{J}$ and $\{t_n\}_{n=1}^{\infty} \subset J$ is such that $\lim_{n\to\infty} t_n = t$, we set $\widetilde{g}(t) = \lim_{n\to\infty} g(t_n)$ in $X$. Defining $\widetilde{f} = \widetilde{g} \circ \widetilde{\varphi}$ on $\mathbb{R}$, we have $\widetilde{f} = f$ on $T_1$ and $V(\widetilde{f}, \mathbb{R}) = V(f, T_1)$, since

$$\begin{aligned} V(\widetilde{f}, \mathbb{R}) &\le L_d(\widetilde{g}, \overline{J}) \mathrm{osc}(\widetilde{\varphi}, \mathbb{R}) \le \mathrm{osc}(\widetilde{\varphi}, \mathbb{R}) = \mathrm{osc}(\varphi, T_1) = V(\varphi, T_1) \\ &= V(f, T_1) = V(\widetilde{f}, T_1) \le V(\widetilde{f}, \mathbb{R}). \end{aligned}$$

Let $\{t_i\}_{i=0}^m$ be an arbitrary partition of $T$. Taking into account the density of $T_1 = T \setminus E$ in $T$ (in fact, any point $t \in E$, $t \neq \inf T$ and $t \neq \sup T$, is a point of density of $T$, and since $\mathrm{meas}(E) = 0$, $t$ is also a point of density of $T \setminus E$, and so, it is a limit point from the left and from the right for $T \setminus E$) and the definition of $\widetilde{\varphi}$, choose points $\{s_i\}_{i=0}^m \subset T \setminus E$ in such a way that $s_i \le t_i$, $i = 1, \dots, m$, $s_0 \le s_1 \le \dots \le s_m$ and

$$0 \le \widetilde{\varphi}(t_i) - \varphi(s_i) \le 2^{-i-1}\varepsilon/3, \qquad i = 0, 1, \dots, m.$$

Then for $i = 0, 1, \dots, m$ we have:

$$\begin{aligned} d(\widetilde{f}(t_i), f(s_i)) &= d\Big((\widetilde{g} \circ \widetilde{\varphi})(t_i), (g \circ \varphi)(s_i)\Big) = d\Big(\widetilde{g}(\widetilde{\varphi}(t_i)), \widetilde{g}(\varphi(s_i))\Big) \\ &\le |\widetilde{\varphi}(t_i) - \varphi(s_i)| \le 2^{-i-1}\varepsilon/3, \end{aligned}$$

which yields

$$\begin{aligned} \sum_{i=1}^m d(\widetilde{f}(t_i), \widetilde{f}(t_{i-1})) \le& \sum_{i=1}^m d(\widetilde{f}(t_i), f(s_i)) + \sum_{i=1}^m d(f(s_i), f(s_{i-1})) \\ &+ \sum_{i=1}^m d(f(s_{i-1}), \widetilde{f}(t_{i-1})) \\ \le& (\varepsilon/6) + V(f, T \setminus E) + (\varepsilon/3). \end{aligned}$$

We get $V(\widetilde{f}|_T, T) \le V(f, T \setminus E) + (\varepsilon/2)$ due to the arbitrariness of partition $\{t_i\}_{i=0}^m$ of $T$, so that together with (2.4) we have $V(\widetilde{f}|_T, T) \le v + \varepsilon$. Since $\widetilde{f}|_T \in \mathrm{BV}(T; X)$ and $\widetilde{f}|_T = f$ a. e. on $T$, from the definition (2.1) and the last inequality we find that $V_{\mathrm{ess}}(f, T) \le v + \varepsilon$ for any $\varepsilon > 0$. □

**Theorem 2.2.** *Suppose that* $f : T \to X$. *Then we have*:

(a) $f \in \mathrm{BV}_{\mathrm{ess}}(T; X)$ *if and only if there exists a set* $E \subset T$ *such that* $\mathrm{meas}(E) = 0$ *and* $f|_{T \setminus E} \in \mathrm{BV}(T \setminus E; X)$; *moreover,* $E$ *can be chosen such that* $V(f, T \setminus E) = V_{\mathrm{ess}}(f, T)$.

(b) *If* $\{f_n\}_{n=1}^\infty \subset \mathrm{BV}_{\mathrm{ess}}(T; X)$ *and* $d(f_n(t), f(t)) \to 0$ *as* $n \to \infty$ *for almost all* $t \in T$, *then* $V_{\mathrm{ess}}(f, T) \le \liminf_{n\to\infty} V_{\mathrm{ess}}(f_n, T)$ (*lower semicontinuity*).

(c) (Structural Theorem) $f \in \mathrm{BV}_{\mathrm{ess}}(T; X)$ *if and only if there exists a nondecreasing bounded function* $\varphi$ *from* $T$ *into* $\mathbb{R}$ *and a mapping* $g \in \mathrm{Lip}(J; X)$, *where* $J = \varphi(T)$ *and* $L_d(g, J) \le 1$, *such that* $f = g \circ \varphi$ *a. e. on* $T$.

(d) (Helly's type Theorem) *If* $\mathcal{F} = \{f_n\}_{n=1}^\infty \subset \mathrm{BV}_{\mathrm{ess}}(T; X)$, $\sup_{n\in\mathbb{N}} V_{\mathrm{ess}}(f_n, T)$ *is finite and the set* $\{f_n(t)\}_{n=1}^\infty$ *is precompact in* $X$ *for almost all* $t \in T$, *then* $\mathcal{F}$ *contains a subsequence which converges in metric* $d$ *a. e. on* $T$ *to a mapping from* $\mathrm{BV}_{\mathrm{ess}}(T; X)$.

**Proof.** (a) Sufficiency is a consequence of Theorem 2.1. Suppose that $f$ is in $\mathrm{BV}_{\mathrm{ess}}(T;X)$. By Theorem 2.1, we find $E_n\subset T$ such that $\mathrm{meas}(E_n)=0$, $n\in\mathbb{N}$, and $V(f,T\setminus E_n)\to V_{\mathrm{ess}}(f,T)$ as $n\to\infty$. The set $E=\bigcup_{n=1}^\infty E_n$ is of measure zero and $T\setminus E\subset T\setminus E_n$ for all $n\in\mathbb{N}$, so applying Theorem 2.1 again and taking into account the monotonicity of $V(\cdot,\cdot)$ in the second variable, we have

$$V_{\mathrm{ess}}(f,T)\le V(f,T\setminus E)\le V(f,T\setminus E_n)\to V_{\mathrm{ess}}(f,T)\quad\text{as}\quad n\to\infty,$$

whence $V(f,T\setminus E)=V_{\mathrm{ess}}(f,T)$.

(b) By the assumption there exists a set $E\subset T$ of Lebesgue measure zero such that $d(f_n(t),f(t))\to 0$ as $n\to\infty$ for all $t\in T\setminus E$. Given arbitrary set $G\subset T$ with $\mathrm{meas}(G)=0$, by the monotonicity and lower semi-continuity of $V(\cdot,\cdot)$, we have:

$$V(f,T\setminus(E\cup G))\le\liminf_{n\to\infty}V(f_n,T\setminus(E\cup G))\le\liminf_{n\to\infty}V(f_n,T\setminus G),$$

so that, by Theorem 2.1, we get

$$V_{\mathrm{ess}}(f,T)\le\liminf_{n\to\infty}V(f_n,T\setminus G)\qquad\forall\,G\subset T,\quad\mathrm{meas}(G)=0.\tag{2.5}$$

By (a), choose $G_n\subset T$ with $\mathrm{meas}(G_n)=0$ such that $f_n|_{T\setminus G_n}\in\mathrm{BV}(T\setminus G_n;X)$ and $V(f_n,T\setminus G_n)=V_{\mathrm{ess}}(f_n,T)$, $n\in\mathbb{N}$. Then the set $G=\bigcup_{n=1}^\infty G_n$ is of Lebesgue measure zero and $V(f_n,T\setminus G)\le V(f_n,T\setminus G_n)=V_{\mathrm{ess}}(f_n,T)$, and it remains to take into account (2.5).

(c) Since $f\in\mathrm{BV}_{\mathrm{ess}}(T;X)$, by (a) there exists $E\subset T$ with $\mathrm{meas}(E)=0$ such that $f|_{T\setminus E}\in\mathrm{BV}(T\setminus E;X)$, and since $X$ is complete, by the extension procedure from the proof of Theorem 2.1 there exists $\widetilde{f}\in\mathrm{BV}(T;X)$ such that $\widetilde{f}|_{T\setminus E}=f|_{T\setminus E}$. It remains to note that, by Lemma 1.2, $\widetilde{f}=g\circ\varphi$ on $T$, where $\varphi:T\to\mathbb{R}$ is a nondecreasing bounded function, $g\in\mathrm{Lip}(J;X)$, $J=\varphi(T)$ and $L_d(g,J)\le 1$. The sufficiency part is a straightforward consequence of Lemma 1.2 and item (a).

(d) By the assumption, there exists a set $E_0\subset T$ of Lebesgue measure zero such that the sequence $\{f_n(t)\}_{n=1}^\infty$ is precompact in $X$ for all $t\in T\setminus E_0$. By (a), for each $n\in\mathbb{N}$ we can find $G_n\subset T$ with $\mathrm{meas}(G_n)=0$ such that $V(f_n,T\setminus G_n)=V_{\mathrm{ess}}(f_n,T)$. Then $T_0=E_0\cup\bigcup_{n=1}^\infty G_n$ is of Lebesgue measure zero,

$$V(f_n,T\setminus T_0)\le V(f_n,T\setminus G_n)=V_{\mathrm{ess}}(f_n,T)\le\sup_{k\in\mathbb{N}}V_{\mathrm{ess}}(f_k,T)<\infty,\quad n\in\mathbb{N},$$

and $\{f_n(t)\}_{n=1}^\infty$ is precompact in $X$ for all $t\in T\setminus T_0$. Theorem 1.3 implies the existence of a subsequence of $\{f_n\}_{n=1}^\infty$ which converges in metric $d$ pointwise on $T\setminus T_0$ to a mapping $f$ from $\mathrm{BV}(T\setminus T_0;X)$. Define $f$ on the set $T_0$ arbitrarily and apply item (a). $\square$

In this section we have made an attempt to develop the preliminaries of the theory of metric space valued mappings of finite essential variation sufficient for the selections problem. If $T = [a, b]$ and $X = \mathbb{R}$, the corresponding theory is well known, e. g., [5], [50]. In particular, in [5] it was proved that for continuous functions the notions of essential variation and Jordan variation coincide. This result is also valid for continuous mappings with values in a metric space.

## 3. The space $\mathrm{GV}_\Phi(T; X)$

Let $\mathcal{N}$ denote the set of all continuous convex functions $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\Phi(\rho) = 0$ if and only if $\rho = 0$, and $\mathcal{N}_\infty$ — the set of all functions $\Phi \in \mathcal{N}$, for which the *Orlicz condition* holds: $\lim_{\rho\to\infty} \Phi(\rho)/\rho = \infty$. In the terminology of [68, § 2] functions from $\mathcal{N}$ are said to be $\varphi$-functions; in [62, Chapter 1, Section 2] functions from $\mathcal{N}_\infty$ are called $N$-functions. Any function $\Phi \in \mathcal{N}$ is strictly increasing, and so, its inverse $\Phi^{-1}$ is continuous and concave; besides, functions $\rho \mapsto \Phi(\rho)/\rho$ and $\rho \mapsto \omega_\Phi(\rho) = \rho\Phi^{-1}(1/\rho)$ are nondecreasing for $\rho > 0$, so the following limits exist:

$$\Phi'(0) = \lim_{\rho\to+0} \Phi(\rho)/\rho \in [0, \infty), \qquad [\Phi] = \lim_{\rho\to\infty} \Phi(\rho)/\rho \in (0, \infty] \tag{3.1}$$

and $\omega_\Phi(0) = \lim_{\rho\to+0} \omega_\Phi(\rho) = 1/[\Phi]$. Moreover, if $\Phi \in \mathcal{N}_\infty$, then

$$\lim_{r\to+0} r\Phi^{-1}(c/r) = c \lim_{\rho\to\infty} \rho/\Phi(\rho) = 0, \qquad c \in [0, \infty) \tag{3.2}$$

and, in particular, $\omega_\Phi(0) = 0$; in this case the function $\omega_\Phi : [0, \infty) \to [0, \infty)$ satisfies conditions: $\omega_\Phi$ is nondecreasing (and concave), $\lim_{\rho\to+0} \omega_\Phi(\rho) = \omega_\Phi(0) = 0$ and $\omega_\Phi(\rho_1 + \rho_2) \le \omega_\Phi(\rho_1) + \omega_\Phi(\rho_2)$ for $\rho_1, \rho_2 \ge 0$.

Given $\Phi \in \mathcal{N}$, $f : T \to X$ and a partition $\xi = \{t_i\}_{i=0}^m$ of $T$, we set

$$V_\Phi[f; \xi] = \sum_{i=1}^m \Phi\left(\frac{d(f(t_i), f(t_{i-1}))}{t_i - t_{i-1}}\right)(t_i - t_{i-1}) \tag{3.3}$$

and

$$V_\Phi(f, T) \equiv V_{\Phi,d}(f, T) = \sup\left\{ V_\Phi[f, \xi] \mid \xi \text{ is a partition of } T \right\}. \tag{3.4}$$

The quantity (3.4) is said to be the *total* $\Phi$*-variation* (*in the sense of Jordan, Riesz and Orlicz*). If it is finite, we say that $f$ is a *mapping of bounded* (or *finite*) $\Phi$*-variation* and write $f \in \mathrm{BV}_\Phi(T; X)$. If $\Phi(\rho) = \rho$, the definition (3.3)–(3.4) gives the classical notion of Jordan variation [58] (see also [86, Chapter 8] and [100, Chapter 4, Section 9]). If $\Phi(\rho) = \rho^q$, where $q > 1$, then (3.3) and (3.4) define the notion of $q$-variation in the sense of Riesz [96] (or [97, Chapter 2, Section 3.36]). Real valued functions of bounded $\Phi$-variation with $\Phi \in \mathcal{N}_\infty$ were extensively studied, e. g., [21], [38], [69], [75] (and references therein).

Note that if $\Phi \in \mathcal{N} \setminus \mathcal{N}_\infty$, so that the value $[\Phi]$ from (3.1), which is also equal to $\sup_{\rho>0} \Phi(\rho)/\rho$, is finite, and $T$ is bounded, then the sets $\mathrm{BV}_\Phi(T;X)$ and $\mathrm{BV}(T;X)$ consist of the same mappings. Thus, the set $\mathrm{BV}_\Phi(T;X)$ is most interesting in the case $\Phi \in \mathcal{N}_\infty$.

Given $\Phi \in \mathcal{N}$, the quantity $V_\Phi[f;\xi]$ does not decrease when we add points to the partition $\xi$: $V_\Phi[f;\xi] \le V_\Phi[f;\xi \cup \{t\}]$ if $\xi = \{t_i\}_{i=0}^m$ is a partition of $T$ and $t \in T \setminus \xi$. This is clear if $t < t_0$ or $t > t_m$, so let us suppose that $t_{k-1} < t < t_k$ for some $k \in \{1, \dots, m\}$. Putting, for the sake of brevity,

$$U(t,s) = U_\Phi(t,s) = \Phi\Big(d(f(t), f(s))/(t-s)\Big)(t-s), \qquad t, s \in T,\ s < t, \tag{3.5}$$

we have:

$$V_\Phi[f;\xi] = \left(\sum_{i=1}^{k-1} U(t_i, t_{i-1})\right) + U(t_k, t_{k-1}) + \left(\sum_{i=k+1}^{m} U(t_i, t_{i-1})\right), \tag{3.6}$$

where first sum is omitted if $k = 1$ or the last sum is omitted if $k = m$. Applying the triangle inequality for $d$, monotonicity and convexity of $\Phi$ and the Jensen inequality for sums (e.g., [86, X.5.4]), we find

$$U(t_k, t_{k-1}) \le U(t_k, t) + U(t, t_{k-1}), \tag{3.7}$$

which together with (3.6) proves our assertion. This fact implies that (3.4) is the extension of (3.3), i.e. $V_\Phi(f,\xi) = V_\Phi[f;\xi]$ for any partition $\xi$ of $T$, and that the value $V_\Phi(f,T)$ does not change if the supremum in (3.4) is taken only over those partitions of $T$, in which a finite number of points is fixed.

The main properties of $V_\Phi$ are gathered in the following lemma (cf. also Lemma 4.3 below).

**Lemma 3.1** ([20, 23, 27]). *Let $\Phi \in \mathcal{N}$ and $f : T \to X$. Then*

(a) *if $E \subset G \subset T$, then $V_\Phi(f,E) \le V_\Phi(f,G)$;*
(b) *if $t,\, s \in T$ and $s < t$, then $d(f(t), f(s)) \le (t-s)\Phi^{-1}\big(V_\Phi(f,T)/(t-s)\big)$;*
(c) *if $t \in T$, then $V_\Phi(f, T \cap (-\infty, t]) + V_\Phi(f, T \cap [t, \infty)) = V_\Phi(f,T)$;*
(d) *if $\{f_n\}_{n=1}^\infty \subset X^T$, $\{\Phi_n\}_{n=1}^\infty \subset \mathcal{N}$, $\lim_{n\to\infty} d(f_n(t), f(t)) = 0$ for $t \in T$ and $\lim_{n\to\infty} \Phi_n(\rho) = \Phi(\rho)$ for $\rho \in [0,\infty)$, then*

$$V_\Phi(f,T) \le \liminf_{n\to\infty} V_{\Phi_n}(f_n, T);$$

(e) $V_\Phi(f,T) = \sup\big\{\, V_\Phi(f, T \cap [a,b]) \mid a,\, b \in T,\ a < b \,\big\}$;
(f) *if $s = \sup T \in (\mathbb{R} \setminus T) \cup \{\infty\}$, then*

$$V_\Phi(f,T) = \lim_{T \ni t \to s} V_\Phi(f, T \cap (-\infty, t]);$$

(g) *if* $i = \inf T \in (\mathbb{R} \setminus T) \cup \{-\infty\}$, *then*

$$V_\Phi(f,T) = \lim_{T \ni t \to i} V_\Phi(f, T \cap [t,\infty));$$

(h) *if* $s$ *and* $i$ *are as in* (f) *and* (g), *then also*

$$V_\Phi(f,T) = \lim_{T^2 \ni (a,b) \to (i,s)} V_\Phi(f, T \cap [a,b]) = \lim_{T \ni b \to s} \lim_{T \ni a \to i} V_\Phi(f, T \cap [a,b])$$
$$= \lim_{T \ni a \to i} \lim_{T \ni b \to s} V_\Phi(f, T \cap [a,b]).$$

For $\Phi \in \mathcal{N}$ and bounded $T$, we have the embeddings:

$$\mathrm{Lip}(T;X) \subset \mathrm{BV}_\Phi(T;X) \subset \mathrm{BV}(T;X), \tag{3.8}$$

and, if $|T| = \sup T - \inf T$, the following inequalities hold:

$$V_\Phi(f,T) \le \Phi\big(L_d(f,T)\big)|T|, \;\; f \in \mathrm{Lip}(T;X), \tag{3.9}$$

$$\Phi\big(V(f,T)/|T|\big) \le V_\Phi(f,T)/|T|, \quad f \in \mathrm{BV}_\Phi(T;X). \tag{3.10}$$

Inequality (3.9) and the first embedding (3.8) follow from a straightforward verification. Inequality (3.10), which is the *Jensen inequality for variations*, is valid, since if $\xi$ is a partition of $T$ of the form $\{t_i\}_{i=0}^m$ and $f \in \mathrm{BV}_\Phi(T;X)$, then by (3.5) and Jensen's inequality for sums, we have:

$$\Phi\left(\frac{\sum_{i=1}^m d(f(t_i), f(t_{i-1}))}{\sum_{i=1}^m (t_i - t_{i-1})}\right) \le \frac{\sum_{i=1}^m U_\Phi(t_i, t_{i-1})}{\sum_{i=1}^m (t_i - t_{i-1})} \le \frac{V_\Phi(f,T)}{\sum_{i=1}^m (t_i - t_{i-1})},$$

whence

$$\sum_{i=1}^m d(f(t_i), f(t_{i-1})) \;\le\; \left(\sum_{i=1}^m (t_i - t_{i-1})\right)\Phi^{-1}\left(\frac{V_\Phi(f,T)}{\sum_{i=1}^m (t_i - t_{i-1})}\right). \tag{3.11}$$

The function $\rho \mapsto \rho\Phi^{-1}(c/\rho)$ is nondecreasing ($\rho > 0$, $c \ge 0$), and $\sum_{i=1}^m (t_i - t_{i-1}) = t_m - t_0 \le |T|$, and so, (3.11) implies the inequality, equivalent to (3.10):

$$V(f,T) \le |T|\Phi^{-1}\big(V_\Phi(f,T)/|T|\big). \tag{3.12}$$

Moreover, if $\Phi \in \mathcal{N}_\infty$ and $T \subset \mathbb{R}$ is arbitrary, then

$$\mathrm{BV}_\Phi(T;X) \subset \mathrm{AC}(T;X); \tag{3.13}$$

in fact, if $\{a_i, b_i\}_{i=1}^n \subset T$ and $a_1 < b_1 \le a_2 < b_2 \le \dots \le a_n < b_n$, then setting $t_i = b_i$, $t_{i-1} = a_i$ and $m = n$ in (3.11), we find

$$\sum_{i=1}^n d(f(b_i), f(a_i)) \le \left(\sum_{i=1}^n (b_i - a_i)\right)\Phi^{-1}\left(\frac{V_\Phi(f,T)}{\sum_{i=1}^n (b_i - a_i)}\right).$$

Taking into account that $\Phi$ from $\mathcal{N}_\infty$ satisfies (3.2), for any $\varepsilon > 0$ we can find $\delta(\varepsilon) > 0$ such that $\rho\Phi^{-1}(V_\Phi(f,T)/\rho) \le \varepsilon$ for all $0 < \rho \le \delta(\varepsilon)$, so that if $\sum_{i=1}^n (b_i - a_i) \le \delta(\varepsilon)$, then the last inequality yields $\sum_{i=1}^n d(f(b_i), f(a_i)) \le \varepsilon$.

Having Jensen's inequality for variations at hand, we can supplement Lemma 1.2 in the following way:

**Lemma 3.2.** *Let $T$ be bounded, $f : T \to X$ and $\Phi \in \mathcal{N}$. Then*: $f \in \mathrm{BV}_\Phi(T;X)$ *if and only if there exist a nondecreasing* (*bounded*) *function* $\varphi \in \mathrm{BV}_\Phi(T;\mathbb{R})$ *and a natural mapping* $g : J = \varphi(T) \to X$ *such that* $f = g\circ\varphi$ *on* $T$. *In the necessity part one can set* $\varphi(t) = V(f, T \cap (-\infty, t])$, $t \in T$, *and then* $V_\Phi(\varphi, T) = V_\Phi(f, T)$.

**Proof.** *Sufficiency.* Let $\Phi \in \mathcal{N}$, $T$ be arbitrary, $\varphi \in \mathrm{BV}_\Phi(T;\mathbb{R})$, $J = \varphi(T)$, $g \in \mathrm{Lip}(J;X)$, $L_d(g, J) \le 1$ and $f = g \circ \varphi$ on $T$. Let us show that $f \in \mathrm{BV}_\Phi(T;X)$ and $V_\Phi(f, T) \le V_\Phi(\varphi, T)$. Indeed, given a partition $\xi = \{t_i\}_{i=0}^m$ of $T$, we have:

$$\begin{aligned} V_\Phi[f;\xi] &= \sum_{i=1}^m \Phi\biggl(\frac{d(g(\varphi(t_i)), g(\varphi(t_{i-1})))}{t_i - t_{i-1}}\biggr)(t_i - t_{i-1}) \\ &\le \sum_{i=1}^m \Phi\biggl(L_d(g,J)\frac{|\varphi(t_i) - \varphi(t_{i-1})|}{t_i - t_{i-1}}\biggr)(t_i - t_{i-1}) \\ &\le V_\Phi(L_d(g,J)\varphi, T) \le V_\Phi(\varphi, T). \end{aligned}$$

*Necessity.* Since $\mathrm{BV}_\Phi(T;X) \subset \mathrm{BV}(T;X)$, the function $\varphi : T \to \mathbb{R}^+$ given by $\varphi(t) = V(f, T\cap(-\infty,t])$, $t \in T$, is well defined. Then the decomposition $f = g \circ \varphi$ with natural $g : J \to X$ follows from Lemma 1.2. Let us show that $\varphi \in \mathrm{BV}_\Phi(T;\mathbb{R})$. If $\xi = \{t_i\}_{i=0}^m$ is a partition of $T$, by the additivity of $V(\cdot,\cdot)$ and inequality (3.12) for $i \in \{1,\dots,m\}$, we find

$$\begin{aligned} \varphi(t_i) - \varphi(t_{i-1}) &= V(f, T \cap (-\infty, t_i]) - V(f, T\cap(-\infty, t_{i-1}]) \\ &= V(f, T \cap [t_{i-1}, t_i]) \\ &\le (t_i - t_{i-1})\Phi^{-1}\Bigl(V_\Phi(f, T\cap[t_{i-1}, t_i])/(t_i - t_{i-1})\Bigr), \end{aligned}$$

and so, the monotonicity of $\Phi$ and Lemma 3.1(c), (a) imply

$$V_\Phi[\varphi;\xi] \le \sum_{i=1}^m V_\Phi(f, T\cap[t_{i-1}, t_i]) = V_\Phi(f, T\cap[t_0, t_m]) \le V_\Phi(f, T).$$

Hence, $V_\Phi(\varphi, T) \le V_\Phi(f, T)$. From the decomposition $f = g \circ \varphi$ and the sufficiency part we get $V_\Phi(\varphi, T) = V_\Phi(f, T)$. □

By virtue of Helly's selection principle (Theorem 1.3 with $\Phi(\rho) = \rho$) and inequality (3.12) one can obtain a variant of Helly's selection principle in the space $\mathrm{BV}_\Phi(T;X)$; and also, if $\Phi \in \mathcal{N}_\infty$, the sequence, extracted from the family $\mathcal{F}$, may be chosen to converge even uniformly on $T$ (if we take into account Lemma 3.1(b), condition (3.2) and the Arzelà-Ascoli Theorem).

Let us consider briefly the case when $T = I = [a, b]$, $|I| = b - a$ and $X$ is a linear normed space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ equipped with the norm $\|\cdot\|$.

**Theorem 3.3** ([20, Theorem 7]). *Let $(X, \|\cdot\|)$ be a reflexive Banach space, $\Phi \in \mathcal{N}$ and $f \in \mathrm{BV}_\Phi(I; X)$. Then $f$ admits a strong derivative $f'(t) \in X$ for almost all $t \in I$ which is strongly measurable and $\int_a^b \Phi(\|f'(t)\|)dt \le V_\Phi(f, [a, b])$. Moreover, if $\Phi \in \mathcal{N}_\infty$, then $f$ is strongly differentiable a. e. on $I$, its derivative $f'$ is strongly measurable and Bochner integrable on $I$, $f$ is represented in the form $f(t) = f(a) + \int_a^t f'(s)ds$ for all $t \in I$, and the following integral formula for the $\Phi$-variation holds*:

$$V_\Phi(f, [a, b]) = \int_a^b \Phi\Big(\|f'(t)\|\Big)dt. \tag{3.14}$$

**Corollary 3.4** ([20, Corollary 9]). *Suppose that $f : I \to X$ and $\Phi \in \mathcal{N}_\infty$.*

(a) *If $X$ is a reflexive Banach space, then $f \in \mathrm{BV}_\Phi(I; X)$ if and only if $f : I \to X$ is absolutely continuous and $\int_I \Phi(\|f'(t)\|)dt < \infty$.*

(b) *If $X$ is a metric space and $\varphi(t) = V(f, [a, t])$, $t \in I$, then $f \in \mathrm{BV}_\Phi(I; X)$ if and only if $\varphi \in \mathrm{BV}_\Phi(I; \mathbb{R})$, i. e., if and only if $\varphi \in \mathrm{AC}(I; \mathbb{R})$ and $\int_I \Phi(|\varphi'(t)|)dt$ is finite. Moreover,*

$$V_\Phi(f, I) = V_\Phi(\varphi, I) = \int_I \Phi\Big(|\varphi'(t)|\Big)dt = \int_I \Phi\Big(|\frac{d}{dt}V(f, [a, t])|\Big)dt. \tag{3.15}$$

For $X = \mathbb{R}$ the criterion in Corollary 3.4(a) is known from Riesz [96] (cf. also [97, Chapter 2, Section 3.36]) if $\Phi(\rho) = \rho^q$ with $q > 1$, and Medvedev [75] and Cybertowicz and Matuszewska [38] if $\Phi \in \mathcal{N}_\infty$; in [38] the integral formula (3.14) is established for $X = \mathbb{R}$.

If $\Phi(\rho) = \rho^q$, $\rho \ge 0$, $q \ge 1$, we denote the space $\mathrm{BV}_\Phi(I; X)$ by $\mathrm{BV}_q(I; X)$, and $V_\Phi$ — by $V_q$. Note that if $(X, d)$ is a metric space and $f \in \mathrm{Lip}(I; X)$, then

$$L_d(f, I) = \lim_{q\to\infty}\Big(V_q(f, I)\Big)^{1/q} = \operatorname*{ess\,sup}_{t\in I}\big|\frac{d}{dt}V(f, [a, t])\big|.$$

In fact, inequality (3.9) implies $V_q(f, I) \le \big(L_d(f, I)\big)^q |I|$, whence

$$\limsup_{q\to\infty}\big(V_q(f, I)\big)^{1/q} \le L_d(f, I),$$

and Lemma 3.1(b) for $t, s \in I$, $t \ne s$, gives

$$d(f(t), f(s)) \le |t - s|^{1-(1/q)}\big(V_q(f, I)\big)^{1/q},$$

so that

$$d(f(t), f(s))/|t - s| \le \liminf_{q\to\infty}\big(V_q(f, I)\big)^{1/q}.$$

The second equality follows from (3.15). In particular, if $X$ is a reflexive Banach space, then $L_d(f,I) = \operatorname{ess\,sup}_{t\in I}\|f'(t)\|$. Thus, if $X$ is a metric space and $\varphi(t) = V(f,[a,t])$, $t \in I$, then $f \in \mathrm{Lip}(I;X)$ if and only if $\varphi \in \mathrm{AC}(I;\mathbb{R})$ and $\operatorname{ess\,sup}_{t\in I}|\varphi'(t)| < \infty$, and if $X$ is a reflexive Banach space, then $f \in \mathrm{Lip}(I;X)$ if and only if $f \in \mathrm{AC}(I;X)$ and $\operatorname{ess\,sup}_{t\in I}\|f'(t)\| < \infty$.

**Example 3.5.** (a) Let us show that there exists a function $f \in \mathrm{BV}_q([0,1];\mathbb{R})$ for all $q \ge 1$, which is not Lipschitzian. We set $f(t) = t(1-\log t)$ if $0 < t \le 1$ and $f(0) = 0$. Since $f'(t) = -\log t$ for $0 < t \le 1$, by (3.14) we have:

$$V_q(f,[0,1]) = \int_0^1 (-\log t)^q dt = \int_0^\infty s^q e^{-s} ds = \Gamma(q+1), \qquad q \ge 1,$$

where $\Gamma$ is the Euler gamma-function: $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$, $x > 0$. On the other hand, $\sup_{0<t\le 1} f(t)/t = \infty$, and so, $f \notin \mathrm{Lip}([0,1];\mathbb{R})$. Note also, that since the main term in the asymptotic expansion of $\Gamma(q+1)$ as $q \to \infty$ is, by Stirling's formula, of the form $\sqrt{2\pi q}\,(q/e)^q$, then $\lim_{q\to\infty}(V_q(f,[0,1]))^{1/q} = \lim_{q\to\infty}(\Gamma(q+1))^{1/q} = \infty$.

(b) This is an example of a function

$$f \in \mathrm{AC}([0,1/2];\mathbb{R}) \setminus \bigcap_{q>1} \mathrm{BV}_q([0,1/2];\mathbb{R}).$$

We set $f(t) = -1/\log t$ if $0 < t \le 1/2$ and $f(0) = 0$. Formula (3.14) yields

$$V_q(f,[0,1/2]) = \int_0^{1/2} |f'(t)|^q dt = \int_0^{1/2} \frac{dt}{t^q(\log t)^{2q}} = \int_{\log 2}^{\infty} \frac{e^{(q-1)s}}{s^{2q}}\, ds, \quad q \ge 1,$$

but the last integral converges if $q = 1$ and diverges for all $q > 1$.

(c) Let $\Phi(\rho) = e^\rho - 1$, $\rho \ge 0$, and $f(t) = t(1 - \log t)$ if $0 < t \le 1$ and $f(0) = 0$. Then for $\lambda > 0$ we have:

$$V_\Phi(f/\lambda,[0,1]) = \int_0^1 \Phi\Big(|f'(t)|/\lambda\Big)dt = \int_0^1 \frac{dt}{t^{1/\lambda}} - 1$$
$$= \begin{cases} 1/(\lambda-1) & \text{if } \lambda > 1, \\ \infty & \text{if } 0 < \lambda \le 1. \end{cases}$$

The importance of the sets $\mathrm{BV}_\Phi(I;X)$ is given by

**Theorem 3.6** ([20, Corollary 11])**.** *Given a metric space* $(X,d)$*, the following equality holds*: $\mathrm{AC}([a,b];X) = \bigcup_{\Phi\in\mathcal{N}_\infty} \mathrm{BV}_\Phi([a,b];X)$.

The sets $\mathrm{BV}_\Phi(T;X)$ generated by different functions $\Phi\in\mathcal{N}$ are related as follows:

**Lemma 3.7** ([27, 37, 66])**.** *Let $\Phi$, $\Psi\in\mathcal{N}$. If $T\subset\mathbb{R}$ is bounded, $(X,d)$ is a metric space and $\limsup_{\rho\to\infty}\Psi(\rho)/\Phi(\rho)<\infty$, i.e.,*

$$\exists\, C>0,\quad \rho_0>0\quad \textit{such that}\quad \Psi(\rho)\le C\Phi(\rho)\quad \forall\,\rho\ge\rho_0, \tag{3.16}$$

*then $\mathrm{BV}_\Phi(T;X)\subset\mathrm{BV}_\Psi(T;X)$. Conversely, if $I=[a,b]$, $(X,\|\cdot\|)$ is a linear normed space and $\mathrm{BV}_\Phi(I;X)\subset\mathrm{BV}_\Psi(I;X)$, then condition (3.16) holds.*

Recall that a function $\Phi\in\mathcal{N}$ *satisfies the $\Delta_2$-condition near infinity* or, in short, *$\Delta_2^\infty$-condition*, if $\limsup_{\rho\to\infty}\Phi(2\rho)/\Phi(\rho)<\infty$ ([62, Chapter 1, Section 4] or [68, Section 3]), which is equivalent to

$$\begin{aligned}&\exists\ \text{numbers } C>0 \text{ and } \rho_0>0 \text{ such that}\\ &\Phi(2\rho)\le C\Phi(\rho)\ \forall\,\rho\ge\rho_0,\end{aligned} \tag{3.17}$$

and this, in turn, as it is known, is equivalent to

$$\begin{aligned}&\forall\,\lambda>1\ \exists\, C(\lambda)>0,\ \ \rho_0(\lambda)>0 \text{ such that}\\ &\Phi(\rho)\le C(\lambda)\Phi(\rho/\lambda)\ \forall\,\rho\ge\rho_0(\lambda).\end{aligned} \tag{3.18}$$

For the sake of brevity we shall write $\mathrm{BV}_\Phi$ instead of $\mathrm{BV}_\Phi(T;X)$, $V_\Phi(f)$ instead of $V_\Phi(f,T)$ and $L_d(f)$ instead of $L_d(f,T)$.

**Lemma 3.8.** *Let $X$ be a linear normed space and $\Phi\in\mathcal{N}$. Then $\mathrm{BV}_\Phi(I;X)$ is a linear space if and only if $\Phi$ satisfies the $\Delta_2^\infty$-condition.*

**Proof.** First observe that the convexity of $\Phi$ implies that the set $\mathrm{BV}_\Phi$ is convex and $f\mapsto V_\Phi(f)$ is a convex functional:

$$\begin{aligned}V_\Phi(\theta f+(1-\theta)g)\le \theta V_\Phi(f)+(1-\theta)V_\Phi(g),&\\ f,g\in\mathrm{BV}_\Phi,\quad \theta\in[0,1].&\end{aligned} \tag{3.19}$$

To prove sufficiency (with $I\subset\mathbb{R}$ an arbitrary subset), let $f$, $g\in\mathrm{BV}_\Phi$ and $c\in\mathbb{K}$. Then $V_\Phi(cf)=V_\Phi(|c|f)$. If $|c|\le1$, by (3.19), $cf\in\mathrm{BV}_\Phi$. If $|c|>1$, by (3.18), there exist $C>0$ and $\rho_0>0$ such that $\Phi(|c|\rho)\le C\Phi(\rho)$ for all $\rho\ge\rho_0$. Setting $\Psi(\rho)=\Phi(|c|\rho)$, $\rho\in[0,\infty)$, and applying Lemma 3.7, we get: $\mathrm{BV}_\Phi\subset\mathrm{BV}_\Psi$, and so, $V_\Phi(|c|f)=V_\Psi(f)<\infty$, whence $cf\in\mathrm{BV}_\Phi$. This and (3.19) yield $f+g\in\mathrm{BV}_\Phi$, since

$$V_\Phi(f+g)=V_\Phi\left(\frac12 2f+\frac12 2g\right)\le\frac12 V_\Phi(2f)+\frac12 V_\Phi(2g)<\infty.$$

Conversely, let $\mathrm{BV}_\Phi$ be a linear space. In particular, this means that if $f\in\mathrm{BV}_\Phi$, then $2f\in\mathrm{BV}_\Phi$, or $\mathrm{BV}_\Phi\subset\mathrm{BV}_\Psi$, where $\Psi(\rho)=\Phi(2\rho)$, $\rho\ge0$. By Lemma 3.7, there exist $C>0$ and $\rho_0>0$ such that $\Phi(2\rho)=\Psi(\rho)\le C\Phi(\rho)$ for all $\rho\ge\rho_0$, i.e., $\Phi$ satisfies (3.17). $\square$

As Lemma 3.8 and Example 3.5(c) show, the set $\mathrm{BV}_\Phi(T;X)$ with $X$ a linear normed space and $\Phi\in\mathcal{N}$, is not, in general, a linear space. On the basis of this set we will define a new space $\mathrm{GV}_\Phi(T;X)$ with better properties, called the *space of mappings of bounded generalized* $\Phi$*-variation*.

Let $(X,d)$ be a metric space and $T\subset\mathbb{R}$. Given $\Phi\in\mathcal{N}$ and $\lambda>0$, we set $\Phi_\lambda(\rho)=\Phi(\rho/\lambda)$, $\rho\geq 0$. By Lemma 3.7, if $T$ is bounded, we have $\mathrm{BV}_{\Phi_\lambda}\subset\mathrm{BV}_\Phi$ if $0<\lambda\leq 1$ and $\mathrm{BV}_\Phi\subset\mathrm{BV}_{\Phi_\lambda}$ if $\lambda>1$. By Example 3.5(c), the latter embedding is, in general, strict. Lemma 3.7 and condition (3.18) imply that for the reverse embedding $\mathrm{BV}_{\Phi_\lambda}\subset\mathrm{BV}_\Phi$ with $\lambda>1$ to hold, it is sufficient, and when $T=I$ and $X$ is a linear normed space it is also necessary, that the function $\Phi$ satisfy the $\Delta_2^\infty$-condition. Given arbitrary $\Phi\in\mathcal{N}$, the space $\mathrm{GV}_\Phi=\mathrm{GV}_\Phi(T;X)$ is defined by

$$\mathrm{GV}_\Phi(T;X)=\bigcup_{\lambda>0}\mathrm{BV}_{\Phi_\lambda}(T;X)=\bigcup_{\lambda>1}\mathrm{BV}_{\Phi_\lambda}(T;X). \tag{3.20}$$

From the above it follows that if $T$ is bounded and $\Phi\in\mathcal{N}$ satisfies the $\Delta_2^\infty$-condition, then $\mathrm{GV}_\Phi(T;X)=\mathrm{BV}_\Phi(T;X)$. Conversely, if $T=I$, $X$ is a linear normed space and $\mathrm{GV}_\Phi(I;X)=\mathrm{BV}_\Phi(I;X)$, then $\Phi$ satisfies the $\Delta_2^\infty$-condition: in fact, since $\mathrm{BV}_{\Phi_2}(I;X)\subset\mathrm{BV}_\Phi(I;X)$, by Lemma 3.7 there exist $C>0$ and $\rho_0>0$ such that $\Phi(\rho)\leq C\Phi(\rho/2)$, $\rho\geq\rho_0$.

If $X$ is linear normed space, the set $\mathrm{GV}_\Phi(T;X)$ coincides with the set of those $f\in X^T$, for which there exists a $\lambda>0$ (depending on $f$) such that $f/\lambda\in\mathrm{BV}_\Phi(T;X)$; moreover, it is a linear space, for if $f$, $g\in\mathrm{GV}_\Phi$, then there exist $\lambda>0$ and $\mu>0$ such that $f/\lambda$, $g/\mu\in\mathrm{BV}_\Phi$, and so, from (3.19), we find

$$V_\Phi\left(\frac{f+g}{\lambda+\mu}\right)\ \leq\ \frac{\lambda}{\lambda+\mu}V_\Phi(f/\lambda)+\frac{\mu}{\lambda+\mu}V_\Phi(g/\mu)\ <\ \infty, \tag{3.21}$$

which implies $f+g\in\mathrm{GV}_\Phi$. It is also clear that $cf\in\mathrm{GV}_\Phi$ if $c\in\mathbb{K}$ and $f\in\mathrm{GV}_\Phi$.

For $T\subset\mathbb{R}$ and a metric space $X$ we define the following nonnegative functional (of Luxemburg-Nakano-Orlicz type) on $\mathrm{GV}_\Phi(T;X)$:

$$\begin{aligned} p_\Phi(f)=p_{\Phi,d}(f,T)=\ &\inf\{\lambda>0\mid V_{\Phi_\lambda}(f,T)\leq 1\},\\ &f\in\mathrm{GV}_\Phi(T;X), \end{aligned} \tag{3.22}$$

which is called the *precise* $\Phi$*-variation* of $f$. The number $p_\Phi(f)$ is well defined, since $V_{\Phi_\lambda}(f)\leq V_\Phi(f)/\lambda$ if $\lambda\geq 1$. For instance, if $\Phi(\rho)=\rho^q$, $q\geq 1$, then $p_\Phi(f)=(V_q(f,T))^{1/q}$ for any $f\in\mathrm{BV}_q(T;X)$.

The main properties of $p_\Phi$ are presented in the following

**Lemma 3.9.** *Let* $\Phi\in\mathcal{N}$ *and* $f\in\mathrm{GV}_\Phi(T;X)$. *We have*:

(a) $d(f(t),f(s))\leq\omega_\Phi(|t-s|)p_\Phi(f,T)$ *for all* $t$, $s\in T$;

(b) *if* $\lambda = p_\Phi(f,T) > 0$, *then* $V_{\Phi_\lambda}(f,T) \le 1$ (*and so, the infimum in* (3.22) *is attained for such* $f$);
(c) *if* $\lambda > 0$, *then* $p_\Phi(f,T) \le \lambda$ *if and only if* $V_{\Phi_\lambda}(f,T) \le 1$;
(d) *if* $\lambda > 0$ *and* $V_{\Phi_\lambda}(f,T) = 1$, *then* $p_\Phi(f,T) = \lambda$;
(e) *if a sequence* $\{f_n\}_{n=1}^\infty \subset \mathrm{GV}_\Phi(T;X)$ *converges pointwise on* $T$ *to* $f : T \to X$ *as* $n \to \infty$, *then* $p_\Phi(f,T) \le \liminf_{n\to\infty} p_\Phi(f_n,T)$;
(f) *for bounded* $T$ *the following inequalities hold*:

$$\Phi^{-1}(1/|T|)p_\Phi(f,T) \le L_d(f,T), \qquad f \in \mathrm{Lip}(T;X), \tag{3.23}$$

$$V(f,T) \le \omega_\Phi(|T|)p_\Phi(f,T), \qquad f \in \mathrm{GV}_\Phi(T;X); \tag{3.24}$$

(g) *if* $t \in T$, *then* $p_\Phi(f,T) \le p_\Phi(f, T\cap(-\infty,t]) + p_\Phi(f, T\cap[t,\infty))$;
(h) *if* $X$ *is a linear normed space, then the functional* $p_\Phi(\cdot,T)$ *is a semi-norm on the linear space* $\mathrm{GV}_\Phi(T;X)$.

**Proof.** (a) Given $t$, $s \in T$, $s < t$, by (3.3), (3.4) and (3.22), we have:

$$\Phi\biggl(\frac{d(f(t),f(s))}{(t-s)\lambda}\biggr)(t-s) \le V_{\Phi_\lambda}(f,T) \le 1 \quad \text{if} \quad \lambda > p_\Phi(f,T),$$

so that dividing by $t-s$ and taking the inverse function $\Phi^{-1}$, we get:

$$d(f(t),f(s)) \le (t-s)\Phi^{-1}(1/(t-s))\lambda, \qquad \lambda > p_\Phi(f,T).$$

(b) Set $\lambda = p_\Phi(f,T) > 0$. Choose numbers $\lambda(n) > \lambda$, $n \in \mathbb{N}$, such that $\lambda(n) \to \lambda$ as $n \to \infty$. By definition (3.22), $V_{\Phi_{\lambda(n)}}(f,T) \le 1$ for all $n \in \mathbb{N}$, and so, by Lemma 3.1(d), we find $V_{\Phi_\lambda}(f,T) \le \liminf_{n\to\infty} V_{\Phi_{\lambda(n)}}(f,T) \le 1$.

(c) If $V_{\Phi_\lambda}(f,T) \le 1$, then $p_\Phi(f,T) \le \lambda$ by virtue of (3.22). Suppose that $p_\Phi(f,T) > 0$ (otherwise, by (a), $f$ is constant and $V_{\Phi_\lambda}(f,T) = 0$). If $p_\Phi(f,T) = \lambda$, then $V_{\Phi_\lambda}(f,T) \le 1$ thanks to item (b). It remains to show that

$$\text{if} \quad p_\Phi(f,T) < \lambda, \quad \text{then} \quad V_{\Phi_\lambda}(f,T) < 1. \tag{3.25}$$

Indeed, setting $\mu = p_\Phi(f,T)$ and taking into account the convexity of $\Phi$ and the result of item (b), we have: $V_{\Phi_\lambda}(f,T) \le (\mu/\lambda)V_{\Phi_\mu}(f,T) \le \mu/\lambda < 1$.

(d) In view of (c) and (3.25) the cases $p_\Phi(f,T) > \lambda$ and $p_\Phi(f,T) < \lambda$ are not possible.

(e) It suffices to suppose that $\lambda = \liminf_{n\to\infty} p_\Phi(f_n,T)$ is finite. Then $p_\Phi(f_{n_k},T) \to \lambda$ as $k \to \infty$ for some subsequence $\{f_{n_k}\}_{k=1}^\infty$ of $\{f_n\}_{n=1}^\infty$, so for any $\varepsilon > 0$ we can find $k_0(\varepsilon) \in \mathbb{N}$ such that $p_\Phi(f_{n_k},T) < \lambda + \varepsilon$ for all $k \ge k_0(\varepsilon)$. The definition of $p_\Phi(f_{n_k},T)$ implies $V_{\Phi_{\lambda+\varepsilon}}(f_{n_k},T) \le 1$ if $k \ge k_0(\varepsilon)$, and since $f_{n_k}$ converges to $f$ pointwise on $T$ as $k \to \infty$, Lemma 3.1(d) yields $V_{\Phi_{\lambda+\varepsilon}}(f,T) \le \liminf_{k\to\infty} V_{\Phi_{\lambda+\varepsilon}}(f_{n_k},T) \le 1$, whence $p_\Phi(f,T) \le \lambda + \varepsilon$, $\varepsilon > 0$.

(f) Set $\lambda=L_d(f,T)/\Phi^{-1}(1/|T|)$. If $L_d(f,T)=0$, then $p_\Phi(f,T)=0$, and so, let $L_d(f,T)>0$. Applying (3.9), we have: $V_{\Phi_\lambda}(f,T)\le\Phi_\lambda(L_d(f,T))|T|=1$, and it follows from (c) that $p_\Phi(f,T)\le\lambda$, which proves (3.23).

To prove (3.24), we set $\lambda=V(f,T)/\big(|T|\Phi^{-1}(1/|T|)\big)$ and suppose that $\lambda>0$. Then (3.10) gives $V_{\Phi_\lambda}(f,T)\ge|T|\Phi_\lambda(V(f,T)/|T|)=1$. This and (3.25) then imply $p_\Phi(f,T)\ge\lambda$.

(g) Set $\lambda=p_\Phi(f,T\cap(-\infty,t])$ and $\mu=p_\Phi(f,T\cap[t,\infty))$. If at least one of the numbers $\lambda$ or $\mu$ is zero, then, by item (a), the inequality (actually, the equality) is obvious. Let $\lambda>0$ and $\mu>0$. By (b), we get $V_{\Phi_\lambda}(f,T\cap(-\infty,t])\le1$ and $V_{\Phi_\mu}(f,T\cap[t,\infty))\le1$. In view of (c), inequality $p_\Phi(f,T)\le\lambda+\mu$ is equivalent to $V_{\Phi_{\lambda+\mu}}(f,T)\le1$. In order to prove the latter, let $\xi=\{t_i\}_{i=0}^m$ be a partition of $T$ such that $t_{k-1}\le t\le t_k$ for some $k\in\{1,\dots,m\}$ (the cases $t<t_0$ or $t>t_m$ are similar). Denote by $U_\lambda(t,s)$ the expression $U(t,s)$ from (3.5), corresponding to function $\Phi_\lambda$. For the quantity $V_{\Phi_{\lambda+\mu}}[f;\xi]$ from (3.3) equality (3.6) holds, where $U$ is replaced by $U_{\lambda+\mu}$. The convexity of $\Phi$ and (3.7) imply

$$\begin{aligned}
U_{\lambda+\mu}(t_i,t_{i-1})&\le\frac{\lambda}{\lambda+\mu}U_\lambda(t_i,t_{i-1}),\qquad i=1,\dots,k-1,\\
U_{\lambda+\mu}(t_k,t_{k-1})&\le U_{\lambda+\mu}(t,t_{k-1})+U_{\lambda+\mu}(t_k,t)\\
&\le\frac{\lambda}{\lambda+\mu}U_\lambda(t,t_{k-1})+\frac{\mu}{\lambda+\mu}U_\mu(t_k,t),\\
U_{\lambda+\mu}(t_i,t_{i-1})&\le\frac{\mu}{\lambda+\mu}U_\mu(t_i,t_{i-1}),\qquad i=k+1,\dots,m,
\end{aligned}$$

which together with (3.6) give

$$V_{\Phi_{\lambda+\mu}}[f;\xi]\le\frac{\lambda}{\lambda+\mu}V_{\Phi_\lambda}(f,T\cap(-\infty,t])+\frac{\mu}{\lambda+\mu}V_{\Phi_\mu}(f,T\cap[t,\infty))\le1.$$

(h) Clearly, $p_\Phi(cf,T)=|c|p_\Phi(f,T)$, $c\in\mathbb{K}$. The triangle inequality, which is of the form $p_\Phi(f+g,T)\le p_\Phi(f,T)+p_\Phi(g,T)$, holds if at least one of the numbers at the right hand side is zero. Now if $\lambda=p_\Phi(f,T)>0$ and $\mu=p_\Phi(g,T)>0$, then from (3.21) and (b) we get $V_\Phi((f+g)/(\lambda+\mu),T)\le1$, and so, $p_\Phi(f+g,T)\le\lambda+\mu$ according to (3.22). □

One of the advantages to define the space $\mathrm{GV}_\Phi(T;X)$ is that it is invariant with respect to equivalent metrics on $X$: if $d$ and $d_0$ are equivalent metrics on $X$, i.e., $C_1d(x,y)\le d_0(x,y)\le C_2d(x,y)$ for some constants $C_1>0$ and $C_2>0$ and all $x$, $y\in X$, and $f\in\mathrm{GV}_\Phi(T;X)$ with respect to metric $d$, then $f\in\mathrm{GV}_\Phi(T;X)$ with respect to metric $d_0$ and the following inequalities hold $C_1p_{\Phi,d}(f,T)\le p_{\Phi,d_0}(f,T)\le C_2p_{\Phi,d}(f,T)$, where $p_{\Phi,d}(f,T)$ is the quantity (3.22), evaluated in metric $d$.

By Lemma 3.9(b), the *structural lemma* 3.2 holds for mappings $f$ from $\mathrm{GV}_\Phi(T;X)$ if we replace $\mathrm{BV}_\Phi$ by $\mathrm{GV}_\Phi$, and the equality $V_\Phi(\varphi,T)=V_\Phi(f,T)$

— by $p_\Phi(\varphi, T) = p_\Phi(f, T)$. To see this, let us follow the notation and proof of that Lemma, making the necessary changes. If $\varphi \in \mathrm{GV}_\Phi(T;\mathbb{R})$, without loss of generality we suppose that $\lambda = p_\Phi(\varphi, T) > 0$, and so, if $f = g \circ \varphi$, we have: $V_{\Phi_\lambda}(f, T) \le V_\Phi(\varphi/\lambda, T) \le 1$ since $L_d(g, J) \le 1$, hence $f \in \mathrm{GV}_\Phi(T;X)$ and $p_\Phi(f, T) \le p_\Phi(\varphi, T)$. To prove the necessity part, we note that if $f \in \mathrm{GV}_\Phi(T;X)$, $\lambda = p_\Phi(f, T) > 0$ and $\varphi(t) = V(f, T \cap (-\infty, t])$, $t \in T$, then, by Lemma 3.2, $V_\Phi(\varphi/\lambda, T) = V_{\Phi_\lambda}(f, T) \le 1$, so that $\varphi \in \mathrm{GV}_\Phi(T;\mathbb{R})$ and $p_\Phi(\varphi, T) \le \lambda = p_\Phi(f, T)$.

In order to establish the relations between spaces $\mathrm{GV}_\Phi(T;X)$, generated by different functions $\Phi \in \mathcal{N}$, let us recall certain definitions ([62, Sections 3, 13], [68, Theorem 3.4]). Given functions $\Phi$, $\Psi \in \mathcal{N}$, we write $\Psi \preccurlyeq \Phi$ and say that $\Phi$ *dominates* $\Psi$ *near infinity* if there exist constants $C > 0$ and $\rho_0 > 0$ such that $\Psi(\rho) \le \Phi(C\rho)$ for all $\rho \ge \rho_0$. For example, if $\Phi(\rho) = \rho^p$ and $\Psi = \rho^q$ with $p, q \ge 1$, then $\Psi \preccurlyeq \Phi$ if and only if $q \le p$. Functions $\Phi$, $\Psi \in \mathcal{N}$ are said to be *equivalent near infinity*, in symbols $\Phi \sim \Psi$, provided $\Psi \preccurlyeq \Phi$ and $\Phi \preccurlyeq \Psi$. Clearly, $\Phi \sim \Psi$ if and only if, for some constants $C_1 > 0$, $C_2 > 0$ and $\rho_0 > 0$, we have $\Phi(C_1\rho) \le \Psi(\rho) \le \Phi(C_2\rho)$ for all $\rho \ge \rho_0$. In particular, if $\lim_{\rho\to\infty} \Phi(\rho)/\Psi(\rho) > 0$ is finite, then $\Phi \sim \Psi$.

**Theorem 3.10.** *Let* $\Phi$*,* $\Psi \in \mathcal{N}$*. If* $T$ *is bounded,* $(X, d)$ *is a metric space and* $\Psi \preccurlyeq \Phi$*, then* $\mathrm{GV}_\Phi(T;X) \subset \mathrm{GV}_\Psi(T;X)$ *and there exists a number* $\kappa = \kappa(\Phi, \Psi, |T|) > 0$*, depending only on* $\Phi$*,* $\Psi$ *and* $|T|$*, such that* $p_\Psi(f, T) \le \kappa\, p_\Phi(f, T)$ *for all* $f \in \mathrm{GV}_\Phi(T;X)$*. Conversely, if* $I = [a, b]$*,* $(X, \|\cdot\|)$ *is a linear normed space and* $\mathrm{GV}_\Phi(I;X) \subset \mathrm{GV}_\Psi(I;X)$*, then* $\Psi \preccurlyeq \Phi$*. Thus, the spaces* $\mathrm{GV}_\Phi(I;X)$ *and* $\mathrm{GV}_\Psi(I;X)$ *consist of the same mappings if and only if* $\Phi \sim \Psi$*, and moreover, functionals* $p_\Phi(\cdot, I)$ *and* $p_\Psi(\cdot, I)$ *are equivalent.*

**Proof.** 1. If $\Psi \preccurlyeq \Phi$, then $\Psi(\rho) \le \Phi(C\rho)$ for some constants $C > 0$ and $\rho_0 > 0$ and all $\rho \ge \rho_0$. Given $f \in \mathrm{GV}_\Phi(T;X)$, there exists $\lambda > 0$ such that $V_{\Phi_\lambda}(f, T) < \infty$, and so, if $\mu = \lambda C$, we have: $V_{\Psi_\mu}(f, T) \le \Psi(\rho_0)|T| + V_{\Phi_\lambda}(f, T)$.

Now, let us prove the inequality. Let $f \in \mathrm{GV}_\Phi(T;X)$ and $\lambda = p_\Phi(f, T)$. If $\lambda = 0$, then $f$ is constant by Lemma 3.9(a), and so, $p_\Psi(f, T) = 0$. Assume that $\lambda > 0$ and set $\rho_1 = \Psi^{-1}(1/(2|T|))$ and $N = \max\{1, \Psi(\rho_0)/\Phi(C\rho_1)\}$. Since $\Psi \preccurlyeq \Phi$, then $\Psi(\rho) \le N\Phi(C\rho)$ for all $\rho \ge \rho_1$: in fact, this is clear if $\rho_1 \ge \rho_0$ or $\rho_1 < \rho_0 \le \rho$, so we suppose that $\rho_1 \le \rho \le \rho_0$, in which case $\Psi(\rho) \le \Psi(\rho_0)$ and $\Phi(C\rho_1) \le \Phi(C\rho)$ by the monotonicity of $\Phi$ and $\Psi$, and so, $\Psi(\rho) \le \Phi(C\rho)\Psi(\rho_0)/\Phi(C\rho_1)$. Let $\xi = \{t_i\}_{i=0}^m$ be an arbitrary partition of $T$. Setting $\mu = 2NC\lambda$ and denoting by $\{i\}$ the set of all indices $i \in \{1, \dots, m\}$, for which $d(f(t_i), f(t_{i-1}))/((t_i - t_{i-1})\mu) < \rho_1$, and by $[i]$ — the set of remaining indices, taking into account the convexity of $\Phi$ and

Lemma 3.9(b), we find

$$V_{\Psi_\mu}[f;\xi] = \Big(\sum_{i\in\{i\}} + \sum_{i\in[i]}\Big)\Psi\bigg(\frac{d(f(t_i), f(t_{i-1}))}{(t_i - t_{i-1})\mu}\bigg)(t_i - t_{i-1})$$
$$\le \Psi(\rho_1)|T| + N\sum_{i\in[i]}\Phi\bigg(\frac{d(f(t_i), f(t_{i-1}))}{(t_i - t_{i-1})2N\lambda}\bigg)(t_i - t_{i-1})$$
$$\le \frac{1}{2} + \frac{1}{2}V_{\Phi_\lambda}(f, T) \le 1.$$

Since $\xi$ is arbitrary, this implies $V_{\Psi_\mu}(f, T) \le 1$, and so, $p_\Psi(f, T) \le \mu = 2NCp_\Phi(f, T)$, and it remains to set $\kappa = 2NC$.

2. Suppose now that condition $\Psi \preccurlyeq \Phi$ does not hold. Then there exists a sequence $\{\rho_n\}_{n=1}^\infty$ of positive numbers such that $\lim_{n\to\infty}\rho_n = \infty$ and $\Psi(\rho_n) > \Phi(n2^n\rho_n)$ for all $n \in \mathbb{N}$. Setting $\theta = 1/2^n$ and $\rho = n2^n\rho_n$ in the (convexity) inequality $\Phi(\theta\rho) \le \theta\Phi(\rho)$, we get $\Phi(n2^n\rho_n) \ge 2^n\Phi(n\rho_n)$; thus,

$$\Psi(\rho_n) > 2^n\Phi(n\rho_n), \qquad n \in \mathbb{N}. \tag{3.26}$$

We define the sequence of points $\{t_n\}_{n=0}^\infty$ in $I$ as follows: $t_0 = a$ and $t_n - t_{n-1} = 2^{-n}|I|\Phi(\rho_1)/\Phi(n\rho_n)$ if $n \in \mathbb{N}$. Put

$$f(t) = \begin{cases} (n\rho_n(t - t_{n-1}) + S_{n-1})x & \text{if } t_{n-1} \le t < t_n, \quad n \in \mathbb{N},\\ S_\infty x & \text{if } \lim_{n\to\infty} t_n \le t \le b, \end{cases}$$

where $S_0 = 0$, $S_k = \sum_{n=1}^k n\rho_n(t_n - t_{n-1})$, $k \in \mathbb{N}\cup\{\infty\}$, $x \in X$, $\|x\| = 1$, and note that $S_\infty < \infty$. Let us show that $f \in \mathrm{BV}_\Phi(I; X)$, and at the same time $f \notin \mathrm{GV}_\Psi(I; X)$. In fact,

$$V_\Phi(f, I) = \sum_{n=1}^\infty U_\Phi(t_n, t_{n-1}) = \sum_{n=1}^\infty \Phi(n\rho_n)(t_n - t_{n-1}) = |I|\Phi(\rho_1) < \infty.$$

Now if $\lambda \ge 1$, then for any $m \in \mathbb{N}$, $m \ge \lambda$, by virtue of (3.26) we have:

$$V_\Psi(f/\lambda, I)s \ge \sum_{n=m}^{2m}\Psi\bigg(\frac{\|f(t_n) - f(t_{n-1})\|}{(t_n - t_{n-1})\lambda}\bigg)(t_n - t_{n-1})$$
$$\ge \sum_{n=m}^{2m}\Psi(\rho_n)(t_n - t_{n-1}) \ge m|I|\Phi(\rho_1).$$

Therefore, $V_\Psi(f/\lambda, I) = \infty$ for all $\lambda > 0$. □

## 4. Metric semigroups of mappings

A triple $(X, d, +)$ is said to be a *metric semigroup* if $(X, d)$ is a metric space with metric $d$, $(X, +)$ is an additive commutative semigroup (i. e., $x + (y + z) = (x + y) + z$ and $x + y = y + x$ for all $x, y, z \in X$) and $d$ is translation invariant in the sense that $d(x, y) = d(x + z, y + z)$ for all $x, y, z \in X$. A metric semigroup $(X, d, +)$ is called *complete* if $(X, d)$ is a complete metric space.

A simple example of a metric semigroup is any linear normed space $(X, \|\cdot\|)$ with induced metric $d(x, y) = \|x - y\|$, $x, y \in X$, and the addition operation $+$ from $X$; this semigroup is complete if $X$ is a Banach space. If $K \subset X$ is a convex cone (i. e., $x + y, \lambda x \in K$ whenever $x, y \in K$, $\lambda \geq 0$), then $(K, d, +)$ is also a metric semigroup, which is complete if $X$ is a Banach space and $K$ is closed in $X$. More examples of metric semigroups relevant for our purposes are presented below in this section (for metric semigroups $\mathrm{cc}(X)$ and $\mathrm{cbc}(X)$ see p. 47 and p. 60).

Note that if $(X, d, +)$ is a metric semigroup, then, by the translation invariance of $d$ and the triangle inequality for $d$, given $x, y, u, v \in X$, we have:

$$d(x, y) \leq d(x + u, y + v) + d(u, v), \tag{4.1}$$

$$d(x + u, y + v) \leq d(x, y) + d(u, v). \tag{4.2}$$

In particular, inequality (4.2) implies that the addition operation $(x, y) \mapsto x + y$ is a continuous mapping from $X \times X$ into $X$: $x_n + y_n \to x + y$ in $X$ as $n \to \infty$ whenever $x_n \to x$ and $y_n \to y$ in $X$ as $n \to \infty$. More generally, if $x_n \to x$, $y_n \to y$, $u_n \to u$ and $v_n \to v$ in $X$ as $n \to \infty$, then

$$\lim_{n\to\infty} d(x_n + y_n, u_n + v_n) = d(x + y, u + v). \tag{4.3}$$

**4.1. The space $\mathrm{GV}_\Phi(T; X)$ as a metric semigroup.** Let $T \subset \mathbb{R}$, $a \in T$ be a given point, $(X, d, +)$ be a metric semigroup, $\Phi \in \mathcal{N}$ and $f$, $g \in \mathrm{GV}_\Phi(T; X)$.

The *addition operation* in $\mathrm{GV}_\Phi(T; X)$ is introduced pointwise: $(f+g)(t) = f(t)+g(t)$, $t \in T$. It is well defined, i. e., $f+g \in \mathrm{GV}_\Phi(T; X)$; indeed, $V_{\Phi_\lambda}(f)$ and $V_{\Phi_\mu}(g)$ are finite for some constants $\lambda > 0$ and $\mu > 0$ and, given $t$, $s \in T$, $s < t$, inequality (4.2) yields

$$\frac{d((f+g)(t), (f+g)(s))}{(t-s)(\lambda+\mu)} \leq \frac{\lambda}{\lambda+\mu} \cdot \frac{d(f(t), f(s))}{(t-s)\lambda} + \frac{\mu}{\lambda+\mu} \cdot \frac{d(g(t), g(s))}{(t-s)\mu},$$

and so, by the monotonicity and convexity of $\Phi$, we get:

$$V_{\Phi_{\lambda+\mu}}(f+g) \leq \frac{\lambda}{\lambda+\mu} V_{\Phi_\lambda}(f) + \frac{\mu}{\lambda+\mu} V_{\Phi_\mu}(g) < \infty.$$

This inequality, Lemma 3.9(b) and (3.22) also imply $p_\Phi(f+g) \le p_\Phi(f) + p_\Phi(g)$.

We define the *metric* $d_\Phi$ on $\mathrm{GV}_\Phi(T;X)$ as follows ([19], [22]):

$$d_\Phi(f,g) = d(f(a),g(a)) + \Delta_\Phi(f,g), \tag{4.4}$$

where

$$\Delta_\Phi(f,g) \equiv \Delta_{\Phi,d}(f,g,T) = \inf\{\lambda > 0 \mid W_{\Phi_\lambda}(f,g) \le 1\} \tag{4.5}$$

and

$$\begin{aligned} W_\Phi(f,g) &\equiv W_{\Phi,d}(f,g,T) \\ &= \sup_\xi \sum_{i=1}^m \Phi\biggl(\frac{d(f(t_i)+g(t_{i-1}), g(t_i)+f(t_{i-1}))}{t_i - t_{i-1}}\biggr)(t_i - t_{i-1}), \end{aligned} \tag{4.6}$$

and the supremum is taken over all partitions $\xi = \{t_i\}_{i=0}^m$ ($m \in \mathbb{N}$) of the set $T$. In the significant particular case $\Phi(\rho)=\rho$, i. e., when $\mathrm{GV}_\Phi(T;X) = \mathrm{BV}(T;X)$, we denote $d_\Phi$ by $d_1$ and $W_\Phi = \Delta_\Phi$ — by $\Delta_1 = \Delta_{1,d}$. In the context of the Hausdorff metric $d$ on the space of all compact convex subsets of a real linear normed space the metric $d_\Phi$ was employed by Zawadzka [113] (with $\Phi(\rho) = \rho$), Merentes and Nikodem [77] (with $\Phi(\rho) = \rho^q$ and $q > 1$) and Chistyakov [23] (in the general case $\Phi \in \mathcal{N}$).

It will be shown below that $\Delta_\Phi$ is a semimetric and $d_\Phi$ is a metric on $\mathrm{GV}_\Phi(T;X)$, which are translation invariant. Now let us verify that the value $\Delta_\Phi(f,g)$ is finite. In fact, since $V_{\Phi_\lambda}(f) < \infty$ and $V_{\Phi_\mu}(g) < \infty$ (see above), given $t$, $s \in T$, $s < t$, (4.2) implies

$$\frac{d(f(t)+g(s), g(t)+f(s))}{(t-s)(\lambda+\mu)} \le \frac{\lambda}{\lambda+\mu}\cdot\frac{d(f(t),f(s))}{(t-s)\lambda} + \frac{\mu}{\lambda+\mu}\cdot\frac{d(g(t),g(s))}{(t-s)\mu},$$

hence (again by the monotonicity and convexity of $\Phi$)

$$W_{\Phi_{\lambda+\mu}}(f,g) \le \frac{\lambda}{\lambda+\mu} V_{\Phi_\lambda}(f) + \frac{\mu}{\lambda+\mu} V_{\Phi_\mu}(g) < \infty. \tag{4.7}$$

Again by the convexity of $\Phi$, for $\nu \ge \lambda + \mu$ we have:

$$W_{\Phi_\nu}(f,g) \le \frac{\lambda+\mu}{\nu} W_{\Phi_{\lambda+\mu}}(f,g) \to 0 \quad \text{as} \quad \nu \to \infty,$$

and so, $\Delta_\Phi(f,g)$ is well defined.

The main properties of $\Delta_\Phi$ and $W_\Phi$ are gathered in Lemmas 4.1 and 4.3. The following lemma is a counterpart of Lemma 3.9(a)–(e) for $\Delta_\Phi$.

**Lemma 4.1.** *Let $T \subset \mathbb{R}$, $(X,d,+)$ be a metric semigroup and $f,g \in \mathrm{GV}_\Phi(T;X)$ where $\Phi \in \mathcal{N}$. Then we have:*

(a) $|d(f(t),g(t)) - d(f(s),g(s))| \le d(f(t)+g(s), g(t)+f(s)) \le \omega_\Phi(|t-s|)\Delta_\Phi(f,g)$ *whenever* $t$, $s \in T$;

(b) *if* $\lambda = \Delta_\Phi(f,g) > 0$, *then* $W_{\Phi_\lambda}(f,g) \le 1$;

(c) *given* $\lambda>0$, $\Delta_\Phi(f,g)\le\lambda$ *if and only if* $W_{\Phi_\lambda}(f,g)\le 1$;
(d) *if* $\lambda>0$ *and* $W_{\Phi_\lambda}(f,g)=1$, *then* $\Delta_\Phi(f,g)=\lambda$;
(e) *if sequences* $\{f_n\}_{n=1}^\infty$, $\{g_n\}_{n=1}^\infty\subset \mathrm{GV}_\Phi(T;X)$ *converge pointwise on* $T$ *to* $f$ *and* $g$ *as* $n\to\infty$, *respectively, then*

$$\Delta_\Phi(f,g)\le\liminf_{n\to\infty}\Delta_\Phi(f_n,g_n);$$

(f) $|p_\Phi(f)-p_\Phi(g)|\le\Delta_\Phi(f,g)\le p_\Phi(f)+p_\Phi(g)$.

**Proof.** (a) By (4.5) and (4.6), we have, for $t$, $s\in T$, $s\ne t$,

$$\Phi\biggl(\frac{d(f(t)+g(s),g(t)+f(s))}{|t-s|\lambda}\biggr)|t-s|\le W_{\Phi_\lambda}(f,g)\le 1\quad\text{if}\quad \lambda>\Delta_\Phi(f,g).$$

Dividing by $|t-s|$ and applying $\Phi^{-1}$, we get the second inequality in (a). The first inequality in (a) is a consequence of (4.1).

(b) First, let us show that if conditions of (e) are satisfied and $\lambda(n)\to\lambda$ as $n\to\infty$, where $\lambda(n)>0$ and $\lambda>0$, then

$$W_{\Phi_\lambda}(f,g)\le\liminf_{n\to\infty}W_{\Phi_{\lambda(n)}}(f_n,g_n).\tag{4.8}$$

The pointwise convergence of $f_n$ to $f$ and $g_n$ to $g$ and property (4.3) imply

$$\lim_{n\to\infty}d(f_n(t)+g_n(s),g_n(t)+f_n(s))=d(f(t)+g(s),g(t)+f(s)),\quad t,\,s\in T.$$

Given $\xi=\{t_i\}_{i=0}^m$ a partition of $T$, by (4.6), for all $n\in\mathbb{N}$ we have

$$\sum_{i=1}^m\Phi\biggl(\frac{d(f_n(t_i)+g_n(t_{i-1}),g_n(t_i)+f_n(t_{i-1}))}{(t_i-t_{i-1})\lambda(n)}\biggr)(t_i-t_{i-1})\le W_{\Phi_{\lambda(n)}}(f_n,g_n).$$

Passing to the limit inferior as $n\to\infty$ and making use of the continuity of $\Phi$, and then taking the supremum over all partitions $\xi$ of $T$ at the left hand side, we arrive at (4.8).

In order to prove (b), let $\lambda(n)>\lambda=\Delta_\Phi(f,g)$, $n\in\mathbb{N}$, be such that $\lim_{n\to\infty}\lambda(n)=\lambda$. Since $W_{\Phi_{\lambda(n)}}(f,g)\le 1$ for all $n\in\mathbb{N}$, (4.8) yields $W_{\Phi_\lambda}(f,g)\le 1$.

(c) As in the proof of Lemma 3.9(c), by virtue of (a) and (b), it suffices to show only that if $0<\Delta_\Phi(f,g)<\lambda$, then $W_{\Phi_\lambda}(f,g)<1$. Setting $\mu=\Delta_\Phi(f,g)$, by the convexity of $\Phi$ and item (b), we have: $W_{\Phi_\lambda}(f,g)\le(\mu/\lambda)W_{\Phi_\mu}(f,g)\le\mu/\lambda<1$.

(d) By the just proved assertion and item (c), it follows that the cases $\Delta_\Phi(f,g)<\lambda$ and $\Delta_\Phi(f,g)>\lambda$ do not hold.

(e) Suppose that $\lambda=\liminf_{n\to\infty}\Delta_\Phi(f_n,g_n)<\infty$. Then there exists a subsequence $\{n_k\}_{k=1}^\infty$ of $\{n\}_{n=1}^\infty$ such that $\Delta_\Phi(f_{n_k},g_{n_k})\to\lambda$ as $k\to\infty$, and so, given $\varepsilon>0$, we find a $k_0(\varepsilon)\in\mathbb{N}$, for which $\Delta_\Phi(f_{n_k},g_{n_k})<\lambda+\varepsilon$ for all $k\ge k_0(\varepsilon)$. By the definition of $\Delta_\Phi(f_{n_k},g_{n_k})$, we have $W_{\Phi_{\lambda+\varepsilon}}(f_{n_k},g_{n_k})\le 1$,

$k \geq k_0(\varepsilon)$. From the pointwise convergence of $f_{n_k}$ and $g_{n_k}$ and (4.8) we get $W_{\Phi_{\lambda+\varepsilon}}(f,g) \leq 1$ and, therefore, $\Delta_\Phi(f,g) \leq \lambda+\varepsilon$ for all $\varepsilon > 0$.

(f) First we establish the second inequality. Set $\lambda = p_\Phi(f)$ and $\mu = p_\Phi(g)$. If $\lambda=0$ or $\mu=0$, then the (in)equality is obvious by virtue of Lemma 3.9(a). Suppose that $\lambda>0$ and $\mu>0$. Then $V_{\Phi_\lambda}(f) \leq 1$ and $V_{\Phi_\mu}(g) \leq 1$ according to Lemma 3.9(b), and so, $W_{\Phi_{\lambda+\mu}}(f,g) \leq 1$ thanks to (4.7). Taking into account (4.5) we get that $\Delta_\Phi(f,g) \leq \lambda+\mu$.

To prove the first inequality, we set $\lambda = \Delta_\Phi(f,g)$ and $\mu = p_\Phi(g)$ and assume that $\lambda>0$ and $\mu>0$. From (4.1),

$$d(f(t),f(s)) \leq d(f(t)+g(s),g(t)+f(s)) + d(g(t),g(s)), \quad t,\, s \in T. \tag{4.9}$$

By the convexity of $\Phi$, (3.3), (3.4), (4.6) and Lemmas 3.9(b) and 4.1(b), we find that

$$V_{\Phi_{\lambda+\mu}}(f) \leq \frac{\lambda}{\lambda+\mu} W_{\Phi_\lambda}(f,g) + \frac{\mu}{\lambda+\mu} V_{\Phi_\mu}(g) \leq 1, \tag{4.10}$$

whence $p_\Phi(f) \leq \lambda+\mu = \Delta_\Phi(f,g) + p_\Phi(g)$, and it remains to take into account the symmetry in $f$ and $g$ in the formulae.

If $\lambda = 0$, by Lemma 4.1(a), (4.9) and the symmetry in $f$ and $g$, we have: $d(f(t),f(s)) = d(g(t),g(s))$ for all $t,s \in T$, and so, $p_\Phi(f) = p_\Phi(g)$. If $\mu = 0$, Lemma 3.9(a) implies that $g$ is a constant mapping and, hence, $d(f(t),f(s)) = d(f(t)+g(s),g(t)+f(s))$, $t,s \in T$, so that $\Delta_\Phi(f,g) = p_\Phi(f)$. □

**Theorem 4.2.** *If $T \subset \mathbb{R}$, $(X,d,+)$ is a (complete) metric semigroup and the function $\Phi \in \mathcal{N}$, then the triple $(\mathrm{GV}_\Phi(T;X), d_\Phi, +)$ is also a (respectively, complete) metric semigroup.*

**Proof.** Let $f,\, g,\, h \in \mathrm{GV}_\Phi(T;X)$. The translation invariance of $d_\Phi$ follows from equality $\Delta_\Phi(f+h,g+h) = \Delta_\Phi(f,g)$, which is a consequence of the translation invariance of $d$ and the following equality for $t,\, s \in T$:

$$d\Big((f+h)(t)+(g+h)(s),(g+h)(t)+(f+h)(s)\Big) = d\Big(f(t)+g(s),g(t)+f(s)\Big).$$

Now let us show that $d_\Phi$ is a metric on $\mathrm{GV}_\Phi(T;X)$. If $d_\Phi(f,g) = 0$, then, by (4.4) and Lemma 4.1(a), $d(f(t),g(t)) = d(f(a),g(a)) = 0$, $t \in T$, $t \neq a$, that is, $f = g$. Clearly, $d_\Phi$ is symmetrical: $d_\Phi(f,g) = d_\Phi(g,f)$. In order to prove the triangle inequality for $d_\Phi$, it suffices to show that $\Delta_\Phi(f,g) \leq \Delta_\Phi(f,h) + \Delta_\Phi(g,h)$. From (4.1) and the translation invariance of $d$ we have, for all $t,\, s \in T$,

$$\begin{aligned} d\Big(f(t)+g(s),g(t)+f(s)\Big) &\leq d\Big(f(t)+h(s),h(t)+f(s)\Big) \\ &\quad + d\Big(g(t)+h(s),h(t)+g(s)\Big). \end{aligned} \tag{4.11}$$

First assume that $\Delta_\Phi(f,h)=0$. By Lemma 4.1(a),

$$d(f(t)+h(s),h(t)+f(s))=0, \qquad t,\, s\in T,$$

and so, $W_{\Phi_\lambda}(f,g)\le W_{\Phi_\lambda}(g,h)$ for all $\lambda>0$ by virtue of (4.11) and (4.6). Then (4.5) implies $\Delta_\Phi(f,g)\le\Delta_\Phi(g,h)$. The symmetry in $f$ and $g$ gives $\Delta_\Phi(f,g)=\Delta_\Phi(g,h)$. Similarly, if $\Delta_\Phi(g,h)=0$, then $\Delta_\Phi(f,g)=\Delta_\Phi(f,h)$.

Let $\lambda=\Delta_\Phi(f,h)>0$ and $\mu=\Delta_\Phi(g,h)>0$. Then, by Lemma 4.1(b), $W_{\Phi_\lambda}(f,h)\le 1$ and $W_{\Phi_\mu}(g,h)\le 1$. Using (4.11), (4.6), the monotonicity and convexity of $\Phi$, we have:

$$W_{\Phi_{\lambda+\mu}}(f,g)\le\frac{\lambda}{\lambda+\mu}W_{\Phi_\lambda}(f,h)+\frac{\mu}{\lambda+\mu}W_{\Phi_\mu}(g,h)\le 1,$$

which proves that $\Delta_\Phi(f,g)\le\lambda+\mu$.

Suppose that $(X,d)$ is a complete metric space and $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathrm{GV}_\Phi(T;X)$, i. e.,

$$d_\Phi(f_n,f_m)\;=\;d(f_n(a),f_m(a))+\Delta_\Phi(f_n,f_m)\;\to\;0\;\text{ as }\;n,\,m\to\infty. \tag{4.12}$$

By Lemma 4.1(a), $\{f_n(t)\}_{n=1}^\infty$ is a Cauchy sequence in $X$ for all $t\in T$. Let $f:T\to X$ be such that $f_n(t)\to f(t)$ in $X$ as $n\to\infty$ for all $t\in T$. From Lemma 4.1(e) we find

$$\Delta_\Phi(f_n,f)\le\liminf_{m\to\infty}\Delta_\Phi(f_n,f_m)\le\lim_{m\to\infty}d_\Phi(f_n,f_m)\in[0,\infty), \quad n\in\mathbb{N}.$$

Again, since $\{f_n\}_{n=1}^\infty$ is Cauchy, then

$$\limsup_{n\to\infty}\Delta_\Phi(f_n,f)\le\lim_{n\to\infty}\lim_{m\to\infty}d_\Phi(f_n,f_m)=0,$$

whence we conclude that $d_\Phi(f_n,f)\to 0$ as $n\to\infty$. It remains to show that $f\in\mathrm{GV}_\Phi(T;X)$. It follows from (4.12) and Lemma 4.1(f) that $\{p_\Phi(f_n)\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$, and so, it is bounded and convergent. Our assertion now follows from Lemma 3.9(e). □

Further properties of $\Delta_\Phi$ $W_\Phi$ are presented in the following

**Lemma 4.3.** *Let* $\emptyset\ne T\subset\mathbb{R}$, $(X,d,+)$ *be a metric semigroup,* $\Phi\in\mathcal{N}$ *and* $f,\,g:T\to X$. *Then we have*:

(a) $W_\Phi(f,g,T_1)\le W_\Phi(f,g,T_2)$ *whenever* $\emptyset\ne T_1\subset T_2\subset T$;

(b) $d\Big(f(t)+g(s),g(t)+f(s)\Big)\le(t-s)\Phi^{-1}\Big(W_\Phi(f,g,T)/(t-s)\Big)$, $t,s\in T$, $s<t$;

(c) *if* $t\in T$, *then* $W_\Phi(f,g,T)=W_\Phi(f,g,T\cap(-\infty,t])+W_\Phi(f,g,T\cap[t,\infty))$, *and also* $\Delta_\Phi(f,g,T)\le\Delta_\Phi(f,g,T\cap(-\infty,t])+\Delta_\Phi(f,g,T\cap[t,\infty))$;

(d) $W_\Phi(f,g,T)=\sup\{\,W_\Phi(f,g,T\cap[a,b])\mid a,\,b\in T,\,a<b\,\}$;

(e) $W_\Phi(f,g,T)=\lim_{T\ni t\to s}W_\Phi(f,g,T\cap(-\infty,t])$ *if* $s=\sup T\in(\mathbb{R}\setminus T)\cup\{\infty\}$;

(f) $W_\Phi(f,g,T)=\lim_{T\ni t\to i}W_\Phi(f,g,T\cap[t,\infty))$ *if* $i=\inf T\in(\mathbb{R}\setminus T)\cup\{-\infty\}$;

(g) *if $s$ and $i$ are as in* (e) *and* (f), *then, in addition, $W_\Phi(f,g,T)$ is equal to*

$$\lim_{T^2\ni(a,b)\to(i,s)} W_\Phi(f,g,T\cap[a,b]) = \lim_{T\ni b\to s}\lim_{T\ni a\to i} W_\Phi(f,g,T\cap[a,b]) = \lim_{T\ni a\to i}\lim_{T\ni b\to s} W_\Phi(f,g,T\cap[a,b]).$$

**Proof.** (a) and (b) are consequences of the definition (4.6).

(c) Let us denote by $W_\Phi[f,g,\xi]$ the sum under the supremum sign in (4.6), corresponding to the partition $\xi=\{t_i\}_{i=0}^m$ of $T$. Let us prove the equality in (c). Given partitions $\xi_1$ of $T\cap(-\infty,t]$ and $\xi_2$ of $T\cap[t,\infty)$, we set $\widetilde{\xi_i}=\xi_i\cup\{t\}$, $i=1$, 2. Then $\widetilde{\xi_1}\cup\widetilde{\xi_2}$ is a partition of $T$, and so,

$$W_\Phi[f,g,\xi_1]+W_\Phi[f,g,\xi_2]\le W_\Phi[f,g,\widetilde{\xi_1}]+W_\Phi[f,g,\widetilde{\xi_2}] = W_\Phi[f,g,\widetilde{\xi_1}\cup\widetilde{\xi_2}]\le W_\Phi(f,g,T),$$

which give the inequality $\ge$. In order to prove the reverse inequality, let $\xi=\{t_i\}_{i=0}^m$ be a partition of $T$. If $t\in\xi$ or $t<t_0$ or $t_m<t$, then $W_\Phi[f,g,\xi]\le W_\Phi(f,g,T\cap(-\infty,t])+W_\Phi(f,g,T\cap[t,\infty))$. So, suppose that $t_{k-1}<t<t_k$ for some $k\in\{1,\dots,m\}$. Inequality (4.1) and the translation invariance of $d$ imply, for $\rho(t,s)=d(f(t)+g(s),g(t)+f(s))$,

$$\begin{aligned}\rho(t_k,t_{k-1}) \le{}& d\Big(f(t_k)+g(t_{k-1})+g(t)+f(t_{k-1}),\, g(t_k)+f(t_{k-1})+f(t)+g(t_{k-1})\Big)\\ &+ d\Big(f(t)+g(t_{k-1}),\, g(t)+f(t_{k-1})\Big)\\ ={}&\rho(t_k,t)+\rho(t,t_{k-1}).\end{aligned}$$

From this, the monotonicity and convexity of $\Phi$ and Jensen's inequality for sums we get that the quantity $U(t,s)=(t-s)\Phi\big(\rho(t,s)/(t-s)\big)$, $s<t$, satisfies inequality (3.7). Taking into account (3.6), where $V_\Phi[f,\xi]$ is replaced by $W_\Phi[f,g,\xi]$, and applying (3.7), we have:

$$\begin{aligned}W_\Phi[f,g,\xi]&\le W_\Phi[f,g,\{t_i\}_{i=0}^{k-1}\cup\{t\}]+W_\Phi[f,g,\{t\}\cup\{t_i\}_{i=k}^m]\\ &\le W_\Phi(f,g,T\cap(-\infty,t])+W_\Phi(f,g,T\cap[t,\infty)),\end{aligned}$$

which due to the arbitrariness of $\xi$ proves the equality in (c).

The inequality in (c) is established similar to Lemma 3.9(g), if we replace $V_{\Phi_\lambda}(f,\cdot)$ there by $W_{\Phi_\lambda}(f,g,\cdot)$, $p_\Phi(f,\cdot)$ — by $\Delta_\Phi(f,g,\cdot)$ and apply Lemma 4.1(a)–(c) instead of Lemma 3.9(a)–(c).

(d) By (a), the left hand side in (d) is not less than the right hand side. Conversely, given a number $\alpha<W_\Phi(f,g,T)$, we find a partition $\xi=\{t_i\}_{i=0}^m$ of $T$ such that $W_\Phi[f,g,\xi]\ge\alpha$, so setting $a=t_0$ and $b=t_m$ and noting that $\xi$ is a partition of $T\cap[a,b]$, we obtain $W_\Phi(f,g,T\cap[a,b])\ge W_\Phi[f,g,\xi]\ge\alpha$.

(e) Since $s=\sup T\notin T$, $s$ is a limit point for $T$. By (a), the function $t\mapsto W_\Phi(f,g,T\cap(-\infty,t])$, mapping $T$ into $[0,\infty]$, is nondecreasing, and so, the limit in (e) exists and does not exceed $W_\Phi(f,g,T)$. On the other hand, by (d), for any $\alpha < W_\Phi(f,g,T)$ there exist $a$, $b \in T$, $a < b < s$, such that $W_\Phi(f,g,T\cap[a,b]) \ge \alpha$. Then (a) implies $W_\Phi(f,g,T\cap(-\infty,t]) \ge W_\Phi(f,g,T\cap[a,b]) \ge \alpha$ for all $t \in T\cap[b,s)$, which proves (e).

Item (f) and the first equality in (g) are proved similarly to (e). The second equality in (g) follows from (e) and (f), since $T\cap[a,b] = T\cap(-\infty,b]\cap[a,\infty)$. The last equality in (g) is established similarly. □

**4.2. The metric semigroup** $\mathrm{Lip}(T;X)$**.** Let $(X,d,+)$ be a metric semigroup and $(T,d_1)$ be a metric space. Given $f : T \to X$, we set

$$L(f) = L(f,T) = \sup\{d(f(t),f(s))/d_1(t,s)\ ;\ t,\, s \in T,\, t \neq s\}$$

and denote by $\mathrm{Lip}(T;X)$ the set of all Lipschitzian mappings $f : T \to X$ (i. e., $L(f)$ is finite). The pointwise addition operation on $\mathrm{Lip}(T;X)$ is well defined, since, for any $f$, $g \in \mathrm{Lip}(T;X)$, by (4.2), we have

$$d\big((f+g)(t),(f+g)(s)\big) \le d(f(t),f(s)) + d(g(t),g(s)),$$

and so, $L(f+g) \le L(f) + L(g)$. Given $a \in T$, the metric $d_L$ on $\mathrm{Lip}(T;X)$ is defined by (cf. Smajdor and Smajdor [105]):

$$d_L(f,g) = d(f(a),g(a)) + d_\ell(f,g), \qquad f,\, g \in \mathrm{Lip}(T;X),$$

with

$$d_\ell(f,g) = \sup\big\{\, d(f(t)+g(s),g(t)+f(s))/d_1(t,s)\,;\, t,\, s \in T,\, t \neq s \,\big\}.$$

Then $d_\ell$ is a semimetric and $d_L$ is a metric on $\mathrm{Lip}(T;X)$, which are translation invariant.

The main properties of $d_\ell$ are contained in the following

**Lemma 4.4.** *Given $(X,d,+)$ and $T$ as above and $f,g \in \mathrm{Lip}(T;X)$ we have:*

(a) $|d(f(t),g(t) - d(f(s),g(s))| \le d\big(f(t)+g(s),g(t)+f(s)\big) \le d_\ell(f,g)d_1(t,s)$ *for all $t$, $s \in T$;*
(b) *if $\{f_n,g_n\}_{n=1}^\infty \subset \mathrm{Lip}(T;X)$, $d(f_n(t),f(t)) \to 0$ and $d(g_n(t),g(t)) \to 0$ as $n \to \infty$ for all $t \in T$, then $d_\ell(f,g) \le \liminf_{n\to\infty} d_\ell(f_n,g_n)$;*
(c) $|L(f) - L(g)| \le d_\ell(f,g) \le L(f) + L(g)$.

We conclude that $(\mathrm{Lip}(T;X), d_L, +)$ is a metric semigroup which, by Lemma 4.4, is complete provided $(X,d,+)$ is complete.

**4.3. Embeddings of metric semigroups.** Here we assume that $T \subset \mathbb{R}$ is bounded, $|T| = \sup T - \inf T$, $a \in T$ is given and $(X, d, +)$ is a metric semigroup. The main result is the following lemma, generalizing inequalities in (3.23), (3.24) and in Theorem 3.10:

**Lemma 4.5.** *Given $\Phi, \Psi \in \mathcal{N}$, we have the following embeddings of metric semigroups*:

(a) $\mathrm{Lip}(T;X) \subset \mathrm{GV}_\Phi(T;X) \subset \mathrm{BV}(T;X)$ *and*

$$\Delta_\Phi(f,g) \le d_\ell(f,g)/\Phi^{-1}(1/|T|),\ f,\, g \in \mathrm{Lip}(T;X), \tag{4.13}$$

$$\Delta_1(f,g) \le \omega_\Phi(|T|)\Delta_\Phi(f,g), \quad f,\, g \in \mathrm{GV}_\Phi(T;X); \tag{4.14}$$

(b) *if $\Psi \preccurlyeq \Phi$, then $\mathrm{GV}_\Phi(T;X) \subset \mathrm{GV}_\Psi(T;X)$ and there exist numbers $\kappa > 0$ and $\kappa_0 > 0$, depending only on $\Phi$, $\Psi$ and $|T|$, such that*

$$\Delta_\Psi(f,g) \le \kappa\Delta_\Phi(f,g), \quad d_\Psi(f,g) \le \kappa_0 d_\Phi(f,g), \quad f,\, g \in \mathrm{GV}_\Phi(T;X).$$

**Proof.** (a) The first embedding follows from (3.9) and (3.20), and the second — from (3.20) and (3.12). To prove (4.13), note that, by Lemma 4.4(a),

$$d(f(t)+g(s), g(t)+f(s)) \le d_\ell(f,g)|t-s|, \quad t, s \in T. \tag{4.15}$$

Set $\lambda = d_\ell(f,g)/\Phi^{-1}(1/|T|)$. If $\lambda = 0$, then the left hand side in (4.15) is zero, and so, $\Delta_\Phi(f,g) = 0$. If $\lambda > 0$, then (4.15) and the monotonicity of $\Phi$ imply, for any partition $\xi = \{t_i\}_{i=0}^m$ of $T$, that

$$\begin{aligned} W_{\Phi_\lambda}[f,g,\xi] &= \sum_{i=1}^m \Phi\biggl(\frac{d(f(t_i)+g(t_{i-1}), g(t_i)+f(t_{i-1}))}{(t_i-t_{i-1})\lambda}\biggr)(t_i-t_{i-1}) \\ &\le \sum_{i=1}^m \Phi\Bigl(d_\ell(f,g)/\lambda\Bigr)(t_i-t_{i-1}) \le \Phi\Bigl(d_\ell(f,g)/\lambda\Bigr)|T| = 1. \end{aligned}$$

Hence, $W_{\Phi_\lambda}(f,g) \le 1$, and (4.5) yields $\Delta_\Phi(f,g) \le \lambda$.

In order to prove (4.14), let us show that

$$\Phi\bigl(W_1(f,g)/|T|\bigr) \le W_\Phi(f,g)/|T|, \quad f, g \in \mathrm{GV}_\Phi(T;X). \tag{4.16}$$

In fact, using the notation $\rho(t,s)$ and $U(t,s)$ from the proof of Lemma 4.3(c) we find that this is a consequence of Jensen's inequality for sums

$$\Phi\biggl(\frac{\sum_{i=1}^m \rho(t_i,t_{i-1})}{\sum_{i=1}^m (t_i-t_{i-1})}\biggr) \le \frac{\sum_{i=1}^m U(t_i,t_{i-1})}{\sum_{i=1}^m (t_i-t_{i-1})} \le \frac{W_\Phi(f,g)}{\sum_{i=1}^m (t_i-t_{i-1})} \quad \forall\, \xi = \{t_i\}_{i=0}^m,$$

inequality $\sum_{i=1}^m (t_i - t_{i-1}) \le |T|$ and the monotonicity of $\omega_\Phi$. Set $\lambda = W_1(f,g)/\omega_\Phi(|T|)$. If $\lambda = 0$, i. e., $W_1(f,g) = \Delta_1(f,g) = 0$, then by Lemma 4.1(a) with $\Phi(\rho) = \rho$, we have $d(f(t)+g(s), g(t)+f(s)) = 0$ for all $t$, $s \in T$, and so, $W_\Phi(f,g) = 0$. If $\lambda > 0$, then (4.16) implies $W_{\Phi_\lambda}(f,g) \ge 1$, and by the assertion in the proof of Lemma 4.1(c), $\Delta_\Phi(f,g) \ge \lambda$.

(b) The first inequality can be proved along the same lines as Theorem 3.10 if we take into account the following changes: apply Lemma 4.1(a) instead of Lemma 3.9(a), replace $p_\Phi$ by $\Delta_\Phi$, $V_{\Psi_\mu}$ — by $W_{\Psi_\mu}$ and $d(f(t_i), f(t_{i-1}))$ — by $\rho(t_i, t_{i-1})$. Finally, by putting $\kappa_0 = \max\{1, 2NC\}$, we have proved the second inequality as well. □

## 5. Selections of bounded variation

Throughout the rest of the paper $\mathrm{c}(X)$ denotes the family of all nonempty compact subsets of a metric space $(X, d)$, equipped with the Hausdorff metric $D$ generated by $d$.

**Theorem 5.1** (on BV selections)**.** *Let $T \subset \mathbb{R}$, $(X, d)$ be a metric space and $F \in \mathrm{BV}(T; \mathrm{c}(X))$. Then for any $t_0 \in T$ and $x_0 \in X$ there exists a selection $f \in \mathrm{BV}(T; X)$ of $F$ such that*

$$d(x_0, f(t_0)) = \mathrm{dist}(x_0, F(t_0)) \quad \textit{and} \quad V_d(f, T) \le V_D(F, T). \tag{5.1}$$

**Proof.** 1. First, let $T$ be bounded, $T \subset [a, b]$ and $a, b \in T$. Since $F$ is of bounded variation (with respect to $D$), by Lemma 1.1(b) the set of points of discontinuity of $F$ on $T$ is at most countable. The set of points from $T$, which are isolated from the left for $T$ (i. e., points $t \in T$ such that $(t - \varepsilon, t) \cap T = \emptyset$ for some $\varepsilon > 0$), is also at most countable, since intervals of "emptiness from the left", corresponding to different points isolated from the left, are disjoint and each such interval contains a rational point. Let us denote by $S$ at most countable dense subset of $T$. Appending to $S$ the set of discontinuity points of $F$, the set of points from $T$ isolated from the left and points $a$, $t_0$ and $b$, let us denote the resulting at most countable dense subset of $T$ by $Q = \{t_i\}_{i=0}^{\infty}$, and assume that all points in $Q$ are different. Then for any $n \in \mathbb{N}$ the set $\xi_n = \{t_i\}_{i=0}^{n}$ is a partition of $T$; ordering the points in $\xi_n$ in ascending order and denoting them by $\xi_n = \{t_i^n\}_{i=0}^{n}$, we have:

$$a = t_0^n < t_1^n < \cdots < t_{n-1}^n < t_n^n = b, \tag{5.2}$$

$$\exists\, k_0(n) \in \{0, 1, \ldots, n\} \text{ such that } t_0 = t_{k_0(n)}^n,$$

$$\forall\, t \in Q \ \exists\, n_0 = n_0(t) \in \mathbb{N} \text{ such that } t \in \bigcap_{n=n_0}^{\infty} \xi_n. \tag{5.3}$$

By the compactness of $F(t_0)$, choose an element $y_0 \in F(t_0)$ such that $d(x_0, y_0) = \mathrm{dist}(x_0, F(t_0))$. We define elements $x_i^n$ from $F(t_i^n)$, where $n \in \mathbb{N}$ and $i = 0, 1, \ldots, n$, inductively as follows. Let $n \in \mathbb{N}$, and suppose first that $a < t_0 < b$, so that $k_0(n) \in \{1, \ldots, n-1\}$.

(a) Set $x^n_{k_0(n)} = y_0$.
(b) If $i \in \{1, \dots, k_0(n)\}$ and $x^n_i \in F(t^n_i)$ is already chosen, pick $x^n_{i-1} \in F(t^n_{i-1})$ such that $d(x^n_i, x^n_{i-1}) = \operatorname{dist}(x^n_i, F(t^n_{i-1}))$.
(c) If $i \in \{k_0(n)+1, \dots, n\}$ and $x^n_{i-1} \in F(t^n_{i-1})$ is already chosen, pick an element $x^n_i \in F(t^n_i)$ such that $d(x^n_{i-1}, x^n_i) = \operatorname{dist}(x^n_{i-1}, F(t^n_i))$.

If $t_0 = a$, so that $k_0(n) = 0$, we define $x^n_i \in F(t^n_i)$, following (a) and (c), and if $t_0 = b$, i. e., $k_0(n) = n$, we define $x^n_i \in F(t^n_i)$ in accordance with (a) and (b).

Given $n \in \mathbb{N}$, we define a mapping $f_n : T \to X$ as follows:

$$f_n(t) = \begin{cases} x^n_i & \text{if } t = t^n_i,\ i = 0, 1, \dots, n, \\ x^n_{i-1} & \text{if } T \cap (t^n_{i-1}, t^n_i) \neq \emptyset \text{ and} \\ & t \in T \cap (t^n_{i-1}, t^n_i),\ i = 1, \dots, n. \end{cases} \tag{5.4}$$

Note that $f_n(t_0) = f_n(t^n_{k_0(n)}) = x^n_{k_0(n)} = y_0$, $n \in \mathbb{N}$, and that, by the additivity of $V(\cdot, \cdot)$, definitions (b) and (c) and definition of the Hausdorff metric $D$,

$$\begin{aligned} V_d(f_n, T) &= \sum_{i=1}^n V_d(f_n, T \cap [t^n_{i-1}, t^n_i]) = \sum_{i=1}^n d(x^n_i, x^n_{i-1}) \\ &\le \sum_{i=1}^n D(F(t^n_i), F(t^n_{i-1})) \le V_D(F, T), \quad n \in \mathbb{N}. \end{aligned} \tag{5.5}$$

In order to apply the generalized Helly selection principle (Theorem 1.3 with $\Phi(\rho) = \rho$), we have to verify that the sequence $\{f_n(t)\}_{n=1}^\infty$ is precompact in $X$ for all $t \in T$. If $t \in Q$, by (5.3) there exists $n_0(t) \in \mathbb{N}$ such that $t \in \xi_n$ for all $n \ge n_0(t)$, and so, by virtue of (5.4), (a), (b) and (c) we have:

$$f_n(t) \in F(t) \quad \text{for all} \quad n \ge n_0(t), \tag{5.6}$$

and it suffices to take into account the compactness of $F(t)$.

Now, if $t \in T \setminus Q$, then $t$ is a point of continuity of $F$, which is a limit point from the left for $T$. So, there exists a sequence of points $\tau_k \in T$, $\tau_k < t$, $k \in \mathbb{N}$, such that $\lim_{k\to\infty} \tau_k = t$. By the density of $S$ in $T$, for any $k \in \mathbb{N}$ there exists $s_k \in S$ such that $|s_k - \tau_k| < t - \tau_k$, and so, $s_k < t$ and $s_k \to t$ as $k \to \infty$. From (5.3) for $k \in \mathbb{N}$ we find a number $n_k \in \mathbb{N}$ (also depending on $t$) such that $s_k \in \xi_{n_k}$ and, therefore, $s_k = t^{n_k}_{j_k}$ for some $j_k \in \{0, 1, \dots, n_k - 1\}$. Thanks to property (5.3), without loss of generality we may assume that the sequence $\{n_k\}_{k=1}^\infty$ is strictly increasing. It follows from (5.2) that there exists a unique number $i_k \in \{j_k, \dots, n_k - 1\}$ such that

$$s_k = t^{n_k}_{j_k} \le t^{n_k}_{i_k} < t < t^{n_k}_{i_k+1}, \qquad k \in \mathbb{N}. \tag{5.7}$$

By definition (5.4), we have $f_{n_k}(t) = x^{n_k}_{i_k} \in F(t^{n_k}_{i_k})$, $k \in \mathbb{N}$. Pick, for each $k \in \mathbb{N}$, an element $x^k_t \in F(t)$ such that $d(x^{n_k}_{i_k}, x^k_t) = \operatorname{dist}(x^{n_k}_{i_k}, F(t))$. Then from the definition of $D$, continuity of $F$ at $t$ and (5.7) we find

$$d(f_{n_k}(t), x_t^k) \le D(F(t_{i_k}^{n_k}), F(t)) \to 0 \quad \text{as} \quad k \to \infty.$$

Since the set $F(t)$ is compact and $\{x_t^k\}_{k=1}^\infty \subset F(t)$, there exists a subsequence of $\{x_t^k\}_{k=1}^\infty$ (which we will denote by the same symbol), which converges in $X$ to an element $x_t \in F(t)$ as $k \to \infty$, so that

$$d(f_{n_k}(t), x_t) \ \le \ d(f_{n_k}(t), x_t^k) + d(x_t^k, x_t) \ \to \ 0 \quad \text{as} \quad k \ \to \ \infty. \tag{5.8}$$

This proves the precompactness of the sequence $\{f_n(t)\}_{n=1}^\infty$ in $X$.

By Theorem 1.3, the family $\mathcal{F} = \{f_n\}_{n=1}^\infty$ contains a subsequence, denoted with no loss of generality again by $\{f_{n_k}\}_{k=1}^\infty$, which converges in $X$ pointwise on $T$ to a mapping $f \in \mathrm{BV}(T; X)$. Clearly, $f(t_0) = y_0$, and so, $d(x_0, f(t_0)) = \mathrm{dist}(x_0, F(t_0))$. The inclusion $f(t) \in F(t)$ for all $t \in T$ is a consequence of the closedness of $F(t)$, (5.6) and (5.8). The lower semicontinuity of $V(\cdot, \cdot)$ and (5.5) ensure that

$$V_d(f, T) \le \liminf_{k\to\infty} V_d(f_{n_k}, T) \le V_D(F, T).$$

*Remark.* We note that if the "initial point" $x_0$ is in $F(t_0)$, the desired selection $f$ of $F$ satisfies the condition $f(t_0) = x_0$.

2. Now, if the set $T$ is arbitrary, we set $a = \inf T \in \mathbb{R} \cup \{-\infty\}$ and $b = \sup T \in \mathbb{R} \cup \{\infty\}$. By step 1 it remains to consider the cases when $T$ is unbounded or $a \notin T$ or $b \notin T$. Let us suppose that $a \notin T$ and $b \notin T$ (the other possibilities may be combined from this case and step 1 by applying properties 1), 4a) and 4b) from Section 1). Choose an increasing sequence $\{t_n\}_{n\in\mathbb{Z}} \subset T$ such that $t_n \to b$ and $t_{-n} \to a$ as $n \to \infty$. Setting $T_n = T \cap [t_n, t_{n+1}]$ for $n \in \mathbb{Z}$ and applying step 1 to the set $T_0 = T \cap [t_0, t_1]$, we find a selection $f_0 \in \mathrm{BV}(T_0; X)$ of $F$ (more precisely, of the restriction $F|_{T_0}$ of $F$ to $T_0$) such that $d(x_0, f_0(t_0)) = \mathrm{dist}(x_0, F(t_0))$ and $V_d(f_0, T_0) \le V_D(F, T_0)$. "Moving along the sets $T_n$ to the right" of point $t_1$, we successively apply the result of step 1: choose a selection $f_1 \in \mathrm{BV}(T_1; X)$ of $F$ on $T_1$ such that $f_1(t_1) = f_0(t_1) \in F(t_1)$ and $V_d(f_1, T_1) \le V_D(F, T_1)$, and, inductively, if a selection $f_{n-1}$ of $F$ on the set $T_{n-1}$ is already chosen, $n \ge 2$, we pick a selection $f_n \in \mathrm{BV}(T_n; X)$ of $F$ on $T_n$ such that

$$f_n(t_n) = f_{n-1}(t_n) \quad \text{and} \quad V_d(f_n, T_n) \le V_D(F, T_n). \tag{5.9}$$

In a similar manner we "move along the sets $T_n$ to the left" of $t_0$. Then for each $n \in \mathbb{Z}$ there exists a selection $f_n \in \mathrm{BV}(T_n; X)$ of $F$ on $T_n$, for which the relations (5.9) hold. Given $t \in T$, so that $t \in T_n$ for some $n \in \mathbb{Z}$, we set $f(t) = f_n(t)$. The mapping $f : T \to X$ is a selection of $F$ on $T$, $d(x_0, f(t_0)) = \mathrm{dist}(x_0, F(t_0))$, and by virtue of properties 4c) and 1) from

Section 1 we have:

$$V_d(f,T)=\lim_{k\to\infty}V_d(f,T\cap[t_{-k},t_k])=\lim_{k\to\infty}\sum_{n=-k}^{k-1}V_d(f_n,T_n)$$

$$\le\lim_{k\to\infty}\sum_{n=-k}^{k-1}V_D(F,T_n)=\lim_{k\to\infty}V_D(F,T\cap[t_{-k},t_k])=V_D(F,T).$$

□

**Example 5.2.** The inequality $V_d(f,T)\le V_D(F,T)$ in Theorem 5.1 may be violated if, at least at one point $t\in T$, the value $F(t)$ is only closed and bounded in $X$, but not compact. To see this, let $X=\ell^1(\mathbb{N})$ be the space of summable sequences from Example 1.4. We set $A=\{c_ne_n\}_{n=1}^k$ and $B=\{c_ne_n\}_{n=k+1}^\infty$ where $k\in\mathbb{N}$, $k\ge 2$ is fixed, and $\{c_n\}_{n=1}^\infty\subset\mathbb{R}$ is any sequence satisfying the following conditions:

$$\{\,|c_n|\,\}_{n=1}^\infty \text{ is strictly decreasing} \quad\text{and}\quad \inf_{n\ge k+1}|c_n|\;>\;0; \tag{5.10}$$

here the first condition guarantees, in particular, that $B$ is bounded in $\ell^1(\mathbb{N})$ and the second one — that $B$ is closed. Clearly, $A$ is compact while $B$ is not. Let us define $F:[0,1]\rightrightarrows\ell^1(\mathbb{N})$ by $F(t)=A$ if $0\le t<1$ and $F(1)=B$. We have $V_D(F,[0,1])=D(A,B)$. In order to find $D(A,B)$ we note that, by (5.10),

$$\mathrm{e}(A,B)=\sup_{1\le i\le k}\Big(|c_i|+\inf_{n\ge k+1}|c_n|\Big)=\sup_{1\le i\le k}|c_i|+\inf_{n\ge k+1}|c_n|=|c_1|+\inf_{n\ge k+1}|c_n|,$$

$$\mathrm{e}(B,A)=\sup_{i\ge k+1}\Big(|c_i|+\inf_{1\le n\le k}|c_n|\Big)=\sup_{i\ge k+1}|c_i|+\inf_{1\le n\le k}|c_n|=|c_{k+1}|+|c_k|.$$

Suppose also that $\{c_n\}_{n=1}^\infty$ satisfies the third condition:

$$|c_1|+\inf_{n\ge k+1}|c_n|\ge|c_{k+1}|+|c_k|. \tag{5.11}$$

Then $D(A,B)=|c_1|+\inf_{n\ge k+1}|c_n|$. Now if $f:[0,1]\to\ell^1(\mathbb{N})$ is any selection of $F$ such that $f(0)=c_1e_1$, then $f(1)=c_je_j$ for some $j\ge k+1$, and so,

$$V_{\|\cdot\|}(f,[0,1])\ge\|f(0)-f(1)\|=|c_1|+|c_j|>|c_1|+\inf_{n\ge k+1}|c_n|=V_D(F,[0,1]).$$

Simple examples of sequences $\{c_n\}_{n=1}^\infty$ satisfying all three conditions (5.10) and (5.11) are $c_n=\alpha\,(n+1)/n$ with $\alpha\ne0$, $n\in\mathbb{N}$. Let us note that the example presented above is more subtle than Example 2 from [6] where all values $F(t)$ are only closed and bounded.

**Remark 5.3.** Multifunctions of bounded variation with noncompact values (such as $\mathcal{F}$ in Example 1.4 or $F$ in Example 5.2) may admit selections of bounded variation as the following observation shows. Suppose that conditions of Theorem 5.1 are satisfied except that the images of $F$ are not necessarily compact, but assume that

$$\forall\, t\in T\ \ \exists\, F_0(t)\in \mathrm{c}(X) \text{ such that } F_0(t)\subset F(t), \text{ and } V_D(F_0,T)<\infty$$

(in particular, one can assume that $V_D(F_0,T)\le V_D(F,T)$). By Theorem 5.1, $F_0$ admits a selection of bounded variation, which is at the same time a selection of $F$.

**Example 5.4.** Here we present an example showing that if $d$ is only a *semimetric* on $X$ (i.e., $d(x,y)=0$ does not necessarily imply $x=y$ in $X$), then there exists a multifunction of bounded variation with compact values in $X$ whose all selections are of unbounded variation. Recall that the *Gromov-Hausdorff distance* $d_{GH}(K',K'')$ between two nonempty compact metric spaces $K'$ and $K''$ ([2], [52], [88]) is the infimum of all $\varepsilon>0$ such that there exist a compact metric space $K$ and isometric embeddings $j':K'\to K$ and $j'':K''\to K$ such that $D\big(j'(K'),j''(K'')\big)<\varepsilon$, where $D$ is the Hausdorff metric on $K$. It is known (Gromov [52]) that $d_{GH}$ is a metric in the isometry class of all nonempty compact metric spaces and $d_{GH}(K',K'')\le(1/2)\max\big\{\mathrm{diam}(K'),\mathrm{diam}(K'')\big\}$. However, $d_{GH}$ is only a semimetric on the family of all nonempty compact metric spaces. Define $F:[0,1]\to\mathrm{c}(\mathbb{R})$ by $F(0)=[0,1]$ and $F(t)=[2n-1,2n]$ if $1/(n+1)<t\le 1/n$, $n\in\mathbb{N}$. Clearly, $d_{GH}\big(F(t),F(s)\big)=0$ for all $t,s\in[0,1]$, and so, $F$ is of bounded variation with respect to $d_{GH}$. On the other hand, it follows from the definition of $F$ that if $f:[0,1]\to\mathbb{R}$ is a selection of $F$, then $V(f,[0,1])=\infty$.

**Remark 5.5.** It is interesting to note (cf. [27, Lemma 11]) that for $F\in\mathrm{BV}(T;\mathrm{c}(X))$ the total image $F(T)=\bigcup_{t\in T}F(t)$ is a totally bounded and separable subset of $X$ and if, moreover, $X$ is complete, then $F(T)$ is precompact (this property is well known for single-valued mappings, e.g., [14, Proposition 2.1]).

As a corollary of Theorem 5.1 and, simultaneously, a motivation why the set $T$ should be arbitrary in $\mathbb{R}$ we get

**Theorem 5.6.** *Let $T\subset\mathbb{R}$ be density-open, $(X,d)$ be a complete metric space, $F\in\mathrm{BV}_{\mathrm{ess}}(T;\mathrm{c}(X))$, $t_0\in T$ and $x_0\in X$. Then there exists a selection $f\in\mathrm{BV}_{\mathrm{ess}}(T;X)$ of $F$ such that $V_{d,\mathrm{ess}}(f,T)\le V_{D,\mathrm{ess}}(F,T)$ and $d(x_0,f(t_0))=\mathrm{dist}(x_0,F(t_0))$.*

**Proof.** Since $V_{D,\mathrm{ess}}(F,T)<\infty$ and $(\mathrm{c}(X),D)$ is a complete metric space (cf. [12, Theorem II-9]), by Theorem 2.2(a) there exists a set $T_0\subset T$ of Lebesgue measure zero such that $F|_{T\setminus T_0}\in\mathrm{BV}(T\setminus T_0;\mathrm{c}(X))$ and $V_D(F,T\setminus T_0)=V_{D,\mathrm{ess}}(F,T)$. Choose an element $y_0\in F(t_0)$ such that $d(x_0,y_0)=\mathrm{dist}(x_0,F(t_0))$, and set $T_1=T_0\cup\{t_0\}$. Theorem 5.1 implies the existence of a selection $f\in\mathrm{BV}(T\setminus T_1;X)$ of $F|_{T\setminus T_1}$, for which $V_d(f,T\setminus T_1)\le V_D(F,T\setminus T_1)$. Let us define $f$ on the set $T_1$ as follows: $f(t_0)=y_0$ and $f(t)=x_t$ if $t\in T_1\setminus\{t_0\}$, where $x_t$ is an arbitrary fixed element of $F(t)$. Clearly, $f(t)\in F(t)$ for all $t\in T$ and

$$V_d(f,T\setminus T_1)\le V_D(F,T\setminus T_1)\le V_D(F,T\setminus T_0)=V_{D,\mathrm{ess}}(F,T).$$

Applying Theorem 2.1, we conclude that $V_{d,\mathrm{ess}}(f,T)\le V_{D,\mathrm{ess}}(F,T)$. □

## 6. More regular selections

**Theorem 6.1** (more regular selections). *Let $T\subset\mathbb{R}$, $(X,d)$ be a metric space, $F:T\rightrightarrows X$ be a multifunction with compact values, $t_0\in T$ and $x_0\in X$. Then*:

(a) *if $F\in\mathrm{Lip}(T;\mathrm{c}(X))$, it admits a selection $f\in\mathrm{Lip}(T;X)$ satisfying conditions* (5.1) *and $L_d(f,T)\le L_D(F,T)$;*

(b) *if $F\in\mathrm{BV}(T;\mathrm{c}(X))$ is also continuous, it admits a continuous selection $f\in\mathrm{BV}(T;X)$ satisfying conditions* (5.1);

(c) *if $T$ is compact and $F:T\to\mathrm{c}(X)$ is $\delta(\cdot)$-absolutely continuous, then there exists a $\delta(\cdot)$-absolutely continuous selection $f:T\to X$ of $F$ satisfying conditions* (5.1);

(d) *if $\Phi\in\mathcal{N}$ and $F\in\mathrm{BV}_\Phi(T;\mathrm{c}(X))$, then there exists $f\in\mathrm{BV}_\Phi(T;X)$, a selection of $F$, satisfying conditions* (5.1) *and $V_{\Phi,d}(f,T)\le V_{\Phi,D}(F,T)$;*

(e) *if $\Phi\in\mathcal{N}$ and $F\in\mathrm{GV}_\Phi(T;\mathrm{c}(X))$, then there exists $f\in\mathrm{GV}_\Phi(T;X)$, a selection of $F$, satisfying* (5.1) *and $p_{\Phi,d}(f,T)\le p_{\Phi,D}(F,T)$.*

**Proof.** (a) 1. Suppose first that $T\subset[a,b]$ and $a$, $b\in T$. Since $F$ is Lipschitzian and $T$ is bounded, $F$ is of bounded variation on $T$, so let $f\in\mathrm{BV}(T;X)$ be a selection of $F$ constructed in step 1 of the proof of Theorem 5.1, and assume that the sequence $\{f_{n_k}\}_{k=1}^\infty$ converges to $f$ pointwise on $T$ as $k\to\infty$. Let us show that $f\in\mathrm{Lip}(T;X)$ and $L_d(f,T)\le L_D(F,T)$. The following three possibilities hold for points $t$, $s\in T$, $t<s$: (i) $t$, $s\in Q$; (ii) $t$, $s\in T\setminus Q$; (iii) $t\in T\setminus Q$, $s\in Q$, or $s\in T\setminus Q$, $t\in Q$.

In case (i), by (5.3) there exists a number $n_0$ depending on $t$ and $s$ such that for those $k\in\mathbb{N}$, for which $n_k\ge n_0$, there exist numbers $i_k$, $j_k\in\{0,1,\dots,n_k\}$, $i_k<j_k$, such that $t=t_{i_k}^{n_k}$ and $s=t_{j_k}^{n_k}$. Then by definition (5.4) we have: $f_{n_k}(t)=x_{i_k}^{n_k}\in F(t_{i_k}^{n_k})$ and $f_{n_k}(s)=x_{j_k}^{n_k}\in F(t_{j_k}^{n_k})$.

From properties (b) and (c) in the proof of Theorem 5.1 we find that, given $i \in \{1, \dots, n_k\}$,

$$d(x_i^{n_k}, x_{i-1}^{n_k}) \le D(F(t_i^{n_k}), F(t_{i-1}^{n_k})) \le L_D(F,T)(t_i^{n_k} - t_{i-1}^{n_k}),$$

and so,

$$\begin{aligned} d(f_{n_k}(t), f_{n_k}(s)) = d(x_{i_k}^{n_k}, x_{j_k}^{n_k}) &\le \sum_{i=i_k+1}^{j_k} d(x_i^{n_k}, x_{i-1}^{n_k}) \qquad (6.1) \\ &\le \sum_{i=i_k+1}^{j_k} L_D(F,T)(t_i^{n_k} - t_{i-1}^{n_k}) = L_D(F,T)(t_{j_k}^{n_k} - t_{i_k}^{n_k}). \end{aligned}$$

Since $f_{n_k}(t) \to f(t)$ and $f_{n_k}(s) \to f(s)$ as $k \to \infty$ and $t_{j_k}^{n_k} = s$ and $t_{i_k}^{n_k} = t$, then $d(f(t), f(s)) \le L_D(F,T)|t - s|$.

If case (ii) holds, then, as is shown in (5.7), for each $k \in \mathbb{N}$ there exist numbers $i_k$, $j_k \in \{0, 1, \dots, n_k - 1\}$ such that $t_{i_k}^{n_k} < t < t_{i_k+1}^{n_k}$ and $t_{j_k}^{n_k} < s < t_{j_k+1}^{n_k}$, and so, (5.4) implies $f_{n_k}(t) = x_{i_k}^{n_k}$ and $f_{n_k}(s) = x_{j_k}^{n_k}$. Since $t < s$, $t_{i_k}^{n_k} \to t - 0$ and $t_{j_k}^{n_k} \to s - 0$ as $k \to \infty$, for sufficiently large $k$ we have $i_k < j_k$, and relations (6.1) hold. As in case (i) it remains to note that $f_{n_k}(t) \to f(t)$ and $f_{n_k}(s) \to f(s)$ as $k \to \infty$.

Case (iii) is treated similarly to cases (i) and (ii).

2. For arbitrary $T$ we argue as in step 2 of the proof of Theorem 5.1, replacing BV there by Lip and $V_d$ — by $L_d$. Having defined the mapping $f : T \to X$ as in that proof, we show that it is Lipschitzian with $L_d(f,T) \le L_D(F,T)$. In fact, given $t, s \in T$ with $t < s$ we find $n$, $m \in \mathbb{Z}$, $n + 1 \le m$ (with no loss of generality), such that $t \in T_n$ and $s \in T_m$, and so by the construction we have:

$$\begin{aligned} d(f(t), f(s)) &\le d(f_n(t), f_n(t_{n+1})) + \sum_{k=n+1}^{m-1} d(f_k(t_k), f_k(t_{k+1})) + d(f_m(t_m), f_m(s)) \\ &\le L_D(F,T)\Big((t_{n+1} - t) + \sum_{k=n+1}^{m-1} (t_{k+1} - t_k) + (s - t_m)\Big) \\ &= L_D(F,T)(s - t). \end{aligned}$$

3. Let us prove that $V_d(f,T) \le V_D(F,T)$ (where, in general, $V_D(F,T) \le \infty$). Indeed, in step (a) 1. it was shown that the second condition in (5.1) holds (since $f$ was a selection of $F$ of bounded variation), and so, in addition, conditions (5.9) of step (a) 2. are satisfied. Then calculations from the end of step 2 of the proof of Theorem 5.1 show that the desired selection $f$ of $F$ is subject to conditions (5.1).

(b), (c) Suppose that $F$ satisfies (b) or (c). Then the nondecreasing (bounded) function $\varphi(t) = V_D(F, T \cap (-\infty, t])$, $t \in T$, is, by Lemma 1.1, continuous on $T$ if (b) is satisfied, or is, by Lemma1.2, $\delta(\cdot)$-absolutely continuous on $T$ if (c) is satisfied, and in both cases the equality holds: $\operatorname{osc}(\varphi, T) = V(\varphi, T) = V_D(F, T)$. By Lemma 1.2, we have the decomposition $F = G \circ \varphi$ on $T$, where $G \in \operatorname{Lip}(J; \mathrm{c}(X))$ with $J = \varphi(T)$ and $L_D(G, J) \le 1$. If $\tau_0 = \varphi(t_0)$, then $G(\tau_0) = F(t_0)$, and by Theorem 6.1(a) there exists a selection $g \in \operatorname{Lip}(J; X)$ of $G$ on $J$ such that $d(x_0, g(\tau_0)) = \operatorname{dist}(x_0, G(\tau_0))$ and $L_d(g, J) \le L_D(G, J) \le 1$. Then $f = g \circ \varphi : T \to X$ is the desired continuous selection of $F$ of bounded variation. In fact, $f$ is continuous as the composition of two continuous mappings if (b) is satisfied, and $f$ is $\delta(\cdot)$-absolutely continuous if (c) is satisfied, since $L_d(g, J) \le 1$; also, $f$ is a selection of $F$:

$$f(t) = g(\varphi(t)) \in G(\varphi(t)) = F(t), \qquad t \in T, \tag{6.2}$$

and $f(t_0) = g(\varphi(t_0)) = g(\tau_0)$, and so, $d(x_0, f(t_0)) = \operatorname{dist}(x_0, F(t_0))$. Again, taking into account that $L_d(g, J) \le 1$, we find

$$V_d(f, T) \le L_d(g, J)\operatorname{osc}(\varphi, T) \le \operatorname{osc}(\varphi, T) = V_D(F, T). \tag{6.3}$$

(d) 1. First assume that $T \subset [a, b]$ with $a, b \in T$. By the second embedding in (3.8) with bounded $T$, the function $\varphi(t) = V_D(F, T \cap (-\infty, t])$, $t \in T$, is well defined, bounded and nondecreasing. According to Lemma 3.2 $V_\Phi(\varphi, T) = V_{\Phi,D}(F, T)$ and there exists a mapping $G \in \operatorname{Lip}(J; \mathrm{c}(X))$ with $J = \varphi(T)$ and $L_D(G, J) \le 1$ such that $F = G \circ \varphi$ on $T$. Setting $\tau_0 = \varphi(t_0)$, we have $G(\tau_0) = F(t_0)$, and by Theorem 6.1(a) there exists a selection $g \in \operatorname{Lip}(J; X)$ of $G$ on $J$ such that $d(x_0, g(\tau_0)) = \operatorname{dist}(x_0, G(\tau_0))$ and $L_d(g, J) \le L_D(G, J) \le 1$. Then $f = g \circ \varphi$ is the desired selection of $F$ on $T$: in fact, by Lemma 3.2 (see sufficiency) $V_{\Phi,d}(f, T) \le V_\Phi(\varphi, T) = V_{\Phi,D}(F, T)$, $f$ is a selection of $F$ (see (6.2)) and conditions (5.1) are satisfied (see (6.3)).

2. In the case of arbitrary $T$ we argue as in step 2 of the proof of Theorem 5.1, replacing BV there by $\mathrm{BV}_\Phi$, $V$ — by $V_\Phi$ and applying Lemma 3.1 instead of properties 1)–4) from Section 1. By doing this, we have proved (d) except for the second condition in (5.1). But in step (d) 1. it was shown that if $T$ is bounded, the second condition in (5.1) holds, and so, in addition, conditions (5.9) are satisfied. Now the calculations from the end of step 2 of the proof of Theorem 5.1 imply that the established selection $f$ satisfies inequality $V_d(f, T) \le V_D(F, T)$ (with the latter variation, possibly, infinite).

(e) Set $\lambda = p_{\Phi,D}(F, T)$. If $\lambda = 0$, $F$ is constant by Lemma 3.9(a), and so, it admits a constant selection satisfying (5.1). If $\lambda > 0$, then $V_{\Phi_\lambda,D}(F, T) \le 1$ by Lemma 3.9(b), and so, Theorem 6.1(d) implies the existence of a selection $f \in \mathrm{BV}_{\Phi_\lambda,d}(T; X)$ of $F$ satisfying conditions (5.1) and inequality $V_{\Phi_\lambda,d}(f, T) \le V_{\Phi_\lambda,D}(F, T) \le 1$. Then $f \in \mathrm{GV}_\Phi(T; X)$ and $p_{\Phi,d}(f, T) \le \lambda$ by definition (3.22), which ends the proof of Theorem 6.1. □

In Theorem 6.1(a) we have seen that a compact-valued Lipschitzian multifunction admits a Lipschitzian selection. Contrary to this, the following example shows that compact-valued Hölder continuous multifunctions of any exponent $\gamma \in (0,1)$ need not have even continuous selections.

**Example 6.2.** Let $B=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2=1\}$ be the unit circumference and

$$A(t) = \{(x,y) \in \mathbb{R}^2 \mid x = \cos\theta,\ y = \sin\theta,\ \alpha(t) < \theta < \alpha(t) + 2\beta(t)\},$$

where $\alpha(t) = 1/|t|$ and $\beta(t) = e^{-1/|t|}$, $t \in \mathbb{R} \setminus \{0\}$. Define a multifunction $F : [-1,1] \to \mathrm{c}(\mathbb{R}^2)$ as follows: $F(t) = B \setminus A(t)$ if $t \neq 0$ and $F(0) = B$. It is shown in [31, Proposition 8.2] that, given $\gamma \in (0,1)$, there exists a $C(\gamma) \in \mathbb{R}^+$ such that $D(F(t), F(s)) \le C(\gamma)|t-s|^\gamma$ for all $t, s \in [-1,1]$. Moreover, it is clear that if $q > 1$, $\mathrm{Var}_\Phi(F,[-1,1]) \le 2\big(C(1/q)\big)^q$ if $\Phi(\rho) = \rho^q$, $\rho \in \mathbb{R}^+$.

On the other hand, $F(t)$ for $t \neq 0$ is the unit circumference in $\mathbb{R}^2$ from which a section from the angle $\alpha(t)$ to the angle $\alpha(t) + 2\beta(t)$ is removed. As $t$ gets smaller, the arclength of the hole decreases while the initial angle increases as $1/|t|$, i. e., the hole spins around the origin with increasing angular speed. Any continuous selection $f(t) = (x(t), y(t))$ defined on $[-1, 0\,[$ or on $]\,0, 1\,]$ cannot be continuously extended to the whole interval $[-1, 1\,]$, for the hole in the circumference would force this selection to rotate around the origin with some angle between $\alpha(t) + 2\beta(t)$ and $\alpha(t) + 2\pi$, and the limits $\lim_{t\to\pm 0} f(t)$ cannot exist. Thus, $F$ admits no continuous selections; moreover, for $\Phi$ as above, any selection $f$ satisfies $\mathrm{Var}_\Phi(f,[-1,1]) = \infty$ for any $q > 1$, since mappings $f$ with bounded $\Phi$-th variation have only simple discontinuities (cf. [31, 4.1, 4.2]).

**Remark 6.3.** Let $\Phi \in \mathcal{N}$, $\xi = \{t_i\}_{i=0}^n$ be a partition of the interval $[a,b]$ with $t_0 = a$ and $t_n = b$, $\{x_i\}_{i=0}^n \subset X$, $f(t) = x_{i-1}$ if $t_{i-1} \le t < t_i$, $i = 1, \dots, n$, and $f(b) = x_n$; then there exists a subpartition $\{t_i'\}_{i=0}^{m'} \subset \xi$ (in general, proper inclusion!) such that $\mathrm{Var}_\Phi(f,[a,b])$ is equal to $\sum_{i=0}^{m'} \Phi\big(d((f(t_i'), f(t_{i-1}'))\big)$ (cf. (5.5)). Taking this into account and that the $\Phi$-th variation $\mathrm{Var}_\Phi$ is semi-additive only (property $1_\Phi$) in Section 1), we see that, in order to obtain the existence of selections of more general bounded variations (as $\mathrm{Var}_\Phi$), it is natural to require $\Phi$ to satisfy the following condition of *generalized subadditivity*: there exists a number $C \in \mathbb{R}^+$ such that for all $n \in \mathbb{N}$ and all $\{\rho_i\}_{i=1}^n \subset \mathbb{R}^+$ we have: $\Phi(\rho_1 + \dots + \rho_n) \le C\big(\Phi(\rho_1) + \dots + \Phi(\rho_n)\big)$. However, it was proved in [37, 3.3] that in this case there exist $C_1 > 0$ and $C_2 > 0$ such that $C_1\rho \le \Phi(\rho) \le C_2\rho$ for all $\rho \in \mathbb{R}^+$. This means that if $f : [a,b] \to X$, then $\mathrm{Var}_\Phi(f,[a,b])$ and $V(f,[a,b])$ are finite or not simultaneously. Thus, the requirement for multifunctions to be of bounded *Jordan* variation is the best possible in order to admit selections "preserving" the (type of) variation.

**Remark 6.4.** Certain extensions of Theorem 6.1(a) are known for a compact-valued Carathéodory type multifunction of two variables which is measurable or continuous in the first variable and Lipschitzian or of bounded variation in the second variable (see [27], [63], [104]).

## 7. Representations of set-valued mappings

Let $T \subset \mathbb{R}$ and $(X, d)$ be a metric space. A family of mappings $\mathcal{G} \subset \mathrm{Lip}(T; X)$ is said to be *equi-Lipschitzian* if there exists $C \in \mathbb{R}^+$ such that $\sup_{g \in \mathcal{G}} L_d(g, T) \le C$. We say that a family $\mathcal{F} \subset \mathrm{BV}(T; X)$ is *of equi-bounded variation* if $\sup_\xi \sum_{i=1}^m \sup_{f \in \mathcal{F}} d\big(f(t_i), f(t_{i-1})\big) \le C$ for some number $C \ge 0$, where the supremum $\sup_\xi$ is taken over all partition $\xi = \{t_i\}_{i=0}^m$ ($m \in \mathbb{N}$) of $T$; similarly, a family $\mathcal{F} \subset \mathrm{GV}_\Phi(T; X)$ is said to be *of equi-bounded generalized* $\Phi$*-variation* (with $\Phi \in \mathcal{N}$) if there exist constants $\lambda > 0$ and $C > 0$ such that

$$\sup_\xi \sum_{i=1}^m \Phi\biggl(\frac{\sup_{f \in \mathcal{F}} d(f(t_i), f(t_{i-1}))}{(t_i - t_{i-1})\lambda}\biggr)(t_i - t_{i-1}) \le C.$$

A family $\mathcal{F} \subset \mathrm{AC}(T; X)$ is said to be *equi-absolutely continuous* if the function $\delta(\cdot)$ from the definition of absolute continuity of mappings can be chosen to be independent of $f \in \mathcal{F}$.

The following theorem is a counterpart for Lipschitzian multifunctions of the Castaing representation [11] established for measurable set-valued mappings.

**Theorem 7.1.** *Given a multifunction $G : T \rightrightarrows X$ with compact images, we have*: $G \in \mathrm{Lip}(T; \mathrm{c}(X))$ *if and only if there exists a pointwise precompact equi-Lipschitzian sequence* $\{g_n\}_{n=1}^\infty \subset \mathrm{Lip}(T; X)$ (*of selections of* $G$) *such that*

$$G(t) = \overline{\{g_n(t)\}_{n=1}^\infty} \quad \textit{for all} \quad t \in T.$$

**Proof.** *Necessity.* Let $G \in \mathrm{Lip}(T; \mathrm{c}(X))$. Set

$$\mathcal{S}(G) = \{g \in \mathrm{Lip}(T; X) \mid L_d(g, T) \le L_D(G, T) \text{ and } g(t) \in G(t)\ \forall\, t \in T\}.$$

By Theorem 6.1(a) $\mathcal{S}(G) \ne \emptyset$, while by Arzelà-Ascoli's Theorem the set $\mathcal{S}(G)$ is totally bounded (if in addition $X$ were complete, then $\mathcal{S}(G)$ would be precompact), and hence, $\mathcal{S}(G)$ is separable. Let $\{g_n\}_{n=1}^\infty \subset \mathcal{S}(G)$ be at most countable dense subset of $\mathcal{S}(G)$. We show that $G(t) = \overline{\{g_n(t)\}_{n=1}^\infty}$, $t \in T$. In fact, if $t \in T$ and $x \in G(t)$, then by Theorem 6.1(a) there exists $g \in \mathcal{S}(G)$ such that $x = g(t)$, but due to the density of $\{g_n\}_{n=1}^\infty$ in $\mathcal{S}(G)$ there exists a subsequence $\{g_{n_k}\}_{k=1}^\infty$ of $\{g_n\}_{n=1}^\infty$ such that $g_{n_k}$ converges to $g$ uniformly on $T$ and, in particular, $g_{n_k}(t) \to g(t) = x$ as $k \to \infty$. So, $x \in \overline{\{g_n(t)\}_{n=1}^\infty}$, and the inclusion $\subset$ is established. Conversely, if $x \in \overline{\{g_n(t)\}_{n=1}^{\infty}}$, then some subsequence $\{g_{n_k}(t)\}_{k=1}^{\infty}$ of $\{g_n(t)\}_{n=1}^{\infty}$ converges to $x$ as $k \to \infty$, but by the construction all $g_{n_k}(t)$ belong to $G(t)$ and $G(t)$ is closed, and so, $x \in G(t)$.

*Sufficiency.* Since the sequence $\{g_n\}_{n=1}^{\infty}$ is equi-Lipschitzian, there exists a constant $C \geq 0$ such that $L_d(g_n, T) \leq C$ for all $n \in \mathbb{N}$, and since, given $t \in T$, $\mathcal{G}(t) = \{g_n(t)\}_{n=1}^{\infty}$ is precompact, the closure $G(t) = \overline{\mathcal{G}(t)}$ is compact in $X$. If $x \in \mathcal{G}(t)$, then $x = g_n(t)$ for some $n \in \mathbb{N}$, and so, if $s \in T$, we have:

$$\inf_{y \in \mathcal{G}(s)} d(x, y) \leq d(g_n(t), g_n(s)) \leq C|t - s|,$$

whence $\mathrm{e}(\mathcal{G}(t), \mathcal{G}(s)) = \sup_{x \in \mathcal{G}(t)} \inf_{y \in \mathcal{G}(s)} d(x, y) \leq C|t - s|$. By symmetry in $t$ and $s$, $\mathrm{e}(\mathcal{G}(s), \mathcal{G}(t)) \leq C|t - s|$, i. e., $D(\mathcal{G}(t), \mathcal{G}(s)) \leq C|t - s|$. Taking into account that

$$D(\overline{A}, \overline{B}) = D(A, B) \quad \text{whenever} \quad \emptyset \neq A,\ B \subset X, \tag{7.1}$$

we find that $D(G(t), G(s)) \leq C|t - s|$, quod erat demonstrandum. □

This theorem implies the following *structural theorem* for multifunctions of bounded variation with compact images:

**Theorem 7.2.** *Let $F : T \rightrightarrows X$ be a given multifunction with compact values. We have:*

(a) $F \in \mathrm{BV}(T; \mathrm{c}(X))$,
(b) $F \in \mathrm{BV}(T; \mathrm{c}(X))$ *is continuous,*
(c) $F$ *is $\delta(\cdot)$-absolutely continuous with $T$ compact,*
(d) $F \in \mathrm{BV}_\Phi(T; \mathrm{c}(X))$ *with $T$ bounded and $\Phi \in \mathcal{N}$, or*
(e) $F \in \mathrm{GV}_\Phi(T; \mathrm{c}(X))$ *with $T$ bounded and $\Phi \in \mathcal{N}$,*

*if and only if there exists*

*in case* (a) *a nondecreasing bounded function $\varphi : T \to \mathbb{R}$,*
*in case* (b) *a continuous nondecreasing bounded function $\varphi : T \to \mathbb{R}$,*
*in case* (c) *a $\delta(\cdot)$-absolutely continuous function $\varphi : T \to \mathbb{R}$,*
*in case* (d) *a function $\varphi \in \mathrm{BV}_\Phi(T; \mathbb{R})$, or*
*in case* (e) *a function $\varphi \in \mathrm{GV}_\Phi(T; \mathbb{R})$,*

*respectively, and a pointwise precompact equi-Lipschitzian sequence $\{g_n\}_{n=1}^{\infty} \subset \mathrm{Lip}(J; X)$, where $J = \varphi(T)$ and $\sup_{n \in \mathbb{N}} L_d(g_n, J) \leq 1$, such that*

$$F(t) = \overline{\{g_n(\varphi(t))\}_{n=1}^{\infty}} \quad \textit{for all} \quad t \in T. \tag{7.2}$$

*Given $F \in \mathrm{BV}_{\mathrm{ess}}(T; \mathrm{c}(X))$, the criterion is formulated as in* (a) *if $T$ is density-open, $X$ is complete and* (7.2) *holds a. e. on $T$.*

**Proof.** (a) *Necessity.* Let $F\in\mathrm{BV}(T;\mathrm{c}(X))$. Set $\varphi(t)=V_D(F,T\cap(-\infty,t])$, $t\in T$. By Lemma 1.2, there exists $G\in\mathrm{Lip}(J;\mathrm{c}(X))$ with $J=\varphi(T)$ such that $L_D(G,J)\le 1$ and $F=G\circ\varphi$ on $T$. Theorem 7.1 implies the existence of a pointwise precompact equi-Lipschitzian sequence $\{g_n\}_{n=1}^\infty\subset\mathrm{Lip}(J;X)$ of selections of $G$ with $L_d(g_n,J)\le 1$ such that $F(t)=G(\varphi(t))=\overline{\{g_n(\varphi(t))\}_{n=1}^\infty}$ for all $t\in T$.

*Sufficiency.* Let $\varphi\in\mathrm{BV}(T;\mathbb{R})$, $J=\varphi(T)$ and $\{g_n\}_{n=1}^\infty\subset\mathrm{Lip}(J;X)$ be such that $\sup_{n\in\mathbb{N}}L_d(g_n,J)\le C$ for some $C\in\mathbb{R}^+$ and the set $\{g_n(\tau)\}_{n=1}^\infty$ be precompact in $X$ for all $\tau\in J$. Given $t\in T$, we set $\mathcal{F}(t)=\{g_n(\varphi(t))\}_{n=1}^\infty$, so that $F(t)=\overline{\mathcal{F}(t)}$, $t\in T$. If $x\in\mathcal{F}(t)$, then $x=g_n(\varphi(t))$ for some $n\in\mathbb{N}$, and so, for $s\in T$ we have:

$$\inf_{y\in\mathcal{F}(s)}d(x,y)\le d(g_n(\varphi(t)),g_n(\varphi(s)))\le C|\varphi(t)-\varphi(s)|,$$

whence $\mathrm{e}(\mathcal{F}(t),\mathcal{F}(s))=\sup_{x\in\mathcal{F}(t)}\inf_{y\in\mathcal{F}(s)}d(x,y)\le C|\varphi(t)-\varphi(s)|$, and the symmetry in $t$ and $s$ gives $D(\mathcal{F}(t),\mathcal{F}(s))\le C|\varphi(t)-\varphi(s)|$. By (7.1), we get, for all $t,\,s\in T$,

$$D(F(t),F(s))=D(\overline{\mathcal{F}(t)},\overline{\mathcal{F}(s)})=D(\mathcal{F}(t),\mathcal{F}(s))\le C|\varphi(t)-\varphi(s)|. \quad (7.3)$$

If $\xi=\{t_i\}_{i=0}^m$ is a partition of $T$, we have:

$$\sum_{i=1}^m D(F(t_i),F(t_{i-1}))\le C\sum_{i=1}^m|\varphi(t_i)-\varphi(t_{i-1})|\le CV(\varphi,T),$$

and so, $F\in\mathrm{BV}(T;\mathrm{c}(X))$ and $V_D(F,T)\le CV(\varphi,T)$.

The proofs of (b)–(e) follow the lines of the proof of (a), so we exhibit the necessary changes only.

(b), (c) In the necessity part the function $\varphi$ is also continuous or $\delta(\cdot)$-absolutely continuous, respectively, and in the sufficiency part the continuity or $\delta(\cdot)$-absolute continuity of $F$ follows from the continuity or $\delta(\cdot)$-absolute continuity of $\varphi$, respectively, and (7.3) with $C=1$.

(d) In the necessity part apply Lemma 3.2 instead of Lemma 1.2, so that $\varphi\in\mathrm{BV}_\Phi(T;\mathbb{R})$. In the sufficiency part the inclusion $F\in\mathrm{BV}_\Phi(T;\mathrm{c}(X))$ is a consequence of $\varphi\in\mathrm{BV}_\Phi(T;\mathbb{R})$ and inequality (7.3).

(e) In order to prove the necessity part, use a variant of Lemma 3.2 for $\mathrm{GV}_\Phi$ (see p. 22), so $\varphi\in\mathrm{GV}_\Phi(T;\mathbb{R})$. Let us prove the sufficiency part. Let $\varphi\in\mathrm{GV}_\Phi(T;\mathbb{R})$, $J=\varphi(T)$ and let the hypotheses of the theorem with respect to $\{g_n\}_{n=1}^\infty$ be fulfilled. As in the proof of (a), we get inequality (7.3) with $C=1$. If $\lambda>0$ is such that $\varphi/\lambda\in\mathrm{BV}_\Phi(T;\mathbb{R})$, then for any partition $\xi=\{t_i\}_{i=0}^m$ of $T$ we find

$$\begin{aligned}\sum_{i=1}^m\Phi\biggl(\frac{D(F(t_i),F(t_{i-1}))}{(t_i-t_{i-1})\lambda}\biggr)(t_i-t_{i-1})&\le\sum_{i=1}^m\Phi\biggl(\frac{|\varphi(t_i)-\varphi(t_{i-1})|}{(t_i-t_{i-1})\lambda}\biggr)(t_i-t_{i-1})\\&\le V_\Phi(\varphi/\lambda,T),\end{aligned}$$

and hence, $V_{\Phi_\lambda,D}(F,T)<\infty$, i. e. $F\in \mathrm{GV}_\Phi(T;\mathrm{c}(X))$.

Finally, similarly to (a), the result for $F\in \mathrm{BV}_{\mathrm{ess}}(T;\mathrm{c}(X))$ follows from Theorem 2.2(c) and Theorem 7.1. □

In other words, Theorem 7.2(a) can be expressed as follows: *a set-valued mapping $F:T\to \mathrm{c}(X)$ is of bounded Jordan variation if and only if there exists a pointwise precompact sequence $\{f_n\}_{n=1}^\infty\subset \mathrm{BV}(T;X)$ (of selections of $F$) of equi-bounded variation such that $F(t)$ is the closure of $\{f_n(t)\}_{n=1}^\infty$ in $X$ for all $t\in T$.* In a similar manner one can express the other assertions of Theorem 7.2.

## 8. Boundary selections

Let $\mathrm{cc}(\mathbb{R}^N)$ denote the family of all nonempty compact convex subsets of $\mathbb{R}^N$ ($N\in\mathbb{N}$). In this section we denote by $\partial A$ the boundary of the set $A\subset\mathbb{R}^N$.

**Theorem 8.1.** *Let $T\subset\mathbb{R}$, $F\in \mathrm{BV}(T;\mathrm{cc}(\mathbb{R}^N))$, $t_0\in T$ and $x_0\in\partial F(t_0)$. Then there exists a selection $f\in \mathrm{BV}(T;\mathbb{R}^N)$ of $F$ satisfying the following conditions: $f(t)\in\partial F(t)$ for all $t\in T$, $f(t_0)=x_0$ and $V_d(f,T)\le V_D(F,T)$.*

**Proof.** Proof of this theorem resembles the proof of Theorem 5.1, but it uses an additional observation (taken from Kikuchi and Tomita [60, Theorem 1]), which we expose as step 1 below. Let $(x,y)$ designate the usual inner product of elements $x$ and $y$ from $\mathbb{R}^N$ and $\|x\|=\sqrt{(x,x)}$ — the Euclidean norm of $x\in\mathbb{R}^N$.

1. Let us show that if $t_0\in T$ and $x_0\in\partial F(t_0)$, then for each $t_1\in T$ there exists a point $x_1\in\partial F(t_1)$ such that $\|x_0-x_1\|\le D(F(t_0),F(t_1))$; here $D$ is the Hausdorff metric on $\mathrm{cc}(\mathbb{R}^N)$ generated by $\|\cdot\|$. It suffices to assume that $t_0\ne t_1$. If $x_0$ is not in the interior of $F(t_1)$, by the closedness and convexity of $F(t_1)$ there exists a unique point $x_1\in\partial F(t_1)$ such that $\|x_0-x_1\|=\mathrm{dist}(x_0,F(t_1))\le D(F(t_0),F(t_1))$; here the inequality follows from the definition of $D$ and the equality is a consequence of the projection theorem (e. g., [61, Lemma I.2.1] and [100, Theorem II.9.36]). Now let $x_0$ be in the interior of the set $F(t_1)$. Since $x_0\in\partial F(t_0)$ and $F(t_0)$ is convex, let $P$ be the supporting hyperplane for $F(t_0)$ passing through the point $x_0$ (cf. [59, III.2.3, Corollary of Theorem 5]). Thus, $P=\{x\in\mathbb{R}^N\mid (x-x_0,w)=0\}$ for some $0\ne w\in\mathbb{R}^N$ and $F(t_0)\subset P(-)$ where $P(-)=\{x\in\mathbb{R}^N\mid (x-x_0,w)\le 0\}$. Let $x_1$ be the point of intersection of the following three sets: $X\setminus P(-)$, $\partial F(t_1)$ and the straight line $\ell$ orthogonal to $P$ and passing through $x_0$ (since

$x_0$ is in the interior of the convex set $F(t_1)$, $\ell$ intersects the boundary $\partial F(t_1)$ in exactly two points). We claim that

$$\|x_0 - x_1\| = \operatorname{dist}(x_1, F(t_0)) \le D(F(t_1), F(t_0)). \tag{8.1}$$

As above, the inequality follows from the definition of $D$. In order to obtain equality in (8.1), it suffices to verify the equivalent condition ([61, Theorem I.2.3]): $(x - x_0, x_1 - x_0) \le 0$ for all $x \in F(t_0)$. In fact, if this condition holds and $x \in F(t_0)$, then

$$\|x_1 - x_0\|^2 = (x_1 - x, x_1 - x_0) + (x - x_0, x_1 - x_0) \le (x_1 - x, x_1 - x_0) \\ \le \|x_1 - x\| \cdot \|x_1 - x_0\|,$$

and so, $\|x_1 - x_0\| \le \|x_1 - x\|$ for all $x \in F(t_0)$. The condition itself can be checked as follows. Since $x_1 \in \ell$, $x_1 = x_0 + \theta w$ for some $\theta \in \mathbb{R}$, and since $x_1 \in X \setminus P(-)$, then $0 < (x_1 - x_0, w) = \theta\|w\|^2$, whence $\theta > 0$. The inclusion $F(t_0) \subset P(-)$ implies now that $(x - x_0, x_1 - x_0) = \theta(x - x_0, w) \le 0$ whenever $x \in F(t_0)$.

2. If $T$ is bounded, the proof of the existence of the desired selection $f$ of $F$ follows, on the whole, step 1 of the proof of Theorem 5.1 if we take into account certain modifications. We define elements $x_i^n \in \partial F(t_i^n)$, $n \in \mathbb{N}$, $i = 0, 1, \dots, n$, inductively as follows. If $n \in \mathbb{N}$ and $a < t_0 < b$, we set:

(a) $x_{k_0(n)}^n = x_0$;

(b) if $i \in \{1, \dots, k_0(n)\}$ and $x_i^n \in \partial F(t_i^n)$ is already chosen, by step 1 of this proof pick $x_{i-1}^n \in \partial F(t_{i-1}^n)$ such that

$$\|x_i^n - x_{i-1}^n\| \le D(F(t_i^n), F(t_{i-1}^n)); \tag{8.2}$$

(c) if $i \in \{k_0(n) + 1, \dots, n\}$ and $x_{i-1}^n \in \partial F(t_{i-1}^n)$ is already chosen, again by step 1 pick $x_i^n \in \partial F(t_i^n)$ satisfying (8.2).

If $t_0 = a$ or $t_0 = b$ we define $x_i^n \in \partial F(t_i^n)$ as in step 1 of the proof of Theorem 5.1.

Using definition (5.4), we get inequality (5.5). If $t \in Q$, the precompactness of $\{f_n(t)\}_{n=1}^\infty$ follows from (a), (b), (c) and the refinement of (5.6):

$$f_n(t) \in \partial F(t) \quad \text{for all} \quad n \ge n_0(t). \tag{8.3}$$

If $t \in T \setminus Q$, we first argue as in step 1 of the proof of Theorem 5.1 up to (5.7). Then by virtue of definition (5.4) we have: $f_{n_k}(t) = x_{i_k}^{n_k} \in \partial F(t_{i_k}^{n_k})$, $k \in \mathbb{N}$. Applying step 1 for each $k \in \mathbb{N}$ choose $x_t^k \in \partial F(t)$ such that $\|x_{i_k}^{n_k} - x_t^k\| \le D(F(t_{i_k}^{n_k}), F(t))$. Thanks to (5.7), $t_{i_k}^{n_k} \to t - 0$ as $k \to \infty$, and since $F$ is continuous at $t$, the last inequality implies $\|f_{n_k}(t) - x_t^k\| \to 0$ as $k \to \infty$. Noting that $\{x_t^k\}_{k=1}^\infty \subset \partial F(t)$ and $\partial F(t)$ is compact, without loss of generality, assume that $x_t^k$ converges in $X$ to an element $x_t \in \partial F(t)$ as $k \to \infty$, and so, in view of (5.8) the sequence $\{f_n(t)\}_{n=1}^\infty$ is precompact in $X$.

Applying Theorem 1.3 and taking into account relations (8.3) and (5.8), we obtain the desired selection $f$ of $F$, for which $f(t_0) = x_0$, $f(t) \in \partial F(t)$ for all $t \in T$ and such that its variation does not exceed that of $F$.

In the case of arbitrary $T$ we repeat with obvious modifications the arguments of step 2 of the proof of Theorem 5.1. □

Making use of refinements similar to those in Theorem 8.1 and its proof we obtain a theorem on the existence of more regular selections for multifunctions with more regular properties. We present only the statement of the corresponding theorem.

**Theorem 8.2.** *Let $T \subset \mathbb{R}$, $F : T \to \mathrm{c}(\mathbb{R}^N)$, $t_0 \in T$ and $x_0 \in \partial F(t_0)$. If $F$ satisfies conditions of Theorem* 5.6 *or one of conditions* (a)–(e) *of Theorem* 6.1*, where $\mathrm{c}(X)$ is replaced by $\mathrm{cc}(\mathbb{R}^N)$, then it admits a selection $f$ with properties from Theorem* 5.6 *or from items* (a)–(e) *of Theorem* 6.1*, respectively, such that $f(t) \in \partial F(t)$ for all $t \in T$ and $f(t_0) = x_0$.*

## 9. Selections with respect to a given mapping

Given $(X, \|\cdot\|)$ a linear normed space (over the field $\mathbb{R}$ or $\mathbb{C}$), we denote by $\mathrm{cc}(X)$ the family of all nonempty compact convex subsets of $X$. The addition operation in $\mathrm{cc}(X)$ (the Minkowski sum) is defined by $A + B = \{a + b \mid a \in A, b \in B\}$ whenever $A, B \in \mathrm{cc}(X)$. If $X$ is real, the triple $(\mathrm{cc}(X), D, +)$ with $D = D_{\|\cdot\|}$ the Hausdorff metric generated by the norm $\|\cdot\|$ in $X$ is a *metric semigroup* in the sense of Section 4: in fact (cf. Rådström [93, Lemma 3]), if $A, B \in \mathrm{cc}(X)$ and $\emptyset \neq C \subset X$ is bounded, then $D(A+C, B+C) = D(A, B)$. In order to use the notations for metrics, we set $d(x, y) = \|x - y\|$, $x, y \in X$, and denote by $\Delta_{\Phi,d}$ the quantity (4.5) evaluated in metric $d$.

**Theorem 9.1.** *Let $T \subset \mathbb{R}$, $(X, \|\cdot\|)$ be a real linear normed space, a multifunction $F$ be in $\mathrm{BV}(T; \mathrm{cc}(X))$, $t_0 \in T$ and $\eta \in \mathrm{BV}(T; X)$. Then there exists a selection $f \in \mathrm{BV}(T; X)$ of $F$ such that*

$$\|\eta(t_0) - f(t_0)\| = \mathrm{dist}(\eta(t_0), F(t_0)) \quad \text{and} \quad \Delta_{1,d}(\eta, f) \leq \Delta_{1,D}(\eta, F). \tag{9.1}$$

**Proof.** 1. First, let $T$ be bounded, $T \subset [a, b]$ and $a, b \in T$. To start with, we argue as in step 1 of the proof of Theorem 5.1 up to (5.3), where in order to obtain the set $Q$ we, in addition, append to the set $S$ the set of all discontinuity points of $\eta$ (which is at most countable as well). Since $F(t_0)$ is compact, choose an element $y_0 \in F(t_0)$ such that $\|\eta(t_0) - y_0\| = \mathrm{dist}(\eta(t_0), F(t_0))$. Now we define elements $x_i^n \in F(t_i^n)$, $n \in \mathbb{N}$, $i = 0, 1, \ldots, n$, inductively as follows. Suppose that $a < t_0 < b$.

(a) Set $x^n_{k_0(n)} = y_0$.

(b) If $i \in \{1, \dots, k_0(n)\}$ and $x^n_i \in F(t^n_i)$ is already chosen, pick $x^n_{i-1} \in F(t^n_{i-1})$ such that

$$\|\eta(t^n_{i-1}) - \eta(t^n_i) + x^n_i - x^n_{i-1}\| = \operatorname{dist}\Big(\eta(t^n_{i-1}) - \eta(t^n_i) + x^n_i, F(t^n_{i-1})\Big).$$

(c) If $i \in \{k_0(n)+1, \dots, n\}$ and $x^n_{i-1} \in F(t^n_{i-1})$ is already chosen, pick an element $x^n_i \in F(t^n_i)$ such that

$$\|\eta(t^n_i) - \eta(t^n_{i-1}) + x^n_{i-1} - x^n_i\| = \operatorname{dist}\Big(\eta(t^n_i) - \eta(t^n_{i-1}) + x^n_{i-1}, F(t^n_i)\Big).$$

If $t_0 = a$, i. e., $k_0(n) = 0$, we define $x^n_i \in F(t^n_i)$ following (a) and (c), and if $t_0 = b$, so that $k_0(n) = n$, we define $x^n_i \in F(t^n_i)$ in accordance with (a) and (b). Then by (b) we have

$$\begin{aligned}&\|\eta(t^n_{i-1}) - \eta(t^n_i) + x^n_i - x^n_{i-1}\| = \operatorname{dist}\Big(x^n_i + \eta(t^n_{i-1}), \eta(t^n_i) + F(t^n_{i-1})\Big)\\ &\le \mathrm{e}\Big(F(t^n_i) + \eta(t^n_{i-1}), \eta(t^n_i) + F(t^n_{i-1})\Big) \le D\Big(F(t^n_i) + \eta(t^n_{i-1}), \eta(t^n_i) + F(t^n_{i-1})\Big),\end{aligned}$$

and similarly, by (c),

$$\|\eta(t^n_i) - \eta(t^n_{i-1}) + x^n_{i-1} - x^n_i\| \le D\Big(\eta(t^n_i) + F(t^n_{i-1}), F(t^n_i) + \eta(t^n_{i-1})\Big), \tag{9.2}$$

so that inequality (9.2) is valid for all $i \in \{1, \dots, n\}$. It follows that

$$\begin{aligned}\|x^n_i - x^n_{i-1}\| &\le \|\eta(t^n_i) - \eta(t^n_{i-1}) + x^n_{i-1} - x^n_i\| + \|\eta(t^n_{i-1}) - \eta(t^n_i)\| \qquad (9.3)\\ &\le D\Big(\eta(t^n_i) + F(t^n_{i-1}), F(t^n_i) + \eta(t^n_{i-1})\Big) + \|\eta(t^n_i) - \eta(t^n_{i-1})\|.\end{aligned}$$

For each $n \in \mathbb{N}$ we define $f_n : T \to X$ by (5.4) and $\eta_n : T \to X$ by

$$\eta_n(t) = \begin{cases} \eta(t^n_i) & \text{if } t = t^n_i,\ i = 0, 1, \dots, n,\\ \eta(t^n_{i-1}) & \text{if } T \cap (t^n_{i-1}, t^n_i) \ne \emptyset \text{ and } t \in T \cap (t^n_{i-1}, t^n_i),\ i = 1, \dots, n.\end{cases}$$

Note that $f_n(t_0) = y_0$, $f_n(t^n_i) = x^n_i$ and $f_n(t^n_{i-1}) = x^n_{i-1}$ for all $n$ and $i$. Note also that

$$\begin{aligned}V_d(\eta_n, T) &= \sum_{i=1}^n V_d(\eta_n, T \cap [t^n_{i-1}, t^n_i]) = \sum_{i=1}^n \|\eta(t^n_i) - \eta(t^n_{i-1})\|\\ &\le V_d(\eta, T),\ n \in \mathbb{N},\end{aligned}$$

and that

$$\eta_n(t) \to \eta(t) \ \text{ in } \ X \ \text{ as } \ n \to \infty \ \text{ for all } \ t \in T \tag{9.4}$$

via some subsequence. In fact, if $t \in Q$, then by (5.3) there exists $n_0(t) \in \mathbb{N}$ such that $t \in \bigcap_{n \ge n_0(t)} \xi_n$, and so, $\eta_n(t) = \eta(t)$ for all $n \ge n_0(t)$. If $t \in T \setminus Q$, then $t$ is a point of continuity of $\eta$, which is a limit point from the left for $T$. In this case definition of $\eta_n$ and (5.7) imply $\eta_{n_k}(t) = \eta(t^{n_k}_{i_k})$, and since

$t_{i_k}^{n_k} \to t-0$, then $\eta_{n_k}(t) \to \eta(t)$ as $k \to \infty$. In what follows without loss of generality we assume that property (9.4) holds.

Taking into account (9.3) and definition of $\Delta_{1,D}$ from Section 4.1, we find

$$\begin{aligned} V_d(f_n, T) &= \sum_{i=1}^{n} V_d(f_n, T \cap [t_{i-1}^n, t_i^n]) = \sum_{i=1}^{n} \|x_i^n - x_{i-1}^n\| \\ &\le \Delta_{1,D}(\eta, F) + V_d(\eta, T), \qquad n \in \mathbb{N}. \end{aligned}$$

Let us show that for each $t \in T$ the sequence $\{f_n(t)\}_{n=1}^{\infty}$ is precompact in $X$. If $t \in Q$, this follows from (5.6). If $t \in T \setminus Q$, we argue as in step 1 of the proof of Theorem 5.1 up to (5.7). Then, given $k \in \mathbb{N}$, we have: $f_{n_k}(t) = x_{i_k}^{n_k} \in F(t_{i_k}^{n_k})$ and there exists $x_t^k \in F(t)$ such that

$$\|\eta(t) - \eta(t_{i_k}^{n_k}) + x_{i_k}^{n_k} - x_t^k\| = \operatorname{dist}\Big(\eta(t) - \eta(t_{i_k}^{n_k}) + x_{i_k}^{n_k}, F(t)\Big).$$

By the compactness of $F(t)$ we assume (passing to a subsequence if necessary) that $x_t^k$ converges in $X$ to an element $x_t \in F(t)$ as $k \to \infty$. This implies that

$$\begin{aligned} \|f_{n_k}(t) - x_t\| &\le \|\eta(t) - \eta(t_{i_k}^{n_k}) + x_{i_k}^{n_k} - x_t^k\| + \|\eta(t_{i_k}^{n_k}) - \eta(t)\| + \|x_t^k - x_t\| \\ &\le D\Big(\eta(t) + F(t_{i_k}^{n_k}), F(t) + \eta(t_{i_k}^{n_k})\Big) + \|\eta(t_{i_k}^{n_k}) - \eta(t)\| + \|x_t^k - x_t\|, \end{aligned}$$

where the last sum tends to zero as $k \to \infty$ thanks to the continuity of $F$ and $\eta$ at $t$ and the fact that $t_{i_k}^{n_k} \to t$ as $k \to \infty$ (cf. (5.7)).

Applying Theorem 1.3 we find a subsequence of $\{f_n\}_{n=1}^{\infty}$, which we still denote by $\{f_{n_k}\}_{k=1}^{\infty}$, which converges in $X$ pointwise on $T$ to a mapping $f \in \mathrm{BV}(T; X)$ as $k \to \infty$. Clearly, $f(t_0) = y_0$, and the first condition in (9.1) is satisfied. By (5.6) and the arguments in the last paragraph, $f(t) \in F(t)$ for all $t \in T$. It remains to verify the second condition in (9.1). For this, we note that, by (9.2),

$$\begin{aligned} V_d(\eta_n - f_n, T) &= \sum_{i=1}^{n} V_d(\eta_n - f_n, T \cap [t_{i-1}^n, t_i^n]) \\ &= \sum_{i=1}^{n} \|\eta_n(t_i^n) - f_n(t_i^n) - \eta_n(t_{i-1}^n) + f_n(t_{i-1}^n)\| \\ &= \sum_{i=1}^{n} \|\eta(t_i^n) - x_i^n - \eta(t_{i-1}^n) + x_{i-1}^n\| \\ &\le \sum_{i=1}^{n} D\Big(\eta(t_i^n) + F(t_{i-1}^n), F(t_i^n) + \eta(t_{i-1}^n)\Big) \\ &\le \Delta_{1,D}(\eta, F), \qquad n \in \mathbb{N}. \end{aligned}$$

Now it suffices to pass to the limit via the subsequence $n = n_k$ as $k \to \infty$ in this inequality and take into account Lemma 4.1(e) (with $\Phi(\rho) = \rho$) and property (9.4).

2. If $T$ is arbitrary, we argue almost as in step 2 of the proof of Theorem 5.1 subject to modifications of step 1. To end the proof we only note that the second inequality in (9.1) is a consequence of Lemma 4.3(g), (c) with $\Phi(\rho) = \rho$. □

The following theorem is connected with the existence of more regular selections (with respect to $\eta$) of multifunctions with convex images from classes $\mathrm{GV}_\Phi$ with $\Phi \in \mathcal{N}_\infty$ and Lip. For simplicity we restrict ourselves to the case of an interval $T = I = [a, b]$ in $\mathbb{R}$.

**Theorem 9.2.** *Let $(X, \|\cdot\|)$ be a real Banach space, $K \subset X$ be a closed convex cone, $F : I \to \mathrm{cc}(K)$ be a set-valued mapping and $t_0 \in I$. We have:*

(a) *if $\Phi \in \mathcal{N}_\infty$, $F \in \mathrm{GV}_\Phi(I; \mathrm{cc}(K))$ and $\eta \in \mathrm{GV}_\Phi(I; K)$, then $F$ admits a selection $f \in \mathrm{GV}_\Phi(I; K)$, for which the first condition in* (9.1) *holds and $\Delta_{\Phi,d}(\eta, f) \le \Delta_{\Phi,D}(\eta, F)$;*

(b) *if $F \in \mathrm{Lip}(I; \mathrm{cc}(K))$ and $\eta \in \mathrm{Lip}(I; K)$, then $F$ admits a selection $f$ in the class $\mathrm{Lip}(I; K)$ satisfying the first condition in* (9.1) *and such that the inequality holds: $d_\ell(\eta, f) \le D_\ell(\eta, F)$.*

**Proof.** (a) By the compactness of $F(t_0)$, there exists an element $y_0 \in F(t_0)$ such that $\|\eta(t_0) - y_0\| = \mathrm{dist}(\eta(t_0), F(t_0))$. Given $n \in \mathbb{N}$, let $\xi_n = \{t_i^n\}_{i=0}^n$ be a partition of $I$ (i. e., $a = t_0^n < t_1^n < \cdots < t_{n-1}^n < t_n^n = b$) satisfying the following two conditions:

1) $t_0 \in \xi_n$, so that $t_0 = t_{k_0(n)}^n$ for some $k_0(n) \in \{0, 1, \dots, n\}$, and
2) $\lim_{n\to\infty} \max_{1\le i\le n}(t_i^n - t_{i-1}^n) = 0$.

We define elements $x_i^n \in F(t_i^n)$, $n \in \mathbb{N}$, $i = 0, 1, \dots, n$, as in step 1(a)–(c) of the proof of Theorem 9.1, and so, inequalities (9.2) and (9.3) hold. If $n \in \mathbb{N}$, define $f_n : I \to K$ by

$$f_n(t) = x_{i-1}^n + \frac{t - t_{i-1}^n}{t_i^n - t_{i-1}^n}(x_i^n - x_{i-1}^n), \quad t \in [t_{i-1}^n, t_i^n], \quad i = 1, \dots, n,$$

and $\eta_n : I \to K$ by

$$\eta_n(t) = \eta(t_{i-1}^n) + \frac{t - t_{i-1}^n}{t_i^n - t_{i-1}^n}\Big(\eta(t_i^n) - \eta(t_{i-1}^n)\Big), \quad t \in [t_{i-1}^n, t_i^n], \quad i = 1, \dots, n.$$

Note that $f_n(t_0) = y_0$, $f_n(t_i^n) = x_i^n$ and $f_n(t_{i-1}^n) = x_{i-1}^n$ for all $n$ and $i$. Also, since $\eta \in \mathrm{GV}_\Phi(I; K)$ and $\Phi \in \mathcal{N}_\infty$, the mapping $\eta$ is (absolutely) continuous on $I$, and so,

$$\eta_n(t) \to \eta(t) \ \text{ in } \ K \ \text{ as } \ n \to \infty \ \text{ for all } \ t \in I. \tag{9.5}$$

Moreover,

$$p_\Phi(\eta_n) \le p_\Phi(\eta), \qquad n \in \mathbb{N}\,;$$

indeed, if $\mu = p_\Phi(\eta) > 0$, by the additivity property of $V_{\Phi_\mu}$ from Lemma 3.1(c), piecewise linearity of mapping $\eta_n$ and Lemma 3.9(b), we find

$$\begin{aligned} V_{\Phi_\mu,d}(\eta_n, I) &= \sum_{i=1}^{n} V_{\Phi_\mu,d}(\eta_n, [t_{i-1}^n, t_i^n]) \\ &= \sum_{i=1}^{n} \Phi_\mu\biggl(\frac{\|\eta(t_i^n) - \eta(t_{i-1}^n)\|}{t_i^n - t_{i-1}^n}\biggr)(t_i^n - t_{i-1}^n) \\ &\le V_{\Phi_\mu,d}(\eta, I) \le 1, \end{aligned}$$

proving the inequality. If $p_\Phi(\eta) = 0$, $\eta$ is constant, so that any $\eta_n$ is constant as well.

Let us show that the following inequality holds:

$$\Delta_{\Phi,d}(\eta_n, f_n) \le \Delta_{\Phi,D}(\eta, F) \quad \text{for all} \quad n \in \mathbb{N}. \tag{9.6}$$

With no loss of generality assume that $\lambda = \Delta_{\Phi,D}(\eta, F) > 0$. In order to calculate $W_{\Phi_\lambda}(\eta_n, f_n, I)$, let us note that for $t, s \in [t_{i-1}^n, t_i^n]$ we have:

$$\begin{aligned} &\|\eta_n(t) - \eta_n(s) + f_n(s) - f_n(t)\| \\ &= \frac{|t-s|}{t_i^n - t_{i-1}^n}\|\eta(t_i^n) - \eta(t_{i-1}^n) + x_{i-1}^n - x_i^n\|. \end{aligned} \tag{9.7}$$

Taking into account Lemma 4.3(c), definition (4.6), piecewise linearity of $\eta_n$ and $f_n$, monotonicity of $\Phi$, inequality (9.2) and Lemma 4.1(b), we get

$$\begin{aligned} W_{\Phi_\lambda}(\eta_n, f_n, I) &= \sum_{i=1}^{n} W_{\Phi_\lambda}(\eta_n, f_n, [t_{i-1}^n, t_i^n]) \\ &= \sum_{i=1}^{n} \Phi_\lambda\biggl(\frac{\|\eta(t_i^n) - \eta(t_{i-1}^n) + x_{i-1}^n - x_i^n\|}{t_i^n - t_{i-1}^n}\biggr)(t_i^n - t_{i-1}^n) \\ &\le \sum_{i=1}^{n} \Phi_\lambda\biggl(\frac{D(\eta(t_i^n) + F(t_{i-1}^n), F(t_i^n) + \eta(t_{i-1}^n))}{t_i^n - t_{i-1}^n}\biggr)(t_i^n - t_{i-1}^n) \\ &\le W_{\Phi_\lambda,D}(\eta, F, I) \le 1, \end{aligned}$$

and so, inequality (9.6) follows.

By Lemma 4.1(f) (cf. (9.3)), we have

$$p_\Phi(f_n) \le p_\Phi(\eta_n) + \Delta_{\Phi,d}(\eta_n, f_n) \le p_\Phi(\eta) + \Delta_{\Phi,D}(\eta, F), \quad n \in \mathbb{N}. \tag{9.8}$$

Since $\Phi \in \mathcal{N}_\infty$, it follows from Lemma 3.9(a) that the sequence $\{f_n\}_{n=1}^\infty$ is equicontinuous.

Let us check that, given $t \in I$, the set $\{f_n(t)\}_{n=1}^\infty$ is precompact in $K$. For $n \in \mathbb{N}$ choose a number $i(n) \in \{1, \dots, n\}$ (also depending on $t$) such that

$t^n_{i(n)-1} \le t \le t^n_{i(n)}$. By condition 2), $t^n_{i(n)} \to t$ and $t^n_{i(n)-1} \to t$ as $n \to \infty$. If $n \in \mathbb{N}$, pick $x_n(t) \in F(t)$ such that

$$\|\eta(t) - \eta(t^n_{i(n)-1}) + x^n_{i(n)-1} - x_n(t)\| = \operatorname{dist}\Big(\eta(t) - \eta(t^n_{i(n)-1}) + x^n_{i(n)-1}, F(t)\Big).$$

By the compactness of $F(t)$ we may assume (taking an appropriate subsequence) that $x_n(t)$ converges in $X$ to an element $x(t) \in F(t)$ as $n \to \infty$. We show that $\|f_n(t) - x(t)\| \to 0$ as $n \to \infty$. Since $\Phi \in \mathcal{N}_\infty$ and $F \in \mathrm{GV}_\Phi(I; \mathrm{cc}(K))$, $F$ is (absolutely) continuous on $I$ with respect to $D$, and so is $\eta$ (with respect to $d$), and hence, by virtue of (9.3) and (4.3),

$$\begin{aligned}
&\|f_n(t) - x(t)\| \le \|f_n(t) - x_n(t)\| + \|x_n(t) - x(t)\| \\
&= \left\| x^n_{i(n)-1} + \frac{t - t^n_{i(n)-1}}{t^n_{i(n)} - t^n_{i(n)-1}} \Big( x^n_{i(n)} - x^n_{i(n)-1} \Big) - x_n(t) \right\| + \|x_n(t) - x(t)\| \\
&\le \|\eta(t) - \eta(t^n_{i(n)-1}) + x^n_{i(n)-1} - x_n(t)\| + \|\eta(t^n_{i(n)-1}) - \eta(t)\| \\
&\quad + \|x^n_{i(n)} - x^n_{i(n)-1}\| + \|x_n(t) - x(t)\| \\
&\le D\Big(\eta(t) + F(t^n_{i(n)-1}), F(t) + \eta(t^n_{i(n)-1})\Big) + \|\eta(t^n_{i(n)-1}) - \eta(t)\| \\
&\quad + D\Big(\eta(t^n_{i(n)}) + F(t^n_{i(n)-1}), F(t^n_{i(n)}) + \eta(t^n_{i(n)-1})\Big) \\
&\quad + \|\eta(t^n_{i(n)}) - \eta(t^n_{i(n)-1})\| + \|x_n(t) - x(t)\|,
\end{aligned}$$

where the sum at the right hand side tends to zero as $n \to \infty$.

Applying the Arzelà-Ascoli Theorem we obtain a subsequence of $\{f_n\}_{n=1}^\infty$ (for which we keep the notation of the whole sequence), which converges in $X$ uniformly on $I$ to a mapping $f \in \mathrm{GV}_\Phi(I; K)$ (here we have used Lemma 3.9(e) and (9.8)). The first equality in (9.1) is valid, since $f(t_0) = y_0$. The proof of the precompactness of $\{f_n(t)\}_{n=1}^\infty$ shows that $f(t) \in F(t)$ for all $t \in I$. By (9.6), (9.5) and Lemma 4.1(e), we get:

$$\Delta_{\Phi,d}(\eta, f) \le \liminf_{n\to\infty} \Delta_{\Phi,d}(\eta_n, f_n) \le \Delta_{\Phi,D}(\eta, F),$$

which completes the proof of (a).

(b) The arguments here are as in the proof of (a). We exhibit only the necessary modifications. Since we have, for $t, s \in [t^n_{i-1}, t^n_i]$,

$$\|\eta_n(t) - \eta_n(s)\| = \frac{|t - s|}{t^n_i - t^n_{i-1}} \|\eta(t^n_i) - \eta(t^n_{i-1})\| \le L_d(\eta, I)|t - s|,$$

then $L_d(\eta_n, I) \le L_d(\eta, I)$ for all $n \in \mathbb{N}$. The equicontinuity of $\{f_n\}_{n=1}^\infty$ follows from the inequality $L_d(f_n, I) \le D_\ell(\eta, F) + L_d(\eta, I)$, which in turn is a consequence of definition of $f_n$, inequality (9.3) and definition of $D_\ell$ from Section 4.2. The inequality from (b) is established as follows: by (9.7) and (9.2), $L_d(\eta_n - f_n, I) \le D_\ell(\eta, F)$, $n \in \mathbb{N}$, so that it suffices to pass to the limit as $n \to \infty$ and take into account Lemma 4.4(b). □

## 10. Multiselections of bounded variation

A *multiselection* of the set-valued mapping $F : T \rightrightarrows X$ is a multifunction $\Gamma : T \rightrightarrows X$ satisfying $\mathrm{Gr}(\Gamma) \subset \mathrm{Gr}(F)$ or, equivalently, $\Gamma(t) \subset F(t)$ for all $t \in T$.

In this section we will prove, for a given set-valued mapping $F$ of bounded variation, the existence of multiselections of bounded variation, which pass through a given compact set in the image of $F$. The following theorem is a generalization of Theorem 5.1.

**Theorem 10.1.** *Let $T \subset \mathbb{R}$, $(X, d)$ be a metric space and $F \in \mathrm{BV}(T; \mathrm{c}(X))$. Then, given $t_0 \in T$ and $K \in \mathrm{c}(X)$, there exists a multiselection $\Gamma \in \mathrm{BV}(T; \mathrm{c}(X))$ of $F$ such that*

$$D(K, \Gamma(t_0)) \;\le\; \mathrm{e}(K, F(t_0)) \quad \text{and} \quad V_D(\Gamma, T) \;\le\; V_D(F, T). \tag{10.1}$$

*In particular, if $K \subset F(t_0)$, then by the first condition in* (10.1)*, $\Gamma(t_0) = K$.*

**Proof.** 1. We need the following assertion:

$$\begin{gathered}\text{if } K_0,\ F_0 \in \mathrm{c}(X) \text{ and } K_0 \subset F_0, \text{ then for any } F_1 \in \mathrm{c}(X)\\ \text{there exists } K_1 \in \mathrm{c}(X) \text{ such that}\\ K_1 \subset F_1 \text{ and } D(K_0, K_1) \le \mathrm{e}(F_0, F_1).\end{gathered} \tag{10.2}$$

To see this, it suffices to define $K_1$ as the metric projection of $K_0$ onto $F_1$:

$$K_1 = \{x_1 \in F_1 \mid \exists\, x_0 \in K_0 \text{ such that } d(x_0, x_1) = \mathrm{dist}(x_0, F_1)\}.$$

In fact, the compactness of nonempty set $K_1$ follows from the compactness of $K_0$, $F_1$ and closedness of $K_1$. Given $x_0 \in K_0$, choose $x_1' \in F_1$ such that $d(x_0, x_1') = \mathrm{dist}(x_0, F_1)$; since $x_1' \in K_1$, then $\inf_{x_1 \in K_1} d(x_0, x_1) \le d(x_0, x_1') \le \mathrm{e}(F_0, F_1)$, whence by the arbitrariness of $x_0 \in K_0$ we get $\mathrm{e}(K_0, K_1) \le \mathrm{e}(F_0, F_1)$. Now if $x_1 \in K_1$, then $x_1 \in F_1$ and there exists $x_0' \in K_0$ with the property $d(x_0', x_1) = \mathrm{dist}(x_0', F_1) \le \mathrm{e}(F_0, F_1)$; so, $\inf_{x_0 \in K_0} d(x_0, x_1) \le d(x_0', x_1) \le \mathrm{e}(F_0, F_1)$ for all $x_1 \in K_1$, and therefore, $\mathrm{e}(K_1, K_0) \le \mathrm{e}(F_0, F_1)$. This proves the inequality in (10.2).

2. Turning to the main part of the proof, first we argue as in step 1 of the proof of Theorem 5.1 up to (5.3). Setting $K_0 = F_0 = K$ and $F_1 = F(t_0)$ in (10.2) we find a compact subset $K_1 \equiv Y_0$ of $X$ such that $Y_0 \subset F(t_0)$ and $D(K, Y_0) \le \mathrm{e}(K, F(t_0))$. Now we define compact sets $K_i^n \subset F(t_i^n)$, $n \in \mathbb{N}$, $i = 0, 1, \ldots, n$, inductively as follows. Let $n \in \mathbb{N}$, and let $a < t_0 < b$, so that $1 \le k_0(n) \le n - 1$.

(a) Set $K_{k_0(n)}^n = Y_0$.

(b) If $i \in \{1, \dots, k_0(n)\}$ and the set $K_i^n \in \mathrm{c}(X)$, $K_i^n \subset F(t_i^n)$, is already chosen, then setting $K_0 = K_i^n$, $F_0 = F(t_i^n)$ and $F_1 = F(t_{i-1}^n)$ in (10.2) pick $K_{i-1}^n \in \mathrm{c}(X)$, $K_{i-1}^n \subset F(t_{i-1}^n)$, such that $D(K_i^n, K_{i-1}^n) \le \mathrm{e}(F(t_i^n), F(t_{i-1}^n))$.

(c) If $i \in \{k_0(n)+1, \dots, n\}$ and the compact set $K_{i-1}^n \subset F(t_{i-1}^n)$ is already chosen, then setting $K_0 = K_{i-1}^n$, $F_0 = F(t_{i-1}^n)$ and $F_1 = F(t_i^n)$ in (10.2) pick a compact set $K_i^n \subset F(t_i^n)$ such that $D(K_{i-1}^n, K_i^n) \le \mathrm{e}(F(t_{i-1}^n), F(t_i^n))$.

If $t_0 = a$, so that $k_0(n) = 0$, we define compact subsets $K_i^n$ of $F(t_i^n)$ following (a) and (c), and if $t_0 = b$, i. e. $k_0(n) = n$, we define $K_i^n \subset F(t_i^n)$ in accordance with (a) and (b).

Given $n \in \mathbb{N}$, we define $\Gamma_n : T \to \mathrm{c}(X)$ by (cf. (5.4))

$$\Gamma_n(t) \quad = \quad \begin{cases} K_i^n & \text{if } t = t_i^n,\ i = 0, 1, \dots, n, \\ K_{i-1}^n & \text{if } T \cap (t_{i-1}^n, t_i^n) \ne \emptyset \text{ and } t \in T \cap (t_{i-1}^n, t_i^n), \\ & i = 1, \dots, n. \end{cases} \tag{10.3}$$

Then $\Gamma_n(t_0) = Y_0$, and by (b) and (c) we have a counterpart of (5.5):

$$V_D(\Gamma_n, T) = \sum_{i=1}^n V_D(\Gamma_n, T \cap [t_{i-1}^n, t_i^n]) = \sum_{i=1}^n D(K_i^n, K_{i-1}^n)$$
$$\le \sum_{i=1}^n D\big(F(t_i^n), F(t_{i-1}^n)\big) \le V_D(F, T), \quad n \in \mathbb{N}.$$

Let us show that $\{\Gamma_n(t)\}_{n=1}^\infty$ is precompact in $\mathrm{c}(X)$ for all $t \in T$. If $t \in Q$, by (5.3) there exists $n_0(t) \in \mathbb{N}$ such that $t \in \xi_n$ for all $n \ge n_0(t)$, i. e., $t = t_{\ell(n)}^n$ for some $\ell(n) \in \{0, 1, \dots, n\}$ (also depending on $t$). Hence, thanks to (10.3), (a), (b) and (c), we have:

$$\Gamma_n(t) = K_{\ell(n)}^n \subset F(t_{\ell(n)}^n) = F(t) \quad \text{for all} \quad n \ge n_0(t).$$

Therefore, $\{\Gamma_n(t)\}_{n=n_0(t)}^\infty \subset \mathrm{c}(F(t))$, but $F(t)$ is compact in $X$, and so (cf. [12, II.1.4]), $\mathrm{c}(F(t))$ is compact in $\mathrm{c}(X)$, which implies that the sequence $\{\Gamma_n(t)\}_{n=1}^\infty$ is precompact.

Now let $t \in T \setminus Q$. Again we argue as in step 1 of the proof of Theorem 5.1 up to (5.7). By definition (10.3), $\Gamma_{n_k}(t) = K_{i_k}^{n_k} \subset F(t_{i_k}^{n_k})$, $k \in \mathbb{N}$. Making use of (10.2), for each $k \in \mathbb{N}$ choose a compact set $K_k(t) \subset F(t)$ such that $D(K_{i_k}^{n_k}, K_k(t)) \le D(F(t_{i_k}^{n_k}), F(t))$. By the compactness of $\mathrm{c}(F(t))$, with no loss of generality we assume that $K_k(t)$ converges in metric $D$ to a compact set $K(t) \subset F(t)$. Since

$$D(\Gamma_{n_k}(t), K(t)) \le D(K_{i_k}^{n_k}, K_k(t)) + D(K_k(t), K(t)) \to 0, \quad k \to \infty,$$

the set $\{\Gamma_n(t)\}_{n=1}^\infty$ is precompact in $\mathrm{c}(X)$.

The rest of the proof (application of Theorem 1.3, etc.) is carried out with obvious modifications as the respective part of the proof of Theorem 5.1. □

A theorem similar to Theorem 10.1 holds for more regular set-valued mappings. We present the corresponding formulation:

**Theorem 10.2.** *Let $T \subset \mathbb{R}$, $(X, d)$ be a metric space, $F : T \rightrightarrows X$ be a multifunction with compact values, $t_0 \in T$ and $K \in \mathrm{c}(X)$. If $F$ satisfies conditions of Theorem* 5.6 *or one of conditions* (a)–(e) *of Theorem* 6.1*, it admits a multiselection $\Gamma : T \to \mathrm{c}(X)$ of the respective class of mappings of bounded variation (if we replace $f$ by $\Gamma$ in those theorems, then $\Gamma$ has the same properties as $f$) such that $D(K, \Gamma(t_0)) \le \mathrm{e}(K, F(t_0))$.*

## 11. Functional inclusion $f(t) \in F(t, f(t))$

Let $\mathrm{cb}(X)$ denote the family of all nonempty closed bounded subsets of the metric space $(X, d)$ equipped with the Hausdorff metric $D$ generated by $d$.

**Theorem 11.1.** *Suppose the following conditions hold for $F : T \times X \to \mathrm{cb}(X)$*:

(i) *there exists a function $\varphi \in \mathrm{BV}(T; \mathbb{R})$ and a number $0 \le \mu < 1$ such that $D(F(t,x), F(s,y)) \le |\varphi(t) - \varphi(s)| + \mu\, d(x, y)$ for all $(t, x)$, $(s, y) \in T \times X$;*

(ii) *$\forall\, t \in T\ \exists\, K(t) \in \mathrm{c}(X)$ such that $F(t, x) \subset K(t)$ for all $x \in X$.*

*Then, given $t_0 \in T$ and $x_0 \in X$, there exists a mapping $f \in \mathrm{BV}(T; X)$ satisfying*:

(a) *$f(t) \in F(t, f(t))$ for all $t \in T$;*

(b) *$d(x_0, f(t_0)) = \mathrm{dist}\Big(x_0, F(t_0, f(t_0))\Big)$, and*

(c) *$V(f, T) \le V(\varphi, T)/(1 - \mu)$.*

*Moreover, if $x_0 \in X$ is such that $x_0 \in F(t_0, x_0)$, then* (b) *can be replaced by $f(t_0) = x_0$.*

**Proof.** The set $F(t, x_0)$ is closed and, by (ii), is contained in $K(t)$, and so, it is compact for all $t \in T$. Condition (i) implies $F(\cdot, x_0) \in \mathrm{BV}(T; \mathrm{c}(X))$ and $V(F(\cdot, x_0), T) \le V(\varphi, T)$. By Theorem 5.1, there exists $f_1 \in \mathrm{BV}(T; X)$ such that $f_1(t) \in F(t, x_0)$ for all $t \in T$,

$$d(x_0, f_1(t_0)) = \mathrm{dist}(x_0, F(t_0, x_0)) \tag{11.1}$$

and $V(f_1,T) \le V(F(\cdot,x_0),T) \le V(\varphi,T)$. Set $F_1(t) = F(t, f_1(t))$, $t \in T$. Then, by (ii), $F_1 : T \to \mathrm{c}(X)$, and for all $t$, $s \in T$ we have, thanks to (i),

$$D(F_1(t), F_1(s)) \le |\varphi(t) - \varphi(s)| + \mu\, d(f_1(t), f_1(s)), \tag{11.2}$$

and so,

$$V(F_1,T) \le V(\varphi,T) + \mu\, V(f_1,T) \le (1+\mu)V(\varphi,T).$$

Again by Theorem 5.1 we find $f_2 \in \mathrm{BV}(T;X)$, $f_2(t) \in F_1(t) = F(t, f_1(t))$ for all $t \in T$,

$$d(x_0, f_2(t_0)) = \mathrm{dist}\Big(x_0, F(t_0, f_1(t_0))\Big) \tag{11.3}$$

and $V(f_2,T) \le V(F_1,T) \le (1+\mu)V(\varphi,T)$. Setting $F_2(t) = F(t, f_2(t))$, $t \in T$, we have $F_2 : T \to \mathrm{c}(X)$ by (ii), and

$$D(F_2(t), F_2(s)) \le |\varphi(t) - \varphi(s)| + \mu\, d(f_2(t), f_2(s)), \qquad t,\ s \in T,$$

by virtue of (i), whence

$$V(F_2,T) \le V(\varphi,T) + \mu\, V(f_2,T) \le (1+\mu+\mu^2)V(\varphi,T).$$

By induction, for each $n \in \mathbb{N}$ there exists $f_n \in \mathrm{BV}(T;X)$ satisfying (where $f_0(t) \equiv x_0$):

$$f_n(t) \in F(t, f_{n-1}(t)) \subset K(t) \quad \text{for all} \quad t \in T, \tag{11.4}$$

$$\begin{aligned} &d(x_0, f_n(t_0)) = \mathrm{dist}\Big(x_0, F(t_0, f_{n-1}(t_0))\Big), \quad \text{and} \\ &V(f_n,T) \le \Big(\sum_{i=0}^{n-1} \mu^i\Big) V(\varphi,T) \le V(\varphi,T)/(1-\mu). \end{aligned} \tag{11.5}$$

Thus, the sequence $\{f_n\}_{n=1}^{\infty} \subset \mathrm{BV}(T;X)$ is pointwise precompact and is of uniformly bounded variation on $T$, and so, by Helly's selection principle (Theorem 1.3) it contains a subsequence, denoted by the same symbol, which converges pointwise on $T$ to a mapping $f \in \mathrm{BV}(T;X)$ as $n \to \infty$. Due to the lower semi-continuity of $V(\cdot,\cdot)$, condition (c) is satisfied. In order to prove (a), we employ the following inequality: if $\emptyset \ne A, B \subset X$ and $x, y \in X$, then

$$|\mathrm{dist}(x,A) - \mathrm{dist}(y,B)| \le d(x,y) + D(A,B). \tag{11.6}$$

Since $\mathrm{dist}\Big(f_n(t), F(t, f_{n-1}(t))\Big) = 0$, $n \in \mathbb{N}$, by assumption (i) for all $t \in T$ we have:

$$\begin{aligned} &\Big|\mathrm{dist}\Big(f(t), F(t, f(t))\Big) - \mathrm{dist}\Big(f_n(t), F(t, f_{n-1}(t))\Big)\Big| \\ &\le d(f(t), f_n(t)) + D\Big(F(t, f(t)), F(t, f_{n-1}(t))\Big) \\ &\le d(f(t), f_n(t)) + \mu\, d(f(t), f_{n-1}(t)) \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

It follows that $\operatorname{dist}\Big(f(t), F(t, f(t))\Big) = 0$, and since the set $F(t, f(t))$ is closed, then $f(t) \in F(t, f(t))$ for all $t \in T$. Passing to the limit as $n \to \infty$ in (11.5) and taking into account that, by (11.6) and (i), $d(x_0, f_n(t_0))$ converges to $d(x_0, f(t_0))$ and $\operatorname{dist}\Big(x_0, F(t_0, f_{n-1}(t_0))\Big)$ converges to $\operatorname{dist}\Big(x_0, F(t_0, f(t_0))\Big)$, we arrive at the equality (b).

Now if $x_0 \in F(t_0, x_0)$, then $f_1(t_0) = x_0$ by (11.1), and so, $x_0 \in F_1(t_0) = F(t_0, f_1(t_0))$, which implies $f_2(t_0) = x_0$ by virtue of (11.3), and so on. It remains to note that (11.5) can now be rewritten in the form $f_n(t_0) = x_0$. □

If $\mu = 0$, i. e., $F(t, x) = F(t)$ is independent of $x \in X$, Theorem 11.1 reduces to Theorem 5.1, since one can set $\varphi(t) = V(F, T \cap (-\infty, t])$ $(t \in T)$ in (i) and define $K(t)$ to be equal to $F(t) \in \mathrm{c}(X)$ in (ii).

An application of Theorem 11.1 and the following Theorem 11.2, corresponding to Theorem 6.1, will be presented in Section 14.

**Theorem 11.2.** *Suppose that assumptions of Theorem* 11.1 *are satisfied. If the function* $\varphi$ *in* (i) *there is such that* ($\alpha$) $\varphi \in \mathrm{Lip}(T; \mathbb{R})$, *or* ($\beta$) $\varphi \in \mathrm{GV}_\Phi(T; \mathbb{R})$ *with* $T$ *bounded and* $\Phi \in \mathcal{N}$, *then there exists a mapping* $f : T \to X$ *satisfying properties* (a), (b) *and* (c) *of that Theorem and* ($\alpha$) $f \in \mathrm{Lip}(T; X)$ *and* $L(f, T) \le L(\varphi, T)/(1-\mu)$, *or* ($\beta$) $f \in \mathrm{GV}_\Phi(T; X)$ *and* $p_\Phi(f, T) \le p_\Phi(\varphi, T)/(1-\mu)$, *respectively.*

**Proof.** ($\alpha$) 1. First let $T \subset [a, b]$ and $a, b \in T$. In this case $\varphi \in \mathrm{BV}(T; \mathbb{R})$, so we can repeat the proof of Theorem 11.1 taking into account that due to Theorem 6.1(a) condition (i) guarantees also that $f_n \in \mathrm{Lip}(T; X)$ and the following estimate holds: $L(f_n, T) \le L(\varphi, T)/(1-\mu)$, $n \in \mathbb{N}$. By Theorem 1.3, a (sub)sequence $\{f_n\}_{n=1}^\infty$ admits a pointwise limit $f \in \mathrm{BV}(T; X)$, for which conditions (a), (b) and (c) of Theorem 11.1 hold, and the above uniform estimate for $L(f_n, T)$ means that $f \in \mathrm{Lip}(T; X)$ and $L(f, T) \le L(\varphi, T)/(1-\mu)$.

2. For arbitrary $T$ we follow the arguments of step 2 of the proof of Theorem 5.1. Consider, for instance, the case when $a = \inf T \notin T$ and $b = \sup T \notin T$. Let $\{t_n\}_{n\in\mathbb{Z}} \subset T$ be an increasing sequence such that $t_n \to b$ and $t_{-n} \to a$ as $n \to \infty$, and set $T_n = T \cap [t_n, t_{n+1}]$ for $n \in \mathbb{Z}$. Applying step 1 for $T_0 = T \cap [t_0, t_1]$ (instead of $T$) we find a mapping $f_{(0)} \in \mathrm{Lip}(T_0; X)$ such that $f_{(0)}(t) \in F(t, f_{(0)}(t))$ for all $t \in T_0$, $d(x_0, f_{(0)}(t_0)) = \operatorname{dist}\Big(x_0, F(t_0, f_{(0)}(t_0))\Big)$ and $L(f_{(0)}, T_0) \le L(\varphi, T_0)/(1-\mu)$. Again apply step 1 to the set $T_1$ and choose a mapping $f_{(1)} \in \mathrm{Lip}(T_1; X)$ such that $f_{(1)}(t) \in F(t, f_{(1)}(t))$ for all $t \in T_1$, $f_{(1)}(t_1) = f_{(0)}(t_1)$ and $L(f_{(1)}, T_1) \le L(\varphi, T_1)/(1-\mu)$, and proceed this way for $T_n$ with $n \ge 2$ and $n \le -1$. Then for each $n \in \mathbb{Z}$ we obtain

a mapping $f_{(n)} \in \mathrm{Lip}(T_n; X)$ such that $f_{(n)}(t) \in F(t, f_{(n)}(t))$ for all $t \in T_n$, $f_{(n)}(t_n) = f_{(n-1)}(t_n)$ and $L(f_{(n)}, T_n) \le L(\varphi, T)/(1-\mu)$. Define $f : T \to X$ as follows: if $t \in T$, so that $t \in T_n$ for some $n \in \mathbb{Z}$, we set $f(t) = f_{(n)}(t)$. Then $f$ satisfies conditions (a), (b) and (c) of Theorem 11.1 and inequality $L(f,T) \le L(\varphi,T)/(1-\mu)$, which can be established as in step 2 of the proof of Theorem 6.1(a).

($\beta$) Without loss of generality assume that $\lambda = p_\Phi(\varphi, T) > 0$. Since $T$ is bounded, by (3.24) $\varphi \in \mathrm{BV}(T;\mathbb{R})$, so the arguments from the proof of Theorem 11.1 can be applied. Condition (i) implies $F(\cdot, x_0) \in \mathrm{GV}_\Phi(T; \mathrm{c}(X))$ and $p_\Phi(F(\cdot,x_0),T) \le \lambda$. By Theorem 6.1(e) there exists a selection $f_1 \in \mathrm{GV}_\Phi(T;X)$ of $F(\cdot,x_0)$ such that (11.1) holds and

$$p_\Phi(f_1,T) \le p_\Phi(F(\cdot,x_0),T) \le \lambda.$$

Set $F_1(t) = F(t, f_1(t))$, $t \in T$. Given a partition $\{t_i\}_{i=0}^m$ of $T$, by estimate (11.2), the convexity of $\Phi$ and Lemma 3.9(b), for $\gamma = p_\Phi(f_1,T)$, $\Lambda = \lambda + \mu\,\gamma$ and $i = 1,\dots,m$ we have:

$$\Phi\left(\frac{D(F_1(t_i),F_1(t_{i-1}))}{(t_i-t_{i-1})\,\Lambda}\right) \le \Phi\left(\frac{|\varphi(t_i)-\varphi(t_{i-1})|}{(t_i-t_{i-1})\,\Lambda} + \frac{\mu\, d(f_1(t_i),f_1(t_{i-1}))}{(t_i-t_{i-1})\,\Lambda}\right)$$
$$\le \frac{\lambda}{\Lambda}\,\Phi\left(\frac{|\varphi(t_i)-\varphi(t_{i-1})|}{(t_i-t_{i-1})\,\lambda}\right) + \frac{\mu\,\gamma}{\Lambda}\,\Phi\left(\frac{d(f_1(t_i),f_1(t_{i-1}))}{(t_i-t_{i-1})\,\gamma}\right),$$

whence

$$V_{\Phi_\Lambda}(F_1,T) \le \frac{\lambda}{\Lambda}\,V_\Phi(\varphi/\lambda,T) + \frac{\mu\,\gamma}{\Lambda}\,V_{\Phi_\gamma}(f_1,T) \le 1.$$

Therefore,

$$p_\Phi(F_1,T) \le \Lambda \le (1+\mu)p_\Phi(\varphi,T).$$

In this way we get a sequence $\{f_n\}_{n=1}^\infty \subset \mathrm{GV}_\Phi(T;X)$ with properties (11.4), (11.5) and such that $p_\Phi(f_n,T) \le p_\Phi(\varphi,T)/(1-\mu)$. This inequality, (3.24), (11.4), Theorem 1.3 and Lemma 3.9(e) imply the existence of $f \in \mathrm{GV}_\Phi(T;X)$ satisfying the desired conditions. □

**Remark 11.3.** (a) If $T = I = [a,b]$ and $\varphi \in \mathrm{AC}(I;\mathbb{R})$ in condition (i) of Theorem 11.1, then the inclusion $f(t) \in F(t,f(t))$ admits a solution $f \in \mathrm{AC}(I;X)$; in fact, in view of Theorem 3.6 and (3.20) we have $\varphi \in \mathrm{GV}_\Phi(I;\mathbb{R})$ for some $\Phi \in \mathcal{N}_\infty$, and so, by Theorem 11.2 and (3.13), $f \in \mathrm{GV}_\Phi(I;X) \subset \mathrm{AC}(I;X)$.

(b) If, under the assumptions of Theorem 11.1, $T$ is density-open, $X$ is complete and $\varphi \in \mathrm{BV}_{\mathrm{ess}}(T;\mathbb{R})$, then there exists $f \in \mathrm{BV}_{\mathrm{ess}}(T;X)$ such that $f(t) \in F(t,f(t))$ for almost all $t \in T$ and $V_{\mathrm{ess}}(f,T) \le V_{\mathrm{ess}}(\varphi,T)/(1-\mu)$. Indeed, since $V_{\mathrm{ess}}(\varphi,T)$ is finite, by Theorem 2.2(a) there exists a set $T_0 \subset T$ of Lebesgue measure zero such that $\varphi|_{T\setminus T_0} \in \mathrm{BV}(T\setminus T_0;\mathbb{R})$ and $V(\varphi, T\setminus$

$T_0) = V_{\mathrm{ess}}(\varphi, T)$. Let $t_0 \in T \setminus T_0$ and $x_0 \in X$. By Theorem 11.1, there exists $f \in \mathrm{BV}(T \setminus T_0; X)$ such that $f(t) \in F(t, f(t))$ for all $t \in T \setminus T_0$ and $V(f, T \setminus T_0) \le V(\varphi, T \setminus T_0)/(1-\mu) \le V_{\mathrm{ess}}(\varphi, T)/(1-\mu)$. Extending $f$ onto $T_0$ arbitrarily we find that $f \in \mathrm{BV}_{\mathrm{ess}}(T; X)$ is the required mapping.

(c) Suppose $F : T \times \mathbb{R}^N \to \mathrm{cc}(\mathbb{R}^N)$ satisfies conditions of Theorem 11.1. If $t_0 \in T$ and $x_0 \in \mathbb{R}^N$ is such that $x_0 \in \partial F(t_0, x_0)$, then there exists $f \in \mathrm{BV}(T; \mathbb{R}^N)$ such that $f(t) \in \partial F(t, f(t))$ for all $t \in T$, $f(t_0) = x_0$ and inequality (c) from Theorem 11.1 holds.

A generalization of Theorems 10.1 and 11.1 is given by

**Theorem 11.4.** *Suppose $F : T \times \mathrm{c}(X) \to \mathrm{cb}(X)$ satisfies conditions*:

(i) *there exists a function $\varphi \in \mathrm{BV}(T; \mathbb{R})$ and a number $0 \le \mu < 1$ such that*
$D(F(t,A), F(s,B)) \le |\varphi(t) - \varphi(s)| + \mu\, D(A,B)$ *for all $t$, $s \in T$ and $A$, $B \in \mathrm{c}(X)$*;

(ii) $\exists\, K : T \to \mathrm{c}(X)$ *such that $F(t,A) \subset K(t)$ for all $A \in \mathrm{c}(X)$.*

*Then for any $t_0 \in T$ and $\mathfrak{X}_0 \in \mathrm{c}(X)$ there exists $\mathfrak{X} \in \mathrm{BV}(T; \mathrm{c}(X))$ such that*

(a) $\mathfrak{X}(t) \subset F(t, \mathfrak{X}(t))$ *for all $t \in T$*;

(b) $D(\mathfrak{X}_0, \mathfrak{X}(t_0)) \le \mathrm{e}\Big(\mathfrak{X}_0, F(t_0, \mathfrak{X}(t_0))\Big)$, *and*

(c) $V(\mathfrak{X}, T) \le V(\varphi, T)/(1-\mu)$.

*Moreover, if $\mathfrak{X}_0 \in \mathrm{c}(X)$ is such that $\mathfrak{X}_0 \subset F(t_0, \mathfrak{X}_0)$, then* (b) *can be replaced by $\mathfrak{X}(t_0) = \mathfrak{X}_0$.*

**Proof.** Proof of this theorem is analogous to the proof of Theorem 11.1 if we apply Theorem 10.1 in place of Theorem 5.1. Only the verification of property (a) is subject to change. We will use the following inequality

$$|\mathrm{e}(\mathfrak{X}, A) - \mathrm{e}(\mathcal{Y}, B)| \le D(\mathfrak{X}, \mathcal{Y}) + D(A, B), \quad \mathfrak{X},\ \mathcal{Y},\ A,\ B \in \mathrm{cb}(X), \tag{11.7}$$

which is a consequence of the following two inequalities:

$$\mathrm{e}(\mathfrak{X}, A) \le \mathrm{e}(\mathfrak{X}, \mathcal{Y}) + \mathrm{e}(\mathcal{Y}, B) + \mathrm{e}(B, A),$$
$$\mathrm{e}(\mathcal{Y}, B) \le \mathrm{e}(\mathcal{Y}, \mathfrak{X}) + \mathrm{e}(\mathfrak{X}, A) + \mathrm{e}(A, B).$$

Let $\{\mathfrak{X}_n\}_{n=1}^{\infty} \subset \mathrm{BV}(T; \mathrm{c}(X))$ be the constructed sequence satisfying for all $n \in \mathbb{N}$ the conditions (with $\mathfrak{X}_0(t) \equiv \mathfrak{X}_0$):

$$\mathfrak{X}_n(t) \subset F(t, \mathfrak{X}_{n-1}(t)) \quad \text{for all} \quad t \in T,$$
$$D(\mathfrak{X}_0, \mathfrak{X}_n(t_0)) \le \mathrm{e}\Big(\mathfrak{X}_0, F(t_0, \mathfrak{X}_{n-1}(t_0))\Big), \quad \text{and}$$
$$V(\mathfrak{X}_n, T) \le \Big(\sum_{i=0}^{n-1} \mu^i\Big) V(\varphi, T) \le V(\varphi, T)/(1-\mu),$$

and such that $D(\mathfrak{X}_n(t),\mathfrak{X}(t))\to 0$ as $n\to\infty$ for all $t\in T$, where $\mathfrak{X}\in\mathrm{BV}(T;\mathrm{c}(X))$. Noting that $\mathrm{e}\Big(\mathfrak{X}_n(t),F(t,\mathfrak{X}_{n-1}(t))\Big)=0$, by (11.7) and assumption (i) for each $t\in T$ we get

$$\begin{aligned}&\Big|\mathrm{e}\Big(\mathfrak{X}(t),F(t,\mathfrak{X}(t))\Big)-\mathrm{e}\Big(\mathfrak{X}_n(t),F(t,\mathfrak{X}_{n-1}(t))\Big)\Big|\\&\le D\Big(\mathfrak{X}(t),\mathfrak{X}_n(t)\Big)+D\Big(F(t,\mathfrak{X}(t)),F(t,\mathfrak{X}_{n-1}(t))\Big)\\&\le D\Big(\mathfrak{X}(t),\mathfrak{X}_n(t)\Big)+\mu\,D\Big(\mathfrak{X}(t),\mathfrak{X}_{n-1}(t)\Big)\to 0\quad\text{as}\quad n\to\infty.\end{aligned}$$

Therefore, $\mathrm{e}\Big(\mathfrak{X}(t),F(t,\mathfrak{X}(t))\Big)=0$ implying (a). □

In a similar manner, making use of Theorem 10.2 one can modify Theorem 11.2 and remarks following it. We leave the details to the interested reader.

## 12. Jensen's functional equation

The main aim of this and the next section is to provide tools and motivation for the results of Section 14. Moreover, the study of the multivalued Jensen functional equation in this Section as well as the characterization of multivalued Lipschitzian superposition operators in the BV setting in Section 13 is important and interesting in its own right.

Let $(Y,\|\cdot\|)$ be a real linear normed space. Denote by $\mathrm{cbc}(Y)$ the family of all nonempty closed bounded convex subsets of $Y$ equipped with the usual Hausdorff metric $D$ (generated by $\|\cdot\|$). We define the binary operation of $(*\text{-})$ addition $\overset{*}{+}$ in $\mathrm{cbc}(Y)$ by (cf. [106])

$$P\overset{*}{+}Q=\overline{P+Q},\qquad P,\,Q\in\mathrm{cbc}(Y),$$

where $P+Q$ is the Minkowski sum and $\overline{P}$ means the closure of $P$ in $Y$. The following equalities hold in $\mathrm{cbc}(Y)$:

$$P\overset{*}{+}Q=\overline{\overline{P}+\overline{Q}},\tag{12.1}$$

$$\lambda(P\overset{*}{+}Q)=\lambda P\overset{*}{+}\lambda Q,\quad(\lambda+\mu)P=\lambda P\overset{*}{+}\mu P,\quad\lambda,\,\mu\in\mathbb{R}^+.$$

The triple $(\mathrm{cbc}(Y),D,\overset{*}{+})$ is a metric semigroup in the sense of Section 4, since for any $P,Q\in\mathrm{cbc}(Y)$ and $\emptyset\ne R\subset Y$ bounded, we have

$$D(P\overset{*}{+}R,Q\overset{*}{+}R)=D(P+R,Q+R)=D(P,Q)\tag{12.2}$$

(cf. (7.1) for the first equality and [39, Lemma 2.2] for the second equality), so that $D$ is translation invariant. This metric semigroup is complete if $Y$ is a Banach space (which follows from properties of $D$, e. g., [12, Theorems II-9, II-14]). Moreover,

$$D(\lambda P, \lambda Q) = |\lambda| D(P, Q), \qquad P,\, Q \in \mathrm{cbc}(Y), \quad \lambda \in \mathbb{R}. \tag{12.3}$$

Now, let $I = [a, b]$ be an interval, $(X, \|\cdot\|)$ be a linear normed space, $d(x, y) = \|x-y\|$ if $x, y \in X$, and $K \subset X$ be a convex cone (i. e., $K+K \subset K$ and $\lambda K \subset K$ for $\lambda \in \mathbb{R}^+$). Given $\Phi, \Psi \in \mathcal{N}$, by Theorem 4.2 the triples

$$(\mathrm{GV}_\Phi(I; K), d_\Phi, +) \quad \text{and} \quad (\mathrm{GV}_\Psi(I; \mathrm{cbc}(Y)), D_\Psi, \overset{*}{+})$$

are metric semigroups with respect to pointwise operations of addition which are denoted by the same symbols as in $K$ and $\mathrm{cbc}(Y)$, respectively (i. e., $(f+g)(t) = f(t) + g(t)$ if $f, g \in \mathrm{GV}_\Phi(I; K)$ and $(F \overset{*}{+} G)(t) = F(t) \overset{*}{+} G(t)$ if $F, G \in \mathrm{GV}_\Psi(I; \mathrm{cbc}(Y))$, $t \in I$). These semigroups are equipped with the respective translation invariant metrics $d_\Phi$ and $D_\Psi$ starting from the induced metric $d$ on $X$ and the Hausdorff metric $D$ on $\mathrm{cbc}(Y)$ according to formulas (4.4)–(4.6) from Section 4.

A set-valued operator $A : K \to \mathrm{cbc}(Y)$ is said to be *linear* if it is $*$-*additive* (i. e., $A(x+y) = A(x) \overset{*}{+} A(y)$ for all $x, y \in K$) and *nonnegatively homogeneous* (i. e., $A(\lambda x) = \lambda A(x)$ for all $x \in K$ and $\lambda \in \mathbb{R}^+$). Observe that $A(0) = \{0\}$.

Let us denote by $\mathrm{L}(K; \mathrm{cbc}(Y))$ the metric semigroup of all linear Lipschitzian set-valued operators from $K$ into $\mathrm{cbc}(Y)$ endowed with the metric (cf. Section 4.2 where $a = 0$):

$$\begin{aligned} D_L(A, B) &\equiv D_\ell(A, B) \\ &= \sup_{x, y \in K,\, x \neq y} D\Big(A(x) \overset{*}{+} B(y), B(x) \overset{*}{+} A(y)\Big) \Big/ \|x - y\| \end{aligned} \tag{12.4}$$

whenever $A,\, B \in \mathrm{L}(K; \mathrm{cbc}(Y))$.

Given $H : I \times K \to \mathrm{cbc}(Y)$, the operator $\mathcal{H} : K^I \to \mathrm{cbc}(Y)^I$ defined by

$$(\mathcal{H} f)(t) \equiv \mathcal{H}(f)(t) := H(t, f(t)) \quad \text{if} \quad f : I \to K \quad \text{and} \quad t \in I \tag{12.5}$$

is said to be a (*set-valued Nemytskii*) *superposition operator* generated by $H$, and the set-valued mapping $H$ is called the *generator* of $\mathcal{H}$.

The main result of Section 13 is Theorem 13.1 characterizing Lipschitzian set-valued superposition operators between metric semigroups $\mathrm{GV}_\Phi(I; K)$ and $\mathrm{GV}_\Psi(I; \mathrm{cbc}(Y))$. In order to prove it, we need the following two lemmas.

**Lemma 12.1.** *If* $A : K \to \mathrm{cbc}(Y)$ *is* $*$-*additive and continuous, it is linear.*

**Proof.** It suffices to show that if $x \in K$ and $\lambda \in \mathbb{R}^+$, then $A(\lambda x) = \lambda A(x)$. Let $\{\lambda_k\}_{k=1}^{\infty}$ be a sequence of positive rational numbers converging to $\lambda$ as $k \to \infty$. By the continuity of $A$ we have:

$$D\Big(A(\lambda_k x), A(\lambda x)\Big) \to 0 \quad \text{as} \quad k \to \infty.$$

Note that $A(\lambda_k x) = \lambda_k A(x)$, $k \in \mathbb{N}$; in fact, if $k$ is fixed, then $\lambda_k = n/m$ for some $n, m \in \mathbb{N}$, and so, the convexity of values of $A$ and the $*$-additivity of $A$ imply $A(x) = A(x/m) \overset{*}{+} \cdots \overset{*}{+} A(x/m) = mA(x/m)$, or $A(x/m) = A(x)/m$, and similarly,

$$\begin{aligned} A(\lambda_k x) = A(n(x/m)) &= \underbrace{A(x/m) \overset{*}{+} \cdots \overset{*}{+} A(x/m)}_{n \text{ times}} \\ &= nA(x/m) = (n/m)A(x) = \lambda_k A(x). \end{aligned}$$

Since the set $A(x)$ is bounded, the mapping $\mu \mapsto \mu\, A(x)$ is continuous from $\mathbb{R}$ into $\operatorname{cbc}(Y)$ (cf. [87, Lemma 3.2]), so

$$D\Big(A(\lambda_k x), \lambda A(x)\Big) = D\Big(\lambda_k A(x), \lambda A(x)\Big) \to 0 \quad \text{as} \quad k \to \infty.$$

As $k \to \infty$ it follows that

$$D\Big(A(\lambda x), \lambda A(x)\Big) \le D\Big(A(\lambda x), A(\lambda_k x)\Big) + D\Big(A(\lambda_k x), \lambda A(x)\Big) \to 0.$$

Since the values of $A$ are closed, $A(\lambda x) = \lambda A(x)$. □

The following lemma was established for operators $F$ with compact convex values in $Y$ by Fifer [43, Theorem 2] (if $K = \mathbb{R}^+$) and Nikodem [87, Theorem 5.6] (if $K$ is a cone). An abstract version of this lemma is due to W. Smajdor [106, Theorem 1]. In the proof of Lemma 12.2 below we follow the paper [106].

**Lemma 12.2.** *Let $Y$ be a Banach space. Then a set-valued operator $F$ from $K$ into* $\operatorname{cbc}(Y)$ *satisfies the Jensen functional equation, i. e.,*

$$F\left(\frac{x+y}{2}\right) = \frac{1}{2}\Big(F(x) \overset{*}{+} F(y)\Big) \quad \textit{for all} \quad x, y \in K, \tag{12.6}$$

*if and only if there exists a $*$-additive operator $A : K \to \operatorname{cbc}(Y)$ and a set $B \in \operatorname{cbc}(Y)$ such that $F(x) = A(x) \overset{*}{+} B$ for all $x \in K$. (The operator $A$ and the set $B$ are determined uniquely by $F$.)*

**Proof.** *Necessity.* First, note that the following two mappings are continuous:

$$\begin{aligned} &\mathbb{R}^+ \times \operatorname{cbc}(Y) \ni (\lambda, A) \longmapsto \lambda A \in \operatorname{cbc}(Y), \\ &\operatorname{cbc}(Y) \times \operatorname{cbc}(Y) \ni (A, B) \longmapsto A \overset{*}{+} B \in \operatorname{cbc}(Y). \end{aligned} \tag{12.7}$$

Setting $F_n^y(x) = F(2^n x + y)/2^n$, $\alpha_n = (2^n - 1)/2^n$, $x, y \in K$, $n \in \mathbb{N}$, by (12.6) and induction we have

$$F(x+y) = F_n^y(x) \overset{*}{+} \alpha_n F(y) \qquad \text{for all} \quad x, y \in K,\, n \in \mathbb{N}. \tag{12.8}$$

Let us show that $\{F_n^y(x)\}_{n=1}^\infty$ is a Cauchy sequence in $\operatorname{cbc}(Y)$ for all $x, y \in K$. Given $n, m \in \mathbb{N}$, $n > m$, by (12.2), (12.8), (12.1) and (12.3) we have:

$$\begin{aligned} D\Big(F_n^y(x), F_m^y(x)\Big) &= D\Big(F_n^y(x) \overset{*}{+} \alpha_n F(y), F_m^y(x) \overset{*}{+} \alpha_n F(y)\Big) \\ &= D\Big(F(x+y), F_m^y(x) \overset{*}{+} \alpha_m F(y) \overset{*}{+} (\alpha_n - \alpha_m) F(y)\Big) \\ &= D\Big(F(x+y), F(x+y) \overset{*}{+} (\alpha_n - \alpha_m) F(y)\Big) \\ &= D\Big(\{0\}, (\alpha_n - \alpha_m) F(y)\Big) \\ &= \left(\frac{1}{2^m} - \frac{1}{2^n}\right) D\Big(\{0\}, F(y)\Big). \end{aligned}$$

Since $\operatorname{cbc}(Y)$ is complete, for each $y \in K$ there exists $A^y : K \to \operatorname{cbc}(Y)$ such that $D(F_n^y(x), A^y(x)) \to 0$ as $n \to \infty$ for all $x \in K$. Passing to the limit as $n \to \infty$ in the equality (12.8), by virtue of (12.7) we get

$$F(x+y) = A^y(x) \overset{*}{+} F(y), \qquad x,\, y \in K. \tag{12.9}$$

For each $y \in K$ the operator $A^y$ is $*$-additive, since, given $x_1, x_2 \in K$, by Jensen's equation we have in $\operatorname{cbc}(Y)$:

$$\begin{aligned} A^y(x_1 + x_2) &= \lim_{n\to\infty} F_n^y(x_1 + x_2) = \lim_{n\to\infty} \Big(F_{n+1}^y(x_1) \overset{*}{+} F_{n+1}^y(x_2)\Big) \\ &= A^y(x_1) \overset{*}{+} A^y(x_2). \end{aligned}$$

Now we show that the operator $A^y$ is independent of $y \in K$, namely,

$$A^y(x) = \lim_{n\to\infty} F_n^0(x) \qquad \text{in } \operatorname{cbc}(Y) \qquad \text{for all } x, y \in K. \tag{12.10}$$

Taking into account (12.6), (12.9) and the $*$-additivity of $A^y$, we find

$$\begin{aligned} F(2x) \overset{*}{+} F(2y) = 2F(x+y) &= 2A^y(x) \overset{*}{+} 2F(y) \\ &= A^y(2x) \overset{*}{+} 2F(y), \end{aligned} \tag{12.11}$$

and so, the second equality in (12.11) with $x = y$ implies $F(2y) = A^y(y) \overset{*}{+} F(y)$, and this together with the third equality in (12.11) gives:

$$F(2x) \overset{*}{+} A^y(y) \overset{*}{+} F(y) = A^y(2x) \overset{*}{+} 2F(y).$$

Cancelling by $F(y)$ here, replacing $x$ by $2^{n-1}x$, dividing by $2^n$ and taking into account the $*$-additivity of $A^y$, we arrive at the equality:

$$F_n^0(x) \overset{*}{+} \frac{1}{2^n} A^y(y) = A^y(x) \overset{*}{+} \frac{1}{2^n} F(y).$$

As $n \to \infty$, thanks to (12.7) we get (12.10). To end this part of the proof, set $y = 0$, $A = A^0$ and $B = F(0)$ in (12.9).

*Sufficiency.* Let $A : K \to \mathrm{cbc}(Y)$ be $*$-additive, $B \in \mathrm{cbc}(Y)$ and $F(x) = A(x) \overset{*}{+} B$, $x \in K$. Then $F$ satisfies the Jensen equation (12.6), for, given $x, y \in K$, we have

$$\begin{aligned} 2F\left(\frac{x+y}{2}\right) &= 2A\left(\frac{x+y}{2}\right) \overset{*}{+} 2B = A(x+y) \overset{*}{+} 2B \\ &= (A(x) \overset{*}{+} B) \overset{*}{+} (A(y) \overset{*}{+} B) = F(x) \overset{*}{+} F(y). \end{aligned}$$

□

## 13. Multivalued superposition operators

We say that $\Phi \in \mathcal{N}$ *grows at infinity significantly slower than* $\Psi \in \mathcal{N}$ (in symbols, $\Phi \lhd \Psi$) if $\lim_{\rho\to\infty} \Phi(C\rho)/\Psi(\rho) = 0$ for all $C > 0$. It is known (e. g., [23, Lemma 4.2]) that $\Phi \lhd \Psi$ if and only if $\lim_{r\to\infty} \Psi^{-1}(r)/\Phi^{-1}(r) = 0$. For instance, if $\Phi(\rho) = \rho^{q_1}$, $\Psi(\rho) = \rho^{q_2}$, $\rho \in \mathbb{R}^+$, $q_1, q_2 \in [1, \infty)$, then $\Phi \lhd \Psi$ if and only if $q_1 < q_2$. Observe also that $\Phi \lhd \Psi$ if and only if $\Phi \preccurlyeq \Psi$ and $\Phi \not\sim \Psi$.

The main result of this section is the following theorem.

**Theorem 13.1.** *Let $I = [a, b]$, $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be two linear normed spaces, $K \subset X$ be a convex cone, $\Phi \in \mathcal{N}$ and $\Psi \in \mathcal{N}_\infty$. Suppose that $\mathcal{H} : K^I \to \mathrm{cbc}(Y)^I$ is a set-valued superposition operator generated by a mapping $H : I \times K \to \mathrm{cbc}(Y)$ via (12.5).*

*If $Y$ is a real Banach space and*

$$\mathcal{H} \in \mathrm{Lip}\Big(\mathrm{GV}_\Phi(I; K); \mathrm{GV}_\Psi(I; \mathrm{cbc}(Y))\Big), \tag{13.1}$$

*then $H(t, \cdot) \in \mathrm{Lip}(K; \mathrm{cbc}(Y))$ for all $t \in I$ and there exist two mappings $H_0 \in \mathrm{GV}_\Psi(I; \mathrm{cbc}(Y))$ and $H_1 : I \to \mathrm{L}(K; \mathrm{cbc}(Y))$ with the property that $H_1(\cdot)x =$*

*$[t \mapsto H_1(t)x] \in \mathrm{GV}_\Psi(I;\mathrm{cbc}(Y))$ for all $x \in K$ such that the Matkowski representation holds*:

$$H(t,x) = H_0(t) \overset{*}{+} H_1(t)x, \qquad t \in I, \quad x \in K. \tag{13.2}$$

*Moreover, under the conditions above if $\Phi \lhd \Psi$, then $H(t,x) = H(t,0)$ for all $t \in I$ and $x \in K$ (so that $\mathcal{H}$ is a constant set-valued operator).*

*Conversely, if $\Psi \preccurlyeq \Phi$, $H_0 \in \mathrm{GV}_\Psi(I;\mathrm{cbc}(Y))$, $H_1 \in \mathrm{GV}_\Psi(I;\mathrm{L}(K;\mathrm{cbc}(Y)))$ and the generator $H$ is of the form* (13.2)*, then the superposition operator $\mathcal{H}$ satisfies* (13.1).

**Proof.** For the sake of clarity we divide the proof into eight steps.

1. Let condition (13.1) be satisfied. Then there exists a number $\mu > 0$ such that if $f_1, f_2 \in \mathrm{GV}_\Phi(I;K)$ and $\lambda = \mu\, d_\Phi(f_1,f_2)$, then $D_\Psi(\mathcal{H}f_1,\mathcal{H}f_2) \le \lambda$ and, hence (cf. (4.4) and (4.5)), $\Delta_{\Psi,D}(\mathcal{H}f_1,\mathcal{H}f_2) \le \lambda$. If $d_\Phi(f_1,f_2) > 0$, by Lemma 4.1(c) the last inequality is equivalent to $W_{\Psi_\lambda,D}(\mathcal{H}f_1,\mathcal{H}f_2) \le 1$, and so, from the definition (4.6) for all $\alpha,\beta \in I$, $\alpha < \beta$, we get, in particular,

$$\Psi\left(\frac{D\Big((\mathcal{H}f_1)(\beta) \overset{*}{+} (\mathcal{H}f_2)(\alpha), (\mathcal{H}f_2)(\beta) \overset{*}{+} (\mathcal{H}f_1)(\alpha)\Big)}{(\beta-\alpha)\lambda}\right)(\beta-\alpha) \le 1.$$

Applying $\Psi^{-1}$ and taking into account (12.5) and the definition of function $\omega_\Psi$ in Section 3 (p. 13), we find

$$\begin{aligned} &D\Big(H(\beta,f_1(\beta)) \overset{*}{+} H(\alpha,f_2(\alpha)), H(\beta,f_2(\beta)) \overset{*}{+} H(\alpha,f_1(\alpha))\Big) \\ &\le \mu\,\omega_\Psi(\beta-\alpha) d_\Phi(f_1,f_2). \end{aligned} \tag{13.3}$$

Now if $d_\Phi(f_1,f_2) = 0$, then $\Delta_{\Psi,D}(\mathcal{H}f_1,\mathcal{H}f_2) = 0$, and so, by Lemma 4.1(a) the left hand side of (13.3) is equal to zero. Thus, inequality (13.3) is valid for all mappings $f_1, f_2 \in \mathrm{GV}_\Phi(I;K)$ and all $\alpha,\beta \in I$, $\alpha < \beta$.

2. Let us show that $H(t,\cdot) \in \mathrm{Lip}(K;\mathrm{cbc}(Y))$ for all $t \in I$. More precisely, we will show that there exists a function $\mu_0 : I \to \mathbb{R}^+$ such that

$$D\Big(H(t,x_1), H(t,x_2)\Big) \le \mu_0(t)\|x_1 - x_2\|, \quad t \in I, \quad x_1, x_2 \in K. \tag{13.4}$$

First, suppose that $a < t \le b$, and let $x_1, x_2 \in K$. Define two mappings $f_j \in \mathrm{Lip}(I;K)$, $j = 1,2$, by

$$f_j(s) = \eta_{\alpha,\beta}(s)x_j, \quad s \in I, \quad j = 1,2, \quad \alpha,\beta \in I, \quad \alpha < \beta, \tag{13.5}$$

where

$$\eta_{\alpha,\beta}(s) = \begin{cases} 0 & \text{if } s \le \alpha, \\ \dfrac{s-\alpha}{\beta-\alpha} & \text{ifn } \alpha \le s \le \beta, \\ 1 & \quad\text{if } \beta \le s. \end{cases} \tag{13.6}$$

Note that $f_j(\beta) = x_j$ and $f_j(\alpha) = 0$ for $j = 1, 2$. Let us calculate $d_\Phi(f_1, f_2)$. It is clear that if $x_1 = x_2$, then $\Delta_{\Phi,d}(f_1, f_2) = 0$. If $x_1 \neq x_2$, choose $\lambda > 0$ such that

$$W_{\Phi_\lambda,d}(f_1, f_2) = \Phi\bigg(\frac{\|x_1 - x_2\|}{(\beta - \alpha)\lambda}\bigg)(\beta - \alpha) = 1.$$

Then by Lemma 4.1(d) we find

$$\Delta_{\Phi,d}(f_1, f_2) = \lambda = \|x_1 - x_2\|/\omega_\Phi(\beta - \alpha).$$

Since $\|f_1(a) - f_2(a)\| = 0$, we have $d_\Phi(f_1, f_2) = \Delta_{\Phi,d}(f_1, f_2)$. Substituting mappings (13.5) into inequality (13.3), by virtue of the translation invariance of $D$ on $\mathrm{cbc}(Y)$ (cf. (12.2)) for all $\alpha, \beta \in I$, $\alpha < \beta$, and $x_1, x_2 \in K$ we get

$$D\Big(H(\beta, x_1), H(\beta, x_2)\Big) \le \mu\,\frac{\omega_\Psi(\beta - \alpha)}{\omega_\Phi(\beta - \alpha)}\|x_1 - x_2\|. \tag{13.7}$$

Setting $\alpha = a$ $\beta = t$, we arrive at (13.4) with $\mu_0(t) = \mu\,\omega_\Psi(t-a)/\omega_\Phi(t-a)$.

Now let $t = a$ and $x_1, x_2 \in K$. Define two Lipschitzian mappings from $I$ into $K$ by

$$f_j(s) = (1-\eta_{\alpha,\beta}(s))x_j, \quad s \in I, \quad j = 1, 2, \quad \alpha, \beta \in I, \quad \alpha < \beta, \tag{13.8}$$

so that $f_j(\beta) = 0$ and $f_j(\alpha) = x_j$, $j = 1, 2$. Substituting them into (13.3), noting that

$$d_\Phi(f_1, f_2) = \Big(1 + \frac{1}{\omega_\Phi(\beta - \alpha)}\Big)\|x_1 - x_2\|,$$

and setting $\alpha = a$ and $\beta = b$, we obtain (13.4) for $t = a$ with the constant $\mu_0(a) = \mu\,\omega_\Psi(b - a)(1 + 1/\omega_\Phi(b - a))$.

3. In order to prove (13.2), let $\alpha, \beta \in I$, $\alpha < \beta$, $x_1, x_2 \in K$, and let $f_j \in \mathrm{Lip}(I; K)$, $j = 1, 2$, be defined by

$$f_j(t) = \frac{1}{2}\Big(\eta_{\alpha,\beta}(t)(x_1 - x_2) + x_j + x_2\Big), \qquad t \in I, \quad j = 1, 2.$$

Substituting these mappings into (13.3), we find

$$\begin{aligned} &D\bigg(H(\beta, x_1) \overset{*}{+} H(\alpha, x_2), H\Big(\beta, \frac{x_1 + x_2}{2}\Big) \overset{*}{+} H\Big(\alpha, \frac{x_1 + x_2}{2}\Big)\bigg) \\ &\le \frac{1}{2}\mu\,\omega_\Psi(\beta - \alpha)\|x_1 - x_2\|. \end{aligned} \tag{13.9}$$

Since constant mappings from $I$ into $K$ belong to $\mathrm{GV}_\Phi(I; K)$, condition (13.1) implies $H(\cdot, x) = \mathcal{H}(x) \in \mathrm{GV}_\Psi(I; \mathrm{cbc}(Y))$, $x \in K$, and so, by Lemma 4.1(a) and assumption $\Psi \in \mathcal{N}_\infty$ the mapping $H(\cdot, x)$ is (absolutely) continuous on $I$ with respect to $D$ for all $x \in K$. If $t \in I$, then passing to the limit as $\beta - \alpha \to 0$ in (13.9) in such a way that $\alpha \le t \le \beta$, we get:

$$D\Big(H(t,x_1) \overset{*}{+} H(t,x_2), H\Big(t, \frac{x_1+x_2}{2}\Big) \overset{*}{+} H\Big(t, \frac{x_1+x_2}{2}\Big)\Big) = 0.$$

Since $D$ is a metric on $\mathrm{cbc}(Y)$ and the images of $H$ are convex (bounded and closed), it follows that

$$H(t,x_1) \overset{*}{+} H(t,x_2) = 2H\Big(t, \frac{x_1+x_2}{2}\Big).$$

Therefore, for each $t \in I$ the set-valued operator $H(t,\cdot)$, mapping $K$ into $\mathrm{cbc}(Y)$, satisfies the following Jensen functional equation:

$$H\Big(t, \frac{x_1+x_2}{2}\Big) = \frac{1}{2}\Big(H(t,x_1) \overset{*}{+} H(t,x_2)\Big), \qquad x_1,\, x_2 \in K.$$

By Lemma 12.2, for each $t \in I$ there exists a set $H_0(t) \in \mathrm{cbc}(Y)$ and a $*$-additive set-valued operator $H_1(t)(\cdot) : K \to \mathrm{cbc}(Y)$ such that

$$H(t,x) = H_0(t) \overset{*}{+} H_1(t)x, \qquad x \in K. \tag{13.10}$$

Thanks to (13.4) and the translation invariance of $D$, the operator $H_1(t)(\cdot)$ is (Lipschitz) continuous, and since it is $*$-additive, by Lemma 12.1 it is also linear, so that $H_1(t) \in \mathrm{L}(K;\mathrm{cbc}(Y))$. Hence, $H_1(t)(0) = \{0\}$, $t \in I$, and (13.10) yields $H(t,0) = H_0(t)$ for all $t \in I$. In this way we have proved that the mapping $H_0$ belongs to the metric semigroup $\mathrm{GV}_\Psi(I;\mathrm{cbc}(Y))$.

Let us prove that $H_1(\cdot)x \in \mathrm{GV}_\Psi(I;\mathrm{cbc}(Y))$ for all $x \in K$. Let $x \in K$. Since, as is shown above in this step, $H(\cdot,x)$ and $H_0$ belong to $\mathrm{GV}_\Psi(I;\mathrm{cbc}(Y))$, there exist numbers $\lambda' > 0$ and $\mu' > 0$ such that $V_{\Psi_{\lambda'}}(H(\cdot,x)) < \infty$ and $V_{\Psi_{\mu'}}(H_0) < \infty$. By the translation invariance of $D$ and (13.10), for all $t,\, s \in I$, $s < t$, we have:

$$\begin{aligned} D\Big(H_1(t)x, H_1(s)x\Big) =& D\Big(H_0(t) \overset{*}{+} H_1(t)x, H_0(t) \overset{*}{+} H_1(s)x\Big) \\ \le& D\Big(H_0(t) \overset{*}{+} H_1(t)x, H_0(s) \overset{*}{+} H_1(s)x\Big) \\ &+ D\Big(H_0(s) \overset{*}{+} H_1(s)x, H_0(t) \overset{*}{+} H_1(s)x\Big), \end{aligned}$$

whence

$$D\Big(H_1(t)x, H_1(s)x\Big) \le D\Big(H(t,x), H(s,x)\Big) + \Big(H_0(t), H_0(s)\Big). \tag{13.11}$$

Noting that

$$\frac{D\Big(H_1(t)x, H_1(s)x\Big)}{(t-s)(\lambda'+\mu')}$$

is nongreater than

$$\frac{\lambda'}{\lambda'+\mu'}\cdot\frac{D\Big(H(t,x),H(s,x)\Big)}{(t-s)\lambda'}+\frac{\mu'}{\lambda'+\mu'}\cdot\frac{D\Big(H_0(t),H_0(s)\Big)}{(t-s)\mu'},$$

by the convexity of $\Psi$ we find

$$V_{\Psi_{\lambda'+\mu'}}(H_1(\cdot)x)\le\frac{\lambda'}{\lambda'+\mu'}V_{\Psi_{\lambda'}}(H(\cdot,x))+\frac{\mu'}{\lambda'+\mu'}V_{\Psi_{\mu'}}(H_0)<\infty.$$

Thus, $H_1(\cdot)x\in\mathrm{GV}_\Psi(I;\mathrm{cbc}(Y))$ for all $x\in K$.

4. Suppose now that (13.1) holds and $\Phi\lhd\Psi$. If $t\in I$, $t\ne a$, and $x\in K$, then setting $\beta=t$, $x_1=x$ and $x_2=0$ in (13.7) and noting that (by remarks before this theorem)

$$\frac{\omega_\Psi(t-\alpha)}{\omega_\Phi(t-\alpha)}=\frac{\Psi^{-1}(1/(t-\alpha))}{\Phi^{-1}(1/(t-\alpha))}\to 0\quad\text{as}\quad\alpha\to t-0$$

and that $H(\cdot,x)$ is continuous on $I$ and passing to the limit as $\alpha\to t-0$ in (13.7), we find that $H(t,x)=H(t,0)$ for all $a<t\le b$. The continuity of $H(\cdot,x)$ and $H(\cdot,0)$ yields $H(t,x)=H(t,0)$ whenever $t\in I$ and $x\in K$.

Now let us prove the reverse assertion. Let $H$ be given by (13.2) and $\Psi\preccurlyeq\Phi$, so that $\mathrm{GV}_\Phi(I;K)\subset\mathrm{GV}_\Psi(I;K)$ and inequalities from Lemma 4.5(b) hold.

5. First of all, let us show that if $H_1\in\mathrm{GV}_\Psi(I;\mathrm{L}(K;\mathrm{cbc}(Y)))$ and $f\in\mathrm{GV}_\Psi(I;K)$, where $\Psi\in\mathcal{N}$, then the mapping $H_1f$ defined by $(H_1f)(t)=H_1(t)f(t)$ for $t\in I$ belongs to the metric semigroup $\mathrm{GV}_\Psi(I;\mathrm{cbc}(Y))$. In fact, by the definition (from Section 4.2) of Lipschitz constant $L(H_1(t))$ of the set-valued operator $H_1(t)\in\mathrm{L}(K;\mathrm{cbc}(Y))$ and the definition of metric $D_L$ from (12.4) for all $t$, $s\in I$ we have:

$$\begin{aligned}&D\Big((H_1f)(t),(H_1f)(s)\Big)\\&\le D\Big(H_1(t)f(t),H_1(t)f(s)\Big)+D\Big(H_1(t)f(s),H_1(s)f(s)\Big)\\&\le L(H_1(t))\|f(t)-f(s)\|+D_L(H_1(t),H_1(s))\|f(s)\|.\end{aligned}\tag{13.12}$$

From Lemma 4.4(c), (12.4) and Lemma 3.9(a) we get (setting $|I|=b-a$):

$$|L(H_1(t))-L(H_1(s))|\le D_L(H_1(t),H_1(s))\le\omega_\Psi(|I|)p_{\Psi,D_L}(H_1),\quad t,s\in I,$$

and so, $\sup_{t\in I}L(H_1(t))<\infty$. Similarly, by Lemma 3.9(a) we get:

$$\|f(t)-f(s)\|\le\omega_\Psi(|I|)p_{\Psi,d}(f),\quad t,s\in I,\quad\text{so that}\quad\sup_{s\in I}\|f(s)\|<\infty.$$

Then due to (13.12) for all $t$, $s\in I$ we have that

$$D\Big((H_1f)(t),(H_1f)(s)\Big)$$

is nongreater than

$$\left(\sup_{\tau\in I} L(H_1(\tau))\right)\|f(t)-f(s)\| + D_L(H_1(t),H_1(s))\left(\sup_{\tau\in I}\|f(\tau)\|\right).$$

By the already *standard procedure* (used, e. g., in the proof of the first inequality of Lemma 4.1(f), see (4.9) and (4.10)), this inequality yields

$$p_{\Psi,D}(H_1 f) \le \left(\sup_{t\in I} L(H_1(t))\right) p_{\Psi,d}(f) + p_{\Psi,D_L}(H_1)\left(\sup_{t\in I}\|f(t)\|\right), \quad (13.13)$$

which proves that $H_1 f \in \mathrm{GV}_\Psi(I;\mathrm{cbc}(Y))$.

6. Now we show that the superposition operator $\mathcal{H}$ generated by (13.2) maps $\mathrm{GV}_\Psi(I;K)$ into $\mathrm{GV}_\Psi(I;\mathrm{cbc}(Y))$. Let $f \in \mathrm{GV}_\Psi(I;K)$. Assumption (13.2) implies

$$(\mathcal{H}f)(t) = H_0(t) \overset{*}{+} H_1(t)f(t), \quad t \in I. \quad (13.14)$$

Given $t$, $s \in I$, by applying property (12.2)) we have

$$\begin{aligned} D\Big((\mathcal{H}f)(t),(\mathcal{H}f)(s)\Big) =& D\Big(H_0(t) \overset{*}{+} H_1(t)f(t), H_0(s) \overset{*}{+} H_1(s)f(s)\Big)\\ \le& D\Big(H_0(t) \overset{*}{+} H_1(t)f(t), H_0(t) \overset{*}{+} H_1(s)f(s)\Big)\\ &+ D\Big(H_0(t) \overset{*}{+} H_1(s)f(s), H_0(s) \overset{*}{+} H_1(s)f(s)\Big)\\ =& D\Big(H_1(t)f(t), H_1(s)f(s)\Big) + D\Big(H_0(t), H_0(s)\Big), \end{aligned}$$

whence by the (mentioned) standard procedure we get

$$p_{\Psi,D}(\mathcal{H}f) \le p_{\Psi,D}(H_1 f) + p_{\Psi,D}(H_0), \quad (13.15)$$

and this means that $\mathcal{H}f \in \mathrm{GV}_\Psi(I;\mathrm{cbc}(Y))$.

7. Let us prove that the superposition operator $\mathcal{H}$ is Lipschitzian. By hypotheses $H_0 \in \mathrm{GV}_\Psi(I;\mathrm{cbc}(Y))$, $H_1 \in \mathrm{GV}_\Psi(I;\mathrm{L}(K;\mathrm{cbc}(Y)))$ and $\mathcal{H}$ acts on mappings $f \in \mathrm{GV}_\Psi(I;K)$ according to (13.14). Let $f_1, f_2 \in \mathrm{GV}_\Psi(I;K)$. By definition (4.4),

$$D_\Psi(\mathcal{H}f_1,\mathcal{H}f_2) = D((\mathcal{H}f_1)(a),(\mathcal{H}f_2)(a)) + \Delta_{\Psi,D}(\mathcal{H}f_1,\mathcal{H}f_2) \equiv Z_1 + Z_2.$$

Let us estimate $Z_1$ and $Z_2$ separately. For $Z_1$ we have:

$$\begin{aligned} Z_1 &= D\Big(H_0(a) \overset{*}{+} H_1(a)f_1(a), H_0(a) \overset{*}{+} H_1(a)f_2(a)\Big)\\ &= D\Big(H_1(a)f_1(a), H_1(a)f_2(a)\Big)\\ &\le L(H_1(a))\|f_1(a)-f_2(a)\|. \end{aligned}$$

In order to estimate $Z_2$, applying definition (4.6), translation invariance of $D$, inequality (4.1), $*$-additivity of $H_1(t)$, definition of $L(H_1(t))$ and (12.4), for all $t$, $s \in I$ we have:

$$\begin{aligned}
&D\Big((\mathcal{H}f_1)(t) \overset{*}{+} (\mathcal{H}f_2)(s), (\mathcal{H}f_2)(t) \overset{*}{+} (\mathcal{H}f_1)(s)\Big)\\
&= D\Big(H_0(t) \overset{*}{+} H_1(t)f_1(t) \overset{*}{+} H_0(s) \overset{*}{+} H_1(s)f_2(s),\\
&\qquad H_0(t) \overset{*}{+} H_1(t)f_2(t) \overset{*}{+} H_0(s) \overset{*}{+} H_1(s)f_1(s)\Big) =\\
&= D\Big(H_1(t)f_1(t) \overset{*}{+} H_1(s)f_2(s), H_1(t)f_2(t) \overset{*}{+} H_1(s)f_1(s)\Big)\\
&\overset{(4.1)}{\le} D\Big(H_1(t)f_1(t) \overset{*}{+} H_1(s)f_2(s) \overset{*}{+} H_1(t)f_2(s) \overset{*}{+} H_1(s)f_1(s),\\
&\qquad H_1(t)f_2(t) \overset{*}{+} H_1(s)f_1(s) \overset{*}{+} H_1(s)f_2(s) \overset{*}{+} H_1(t)f_1(s)\Big)\\
&\qquad + D\Big(H_1(t)f_2(s) \overset{*}{+} H_1(s)f_1(s), H_1(s)f_2(s) \overset{*}{+} H_1(t)f_1(s)\Big)\\
&= D\Big(H_1(t)f_1(t) \overset{*}{+} H_1(t)f_2(s), H_1(t)f_2(t) \overset{*}{+} H_1(t)f_1(s)\Big)\\
&\qquad + D\Big(H_1(t)f_1(s) \overset{*}{+} H_1(s)f_2(s), H_1(s)f_1(s) \overset{*}{+} H_1(t)f_2(s)\Big)\\
&\overset{*\text{-add.}}{=} D\Big(H_1(t)(f_1(t) + f_2(s)), H_1(t)(f_2(t) + f_1(s))\Big)\\
&\qquad + D\Big(H_1(t)f_1(s) \overset{*}{+} H_1(s)f_2(s), H_1(s)f_1(s) \overset{*}{+} H_1(t)f_2(s)\Big)\\
&\overset{L(H_1(t)),(12.4)}{\le} L(H_1(t))\|f_1(t) + f_2(s) - f_2(t) - f_1(s)\|\\
&\qquad + D_L(H_1(t), H_1(s))\|f_1(s) - f_2(s)\|\\
&= L(H_1(t))\|(f_1 - f_2)(t) - (f_1 - f_2)(s)\| + D_L(H_1(t), H_1(s))\|(f_1 - f_2)(s)\|.
\end{aligned}$$

Thus, for all $t$, $s \in I$ we obtain the inequality:

$$\begin{aligned}
&D\Big((\mathcal{H}f_1)(t) \overset{*}{+} (\mathcal{H}f_2)(s), (\mathcal{H}f_2)(t) \overset{*}{+} (\mathcal{H}f_1)(s)\Big)\\
&\le \Big(\sup_{\tau \in I} L(H_1(\tau))\Big)\|(f_1 - f_2)(t) - (f_1 - f_2)(s)\|\\
&\qquad + D_L(H_1(t), H_1(s))\Big(\sup_{\tau \in I}\|(f_1 - f_2)(\tau)\|\Big).
\end{aligned}$$

By the standard procedure this implies the estimate:

$$Z_2 = \Delta_{\Psi,D}(\mathcal{H}f_1, \mathcal{H}f_2) \tag{13.16}$$

$$\le \Big(\sup_{t\in I} L(H_1(t))\Big)\Delta_{\Psi,d}(f_1,f_2) + p_{\Psi,D_L}(H_1)\Big(\sup_{t\in I}\|(f_1-f_2)(t)\|\Big).$$

Noting that $\Delta_{\Psi,d}(f_1,f_2) = p_{\Psi,d}(f_1-f_2)$, by Lemma 3.9(a) we find

$$\sup_{t\in I}\|(f_1-f_2)(t)\| \le \|(f_1-f_2)(a)\| + \omega_\Psi(|I|)\Delta_{\Psi,d}(f_1,f_2).$$

Making use of Lemma 4.4(c), definition (12.4) and Lemma 3.9(a), for $t\in I$ we have:

$$L(H_1(t)) \le L(H_1(a)) + D_L(H_1(t),H_1(a)) \le L(H_1(a)) + \omega_\Psi(|I|)p_{\Psi,D_L}(H_1),$$

and so,

$$\sup_{t\in I} L(H_1(t)) \le L(H_1(a)) + \omega_\Psi(|I|)p_{\Psi,D_L}(H_1). \tag{13.17}$$

Therefore, if we set

$$\begin{aligned} &\gamma(\Psi) = \max\{1, 2\omega_\Psi(|I|)\} \quad \text{and} \\ &|||H_1|||_\Psi = L(H_1(a)) + p_{\Psi,D_L}(H_1), \end{aligned} \tag{13.18}$$

then by (13.16) we arrive at the estimate:

$$\begin{aligned} D_\Psi(\mathcal{H}f_1,\mathcal{H}f_2) &= Z_1+Z_2 \le L(H_1(a))\|(f_1-f_2)(a)\| \\ &+ \Big(L(H_1(a)) + \omega_\Psi(|I|)p_{\Psi,D_L}(H_1)\Big)\Delta_{\Psi,d}(f_1,f_2) \\ &+ p_{\Psi,D_L}(H_1)\Big(\|(f_1-f_2)(a)\| + \omega_\Psi(|I|)\Delta_{\Psi,d}(f_1,f_2)\Big) \\ &\le \max\{1,2\omega_\Psi(|I|)\}\Big(L(H_1(a)) + p_{\Psi,D_L}(H_1)\Big)\Big(\|(f_1-f_2)(a)\| \\ &+ \Delta_{\Psi,d}(f_1,f_2)\Big), \end{aligned}$$

or, finally,

$$D_\Psi(\mathcal{H}f_1,\mathcal{H}f_2) \le \gamma(\Psi)\,|||H_1|||_\Psi\, d_\Psi(f_1,f_2). \tag{13.19}$$

8. To end the proof, if $\Psi \preccurlyeq \Phi$, then $\mathrm{GV}_\Phi(I;K) \subset \mathrm{GV}_\Psi(I;K)$, and so, $\mathcal{H}$ maps $\mathrm{GV}_\Phi(I;K)$ into $\mathrm{GV}_\Psi(I;\mathrm{cbc}(Y))$ and is Lipschitzian (i. e., satisfies (13.1)), since by virtue of (13.19) and Lemma 4.5(b), given $f_1,f_2\in \mathrm{GV}_\Phi(I;K)$, we have:

$$D_\Psi(\mathcal{H}f_1,\mathcal{H}f_2) \le \gamma(\Psi)\,|||H_1|||_\Psi\,\kappa_0(\Phi,\Psi,|I|)\,d_\Phi(f_1,f_2).$$

This completes the proof of Theorem 13.1. □

**Remark 13.2.** (a) The result of Theorem 13.1 is valid if we replace the semigroup $\mathrm{GV}_\Phi(I;K)$ by the semigroup $\mathrm{Lip}(I;K)$. We omit the details.

(b) If in Theorem 13.1 $K$ is a linear subspace of $X$, the operator $H_1(t)(\cdot)$ from (13.10) is single-valued for all $t \in I$, since it is $*$-additive, and if $x \in K$, then $(-x) \in K$, and so,

$$H_1(t)(x) \overset{*}{+} H_1(t)(-x) = H_1(t)(x + (-x)) = H_1(t)(0) = \{0\}.$$

(c) The representation of the form (13.2), $H(t,x) = H_0(t) + H_1(t)x$, for generators of Lipschitzian superposition operators on the classical space of Lipschitzian functions was found by Matkowski [71], [72]. In different spaces of functions and mappings it was shown to be valid for single-valued superposition operators ([73], [74], [76], [78], [21], [22], [29], [30], [34]) and set-valued superposition operators ([105], [77], [106], [19], [23], [25]). The above Theorem 13.1 extends the results of [77], [78] and [23]. Theorem 13.3 below generalizes the results of [74], [113] and [23].

Let $(Y, \|\cdot\|)$ be a Banach space. Then, by Theorem 4.2, the set $\mathrm{BV}(I;\mathrm{cbc}(Y))$ is a complete metric semigroup equipped with metric $D_1$ defined by (4.4)–(4.6) with $\Phi(\rho)=\rho$. Suppose that a multivalued mapping $H : I \times K \to \mathrm{cbc}(Y)$ is such that $H(\cdot, x)$ is in $\mathrm{BV}(I;\mathrm{cbc}(Y))$ for all $x \in K$. Since $Y$ is complete, $(\mathrm{cbc}(Y), D)$ is a complete metric space (cf. [12, Theorem II-14]), so that any mapping from $\mathrm{BV}(I;\mathrm{cbc}(Y))$ has one-sided limits at each point of $I$. The *left regularization* $H^- : I \times K \to \mathrm{cbc}(Y)$ of $H$ is defined by

$$H^-(t,x) = \lim_{s\to t-0} H(s,x) \text{ if } a < t \le b \quad \text{and} \quad H^-(a,x) = \lim_{t\to a+0} H^-(t,x)$$

for all $x \in K$, where the limits are taken with respect to the Hausdorff metric $D$ on $\mathrm{cbc}(Y)$. Let $\mathrm{BV}^-(I;\mathrm{cbc}(Y))$ denote the subspace of $\mathrm{BV}(I;\mathrm{cbc}(Y))$ consisting of all mappings which are left continuous on $(a,b]$. Then $H^-(\cdot,x) \in \mathrm{BV}^-(I;\mathrm{cbc}(Y))$ for all $x \in K$.

**Theorem 13.3.** *Suppose the hypotheses of Theorem* 13.1 *are fulfilled. If $Y$ is a real Banach space and $\mathcal{H}$ maps $\mathrm{Lip}(I;K)$ or $\mathrm{GV}_\Phi(I;K)$ with $\Phi \in \mathcal{N}$ into $\mathrm{BV}(I;\mathrm{cbc}(Y))$ and is Lipschitzian, then $H(t,\cdot) \in \mathrm{Lip}(K;\mathrm{cbc}(Y))$ for all $t \in I$ and there exist two mappings $H_0 \in \mathrm{BV}^-(I;\mathrm{cbc}(Y))$ and $H_1 : I \to \mathrm{L}(K;\mathrm{cbc}(Y))$ with the property that $H_1(\cdot)x \in \mathrm{BV}^-(I;\mathrm{cbc}(Y))$ for all $x \in K$ such that $H^-(t,x) = H_0(t) \overset{*}{+} H_1(t)x$ whenever $t \in I$, $x \in K$. Conversely, if $H_0 \in \mathrm{BV}(I;\mathrm{cbc}(Y))$, $H_1 \in \mathrm{BV}(I;\mathrm{L}(K;\mathrm{cbc}(Y)))$ and $H$ is given by* (13.2), *then $\mathcal{H}$ maps $\mathrm{BV}(I;K)$ into $\mathrm{BV}(I;\mathrm{cbc}(Y))$ and is Lipschitzian.*

**Proof.** Let $\mathcal{H} \in \mathrm{Lip}\Big(\mathrm{Lip}(I;K);\mathrm{BV}(I;\mathrm{cbc}(Y))\Big)$. As in the proof of Theorem 13.1 (with $\Psi(\rho) = \rho$) we get the inequality (13.3) where $\omega_\Psi(\beta-\alpha)$ is replaced by 1 and $d_\Phi(f_1,f_2)$ is replaced by $d_L(f_1,f_2)$. Substituting Lipschitzian mappings $f_j$ from (13.5) with $\alpha = a$ and $\beta = t \in (a,b]$ and from

(13.8) with $\alpha \in [a, b)$ and $\beta = b$ into (13.3) we have:

$$D\big(H(t,x_1), H(t,x_2)\big) \le \mu\|x_1 - x_2\|/(t-a), \quad a < t \le b, \tag{13.20}$$

$$D\big(H(\alpha,x_1), H(\alpha,x_2)\big) \le \mu\Big(1 + \frac{1}{b-\alpha}\Big)\|x_1 - x_2\|, \ a \le \alpha < b, \tag{13.21}$$

respectively. This proves that $H(t,\cdot) \in \mathrm{Lip}(K; \mathrm{cbc}(Y))$ for all $t \in I$. Passing to the left limits in (13.20) and to the limits as $\alpha \to t-0$ with $t > a$ and, then, as $t \to a+0$ in (13.21), we obtain inequalities (13.20) and (13.21) with $H$ replaced by $H^-$ and $\alpha$ by $a$, and so, $H^-(t,\cdot) \in \mathrm{Lip}(K; \mathrm{cbc}(Y))$ for all $t \in I$.

In order to prove the validity of the representation for $H^-(t,x)$, let $a < t \le b$, $n \in \mathbb{N}$ and $a < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_n < \beta_n < t$. Since $\mathcal{H}$ is Lipschitzian, we have

$$\sum_{i=1}^{n} D\Big(H(\beta_i, f_1(\beta_i)) \overset{*}{+} H(\alpha_i, f_2(\alpha_i)), H(\beta_i, f_2(\beta_i)) \overset{*}{+} H(\alpha_i, f_1(\alpha_i))\Big)$$
$$\le \mu\, d_L(f_1, f_2)$$

whenever $f_1, f_2 \in \mathrm{Lip}(I; K)$. Substituting into this inequality Lipschitzian mappings $f_j : I \to K$, $j = 1, 2$, defined by

$$f_j(s) = \frac{1}{2}\Big(\eta_n(s)(x_1 - x_2) + x_j + x_2\Big), \quad s \in I, \ x_j \in K, \ j = 1, 2,$$

where $\eta_n \in \mathrm{Lip}([a,b]; [0,1])$ is given by

$$\eta_n(s) = \begin{cases} 0 & \text{if } a \le s \le \alpha_1, \\ \eta_{\alpha_i,\beta_i}(s) & \text{if } \alpha_i \le s \le \beta_i, \ i = 1, \dots, n, \\ 1 - \eta_{\beta_i,\alpha_{i+1}}(s) & \text{if } \beta_i \le s \le \alpha_{i+1}, \ i = 1, \dots, n-1, \\ 1 & \text{if } \beta_n \le s \le b, \end{cases}$$

and $\eta_{\alpha,\beta}$ is defined in (13.6), we get

$$\sum_{i=1}^{n} D\Big(H(\beta_i, x_1) \overset{*}{+} H(\alpha_i, x_2), H\Big(\beta_i, \frac{x_1 + x_2}{2}\Big) \overset{*}{+} H\Big(\alpha_i, \frac{x_1 + x_2}{2}\Big)\Big)$$
$$\le \mu\, \|x_1 - x_2\|/2.$$

Since $H(\cdot, x) = \mathcal{H}(x) \in \mathrm{BV}(I; \mathrm{cbc}(Y))$, $H^-(\cdot, x) \in \mathrm{BV}^-(I; \mathrm{cbc}(Y))$ for all $x \in K$. By the continuity of $\overset{*}{+}$ (cf. (12.7)) on $\mathrm{cbc}(Y)$ and definition of $H^-$, passing to the limit as $\alpha_1 \to t-0$ in the last inequality we have, for all $t \in (a, b]$,

$$D\Big(H^-(t, x_1) \overset{*}{+} H^-(t, x_2), H^-\Big(t, \frac{x_1 + x_2}{2}\Big) \overset{*}{+} H^-\Big(t, \frac{x_1 + x_2}{2}\Big)\Big)$$
$$\le \mu\, \|x_1 - x_2\|/2n,$$

and so, as $n \to \infty$, we get:

$$D\Big(H^-(t,x_1) \overset{*}{+} H^-(t,x_2), H^-\Big(t, \frac{x_1+x_2}{2}\Big) \overset{*}{+} H^-\Big(t, \frac{x_1+x_2}{2}\Big)\Big) = 0.$$

By definition of $H^-$ this equality also holds at $t = a$. Making use of the arguments between (13.9) and (13.10), we arrive at the representation $H^-(t,x) = H_0(t) \overset{*}{+} H_1(t)x$ with $H_0(t) \in \mathrm{cbc}(Y)$ and $*$-additive set-valued operator $H_1(t)(\cdot) : K \to \mathrm{cbc}(Y)$, $t \in I$. From this representation, since $H^-(t,\cdot)$ is Lipschitzian, by virtue of translation invariance of $D$ we find that the operator $H_1(t)(\cdot)$ is continuous, and since it is also $*$-additive, it is linear, so that $H_1$ maps $I$ into $\mathrm{L}(K;\mathrm{cbc}(Y))$. Since $H_1(t)(0) = \{0\}$, the above representation implies $H^-(t,0) = H_0(t)$ for all $t \in I$. Hence, $H_0 \in \mathrm{BV}^-(I;\mathrm{cbc}(Y))$.

That $H_1(\cdot)x \in \mathrm{BV}^-(I;\mathrm{cbc}(Y))$ for all $x \in K$ follows in a similar manner as in the last paragraph of step 3 of the proof of Theorem 13.1; observe only that instead of (13.11) we get the inequality

$$D\Big(H_1(t)x, H_1(s)x\Big) \le D\Big(H^-(t,x), H^-(s,x)\Big) + D\Big(H_0(t), H_0(s)\Big),$$

in which $H^-(\cdot,x)$ and $H_0$ belong to $\mathrm{BV}^-(I;\mathrm{cbc}(Y))$. The case when $\mathrm{Lip}(I;K)$ above is replaced by $\mathrm{GV}_\Phi(I;K)$ is treated similarly.

The converse assertion is a consequence of steps 5–8 of the proof of Theorem 13.1: replace $\mathrm{GV}_\Psi$ by BV, $\omega_\Psi(|I|)$ — by 1 and $p_\Psi(\cdot)$ — by $V(\cdot)$. $\square$

## 14. Linear functional operator inclusion

**Example 14.1.** Let $I = [a,b]$, $(X, \|\cdot\|)$ be a linear normed space, $K \subset X$ be a closed convex cone, $\Psi \in \mathcal{N}$, $H_0 \in \mathrm{GV}_\Psi(I;\mathrm{cc}(K))$, $H_1 \in \mathrm{L}(K;\mathrm{cc}(K))$ with $L(H_1) < 1$, and there exists $\mathcal{K} \in \mathrm{c}(X)$ such that $H_1 x \subset \mathcal{K}$ for all $x \in K$. Set $F(t,x) = H_0(t) + H_1 x$, $t \in I$, $x \in K$. Then $F$ satisfies conditions of Theorem 11.1. In fact, if $t$, $s \in I$ and $x$, $y \in K$, then, by (4.2), we have:

$$\begin{aligned} D(F(t,x), F(s,y)) &= D\Big(H_0(t) + H_1 x, H_0(s) + H_1 y\Big) \\ &\le D(H_0(t), H_0(s)) + D(H_1 x, H_1 y) \\ &\le D(H_0(t), H_0(s)) + L(H_1)\|x - y\|. \end{aligned}$$

Setting $\varphi(t) = V(H_0, [a,t])$, $t \in I$, from remarks on the structural theorem for $\mathrm{GV}_\Phi$ on p. 22 we find that $\varphi \in \mathrm{GV}_\Psi(I;\mathbb{R}) \subset \mathrm{BV}(I;\mathbb{R})$, which provides the estimate $D(H_0(t), H_0(s)) \le |\varphi(t) - \varphi(s)|$. In order to verify condition (ii), it suffices to put $K(t) = H_0(t) + \mathcal{K}$, $t \in I$. Thus, under the hypotheses above, if $x_0 \in K$ is such that $x_0 \in H_0(a) + H_1 x_0$, then by Theorems 11.1–11.2 there exists a mapping $f \in \mathrm{GV}_\Psi(I;K)$ such that $f(t) \in H_0(t) + H_1 f(t)$ for all $t \in I$ and $f(a) = x_0$. (Since $L(H_1) < 1$, a point $x_0$ satisfying

$x_0 \in H_0(a)+H_1x_0$ always exists by virtue of Banach's contraction mapping principle for set-valued mappings, cf. [70], [85], [49, Theorem 15.1].)

The purpose of this section is to prove the existence of solutions $f$ to the linear functional operator inclusion $f(t) \in H_0(t) + H_1(t)f(t)$, $t \in I$, for a *variable* set-valued operator $H_1$ such that $H_1 \in \mathrm{GV}_\Psi(I; \mathrm{L}(K; \mathrm{cc}(K)))$.

**Theorem 14.2.** *Let $I = [a, b]$, $(X, \|\cdot\|)$ be a real Banach space, $K \subset X$ a closed convex cone, $D$ the Hausdorff metric generated by $d(x, y) = \|x - y\|$ $(x,\, y \in X)$, $\Psi \in \mathcal{N}$, $H_0 \in \mathrm{GV}_\Psi(I; \mathrm{cc}(K))$ and $H_1 \in \mathrm{GV}_\Psi(I; \mathrm{L}(K; \mathrm{cc}(K)))$. Let $\gamma(\Psi)|||H_1|||_\Psi < 1$ (see notation (13.18)). Suppose that for each $t \in I$ there exists $\mathcal{K}(t) \in \mathrm{c}(X)$ such that $H_1(t)x \subset \mathcal{K}(t)$ for all $x \in K$. Let $t_0 = a \in I$ and $x_0 \in K$ be such that $x_0 \in H_0(a) + H_1(a)x_0$. Then there exists a mapping $f \in \mathrm{GV}_\Psi(I; K)$ satisfying*:

(a) *$f(t) \in H_0(t) + H_1(t)f(t)$ for all $t \in I$;*
(b) *$f(a) = x_0$, and*
(c) *$p_{\Psi,d}(f) \le \big(p_{\Psi,D}(H_0) + p_{\Psi,D_L}(H_1)\|x_0\|\big)\big/\big(1 - \gamma(\Psi)|||H_1|||_\Psi\big)$.*

**Proof.** Given $f \in K^I$, define a superposition operator by $(\mathcal{H}f)(t) = H_0(t)+ H_1(t)f(t)$, $t \in I$. As is shown in steps 5–7 of the proof of Theorem 13.1, $\mathcal{H}$ maps $\mathrm{GV}_\Psi(I;K)$ into $\mathrm{GV}_\Psi(I; \mathrm{cc}(K))$, and the estimates (13.13) and (13.15) hold. Taking into account inequalities (13.15), (13.13), (13.17), a consequence of (3.24):

$$\sup_{t\in I} \|f(t)\| \le \|f(a)\| + V(f, I) \le \|f(a)\| + \omega_\Psi(|I|)p_{\Psi,d}(f),$$

and notation (13.18), for each $f \in \mathrm{GV}_\Psi(I;K)$ we find

$$\begin{aligned}
p_{\Psi,D}(\mathcal{H}f) &\le p_{\Psi,D}(H_0) + p_{\Psi,D}(H_1f)\\
&\le p_{\Psi,D}(H_0) + \Big(\sup_{t\in I} L(H_1(t))\Big)p_{\Psi,d}(f) + p_{\Psi,D_L}(H_1)\Big(\sup_{t\in I}\|f(t)\|\Big)\\
&\le p_{\Psi,D}(H_0) + \Big(L(H_1(a)) + \omega_\Psi(|I|)p_{\Psi,D_L}(H_1)\Big)\cdot p_{\Psi,d}(f)\\
&\quad + p_{\Psi,D_L}(H_1)\cdot\Big(\|f(a)\| + \omega_\Psi(|I|)p_{\Psi,d}(f)\Big)\\
&= p_{\Psi,D}(H_0) + p_{\Psi,D_L}(H_1)\|f(a)\| + L(H_1(a))p_{\Psi,d}(f)\\
&\quad + 2\omega_\Psi(|I|)p_{\Psi,D_L}(H_1)p_{\Psi,d}(f)\\
&\le p_{\Psi,D}(H_0) + p_{\Psi,D_L}(H_1)\|f(a)\| + \gamma(\Psi)|||H_1|||_\Psi\, p_{\Psi,d}(f).
\end{aligned}$$

Therefore, the following *a priori* estimate holds:

$$p_{\Psi,D}(\mathcal{H}f) \;\le\; C_0 + C_1\|f(a)\| + \mu\, p_{\Psi,d}(f), \quad f \;\in\; \mathrm{GV}_\Psi(I;K), \tag{14.1}$$

where $C_0 = p_{\Psi,D}(H_0)$, $C_1 = p_{\Psi,D_L}(H_1)$ and $\mu = \gamma(\Psi)|||H_1|||_\Psi < 1$.

Since $(\mathcal{H}x_0)(t)=H_0(t)+H_1(t)x_0$, $t\in I$, then $\mathcal{H}x_0\in\mathrm{GV}_\Psi(I;\mathrm{cc}(K))$, $x_0\in(\mathcal{H}x_0)(a)$, and by (14.1) we have

$$p_{\Psi,D}(\mathcal{H}x_0)\le C_0+C_1\|x_0\|\equiv C.$$

By Theorem 9.2 (if $\Psi\in\mathcal{N}_\infty$) or by Theorem 9.1 (if $\Psi\in\mathcal{N}\setminus\mathcal{N}_\infty$) there exists a mapping $f_1\in\mathrm{GV}_\Psi(I;K)$ such that $f_1(t)\in(\mathcal{H}x_0)(t)\subset H_0(t)+\mathcal{K}(t)$ for all $t\in I$, $f_1(a)=x_0$ and $p_{\Psi,d}(f_1)\le p_{\Psi,D}(\mathcal{H}x_0)\le C$. Now for mapping $\mathcal{H}f_1$ we have $(\mathcal{H}f_1)(t)=H_0(t)+H_1(t)f_1(t)$, $t\in I$, and so, $\mathcal{H}f_1\in\mathrm{GV}_\Psi(I;\mathrm{cc}(K))$, $x_0\in(\mathcal{H}x_0)(a)=(\mathcal{H}f_1)(a)$, and again by (14.1),

$$\begin{aligned}p_{\Psi,D}(\mathcal{H}f_1)&\le C_0+C_1\|f_1(a)\|+\mu\,p_{\Psi,d}(f_1)\\&\le C+\mu\,C=(1+\mu)C.\end{aligned}$$

Applying Theorems 9.1 and 9.2, we find a mapping $f_2\in\mathrm{GV}_\Psi(I;K)$ such that $f_2(t)\in(\mathcal{H}f_1)(t)\subset H_0(t)+\mathcal{K}(t)$ for all $t\in I$, $f_2(a)=x_0$ and

$$p_{\Psi,d}(f_2)\le p_{\Psi,D}(\mathcal{H}f_1)\le(1+\mu)C.$$

Similarly, $\mathcal{H}f_2\in\mathrm{GV}_\Psi(I;\mathrm{cc}(K))$, $x_0\in(\mathcal{H}f_2)(a)$, and by (14.1),

$$\begin{aligned}p_{\Psi,D}(\mathcal{H}f_2)&\le C_0+C_1\|f_2(a)\|+\mu\,p_{\Psi,d}(f_2)\\&\le C+\mu\,(1+\mu)C=(1+\mu+\mu^2)C.\end{aligned}$$

By induction, for each $n\in\mathbb{N}$ there exists $f_n\in\mathrm{GV}_\Psi(I;K)$, satisfying (where $f_0(t)\equiv x_0$):

$f_n(t)\in(\mathcal{H}f_{n-1})(t)=H_0(t)+H_1(t)f_{n-1}(t)\subset H_0(t)+\mathcal{K}(t),\quad t\in I,$

$f_n(a)=x_0$, and

$p_{\Psi,d}(f_n)\le\Bigl(\sum_{i=0}^{n-1}\mu^i\Bigr)C\le C/(1-\mu).$

It follows that the sequence $\{f_n\}_{n=1}^\infty\subset\mathrm{GV}_\Psi(I;K)$ is pointwise precompact and is of uniformly bounded $\Psi$-variation on $I$. By virtue of (3.24), Theorem 1.3 and Lemma 3.9(e) we may assume (passing to a subsequence if necessary) that the sequence converges pointwise on $I$ to a mapping $f\in\mathrm{GV}_\Psi(I;K)$. It remains to verify condition (a). Applying inequality (11.6), for $t\in I$ we have:

$$\begin{aligned}&\Bigl|\mathrm{dist}\Bigl(f(t),H_0(t)+H_1(t)f(t)\Bigr)\Bigr|\\&=\Bigl|\mathrm{dist}\Bigl(f(t),(\mathcal{H}f)(t)\Bigr)-\mathrm{dist}\Bigl(f_n(t),(\mathcal{H}f_{n-1})(t)\Bigr)\Bigr|\\&\le\|f(t)-f_n(t)\|+D\Bigl((\mathcal{H}f)(t),(\mathcal{H}f_{n-1})(t)\Bigr)\\&=\|f(t)-f_n(t)\|+D\Bigl(H_0(t)+H_1(t)f(t),H_0(t)+H_1(t)f_{n-1}(t)\Bigr)\\&\le\|f(t)-f_n(t)\|+L(H_1(t))\|f(t)-f_{n-1}(t)\|\to0,\quad n\to\infty,\end{aligned}$$

whence $\operatorname{dist}\Big(f(t), H_0(t)+H_1(t)f(t)\Big)=0$, which was to be proved. □

Observe that the point $t_0 \in I$ in Theorem 14.2 can be arbitrarily chosen since in the respective definitions (4.4)–(4.6) we may set $a = t_0$.

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