

Self attracting diffusions on a sphere and application to a periodic case*

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Abstract

This paper proves almost-sure convergence for the self attracting diffusion on the unit sphere

$$dX_t = \nu \circ dW_t(X_t) - a \int_0^t \nabla_{\mathbb{S}^n} V_{X_s}(X_t) ds dt, \quad X_0 = x \in \mathbb{S}^n,$$

where $\nu > 0$, $a < 0$, $V_y(x) = \langle x, y \rangle$ is the usual scalar product on \mathbb{R}^{n+1} , \circ stands for the Stratonovich differential and $(W_t(\cdot))_{t \geq 0}$ is a Brownian vector field on \mathbb{S}^n . From this we deduce the almost-sure convergence of the real-valued self attracting diffusion

$$d\vartheta_t = \nu dW_t + a \int_0^t \sin(c(\vartheta_t - \vartheta_s)) ds dt,$$

where $(W_t)_{t \geq 0}$ is a real Brownian motion and $c > 0$.

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1 Introduction

In this paper, we are interested in the asymptotic behaviour of the solution of the stochastic differential equation (SDE)

$$dX_t = \nu \circ dW_t(X_t) - a \int_0^t \nabla_{\mathbb{S}^n} V_{X_s}(X_t) ds dt, \quad X_0 = x \in \mathbb{S}^n, \quad (1.1)$$

where $\nu > 0$, $a \in \mathbb{R}$, \circ stands for the Stratonovich differential, $(W_t(\cdot))_{t \geq 0}$ is a Brownian vector field on \mathbb{S}^n , $\nabla_{\mathbb{S}^n}$ is the gradient on \mathbb{S}^n and $V_y(x) = \langle x, y \rangle$ where $\langle \cdot, \cdot \rangle$ is the canonical scalar product on \mathbb{R}^{n+1} .

Let us start with a short heuristic description of the process. First of all, observe that for $x, y \in \mathbb{S}^n$, we have

$$\|x - y\|^2 = 2 - 2\langle x, y \rangle = 2 - 2\cos(D(x, y)), \quad (1.2)$$

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where $D(\cdot, \cdot)$ is the geodesic distance on S^n and $\|\cdot\|$ is the standard Euclidean norm on \mathbb{R}^{n+1} . If $a > 0$, the drift term points in a direction that tends to increase the distance between X_t and its past positions. In other words X_t is repelled by its past. It follows from a more general result proved in [2] dealing with self repelling diffusions on a compact manifold that

Theorem 1.1 (Theorem 5, [2], Benaïm, Gauthier). *If $a > 0$, the law of X_t converges to the uniform law on S^n .*

If $a < 0$, X_t is attracted by its past and one may expect localization. The goal of this paper is to prove such a result.

Theorem 1.2. *If $a < 0$, there exists a random variable $X_\infty \in S^n$ such that*

$$\|X_t - X_\infty\| = \begin{cases} O(t^{-1/2} \sqrt{\ln(t)}) & \text{if } n = 1 \\ O((\frac{\ln(t)}{t})^{-1/4}) & \text{otherwise} \end{cases} .$$

We point out that the self interacting diffusion (1.1) has already received some attention in 2002 by M.Benaïm, M.Ledoux and O.Raimond ([4]), but in the normalized case; that is, when $\int_0^t V_{X_s}(X_t) ds$ is replaced by $\frac{1}{t} \int_0^t V_{X_s}(X_t) ds$. The interpretation is therefore different. While the drift term of (1.5) can be “seen” as a summation over $[0, t]$ of the interaction between the current position X_t and its position at time s and thus an accumulation of the interacting force, their drift is then an average of the interacting force. The asymptotic behaviour is then given by the following Theorem.

Theorem 1.3 (Theorem 4.5, [4], Benaïm, Ledoux, Raimond). *For $a \neq 0$, let $(X_t)_{t \geq 0}$ be the solution of the SDE*

$$dX_t = \circ dW_t(X_t) - \frac{a}{t} \int_0^t \nabla_{S^n} V_{X_s}(X_t) ds dt, \quad X_0 = x \in S^n. \tag{1.3}$$

Set $\mu_t = \frac{1}{t} \int_0^t \delta_{X_s} ds$.

1. *If $a \geq -(n + 1)/2$, then μ_t converges almost surely (for the topology of weak* convergence) toward the Riemannian probability measure on S^n .*
2. *If $a < -(n + 1)/2$, then there exists a random variable $\zeta \in S^n$ such that μ_t converges almost surely toward the measure*

$$\mu_{c,\zeta}(dx) = \frac{\exp(\beta(a)\langle x, \zeta \rangle)}{Z_a},$$

where Z_a is the normalization constant, $\beta(a)$ is the unique positive solution to the implicit equation

$$2a\Lambda'_n(\beta) + \beta = 0,$$

where $\Lambda_n(\beta) = \log(\int_0^\pi \exp(-\beta \cos(x)) \lambda_n(dx))$ and $\lambda_n(dx) = \frac{(\sin(x))^{n-1}}{\int_0^\pi (\sin(x))^{n-1} dx} dx$.

An intermediate framework between those considered in Theorem 1.2 and Theorem 1.3 is to add a time-dependent weight $g(t)$ to the normalized case that increases to infinity, but “not too fast”, when time increases. In that case, O.Raimond proved the following Theorem.

Theorem 1.4 (Theorem 3.1, [12], Raimond). *Let $(X_t)_{t \geq 0}$ be the solution of the SDE*

$$dX_t = \circ dW_t(X_t) - \frac{g(t)}{t} \int_{S^n} V_{X_s}(X_t) ds dt, \quad X_0 = x \in S^n, \tag{1.4}$$

where g is an increasing function such that $\lim_{t \rightarrow \infty} g(t) = \infty$. Assume that there exists positive constants c, t_0 such that for $t \geq t_0$, $g(t) \leq c \log(t)$ and $|g'(t)| = O(t^{-\gamma})$, with $\gamma \in]0, 1]$.

Then, there exists a random variable X_∞ in \mathbb{S}^n such that almost surely, $\mu_t = \frac{1}{t} \int_0^t \delta_{X_s} ds$ converges weakly towards δ_{X_∞} .

As an easy application of Theorem 1.2, we obtain the almost-sure convergence with a rate of convergence of the solution of the real-valued SDE

$$d\vartheta_t = \nu dW_t + a \int_0^t \sin(c(\vartheta_t - \vartheta_s)) ds dt, \quad \vartheta_0 = 0, \tag{1.5}$$

where $(W_t)_t$ is a real Brownian motion, $a < 0$ and $\nu, c > 0$.

In 1995, M.Cranston and Y. Le Jan proved an almost-sure convergence result in [5] in the cases where $a \sin(cx)$ is replaced by $f(x) = ax$ (linear case) or $f(x) = a \times \text{sgn}(x)$ (constant case) with $a < 0$. This last case was extended in all dimension by O.Raimond in [11] in 1997. A few years later, S.Herrmann and B.Roynette weakened the condition of the profile function f around 0 and were still able to get almost-sure convergence (see [6]) for the solution of the stochastic differential equation

$$d\vartheta_t = \nu dW_t + \int_0^t f(\vartheta_t - \vartheta_s) ds dt. \tag{1.6}$$

Rate of convergence were given in [7] by S.Herrmann and M.Scheutzow. For the linear case, they proved that the optimal rate of convergence is $O(t^{-1/2} \sqrt{\log(t)})$ (Proposition 4 in [7]).

However, a common fundamental property of these three papers lies in the fact that the associated profile function f is monotone.

1.1 Reformulation of the problem

From now on, we assume that $a < 0$ and that n is fixed. Since the values of ν and a do not play any particular role, we assume without loss of generality that $\nu = 1$ and $a = -1$. Thus (1.1) becomes

$$dX_t = \circ dW_t(X_t) + \int_0^t \nabla_{\mathbb{S}^n} V_{X_s}(X_t) ds dt, \quad X_0 = x \in \mathbb{S}^n \tag{1.7}$$

with $V_y(x) = \langle x, y \rangle =: V(x, y)$. Because the law of the process $(X_t)_{t \geq 0}$ is the same for any Brownian vector field on \mathbb{S}^n , we assume from now on and without loss of generality that $W_t(x) = B_t - \langle x, B_t \rangle x$, where $(B_t)_{t \geq 0}$ is a standard Brownian motion on \mathbb{R}^{n+1} .

Since V satisfies Hypothesis 1.3 and 1.4 in [4], then (1.7) admits a unique strong solution by Proposition 2.5 in [4]. We recall that for a function $F : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$, we have

$$\nabla_{\mathbb{S}^n}(F|_{\mathbb{S}^n})(x) = \nabla_{\mathbb{R}^{n+1}} F(x) - \langle x, \nabla_{\mathbb{R}^{n+1}} F(x) \rangle x; \quad x \in \mathbb{S}^n. \tag{1.8}$$

For $x \in \mathbb{S}^n$, we let $u \mapsto P(x, u)$ be the orthogonal projection on $T_x \mathbb{S}^n$ given by

$$P(x, u) = u - \langle x, u \rangle x.$$

Following the same idea as in [2], we set $U_t := \int_0^t X_s ds \in \mathbb{R}^{n+1}$ in order to get the SDE on $\mathbb{S}^n \times \mathbb{R}^{n+1}$:

$$\begin{cases} dX_t = P(X_t, \circ dB_t + U_t dt) \\ dU_t = X_t dt \end{cases} \tag{1.9}$$

with initial condition $(X_0, U_0) = (x, 0)$.

Remark 1.5. We have $P(X_t, \circ dB_t + U_t dt) = P(X_t, dB_t + U_t dt) - \frac{n}{2} X_t dt$,

The paper is organised as follows. In Section 2, we present the detailed strategy used for proving Theorem 1.2 and prove the application to a periodic case whereas the more technical proofs are presented in Section 3.

2 Guideline of the proof of Theorem 1.2

Set $R_t = \|U_t\|$ and define $V_t \in \mathbb{S}^n$ and $C_t \in [-1, 1]$ as follows:

$$V_t = \begin{cases} U_t/R_t & \text{if } R_t > 0 \\ X_t & \text{otherwise} \end{cases} \quad (2.1)$$

and

$$C_t = \langle V_t, X_t \rangle. \quad (2.2)$$

With these notations, we have

$$R_t C_t = \langle U_t, X_t \rangle. \quad (2.3)$$

Since the coordinates functions

$$e_j : \mathbb{S}^n \subset \mathbb{R}^{n+1} \rightarrow \mathbb{R} : x \mapsto x_j, \text{ for } j = 1, \dots, n+1,$$

are eigenfunctions for the Laplacian operator on \mathbb{S}^n associated to the eigenvalue $-n$ (see Chapter 3, Section C in [1]), then by Lemma 3 and Lemma 5 in section 4 of [2], the system (1.9) satisfies the Hörmander condition (also called condition (E) in [2] and [8]).

Thus, for all $t > 0$, the law of (X_t, U_t) has a smooth density with respect to the Lebesgue measure on $\mathbb{S}^n \times \mathbb{R}^{n+1}$ (see Theorem 3.(i) in [8]). Hence for all $t > 0$,

$$\mathbb{P}(C_t^2 = 1 \text{ or } R_t = 0) = \mathbb{P}(U_t \text{ is parallel to } X_t) = 0. \quad (2.4)$$

Since $\int_0^t \langle P(X_s, dB_s), V_s \rangle$ is a martingale whose quadratic variation is $\int_0^t (1 - C_s^2) ds$, then the process $(W_t)_{t \geq 0}$ defined by $W_0 = 0$ and, for $t > 0$, by

$$W_t = \int_0^t \mathbf{1}_{\{C_s^2 < 1 \text{ and } R_s > 0\}} \frac{\langle P(X_s, dB_s), V_s \rangle}{\sqrt{1 - C_s^2}}, \quad (2.5)$$

is a standard Brownian motion on \mathbb{R} .

Lemma 2.1. $((C_t, R_t))_{t \geq 0}$ is solution to

$$\begin{cases} dC_t = \sqrt{1 - C_t^2} dW_t + [(R_t + \frac{1}{R_t})(1 - C_t^2) - \frac{n}{2} C_t] dt \\ dR_t = C_t dt \end{cases} \quad (2.6)$$

whenever $R_t > 0$.

Proof. Since $dR_t^2 = 2\langle U_t, dU_t \rangle = 2R_t C_t dt$, then, as long as $R_t > 0$, we have

$$dR_t = C_t dt. \quad (2.7)$$

Hence,

$$dV_t = \frac{1}{R_t} (X_t - C_t V_t) dt. \quad (2.8)$$

Therefore, by Itô's formula

$$\begin{aligned} dC_t &= \langle X_t, dV_t \rangle + \langle V_t, P(X_t, dB_t + U_t dt) \rangle - \frac{n}{2} \langle V_t, X_t \rangle dt \\ &= \sqrt{1 - C_t^2} dW_t + (R_t + \frac{1}{R_t})(1 - C_t^2) dt - \frac{n}{2} C_t dt. \end{aligned} \quad (2.9)$$

□

A first important result, whose proof is postponed to Section 3, is

Lemma 2.2. One has $\liminf_{t \rightarrow \infty} \frac{R_t}{\sqrt{t}} \geq \frac{2}{\sqrt{n}}$ almost surely.

From this lemma, we prove in Section 3

Lemma 2.3. *The processes $(C_t)_{t \geq 0}$ and $(\frac{R_t}{t})_{t > 0}$ converge almost surely to 1. Furthermore*

$$|C_t - 1| = \begin{cases} O(\frac{\ln(t)}{t}) & \text{if } n = 1 \\ O(\sqrt{\frac{\ln(t)}{t}}) & \text{otherwise} \end{cases} .$$

Thanks to this lemma, we obtain

Lemma 2.4. *V_t converges almost surely.*

Proof. Since

$$\|X_t - C_t V_t\| = \sqrt{1 - C_t^2}, \tag{2.10}$$

it follows from Lemma 2.3

$$\frac{1}{R_t} \|X_t - C_t V_t\| = O(t^{-5/4} \ln^{1/4}(t)), \tag{2.11}$$

which is an integrable quantity. Hence the result follows from (2.8) and (2.11). □

We can now prove the main result.

Proof of Theorem 1.2

By Lemmas 2.3 and 2.4, there exists a random variable $X_\infty \in \mathbb{S}^n$ such that $\lim_{t \rightarrow \infty} C_t V_t = X_\infty$.

The rate of convergence follows from the triangle inequality, (2.8), (2.10), (2.11) and Lemma 2.3.

As an application of Theorem 1.2, we have the following result.

Theorem 2.5. *Let $(\vartheta_t)_{t \geq 0}$ be the solution of the SDE*

$$d\vartheta_t = \nu dW_t + a \int_0^t \sin(c(\vartheta_t - \vartheta_s)) ds dt, \quad \vartheta_0 = 0, \tag{2.12}$$

where $(W_t)_t$ is a real Brownian motion, $a < 0$ and $\nu, c > 0$. Then there exists a random variable ϑ_∞ such that $|\vartheta_t - \vartheta_\infty| = O(\sqrt{\frac{\ln(t)}{t}})$.

Proof. First of all (2.12) admits a unique strong solution because the function $\sin(\cdot)$ is Lipschitz continuous (see for example Proposition 1 in [6]).

Set $\vartheta_t^{(c)} = c\vartheta_t$. Hence $(\vartheta_t^{(c)})_{t \geq 0}$ solves the SDE

$$d\vartheta_t^{(c)} = c\nu dW_t + ac \int_0^t \sin(\vartheta_t^{(c)} - \vartheta_s^{(c)}) ds dt, \quad \vartheta_0^{(c)} = 0. \tag{2.13}$$

Letting $X_t = (\cos(\vartheta_t^{(c)}), \sin(\vartheta_t^{(c)}))$, it follows that $(X_t)_{t \geq 0}$ is a solution of (1.1) when $n = 1$. Because $ac < 0$, there exists, by Theorem 1.2, $X_\infty \in \mathbb{S}^1$ such that

$$\|X_t - X_\infty\| = O(t^{-1/2} \sqrt{\log(t)}).$$

The result follows from the continuity of $t \mapsto \vartheta_t^{(c)}$. □

3 Proofs of Lemma 2.2 and Lemma 2.3

3.1 Proof of Lemma 2.2

Set $M_t = -2 \int_0^t R_s \sqrt{1 - C_s^2} dW_s$, where W_t is defined by (2.5), and let

$$\mathcal{E}_M(t) = \exp \left(M_t - \frac{1}{2} \langle M \rangle_t \right). \tag{3.1}$$

Because

$$\langle M \rangle_t = 4 \int_0^t R_s^2 (1 - C_s^2) ds \leq 4 \int_0^t R_s^2 ds \leq 4 \int_0^t s^2 ds = \frac{4t^3}{3}, \tag{3.2}$$

M_t satisfies the Novikov Condition (see [10], Chapter V, section D, page 198). Therefore $\mathcal{E}_M(t)$ is a positive martingale having 1 as expectation. Thus, it converges almost surely to a nonnegative integrable random variable $\mathcal{E}_M(\infty)$.

Hence, there exists a random variable $K < \infty$, such that almost surely, for all $t \geq 0$,

$$\ln(\mathcal{E}_M(t)) \leq 2K.$$

By Itô's formula and Lemma 2.1, we have

$$\begin{aligned} d(R_t C_t) &= C_t^2 dt + R_t \sqrt{1 - C_t^2} dW_t + (R_t^2 + 1)(1 - C_t^2) dt - \frac{n}{2} R_t C_t dt \\ &= -\frac{1}{2} dM_t + \frac{1}{4} d\langle M \rangle_t - \frac{n}{2} R_t C_t dt + dt. \end{aligned} \tag{3.3}$$

Since $dR_t^2 = 2R_t C_t dt$, we obtain

$$\begin{aligned} R_t C_t + \frac{n}{4} R_t^2 &= -\frac{1}{2} \ln(\mathcal{E}_M(t)) + t \\ &\geq t - K. \end{aligned} \tag{3.4}$$

Because $C_t \leq 1$, we have for $t > K$

$$\frac{n}{2} R_t \geq -1 + \sqrt{n(t - K)}. \tag{3.5}$$

This completes the proof.

3.2 Proof of Lemma 2.3

Before starting the proof of Lemma 2.3, let us recall the Definition of an asymptotic pseudotrajectory introduced by Benaïm and Hirsch in [3].

Definition 3.1. Let (M, d) be a metric space and Φ a semiflow; that is

$$\Phi : \mathbb{R}_+ \times M \rightarrow M : (t, x) \mapsto \Phi(t, x) = \Phi_t(x)$$

is a continuous map such that

$$\Phi_0 = Id \text{ and } \Phi_{t+s} = \Phi_t \circ \Phi_s$$

for all $s, t \in \mathbb{R}_+$.

A continuous function $X : \mathbb{R}_+ \rightarrow M$ is an asymptotic pseudotrajectory for Φ if

$$\lim_{t \rightarrow \infty} \sup_{0 \leq h \leq T} d(X_{t+h}, \Phi_h(X_t)) = 0 \tag{3.6}$$

for any $T > 0$. In other words, it means that for each fixed $T > 0$, the curve $X : [0, T] \rightarrow M : h \mapsto X_{t+h}$ shadows the Φ -trajectory over the interval $[0, T]$ with arbitrary accuracy for sufficiently large t .

If X is a continuous random process, then X is an almost sure asymptotic pseudotrajectory for Φ if (3.6) holds almost surely.

Theorem 3.2 (Theorem 1.2 in [3]). Suppose that $X([0, \infty))$ has compact closure in M and set $L(X) = \bigcap_{t \geq 0} \overline{X([t, \infty))}$. Let A be an attractor for Φ with basin \mathcal{W} . If $X_{t_k} \in \mathcal{W}$ for some sequence $t_k \rightarrow \infty$, then $L(X) \subset A$.

Let $(x_t^z)_{t \geq 0}$ be the solution of the SDE

$$dx_t^z = h(t, x_t^z)dB_t + g(x_t^z)dt, \quad x_0^z = z \in \mathbb{R} \tag{3.7}$$

where $(B_t)_{t \geq 0}$ is a Brownian motion, $g : \mathbb{R} \rightarrow \mathbb{R}$ is a Lipschitz function and $h : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ a continuous function.

The next Theorem gives a sufficient condition on h to ensure that $(x_t^z)_{t \geq 0}$ is an almost sure asymptotic pseudotrajectory for the flow induced by the ODE

$$\dot{y} = g(y). \tag{3.8}$$

Theorem 3.3 (Proposition 4.1 in [3]). *Assume there exists a non-increasing function $\varepsilon : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $h^2(t, x) \leq \varepsilon(t)$ for all (t, x) and such that*

$$\forall k > 0, \int_0^\infty \exp(-k/\varepsilon(t))dt < \infty^1. \tag{3.9}$$

Then, for all $z \in \mathbb{R}$, $(x_t^z)_{t \geq 0}$ is an almost sure asymptotic pseudotrajectory for the flow induced by (3.8).

Remark 3.4. The same result holds if $(x_t)_{t \geq 0}$ solves the SDE

$$dx_t = h(t, x_t)dB_t + \delta(t)h_2(x_t)dt + g(x_t)dt,$$

where h_2 is a bounded function and δ is a random adapted function with $\lim_{t \rightarrow \infty} \delta(t) = 0$ almost surely.

Proof of Lemma 2.3

The proof is divided into two parts.

Proof of the convergence First we prove that C_t converges almost surely to 1. Recall that

$$dC_t = \sqrt{1 - C_t^2}dW_t + [(R_t + \frac{1}{R_t})(1 - C_t^2) - \frac{n}{2}C_t]dt. \tag{3.10}$$

Define $\alpha(t) = (\frac{3}{2}t)^{\frac{2}{3}}$ so that $\dot{\alpha}(t) = \alpha^{-\frac{1}{2}}(t)$. Set $Z_t = C_{\alpha(t)}$ and $M_t = W_{\alpha(t)}$. Thus $(M_t)_{t \geq 0}$ is a martingale with respect to the filtration $\mathcal{G}_t = \sigma\{W_s \mid 0 \leq s \leq \alpha(t)\}$, whose quadratic variation at time t is $\alpha(t) = \int_0^t (\sqrt{\dot{\alpha}(s)})^2 ds$.

Define $B_t^{(\alpha)} = \int_0^t \frac{dM_s}{\sqrt{\dot{\alpha}(s)}}$, so that $(B_t^{(\alpha)})_{t \geq 0}$ is a Brownian motion. Then

$$Z_t = \int_0^t \sqrt{\dot{\alpha}(s)}\sqrt{1 - Z_s^2}dB_s^{(\alpha)} + \int_0^t \frac{R_{\alpha(s)} + \frac{1}{R_{\alpha(s)}}}{\sqrt{\alpha(s)}}(1 - Z_s^2)ds - \frac{n}{2} \int_0^t \dot{\alpha}(s)Z_s ds. \tag{3.11}$$

For $y \in [-1, 1]$ and $\sigma \geq 0$, let $(Y_t^{\sigma, y})_{t \geq \sigma}$ be the solution to the SDE on $[-1, 1]$

$$\begin{cases} dY_t^{\sigma, y} = \sqrt{\dot{\alpha}(t)}\sqrt{1 - (Y_t^{\sigma, y})^2}dB_t^{(\alpha)} + [\frac{1}{\sqrt{n}}(1 - (Y_t^{\sigma, y})^2) - \frac{n}{2}\dot{\alpha}(t)Y_t^{\sigma, y}]dt \\ Y_{\sigma}^{\sigma, y} = y \end{cases}. \tag{3.12}$$

We divide the proof of the convergence in two steps. In the first one, we prove that for all $y \in [-1, 1]$ and $\sigma \geq 0$, $Y_t^{\sigma, y}$ converges almost surely to 1; and then prove the convergence of Z_t to 1 in the second one.

Step I: Let $y \in [-1, 1]$ and assume without loss of generality $\sigma = 0$. In order to lighten the notation, we omit the superscripts y and σ in $Y_t^{\sigma, y}$ during this step. We start by

¹For example $\varepsilon(t) = O(1/(\log(t))^\alpha)$ with $\alpha > 1$.

proving that Y_t is an almost sure asymptotic pseudotrajectory for the flow induced by the ODE

$$\dot{x} = \frac{1}{\sqrt{n}}(1 - x^2). \tag{3.13}$$

In order to achieve it, we use Theorem 3.3. Since $x \mapsto (1 - x^2)$ is Lipschitz continuous on $[-1, 1]$ and that $Y_t \in [-1, 1]$ for all $t \geq 0$, it remains to verify the hypothesis concerning the noise term.

Set

$$\varepsilon(t) = \dot{\alpha}(t) = \left(\frac{3}{2}t\right)^{-\frac{1}{3}}. \tag{3.14}$$

It is then obvious that $\varepsilon(t)$ satisfies (3.9). Because $Y_t \in [-1, 1]$ for all $t \geq 0$, it is clear that the conditions in Remark 3.4 are satisfied. Consequently, by Theorem 3.3, $(Y_t)_t$ is an almost sure asymptotic pseudotrajectory for the flow induced by (3.13).

Because $\{1\}$ is an attractor for the flow induced by (3.13) with basin $] - 1, 1]$ and that almost surely $Y_t \in] - 1, 1]$ infinitely often, then

$$\lim_{t \rightarrow \infty} Y_t = 1 \text{ a.s.} \tag{3.15}$$

by Theorem 3.2.

Step II: Our goal is to prove

$$\mathbb{P}(\lim_{t \rightarrow \infty} Z_t = 1) = 1. \tag{3.16}$$

Define the stopping times $\sigma_0 = 0$,

$$\tau_j = \inf \left(t > \sigma_{j-1} \mid \frac{R_{\alpha(t)}}{\sqrt{\alpha(t)}} = \frac{1}{\sqrt{n}} \right), \quad j \geq 1 \tag{3.17}$$

and

$$\sigma_j = \inf \left(t > \tau_j \mid \frac{R_{\alpha(t)}}{\sqrt{\alpha(t)}} = \frac{3}{2\sqrt{n}} \right), \quad j \geq 1 \tag{3.18}$$

with the convention $\inf \emptyset = +\infty$.

By Lemma 2.2, we have

$$\mathbb{P}\left(\bigcup_{j \geq 1} \{\tau_j = \infty\}\right) = 1 \text{ and for all } j \geq 1, \mathbb{P}(\sigma_j < \infty \mid \tau_j < \infty) = 1. \tag{3.19}$$

Let us start by estimating $\mathbb{P}(\lim_{t \rightarrow \infty} Z_t = 1, \tau_{j+1} = \infty \mid \sigma_j < \infty)$. For $s \in [\sigma_j, \tau_{j+1}]$, we have

$$\frac{R_{\alpha(s)} + \frac{1}{R_{\alpha(s)}}}{\sqrt{\alpha(s)}} \geq \frac{1}{\sqrt{n}}.$$

So, by Ikeda-Watanabe's comparison result (see Theorem 1.1, Chapter VI in [9]),

$$\mathbb{P}(Z_{(t+\sigma_j) \wedge \tau_{j+1}} \geq Y_{(t+\sigma_j) \wedge \tau_{j+1}}^{\sigma_j, Z_{\sigma_j}}, \forall t \geq 0 \mid \sigma_j < \infty) = 1. \tag{3.20}$$

As a consequence, we have

$$\begin{aligned} \mathbb{P}\left(\lim_{t \rightarrow \infty} Z_t = 1, \tau_{j+1} = \infty \mid \sigma_j < \infty\right) &\geq \mathbb{P}\left(\lim_{t \rightarrow \infty} Y_{t+\sigma_j}^{\sigma_j, Z_{\sigma_j}} = 1, \tau_{j+1} = \infty \mid \sigma_j < \infty\right) \\ &= \mathbb{P}(\tau_{j+1} = \infty \mid \sigma_j < \infty). \end{aligned} \tag{3.21}$$

where the last equality follows from Step I.

Since $(\{\tau_j = \infty\})_{j \geq 1}$ is an increasing family of events such that

$$\{\tau_j = \infty\} = \mathcal{N}_j \cup \bigcup_{k=0}^{j-1} \{\tau_{k+1} = \infty, \sigma_k < \infty\},$$

where \mathcal{N}_j is an event of probability 0, we obtain from (3.19) and (3.21)

$$\begin{aligned} \mathbb{P}\left(\lim_{t \rightarrow \infty} Z_t = 1\right) &= \sum_{j \geq 0} \mathbb{P}\left(\lim_{t \rightarrow \infty} Z_t = 1, \tau_{j+1} = \infty \text{ and } \sigma_j < \infty\right) \\ &\geq \sum_{j \geq 0} \mathbb{P}\left(\tau_{j+1} = \infty \text{ and } \sigma_j < \infty\right) \\ &= 1. \end{aligned} \tag{3.22}$$

Consequently, C_t converges almost surely to 1. Therefore, so does

$$\frac{R_t}{t} = \frac{1}{t} \int_0^t C_s ds. \tag{3.23}$$

Proof of the rate of convergence Set $\sigma_0 = 0$ and define the stopping times

$$\tau_j = \inf\left(t > \sigma_{j-1} \mid C_t = 0 \text{ or } \frac{R_t}{t} = \frac{1}{2}\right), \quad j \geq 1 \tag{3.24}$$

and

$$\sigma_j = \inf\left(t > \tau_j \mid C_t \geq \frac{1}{2} \text{ and } \frac{R_t}{t} \geq \frac{3}{4}\right), \quad j \geq 1 \tag{3.25}$$

with the convention $\inf \emptyset = +\infty$. So, by the previous part,

$$\mathbb{P}\left(\bigcup_{j \geq 1} \{\tau_j = \infty\}\right) = 1 \text{ and for all } j \geq 1, \mathbb{P}(\sigma_j < \infty \mid \tau_j < \infty) = 1. \tag{3.26}$$

Case $n \geq 2$: Set $Z_t = 1 - C_t$ and define the process $(\vartheta_t)_{t \geq 0}$ by $\vartheta_0 = 2, \vartheta_t = Z_t - Z_{\tau_j} + \vartheta_{\tau_j}$ for $t \in [\tau_j, \sigma_j]$ and

$$\vartheta_t = \vartheta_{\sigma_j} - \int_{\sigma_j}^t \sqrt{1 - C_s^2} dW_s - \frac{1}{2} \int_{\sigma_j}^t s \vartheta_s ds + \frac{n}{2}(t - \sigma_j) \tag{3.27}$$

for $t \in [\sigma_j, \tau_{j+1}]$. Thanks to (3.10), one can also write Z_t , for $\sigma_j \leq t \leq \tau_{j+1}$,

$$Z_t = Z_{\sigma_j} - \int_{\sigma_j}^t \sqrt{1 - C_s^2} dW_s - \int_{\sigma_j}^t \left(\left(R_s + \frac{1}{R_s}\right)(1 + C_s) + \frac{n}{2} \right) Z_s ds + \frac{n}{2}(t - \sigma_j). \tag{3.28}$$

Moreover, for such times t , we have

$$\left(R_t + \frac{1}{R_t}\right)(1 + C_t) \geq \frac{t}{2}.$$

Hence, from Ikeda-Watanabe comparison's result

$$\mathbb{P}(Z_t \leq \vartheta_t, \forall t \geq 0) = 1. \tag{3.29}$$

Since $1 - C_t^2 \in [0, 1]$, we have by Proposition A.1

$$\mathbb{P}\left(\vartheta_t = O(t^{-1/2} \sqrt{\ln(t)}), \tau_{j+1} = \infty \mid \sigma_j < \infty\right) = \mathbb{P}\left(\tau_{j+1} = \infty \mid \sigma_j < \infty\right). \tag{3.30}$$

By the same argumentation as in Step II of the proof of the convergence, one obtains

$$1 - C_t = O(t^{-1/2} \sqrt{\ln(t)}) \text{ a.s.} \tag{3.31}$$

Case $n = 1$: Set $\theta_t = \arccos(C_t) \in [0, \pi]$. Then, as long as $R_t > 0$, θ_t solves

$$d\theta_t = -dW_t - \left(R_t + \frac{1}{R_t}\right) \sin(\theta_t) dt + dL_t(0) - dL_t(\pi), \tag{3.32}$$

where $(L_t(0))_{t \geq 0}$ (resp. $(L_t(\pi))_{t \geq 0}$) is a process of finite variation that increases when $\theta_t = 0$ (resp. $\theta_t = \pi$).

Because C_t converges almost surely to 1, then, from the second order Taylor expansion of $\cos(\cdot)$ about 0, an estimate of its rate of convergence is given by the one of θ_t^2 to 0.

Set $\Theta_t = \theta_t^2$. Then, as long as $R_t > 0$, it solves

$$d\Theta_t = -2\sqrt{\Theta_t} dW_t - 2\left(R_t + \frac{1}{R_t}\right) \sqrt{\Theta_t} \sin(\sqrt{\Theta_t}) dt + dt - 2\sqrt{\Theta_t} dL_t(\pi). \tag{3.33}$$

Note that $L_t(\pi)$ increases only when $\Theta_t = \pi^2$.

Following the same methodology as for the case $n \geq 2$, define a process $(\Psi_t)_{t \geq 0}$ as follows: $\Psi_0 = \pi^2$, $\Psi_t = \Theta_t - \Theta_{\tau_j} + \Psi_{\tau_j}$ if $t \in [\tau_j, \sigma_j]$ and for $t \in [\sigma_j, \tau_{j+1}]$,

$$\Psi_t = \Psi_{\sigma_j} - 2 \int_{\sigma_j}^t \sqrt{\Psi_s} dW_s - 2 \int_{\sigma_j}^t \frac{s}{\pi} \Psi_s ds + (t - \sigma_j). \tag{3.34}$$

Because for $t \in [\sigma_j, \tau_{j+1}]$, we have

$$\left(R_t + \frac{1}{R_t}\right) \geq \frac{t}{2} \text{ and } \sqrt{\Theta_t} \sin(\sqrt{\Theta_t}) \geq \frac{2}{\pi} \Theta_t,$$

it follows from Ikeda-Watanabe's comparison result

$$\mathbb{P}(\Theta_t \leq \Psi_t, \forall t \geq 0) = 1. \tag{3.35}$$

Since $(\Psi_{t \wedge \tau_{j+1}})_{t \geq \sigma_j}$ has the same law as $(Z_{t \wedge \tau_{j+1}}^2)_{t \geq \sigma_j}$, where $(Z_t)_{t \geq \sigma_j}$ is the solution of the SDE

$$dZ_t = dW_t - \frac{t}{\pi} Z_t dt, \quad Z_{\sigma_j} = \theta_{\sigma_j}, \tag{3.36}$$

it follows from Proposition A.1

$$\mathbb{P}\left(\Psi_t = O(t^{-1} \ln(t)), \tau_{j+1} = \infty \mid \sigma_j < \infty\right) = \mathbb{P}\left(\tau_{j+1} = \infty \mid \sigma_j < \infty\right).$$

Arguing like in Step II of the proof of the convergence, one obtains

$$\Theta_t = O(t^{-1} \ln(t)) \text{ a.s.}$$

Thus

$$1 - C_t = O(t^{-1} \ln(t)) \text{ a.s.}$$

Remark 3.5. Following the proof of the rate of convergence from the case $n = 1$, one proves that the rate of convergence to 1 of the solution of the SDE

$$\begin{cases} dC_t^{(n)} = \sqrt{n} \sqrt{1 - (C_t^{(n)})^2} dW_t + \left[\left(R_t + \frac{1}{R_t}\right) (1 - (C_t^{(n)})^2) - \frac{n}{2} C_t^{(n)}\right] dt \\ C_0^{(n)} = y \end{cases} \tag{3.37}$$

is $O(t^{-1} \ln(t))$. Therefore, we conjecture that so does C_t for any $n \geq 2$.

4 Conclusion

The motivating model of this work was the real-valued self attracting diffusion

$$dX_t = \nu dW_t + a \int_0^t \sin(X_t - X_s) ds dt, \quad X_0 = 0$$

with $\nu > 0$ and $a < 0$. Identifying it with $(\cos(X_t), \sin(X_t))$, we had that the almost sure convergence of X_t was an immediate consequence of the more general diffusion on the n -dimensional unit sphere \mathbb{S}^n

$$dX_t = \nu \circ dW_t(X_t) - a \int_0^t \nabla_{\mathbb{S}^n} V_{X_s}(X_t) ds dt, \quad X_0 = x \in \mathbb{S}^n$$

with $V_y(x) = \langle x, y \rangle$.

It would now be interesting to study the self interacting diffusion

$$dX_t = \nu dW_t + \sum_{k=1}^n k a_k \int_0^t \sin(k(X_t - X_s)) ds dt,$$

where the coefficient $a_k \neq 0$ are such that $\sum_{k=1}^n k^2 a_k < 0$.

Because $\sum_{k=1}^n k^2 a_k = (\sum_{k=1}^n k a_k \sin(k \cdot))'(0)$ and that it has to play a more and more important role if $(X_t)_t$ localizes, it sounds reasonable to formulate the following conjecture:

Conjecture 4.1. *Let $(X_t)_{t \geq 0}$ be the solution of the self interacting diffusion*

$$dX_t = \nu dW_t + \sum_{k=1}^n k a_k \int_0^t \sin(k(X_t - X_s)) ds dt, \quad X_0 = x.$$

If $\sum_{k=1}^n k^2 a_k < 0$ (resp. $\sum_{k=1}^n k^2 a_k > 0$), then X_t converges almost-surely (resp. $\limsup_t X_t > \liminf_t X_t$).

A Almost sure convergence for a time-inhomogeneous linear SDE

In this paper, we needed to use the rate of convergence to 0 for the solution of the SDE

$$dX_t = g_t dW_t + \mu_t dt - (1 + \alpha) \lambda t^\alpha X_t dt, \tag{A.1}$$

when $t \mapsto \mu_t$ is a deterministic constant, $\alpha = 1$ and $(g_t)_{t \geq 0}$ is an adapted process bounded by 1. Here, $(W_t)_{t \geq 0}$ stands for a real Brownian motion and $\lambda > 0$.

Proposition A.1. *Let X_t be the solution of (A.1) with initial condition $X_0 = x$. Assume that $(g_t)_{t \geq 0}$ and $(\mu_t)_{t \geq 0}$ are adapted processes bounded by some deterministic constant K and let $\alpha \geq 0$. Then*

$$X_t = O\left(t^{-\alpha/2} \sqrt{\log(t)}\right) \text{ a.s.}$$

Proof. The solution of Equation (A.1) with initial condition $X_0 = x$ is

$$\begin{aligned} X_t &= e^{-\lambda t^{1+\alpha}} \left(x + \int_0^t e^{\lambda s^{1+\alpha}} g_s dW_s + \int_0^t e^{\lambda s^{1+\alpha}} \mu_s ds \right) \\ &= e^{-\lambda t^{1+\alpha}} \left(x + M_t + \int_0^t e^{\lambda s^{1+\alpha}} \mu_s ds \right). \end{aligned} \tag{A.2}$$

Since

$$e^{-\lambda t^{1+\alpha}} \left| \int_0^t e^{\lambda s^{1+\alpha}} \mu_s ds \right| \leq K e^{-\lambda t^{1+\alpha}} \int_0^t e^{\lambda s^{1+\alpha}} ds = O\left(t^{-\alpha}\right), \tag{A.3}$$

then, on the event $\{\langle M \rangle_\infty := \int_0^\infty e^{2\lambda s^{1+\alpha}} g_s^2 ds < \infty\}$, the result is immediate because in that case, M_t converges almost surely. In the sequel, we assume that we are on the event $\{\langle M \rangle_\infty = \infty\}$.

By the Dubins-Schwarz Theorem (see Theorem 4.6 in [10], Chapter 3) with the law of Iterated Logarithm for Brownian motion (see Theorem 9.23 in [10], Chapter 2), we have

$$M_t = O\left(\sqrt{\langle M \rangle_t \log \log(\langle M \rangle_t)}\right). \quad (\text{A.4})$$

By the hypothesis on g_t , we have

$$\langle M \rangle_t = \int_0^t e^{2\lambda s^{1+\alpha}} g_s^2 ds \leq K^2 \int_0^t e^{2\lambda s^{1+\alpha}} ds.$$

Therefore,

$$\langle M \rangle_t = O\left(e^{2\lambda t^{1+\alpha}} t^{-\alpha}\right). \quad (\text{A.5})$$

The desired result follows from (A.2)–(A.5). \square

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