SELF-AVERAGING FROM LATERAL DIVERSITY IN THE ITÔ-SCHRÖDINGER EQUATION

GEORGE PAPANICOLAOU *, LENYA RYZHIK †, AND KNUT SøLNA ‡

Abstract. We consider the random Schrödinger equation as it arises in the paraxial regime for wave propagation in random media. In the white noise limit it becomes the Itô-Schrödinger stochastic partial differential equation (SPDE) which we analyze here in the high frequency regime. We also consider the large lateral diversity limit where the typical width of the propagating beam is large compared to the correlation length of the random medium. We use the Wigner transform of the wave field and show that it becomes deterministic in the large diversity limit when integrated against test functions. This is the self-averaging property of the Wigner transform. It follows easily when the support of the test functions is of the order of the beam width. We also show with a more detailed analysis that the limit is deterministic when the support of the test functions tends to zero but is large compared to the correlation length.

Key words. Random media, Parabolic approximation, Stochastic partial differential equations.

AMS subject classifications. 35L05, 60H15, 35Q60

1. Introduction.

In the study of wave propagation in random media, the parabolic or paraxial approximation is used often when waves propagate mostly in one direction and there is little backscattering [27]. The scattering problem is reduced to an initial value problem in a random medium in which distance along the direction of propagation plays the role of time. This is a very significant simplification that has been adopted in many physical applications [27, 28, 29, 24]. The study of waves in the parabolic approximation is also very useful in the analysis of time reversal and imaging in random media [9, 5, 6, 10, 11, 12]. Self-averaging is the property of some physical quantities to be statistically stable, that is, independent of the random fluctuations in the medium properties. For this reason, self-averaging functionals of the wave field play an important role in imaging and time reversal. They were analyzed in the regime of the parabolic approximation in [5, 25, 2] and in the random geometrical optics regime in [4]. In this paper we extend and simplify the analysis in [25]. We show that in the parabolic approximation, in a variety of scaling regimes local averages of the wave field in phase space are self-averaging when there is substantial lateral diversity. This means that the correlation length of the inhomogeneities is small compared to the width of the propagating beam.

In the parabolic approximation the wave equation reduces to the Schrödinger equation in a random medium. When the propagation distance is large compared to the correlation length, then the random potential in the Schrödinger equation tends to white noise in the propagation direction [13, 1, 16, 17]. We begin here with this white noise, Itô-Schrödinger equation. In Section 2 we formulate the problem and introduce the scaling. In Section 3 we introduce the Wigner transform of the wave field and state the main results. They characterize the behavior of the Wigner transform in the high frequency and large diversity limits. In Section 4 we prove the weak convergence of the Wigner transforms in law, in the various asymptotic limits. This is done in a simple way using infinitesimal generators, which is a general approach that identifies the limit problem in an efficient way. In Section 5 we extend the analysis of weak convergence in law to test functions with asymptotically diminishing support. Such results were also obtained in [2] using convergence of the second moments but the rate of diminishing support is faster in the analysis of Section 5.

2. The random Schrödinger equation.

2.1. Characteristic scales.

We consider wave propagation in a random medium in the regime when the paraxial approximation is valid and waves propagate over distances that are much larger than both the typical wavelength and the correlation length of the random inhomogeneities. We introduce several characteristic scales in order to identify the regimes for the asymptotic analysis that we want to carry out. They are

- L_z, the characteristic distance in the direction of propagation.
- $L_{\mathbf{x}}$, the length scale in the directions transverse to the direction of propagation. This is typically taken to be the width of the propagating beam.

*Department of Mathematics, Stanford University, Stanford CA, 94305; papanico@math.stanford.edu, supported by grants ONR N00014-02-1-0088, 02-SC-ARO-1067-MOD 1 and NSF DMS-0354674-001.

†Department of Mathematics, University of Chicago, Chicago IL, 60637; ryzhik@math.uchicago.edu, supported by an Alfred P. Sloan fellowship, ONR grant N00014-04-1-0224 and NSF grant DMS-0604687.

‡ Department of Mathematics, University of California, Irvine CA, 92697; ksolna@math.uci.edu, supported by an Alfred P. Sloan fellowship, ONR grant N00014-02-1-0090 and NSF grant DMS-0307011, DARPA grant N00014-02-1-0603.

- $k_0 = 2\pi/\lambda_0$, the central wavenumber corresponding to the central wavelength λ_0.
- l, the correlation length of the random medium. It characterizes the dominant spatial scale of the random fluctuations.
- σ_0, the dimensionless standard deviation of the random fluctuations in the medium.

In the asymptotic regimes that we consider here L_z and L_x are large compared to l and λ_0, and σ_0 is small.

2.2. The parabolic approximation. We consider the wave equation in a random medium

$$\frac{1}{c^2(\vec{\mathbf{x}})}\frac{\partial^2 u}{\partial t^2} - \triangle u = 0\ ,\quad t>0\ ,\quad \vec{\mathbf{x}}\in\mathbb{R}^{d+1}\ , \tag{2.1}$$

with $d = 1, 2$ and the local wave speed $c(z, \mathbf{x})$ such that

$$c^{-2}(z,\mathbf{x}) = c_0^{-2}\left[1+\sigma_0\mu\left(\frac{z}{l},\frac{\mathbf{x}}{l}\right)\right].$$

Here z and $\mathbf{x}\in\mathbb{R}^d$ are, respectively, the coordinates along and transverse to the direction of propagation, and $\vec{\mathbf{x}} = (z, \mathbf{x})$. The random function μ models the fluctuations in the propagation speed. Solutions of the wave equation (2.1) may be written in the form

$$u(t,\mathbf{x},z) = \frac{1}{2\pi}\int e^{i\omega(z/c_0-t)}\psi(z,\mathbf{x};\omega/c_0)d\omega\ , \tag{2.2}$$

where the complex amplitude $\psi(z, \mathbf{x}; k)$ satisfies the Helmholtz equation

$$2ik\psi_z + \Delta_{\mathbf{x}}\psi + k^2(n^2-1)\psi = -\psi_{zz}. \tag{2.3}$$

Here $k = \omega/c_0$ is the wavenumber and $n(\mathbf{x}, z) = c_0/c(\mathbf{x}, z)$ is the random index of refraction relative to a reference speed c_0. The fluctuations of the refraction index have the form

$$n^2(z,\mathbf{x}) - 1 = \sigma_0\mu\left(\frac{z}{l},\frac{\mathbf{x}}{l}\right)\ . \tag{2.4}$$

The fluctuations are modeled by a stationary random field with mean zero, variance σ_0^2 and correlation length l. The normalized and dimensionless covariance is given by

$$R(z,\mathbf{x}) = E\{\mu(z+z',\mathbf{x}+\mathbf{x}')\mu(z',\mathbf{x}')\} \tag{2.5}$$

with the normalization $R(0, 0) = 1$.

We obtain the dimensionless form of (2.3) by introducing dimensionless variables by $\mathbf{x} = L_{\mathbf{x}}\mathbf{x}'$, $z = L_z z'$, $k = k_0 k'$ and rewriting it as

$$2ik\frac{\partial\psi}{\partial z} + \frac{L_z}{k_0L_{\mathbf{x}}^2}\Delta_{\mathbf{x}}\psi + k^2k_0L_z\sigma_0\mu\left(\frac{zL_z}{l},\frac{\mathbf{x}L_{\mathbf{x}}}{l}\right)\psi = -\frac{1}{L_zk_0}\frac{\partial^2\psi}{\partial z^2}, \tag{2.6}$$

after dropping the primes. We identify now the following three, usually small, dimensionless parameters in the problem:

- $\varepsilon = \dfrac{l}{L_z}$, the ratio of the correlation length to the propagation distance,
- $\delta = \dfrac{l}{L_{\mathbf{x}}}$, the ratio of the correlation length to the transverse length scale, which is usually the beam width,
- $\theta = \dfrac{L_z}{k_0L_{\mathbf{x}}^2} = \dfrac{\lambda_0L_z}{2\pi L_{\mathbf{x}}^2}$, the inverse of the Fresnel number, the ratio of the diffraction focal spot of the beam to its width.

In terms of these parameters (2.6) has the form

$$2ik\psi_z + \theta\Delta_{\mathbf{x}}\psi + \frac{k^2\sigma_0\delta^2}{\theta\varepsilon^2}\mu(\frac{z}{\varepsilon},\frac{\mathbf{x}}{\delta})\psi = -\frac{\theta\varepsilon^2}{\delta^2}\psi_{zz}. \tag{2.7}$$

We will assume here that ε is the smallest parameter in the problem. This assumption is satisfied by wave fields propagating mainly in the z direction. It then follows formally, but it is quite difficult to prove, that the ψ_{zz} term on the right in (2.7) is a lower order term and can be neglected. The parabolic wave equation, or the random Schrödinger equation, is what results when the right side of (2.7) is zero. The validity of this approximation for underwater sound propagation is discussed in [27] and a more recent analysis is found in [1]. We will thus consider the initial value problem for the random Schrödinger equation

$$2ik\psi_z + \theta\Delta_{\mathbf{x}}\psi + \frac{k^2\sigma\delta}{\theta\sqrt{\varepsilon}}\mu\left(\frac{z}{\varepsilon},\frac{\mathbf{x}}{\delta}\right)\psi = 0\,, z > 0 \tag{2.8}$$

with ψ at $z = 0$ given and where

$$\sigma = \frac{\sigma_0\delta}{\varepsilon^{3/2}}.$$

This scaled noise strength parameter will be assumed to be independent of ε and δ as these parameters tend to zero in the asymptotic analysis that follows.

2.3. The white noise limit. We are interested in the behavior of the solution of (2.8) in the limit $\varepsilon \to 0$ while δ and θ are fixed. This means that ε is the smallest of the three parameters $\varepsilon, \theta, \delta$ in (2.8). We note that under suitable mixing conditions [20] on the random field μ the central limit theorem holds and

$$\lim_{\varepsilon\to 0}\frac{1}{\sqrt{\varepsilon}}\int_0^z \mu\left(\frac{s}{\varepsilon},\mathbf{x}\right)ds = B(z,\mathbf{x}),$$

weakly in law, where B is a Brownian random field parameterized by $\mathbf{x}$. This means that for any test function $h(\mathbf{x})$

$$\frac{1}{\sqrt{\varepsilon}}\int_0^z \mu_h(s/\varepsilon)ds \mapsto B_h(z), z \geq 0,$$

in law, where

$$\mu_h(z) = \int_{\mathbb{R}^d}\mu(z,\mathbf{x})h(\mathbf{x})d\mathbf{x}\ ,\quad B_h(z) = \int_{\mathbb{R}^d}B(z,\mathbf{x})h(\mathbf{x})d\mathbf{x}.$$

The Brownian random field $B(z, \mathbf{x})$ is a Gaussian process with mean zero and covariance

$$E\{B(z_1,\mathbf{x}_1)B(z_2,\mathbf{x}_2)\} = R_0(|\mathbf{x}_1 - \mathbf{x}_2|)\min\{z_1,z_2\}. \tag{2.9}$$

Here R_0 is the integrated in z transverse correlation function defined by

$$R_0(\mathbf{x}) = \int_{-\infty}^{\infty} R(z,\mathbf{x})dz.$$

We assume that it is smooth, rapidly decaying and isotropic.

In the white noise limit $\varepsilon \to 0$ the solution of the random partial differential equation (2.8) converges in law to the process defined by the stochastic partial differential equation

$$2ikd_z\psi + \theta\Delta_{\mathbf{x}}\psi dz + \frac{k^2\sigma\delta}{\theta}\psi\circ d_zB\left(\frac{\mathbf{x}}{\delta},z\right) = 0 \tag{2.10}$$

given here in the Stratonovich form. The Itô form of (2.10) is given by

$$2ikd_z\psi + \theta\Delta_{\mathbf{x}}\psi dz + \frac{ik^3\sigma^2\delta^2}{4\theta^2}R_0(0)\psi dz + \frac{k^2\sigma\delta}{\theta}\psi d_zB\left(\frac{\mathbf{x}}{\delta},z\right) = 0. \tag{2.11}$$

When the fluctuation process $\mu(z, \mathbf{x})$ is Markovian in z with values in a suitable function space, then the above white noise limit for the random Schrödinger equation can be analyzed with the perturbed test function methods presented in [22]. More generally, white noise limits for random ordinary differential equations are studied in [8] and for partial differential equations in [13]. A recent study of white noise limits for Schrödinger and Wigner equations is given in [16, 17].

2.4. Scaling limits. There are two small parameters left in the Itô-Schrödinger equation (2.11) after we have taken the white-noise limit – the inverse Fresnel number θ and the non-dimensional correlation length δ. The purpose of this paper is to analyze the stochastic partial differential equation (2.10) or (2.11) in the following scaling limits.

- The low frequency limit and large lateral diversity limit: $\delta \to 0$ with θ fixed,
- the high frequency or geometric asymptotics limit followed by the large lateral diversity limit: $\theta \ll \delta \ll 1$, that is, $\theta \to 0$ followed by $\delta \to 0$, and
- the combined scaling limit: $\theta \sim \delta \ll 1$ with $\theta \to 0$ and $\delta \to 0$ simultaneously.

We refer to the limit $\theta \to 0$ in (2.11) as the high frequency limit and to the limit $\delta \to 0$ as the limit of large lateral diversity.

2.5. The low frequency limit. It follows immediately from (2.11) that if we pass to the limit $\delta \to 0$ with a fixed $\theta > 0$ we arrive at the homogeneous Schrödinger equation

$$2ik\psi_z + \theta\Delta_{\mathbf{x}}\psi = 0. \tag{2.12}$$

This is because, first, we have an a priori bound $\|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2}$, as the L^2-norm of ψ is preserved by the Schrödinger equation, and, second, for any deterministic test function $\eta(z, \mathbf{x})$ we have by the Itô isometry

$$E\left[\frac{k^2\sigma\delta}{\theta}\int_0^z\int \eta(s,\mathbf{x})\psi(s,\mathbf{x})d_zB\left(\frac{\mathbf{x}}{\delta},s\right)d\mathbf{x}\right]^2$$
$$= \left(\frac{k^2\sigma\delta}{\theta}\right)^2 E\int_0^z\int \eta(s,\mathbf{x})\eta(s,\mathbf{x}')\psi(s,\mathbf{x})\psi(s,\mathbf{x}')R_0\left(\frac{\mathbf{x}-\mathbf{x}'}{\delta}\right)d\mathbf{x}d\mathbf{x}'ds \to 0 \text{ as } \delta \to 0.$$

A similar bound holds for the third term on the left side of (2.11) – therefore, convergence in probability of the solution to (2.11) to the solution of (2.12) follows.

The other regimes – when $\theta \ll \delta$ and $\theta \sim \delta$ are more involved. Their analysis is easier to perform in phase space and not for the solution of the Ito-Schrödinger equation itself. For this purpose we introduce the Wigner transform.

3. The Itô-Wigner equation. In the high frequency limit $\theta \to 0$ (whether coupled with the limit $\delta \to 0$, or not) solutions of the Itô-Schrödinger equation become oscillatory in time and space. Therefore, rather than studying the limit of the solution itself we consider the limits of its Wigner transform which resolves the wave energy of oscillatory fields in the phase space and (unlike the spatial energy density) satisfies a closed evolution equation.

We define the spatial Fourier transform by

$$\hat{f}(\mathbf{p}) = \int d\mathbf{x}e^{-i\mathbf{p}\cdot\mathbf{x}}f(\mathbf{x}),$$

so that the inverse transform is given by

$$f(\mathbf{x}) = \int \frac{d\mathbf{p}}{(2\pi)^d}e^{i\mathbf{p}\cdot\mathbf{x}}\hat{f}(\mathbf{p}),$$

where $d = 1$ or 2 is the number of transverse spatial dimensions.

3.1. The Wigner transform. A convenient tool for the analysis of wave propagation in a random medium is the Wigner distribution [26]. We define it here relative to the scale θ by

$$W_\theta(z,\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\mathbf{p}\cdot\mathbf{y}}\psi(\mathbf{x}-\frac{\theta\mathbf{y}}{2},z)\overline{\psi(\mathbf{x}+\frac{\theta\mathbf{y}}{2},z)}d\mathbf{y}, \tag{3.1}$$

where the bar denotes complex conjugate. The Wigner distribution is real, may be interpreted as phase space wave energy and is particularly well suited for the high frequency asymptotics in random media [26]. Using the Itô calculus we find from (2.11) that the scaled Wigner distribution satisfies the stochastic transport equation

$$dW_\theta(z,\mathbf{x},\mathbf{p}) + \frac{\mathbf{p}}{k}\cdot\nabla_{\mathbf{x}}W_\theta(z,\mathbf{x},\mathbf{p})dz = \frac{k^2\sigma^2\delta^2}{4\theta^2}\int\left(W_\theta\left(z,\mathbf{x},\mathbf{p}+\frac{\theta\mathbf{q}}{\delta}\right) - W_\theta(z,\mathbf{x},\mathbf{p})\right)\frac{\hat{R}_0(\mathbf{q})d\mathbf{q}}{(2\pi)^d}dz$$

$$+\frac{ik\sigma\delta}{2\theta}\int\frac{d\mathbf{q}}{(2\pi)^d}e^{i\mathbf{q}\cdot\mathbf{x}/\delta}\left(W_\theta\left(z,\mathbf{x},\mathbf{p}-\frac{\theta\mathbf{q}}{2\delta}\right) - W_\theta\left(z,\mathbf{x},\mathbf{p}+\frac{\theta\mathbf{q}}{2\delta}\right)\right)d\hat{B}(\mathbf{q},z). \tag{3.2}$$

We will consider the high frequency and large diversity limits with the Itô-Wigner equation (3.2) as our starting point.

We will use the fact that the L^2 norm of the Wigner distribution is conserved

$$\|W_\theta(z)\|_{L^2(\mathbb{R}^{2d})} = \|W_\theta(0)\|_{L^2(\mathbb{R}^{2d})},$$

which follows from the definition (3.1) and the invariance of the $L^2(\mathbb{R}^d)$ norm of $\psi(z,\cdot)$. In the asymptotic analysis we will assume that the initial Wigner transform is a square integrable function independent of θ. The way such initial data can arise from the corresponding ones for the Schrödinger equation is by assuming that we have a suitable mixture of states [3].

3.2. The high frequency limit. We first discuss (2.11) in the high frequency limit $\theta \to 0$ followed by the limit of large lateral diversity, $\delta \to 0$. When we take the high frequency limit in (3.2) we find that W_θ converges weakly to W_δ satisfying the Itô-Liouville equation

$$dW_\delta(z,\mathbf{x},\mathbf{p}) + \frac{\mathbf{p}}{k}\cdot\nabla_{\mathbf{x}}W_\delta(z,\mathbf{x},\mathbf{p})dz + \frac{k^2\sigma^2}{8}R_0''(0)\triangle_{\mathbf{p}}W_\delta dz = -\frac{k\sigma}{2}d\nabla_{\mathbf{x}}B\left(\frac{\mathbf{x}}{\delta},z\right)\cdot\nabla_{\mathbf{p}}W_\delta. \tag{3.3}$$

We state this in the following theorem:

THEOREM 3.1. *The solution W_θ of (3.2) converges in the limit $\theta \to 0$ weakly in law to the process W_δ solving (3.3).* We remark that $R''(0) < 0$ so that (3.3) is well-posed. Existence and uniqueness of solutions of the stochastic equation (3.3) follows from the general theory of stochastic flows [21].

3.3. The large diversity limit. The limiting Wigner distribution in Theorem 3.1 solves a stochastic PDE (3.3), in which the coefficient of the random term fluctuates on the small scale δ. When we subsequently take the limit of large lateral diversity we find that the limiting Wigner distribution actually becomes deterministic. We refer to this as the stabilization of the Wigner distribution. Define W as the deterministic solution of

$$\frac{\partial W}{\partial z}(z,\mathbf{x},\mathbf{p}) + \frac{\mathbf{p}}{k}\cdot\nabla_{\mathbf{x}}W(z,\mathbf{x},\mathbf{p}) + \frac{k^2\sigma^2}{8}R_0''(0)\triangle_{\mathbf{p}}W = 0. \tag{3.4}$$

Then we have the following theorem.

THEOREM 3.2. *The solution W_δ of (3.3) converges in the limit $\delta \to 0$ weakly in $\mathcal{S}'(\mathbb{R}^{2d})$, in probability to W solving (3.4).* We prove Theorems 3.1 and 3.2 in Section 4.

3.4. The combined high frequency and large diversity limit.

3.4.1. Weak limit. Next, consider the case where the parameters θ and δ are small and comparable, with the ratio $\xi = \delta/\theta$ kept fixed, and $\delta \to 0$. We introduce the solution $\widetilde{W}$ of the deterministic part of (3.2)

$$\frac{\partial\widetilde{W}}{\partial z}(z,\mathbf{x},\mathbf{p}) + \frac{\mathbf{p}}{k}\cdot\nabla_{\mathbf{x}}\widetilde{W}(z,\mathbf{x},\mathbf{p}) = \frac{k^2\sigma_\xi^2}{4}\int\frac{d\mathbf{q}}{(2\pi)^d}\hat{R}_0(\mathbf{q})\left(\widetilde{W}(z,\mathbf{x},\mathbf{p}+\mathbf{q}) - \widetilde{W}(z,\mathbf{x},\mathbf{p})\right), \tag{3.5}$$

with $\sigma_\xi = \sigma\xi$. The limiting Wigner distribution is now $\widetilde{W}$.

THEOREM 3.3. *The solution W_δ of (3.2) converges in the limit $\delta = \xi\theta \to 0$ weakly (in $\mathcal{S}'(\mathbb{R}^{2d})$) and in probability to $\widetilde{W}$ solving (3.5).* We prove this theorem also in Section 4.

Note that when θ is comparable to δ the limit Wigner distribution is again deterministic. However, unlike the limit in Theorem 3.2, the full lateral correlation function affects the limiting Wigner distribution, not only its form for small displacements.

3.4.2. Localized test functions. All of the above theorems deal with the weak limit of the Wigner distribution as a distribution in $\mathcal{S}'(\mathbb{R}^{2d})$ with the test functions independent both of θ and δ. This introduces additional averaging that makes the proof of the stabilization of the Wigner distribution in the limit fairly straightforward. The next result shows that the averaging may be performed essentially on an arbitrary scale that is larger than the non-dimensional correlation length δ but still much smaller than the macroscopic scale. That means that we have stabilization with much less averaging.

Let $\lambda \in C_c^\infty(\mathbb{R}^{2d})$ be a given smooth test function of compact support, then we define a stretched test function

$$\lambda_\delta(\mathbf{x},\mathbf{p}) = \frac{\lambda(\mathbf{x}/\delta^{a_1},\mathbf{p}/\delta^{a_2})}{\delta^{(a_1+a_2)d}} \tag{3.6}$$

This is an approximate δ-function on the spacial scale δ^{a_1} and wave vector scale δ^{a_2}.

THEOREM 3.4. *Let W_δ be the (random) solution of (3.2) with $\xi=\delta/\theta$ and let $\widetilde{W}$ satisfy (3.5). Then the difference process*

$$Z_\delta(z) = \int \left[W_\delta(z,\mathbf{x},\mathbf{p}) - \widetilde{W}(z,\mathbf{x},\mathbf{p})\right]\lambda_\delta(\mathbf{x},\mathbf{p})d\mathbf{x}d\mathbf{p}$$

converges to zero in probability as $\delta\to 0$, provided that $a_1+2a_2<1$ if $d=1$ and $a_1+a_2<2/3$ if $d=2$. We prove this theorem in Section 5. In the case $d=1$ one possible choice is $a_2=0$, $a_1<1$ – which means that in the case of one transverse direction we may actually average the Wigner transform on any spatial scale larger than the correlation length provided we average over $\mathbf{p}$. A result similar to Theorem 3.4 with the weaker condition $a_1+a_2<1/2$ is proved in [2].

4. Generators and weak limits for the Itô-Wigner process.

4.1. A general convergence result. Theorems 3.1, 3.2 and 3.3 can be put in a unified framework which we now describe. Consider a family of distributions $W_h(t,\mathbf{x},\mathbf{p})$ which satisfy a stochastic differential equation

$$dW_h = \mathcal{L}_h W_h dz + \int_{\mathbb{R}^d} M_h(\mathbf{q})[W_h]d\hat{B}(z,\mathbf{q})d\mathbf{q},\quad W_h(0,\mathbf{x},\mathbf{p}) = W_0(\mathbf{x},\mathbf{p}), \tag{4.1}$$

in the sense of the associated weak martingale problem. The Brownian fields $\hat{B}(z,\mathbf{q})$ are the Fourier transforms of the corresponding ones $B(z,\mathbf{x})$ with covariance (2.9). We will assume that the operators $M_h(\mathbf{q})$ are anti-symmetric and the operators $\mathcal{L}_h$ are non-positive: $\langle\mathcal{L}_h\lambda,\lambda\rangle\le 0$ for any smooth test function $\lambda(\mathbf{x},\mathbf{p})$. We also assume that the family $W_h(z)$ is uniformly bounded in $L^2(\mathbb{R}^d\times\mathbb{R}^d)$ as is expected from the skew-symmetry of M_h:

$$\|W_h(z)\|_{L^2(\mathbb{R}^{2d})} \le \|W_0\|_{L^2(\mathbb{R}^{2d})}. \tag{4.2}$$

In addition, we assume that for any such $\lambda(\mathbf{x},\mathbf{p})$ we have

$$\mathcal{L}_h\lambda \to \mathcal{L}_0\lambda \text{ in } L^2(\mathbb{R}^{2d}) \text{ as } h\to 0. \tag{4.3}$$

Regarding the operators M_h we ask that

$$\|M_h(\mathbf{q})\lambda\|_{L^2(\mathbb{R}^{2d})} \le C(\lambda) \tag{4.4}$$

with the constant $C(\lambda)$ independent of $h\in(0,1)$ and $\mathbf{q}\in\mathbb{R}^d$, and we require that the following quadratic forms converge:

$$\langle W, M_h(\mathbf{q})\lambda\rangle\langle W, M_h(-\mathbf{q})\lambda\rangle \to \langle W, M_0(\mathbf{q})\lambda\rangle\langle W, M_0(-\mathbf{q})\lambda\rangle, \tag{4.5}$$

weakly in $\mathcal{S}'(\mathbb{R}^d)$ (as functions of the variable $\mathbf{q}$), uniformly in the ball $\{\|W\|_{L^2}\le C\}$ for each smooth test function λ. Condition (4.5) is needed to ensure that the infinitesimal generators for the process W_h converge.

Let us introduce the process W which is a solution of

$$dW = \mathcal{L}_0 W dz + \int_{\mathbb{R}^d} M_0(\mathbf{q})[W]d\hat{B}(z,\mathbf{q})d\mathbf{q},\quad W(0,\mathbf{x},\mathbf{k}) = W_0(\mathbf{x},\mathbf{k}), \tag{4.6}$$

in the sense of the associated weak martingale problem. Let also $\mathcal{A}$ be the infinitesimal generator for the process $W(t)$. We assume that the functions of W of the form

$$F(W) = f(\langle W, \lambda_1\rangle, \dots, \langle W, \lambda_N\rangle), \tag{4.7}$$

where $\lambda_1, \dots, \lambda_N, \dots \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ form a convergence determining class for $\mathcal{A}$. The suitability of test functions of the form (4.7) is addressed in [15].

Theorems 3.1, 3.2 and 3.3 follow from the following result.

THEOREM 4.1. *Under the above assumptions W_h converges weakly in $\mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$ to W, solution of the martingale problem associated to (4.6).* The existence and uniqueness of the solution of the martingale problem for W_h and W depends in an essential way on the particular form of the operators $\mathcal{L}_h$, $\mathcal{L}_0$, M_h and M_0. We address this issue in the specific applications of this result in the following sections.

The method of the proof of Theorem 4.1 is quite standard [22]. Let us recall a general strategy for the proof of weak convergence of a family of distributions $W_h(z) \in C([0,Z]; \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d))$. First, one has to establish tightness for the family $W_h(z)$. This shows that a weak limit along a subsequence exists. The second step is to verify that the infinitesimal generators $\mathcal{A}^h$ of the Markov processes $W_h(z)$ converge to the infinitesimal generator $\mathcal{A}$ for a process $W(z)$. This identifies the limit as a solution of the martingale problem for $\mathcal{A}$. As we are dealing with infinite-dimensional processes, convergence of generators is easier to check on special test functions which nevertheless should determine the generator uniquely.

4.2. Tightness. We consider the processes $W_h(z)$ in the space $C([0,Z]; \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d))$. The sequence $W_h(z)$ induces a sequence of probability measures $\mathbb{P}_h$ on the space $D([0,Z]; L^2(\mathbb{R}^{2d}))$. We then have

LEMMA 4.2. *The family of measures $\mathbb{P}_h$ is tight.* **Proof.** It follows from the results of Fouque in [18] and Mitoma in [23] that in order to verify the tightness of the family of distributions $W_h(z)$ it is sufficient to establish tightness of the processes

$$X_h[\lambda](z) = \langle W_h(z), \lambda\rangle = \int W_h(z, \mathbf{x}, \mathbf{k})\lambda(\mathbf{x}, \mathbf{k})d\mathbf{x}d\mathbf{k}$$

for each test function $\lambda \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$. We use the following tightness criterion:

$$E\left\{ |X_h[\lambda](z) - X_h[\lambda](z')|^2 \middle| \mathcal{F}_{z'} \right\} \le C(z - z') \tag{4.8}$$

for all $0 \le z' \le z$. While (4.8) establishes tightness of the process $X_h[\lambda](z)$ in the space $D([0,Z])$ of right continuous functions with left limits as the processes $X_0[\lambda](z)$ are themselves continuous, tightness in $C([0,Z])$ also follows [7].

Using the stochastic equation (4.1) we compute that (dropping λ in the notation for $X_h[\lambda](z)$)

$$X_h(z) = X_h(0) + \int_0^z \langle W_h(s), \mathcal{L}_h^\star \lambda\rangle ds + \int_0^z \int \langle W_h(s), M_h^\star(\mathbf{q})\lambda\rangle d\hat{B}(s,\mathbf{q})d\mathbf{q}. \tag{4.9}$$

It follows from (4.9) that the stochastic process

$$G_h(z) = X_h(z) - \int_0^z \langle W_h(s), \mathcal{L}_h^\star \lambda\rangle\, ds = \int_0^z \int \langle W_h(s), M_h^\star(\mathbf{q})\lambda\rangle d\hat{B}(s,\mathbf{q})d\mathbf{q}$$

is a martingale. In addition, it has a bounded quadratic variation:

$$\begin{aligned} &E\left[(G_h(z) - G_h(z'))^2 | \mathcal{F}_{z'}\right] \\ &= E\left[\int_{z'}^z \int_{z'}^z \int \langle W_h(s), M_h^\star(\mathbf{q})\lambda\rangle \langle W_h(s'), M_h^\star(\mathbf{q}')\lambda\rangle\, d\hat{B}(s,\mathbf{q})d\hat{B}(s',\mathbf{q}')d\mathbf{q}d\mathbf{q}' \middle| \mathcal{F}_{z'}\right] \\ &= \int_{z'}^z \int E\left[\langle W_h(s), M_h^\star(\mathbf{q})\lambda\rangle \langle W_h(s), M_h^\star(-\mathbf{q})\lambda\rangle\, \hat{R}(s,\mathbf{q})d\mathbf{q}ds \middle| \mathcal{F}_{z'}\right] \le C(\lambda)(z - z'). \end{aligned} \tag{4.10}$$

The last inequality above follows from the fact that $\|W_h(z)\|_{L^2}$ is uniformly bounded by a deterministic constant and (4.4). It follows from (4.10) and the uniform bounds (4.2) and (4.3) that we have the moment bound

$$E\left[(X_h(z) - X_h(z'))^2 \middle| \mathcal{F}_{z'}\right] = E\left[(G_h(z) - G_h(z') + I(z,z'))^2 \middle| \mathcal{F}_{z'}\right] \le C(\lambda)(z - z'). \tag{4.11}$$

Here we have set

$$I(z,z') = \int_{z'}^{z} \langle W_h(s), \mathcal{L}_h^\star \lambda\rangle\, ds.$$

The tightness of the family $W_h(z)$ is a consequence of (4.11).

4.3. Generators for a determining class of test functions. As the processes W_h are infinite-dimensional Markov processes, the action of the corresponding infinitesimal generators on an arbitrary function of W is somewhat difficult to write down explicitly. However, it is sufficient to consider special continuous functions $F \in C(\mathcal{S}'; \mathbb{R})$ of the form (4.7). We have to verify that for such test functions of the form (4.7) we have

$$\mathcal{A}^h F \to \mathcal{A}F, \tag{4.12}$$

uniformly in the balls $\{\|W\|_{L^2} \le C\}$. As we have explained above, (4.12) together with the uniqueness of the Markov process $W(t)$ with the generator $\mathcal{A}$ would prove the weak convergence of W_h to W.

Let $\lambda_1, \dots, \lambda_N \in C_c^\infty(\mathbb{R}^{2d})$ be a collection of smooth test functions of compact support and define the corresponding stochastic processes

$$X_n^h(z) = \langle W_h, \lambda_n\rangle(z) = \int W_h(z,\mathbf{x},\mathbf{p})\lambda_n(\mathbf{x},\mathbf{p})\, d\mathbf{x}d\mathbf{p}. \tag{4.13}$$

Let also $f \in C^\infty(\mathbb{R}^N)$ and define the process

$$f(X_h(z)) = f(X_1^h(z), \dots, X_N^h(z)). \tag{4.14}$$

To keep the presentation simple we will consider in detail the action of the generator $\mathcal{A}_h$ only in the special case $N = 1$ – the generalization to an arbitrary N is immediate at the expense of a greater number of indices. We drop the subscript n and use the Itô formula to obtain

$$\begin{aligned} df(X_h) &= f'(X_h)dX_h + \frac{1}{2}f''(X_h)[dX_h]^2 = f'(X_h)\langle W_h, \mathcal{L}_h^\star\lambda\rangle dz \\ &+ \frac{1}{2}f''(X_h)\int\limits_{\mathbb{R}^d} \langle W_h, [M_h^\star(\mathbf{q})\lambda]\rangle\langle W_h, [M_h^\star(-\mathbf{q})\lambda]\rangle \hat{R}(\mathbf{q})d\mathbf{q}dz + f'(X_h)\int\limits_{\mathbb{R}^d}\langle W_h, [M_h(\mathbf{q})\lambda]\rangle d\hat{B}(z,\mathbf{q})d\mathbf{q}. \end{aligned}$$

Therefore, the generator $\mathcal{A}_h$ acts on $f(X)$ as

$$\mathcal{A}_h f = \langle W, \mathcal{L}_h^\star\lambda\rangle f'(X) + \frac{1}{2}\left[\int_{\mathbb{R}^d} \langle W, [M_h^\star(\mathbf{q})\lambda]\rangle\langle W, [M_h^\star(-\mathbf{q})\lambda]\rangle\hat{R}(\mathbf{q})d\mathbf{q}\right] f''(X). \tag{4.15}$$

Similarly, the analogous infinitesimal generator $\mathcal{A}$ in the case corresponding to the process W solving (4.6) acts on $f(X)$ as

$$\mathcal{A} f = \langle W, \mathcal{L}^\star\lambda\rangle f'(X) + \frac{1}{2}\left[\int_{\mathbb{R}^d} \langle W, [M_0^\star(\mathbf{q})\lambda]\rangle\langle W, [M_0^\star(-\mathbf{q})\lambda]\rangle\hat{R}(\mathbf{q})d\mathbf{q}\right] f''(X). \tag{4.16}$$

Therefore, $\mathcal{A}_h f(X)$ converges to $\mathcal{A}f(X)$ as follows from the assumptions (4.3), (4.5) and (4.4). This finishes the proof of Theorem 4.1.

4.4. The High Frequency Limit. We now prove Theorem 3.1. Equation (3.2) may be written in the form (4.1) as follows:

$$dW_\theta = \mathcal{L}_\theta W dz + \int [M_\theta(\mathbf{q})W] d\hat{B}(z,\mathbf{q})d\mathbf{q} \tag{4.17}$$

with

$$\mathcal{L}_\theta\lambda(\mathbf{x},\mathbf{p}) = -\frac{\mathbf{p}}{k}\cdot\nabla_{\mathbf{x}}\lambda(z,\mathbf{x},\mathbf{p}) + \frac{k^2\sigma^2\delta^2}{4\theta^2}\int\left(\lambda\left(z,\mathbf{x},\mathbf{p}+\frac{\theta\mathbf{q}}{\delta}\right) - \lambda(z,\mathbf{x},\mathbf{p})\right)\frac{\hat{R}_0(\mathbf{q})d\mathbf{q}}{(2\pi)^d}$$

and

$$M_\theta(\mathbf{q})\lambda = \frac{1}{(2\pi)^d}\frac{ik\sigma\delta}{2\theta}e^{i\mathbf{q}\cdot\mathbf{x}/\delta}\left(\lambda\left(z,\mathbf{x},\mathbf{p}-\frac{\theta\mathbf{q}}{2\delta}\right)-\lambda\left(z,\mathbf{x},\mathbf{p}+\frac{\theta\mathbf{q}}{2\delta}\right)\right).$$

Using the Taylor formula for small θ it is straightforward to verify that the operators $\mathcal{L}_\theta$ and $M_\theta(\mathbf{q})$ satisfy the assumptions of Theorem 4.1 with the limits (recall that we let $\theta \to 0$ with $\delta > 0$ fixed)

$$\mathcal{L}_0\lambda = -\frac{\mathbf{p}}{k}\cdot\nabla\lambda + \frac{k^2\sigma^2}{8}(-R_0^{''}(0))\Delta_{\mathbf{x}}\lambda, \qquad M_0(\mathbf{q})\lambda = -\frac{ik\sigma e^{i\mathbf{q}\cdot\mathbf{x}/\delta}}{2}\mathbf{q}\cdot\nabla_{\mathbf{p}}\lambda.$$

Therefore, Theorem 4.1 applies and the solution of (4.17) converges to the solution of

$$dW + \frac{\mathbf{p}}{k}\cdot\nabla W dz - \frac{k^2\sigma^2}{8}(-R_0^{''}(0))\Delta_{\mathbf{x}}W dz = -\frac{k\sigma}{2}\nabla_{\mathbf{x}}dB\left(z,\frac{\mathbf{x}}{\delta}\right)\cdot\nabla_{\mathbf{p}}W. \tag{4.18}$$

Uniqueness of the solution of the martingale problem for (4.18) follows from the general theory of stochastic flows [21]. The conclusion of Theorem 3.1 follows.

4.5. The Large Diversity Limit. In this section we take the Itô-Liouville equation (4.18) as our starting point and derive the large diversity limit $\delta \to 0$ in Theorem 3.2. This is also an easy consequence of Theorem 4.1. Indeed, (4.18) has the form (4.1) with

$$\begin{aligned}\mathcal{L}\lambda &= -\mathbf{p}\cdot\nabla_{\mathbf{x}}\lambda + \frac{k^2\sigma^2}{8}(-R_0^{''}(0))\triangle_{\mathbf{p}}\lambda\,,\\ M_\delta(\mathbf{q})\lambda &= -\frac{ike^{i\mathbf{q}\cdot\mathbf{x}/\delta}}{2}\mathbf{q}\cdot\nabla_{\mathbf{p}}\lambda.\end{aligned}$$

In order to verify that Theorem 4.1 applies with $M_0 = 0$ we only need to check condition (4.5): for any test function $\phi(\mathbf{q})$ we have

$$\begin{aligned}I &= \int_{\mathbb{R}^{2d}}\langle W,[\mathcal{M}_\delta(\mathbf{q})\lambda]\rangle\langle W,[\mathcal{M}_\delta(-\mathbf{q})\lambda]\rangle\phi(\mathbf{q})d\mathbf{q}\\ &= \frac{k^2}{4}\int_{\mathbb{R}^{5d}} e^{i\mathbf{q}\cdot\mathbf{x}/\delta - i\mathbf{q}\cdot\mathbf{y}/\delta}\phi(\mathbf{q})W(\mathbf{x},\mathbf{p})q_j\frac{\partial\lambda(\mathbf{x},\mathbf{p})}{\partial p_j}W(\mathbf{y},\mathbf{r})q_m\frac{\partial\lambda(\mathbf{y},\mathbf{r})}{\partial r_m}d\mathbf{x}d\mathbf{p}d\mathbf{y}d\mathbf{r}d\mathbf{q}\\ &= \frac{k^2}{4}\int_{\mathbb{R}^{4d}}\frac{\partial^2\hat{\phi}((\mathbf{y}-\mathbf{x})/\delta)}{\partial x_m\partial x_j}W(\mathbf{x},\mathbf{p})\frac{\partial\lambda(\mathbf{x},\mathbf{p})}{\partial p_j}W(\mathbf{y},\mathbf{r})\frac{\partial\lambda(\mathbf{y},\mathbf{r})}{\partial r_m}d\mathbf{x}d\mathbf{p}d\mathbf{y}d\mathbf{r},\end{aligned}$$

so that

$$|I| \le C\|W\|_{L^2}^2\left(\int\left|\frac{\partial^2\hat{\phi}((\mathbf{y}-\mathbf{x})/\delta)}{\partial x_m\partial x_j}\right|^2\left|\frac{\partial\lambda(\mathbf{x},\mathbf{p})}{\partial p_j}\right|^2\left|\frac{\partial\lambda(\mathbf{y},\mathbf{r})}{\partial r_m}\right|^2 d\mathbf{x}d\mathbf{p}d\mathbf{y}d\mathbf{r}\right)^{1/2} \le C\delta^{d/2}.$$

Hence, Theorem 4.1 applies and the conclusion of Theorem 3.2 indeed follows.

4.6. The combined high frequency and large diversity limit. We now show that Theorem 3.3 is also a corollary of Theorem 4.1. We find from the transport equation (3.2) that in the case $\xi\theta = \delta$ the Wigner distribution W_δ solves

$$\begin{aligned}&dW_\delta(z,\mathbf{x},\mathbf{p}) + \frac{\mathbf{p}}{k}\cdot\nabla_{\mathbf{x}}W_\delta(z,\mathbf{x},\mathbf{p})dz = \frac{k^2\sigma_\xi^2}{4}\int\hat{R}_0(\mathbf{q})\left(W_\delta(z,\mathbf{x},\mathbf{p}+\mathbf{q}) - W_\delta(z,\mathbf{x},\mathbf{p})\right)dz\frac{d\mathbf{q}}{(2\pi)^d}\\ &+\frac{ik\sigma_\xi}{2}\int e^{i\mathbf{q}\cdot\mathbf{x}/\delta}\left(W_\delta\left(z,\mathbf{x},\mathbf{p}-\frac{\mathbf{q}}{2}\right)-W_\delta\left(z,\mathbf{x},\mathbf{p}+\frac{\mathbf{q}}{2}\right)\right)d\hat{B}(\mathbf{q},z)\frac{d\mathbf{q}}{(2\pi)^d}.\end{aligned} \tag{4.19}$$

This equation is also of the form (4.1) with

$$\mathcal{L}\lambda = -\frac{\mathbf{p}}{k}\cdot\nabla_{\mathbf{x}}\lambda + \frac{k^2\sigma_\xi}{4}\int\hat{R}_0(\mathbf{q})\left(\lambda(\mathbf{x},\mathbf{p}+\mathbf{q})-\lambda(\mathbf{x},\mathbf{p})\right)\frac{d\mathbf{q}}{(2\pi)^d}$$

and

$$M_\delta(\mathbf{q})\lambda(\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi)^d}\frac{ik\sigma_\xi}{2}e^{i\mathbf{q}\cdot\mathbf{x}/\delta}\left[\lambda\left(\mathbf{x},\mathbf{p}-\frac{\mathbf{q}}{2}\right)-\lambda\left(\mathbf{x},\mathbf{p}+\frac{\mathbf{q}}{2}\right)\right].$$

In order to verify that (4.5) holds with $M_0 = 0$ we take a smooth test function $\phi(\mathbf{q})$ and compute

$$\begin{aligned}
I &= \int \langle W, [\mathcal{M}_\delta^\star(\mathbf{q})\lambda]\rangle\, \phi(\mathbf{q})\, \langle W, [\mathcal{M}_\delta^\star(-\mathbf{q})\lambda]\rangle\, d\mathbf{q}\\
&= -\frac{k^2}{4}\int W(\mathbf{x},\mathbf{p})e^{i\mathbf{q}\cdot\mathbf{x}/\delta}\left[\lambda\left(z,\mathbf{x},\mathbf{p}-\frac{\mathbf{q}}{2}\right)-\lambda\left(z,\mathbf{x},\mathbf{p}+\frac{\mathbf{q}}{2}\right)\right]W(\mathbf{y},\mathbf{r})e^{-i\mathbf{q}\cdot\mathbf{y}/\delta}\\
&\times\left[\lambda\left(z,\mathbf{y},\mathbf{r}+\frac{\mathbf{q}}{2}\right)-\lambda\left(z,\mathbf{y},\mathbf{r}-\frac{\mathbf{q}}{2}\right)\right]\phi(\mathbf{q})d\mathbf{q}d\mathbf{x}d\mathbf{p}d\mathbf{y}d\mathbf{r}\\
&= -\frac{1}{(2\pi)^{4d}}\frac{k^2}{4}\int \hat\lambda(z,\mathbf{x},\eta)\hat\lambda(z,\mathbf{y},\eta')W(\mathbf{x},\mathbf{p})e^{i\mathbf{q}\cdot\mathbf{x}/\delta}\left[e^{i(\mathbf{p}-\mathbf{q}/2)\cdot\eta}-e^{i(\mathbf{p}+\mathbf{q}/2)\cdot\eta}\right]W(\mathbf{y},\mathbf{r})e^{-i\mathbf{q}\cdot\mathbf{y}/\delta}\\
&\times\left[e^{i(\mathbf{r}+\mathbf{q}/2)\cdot\eta'}-e^{i(\mathbf{r}-\mathbf{q}/2)\cdot\eta'}\right]\phi(\mathbf{q})d\mathbf{q}d\mathbf{x}d\mathbf{p}d\mathbf{y}d\mathbf{r}d\eta d\eta'\\
&= -\frac{1}{(2\pi)^{4d}}\frac{k^2}{4}\int e^{i\mathbf{p}\cdot\eta+i\mathbf{r}\cdot\eta'}\hat\lambda(z,\mathbf{x},\eta)\hat\lambda(z,\mathbf{y},\eta')\left[\hat\phi\left(-\frac{\mathbf{x}-\mathbf{y}}{\delta}+\frac{\eta-\eta'}{2}\right)-\hat\phi\left(-\frac{\mathbf{x}-\mathbf{y}}{\delta}+\frac{\eta+\eta'}{2}\right)\right.\\
&\left.+\hat\phi\left(-\frac{\mathbf{x}-\mathbf{y}}{\delta}-\frac{\eta-\eta'}{2}\right)-\hat\phi\left(-\frac{\mathbf{x}-\mathbf{y}}{\delta}+\frac{\eta+\eta'}{2}\right)\right]W(\mathbf{x},\mathbf{p})W(\mathbf{y},\mathbf{r})d\mathbf{x}d\mathbf{p}d\mathbf{y}d\mathbf{r}d\eta d\eta'\\
&= I_1+I_2+I_3+I_4.
\end{aligned}$$

Let us look, for instance, at I_1:

$$|I_1| \le C\int\left|\hat\lambda(z,\mathbf{x},\eta)\hat\lambda(z,\mathbf{y},\eta')\right|\left|\hat\phi\left(-\frac{\mathbf{x}-\mathbf{y}}{\delta}+\frac{\eta-\eta'}{2}\right)\right|\left|\tilde W(\mathbf{x},\eta)\tilde W(\mathbf{y},\eta')\right|d\mathbf{x}d\mathbf{y}d\eta d\eta'.$$

Here $\tilde W$ is the Fourier transform of $W(\mathbf{x},\mathbf{k})$ in the second variable only. We may assume without loss of generality that $\hat\lambda(z,\mathbf{x},\eta)$ is compactly supported in η. Then, for almost every $(\mathbf{x},\eta),(\mathbf{y},\eta')\in \mathrm{supp}\hat\lambda$ we have

$$\hat\phi\left(-\frac{\mathbf{x}-\mathbf{y}}{\delta}+\frac{\eta-\eta'}{2}\right)\to 0 \text{ as } \delta\to 0.$$

The Lebesgue dominated convergence theorem implies that $I_1 \to 0$. Similarly we may show that $I_{2,3,4} \to 0$ and thus the proof of Theorem 3.3 is complete.

5. The Local Weak Convergence. We consider here the Itô-Wigner equation (3.2) in the limit $\theta \sim \delta \to 0$ and prove the local weak convergence result stated in Theorem 3.4.

5.1. The Integral Formulation of the Itô-Wigner Equation . Let us recall the Itô-Wigner equation (3.2) in the regime $\delta = \theta$: we will set $\sigma_\xi = 1$ without any loss of generality

$$\begin{aligned}
dW_\delta + \frac{\mathbf{p}}{k}\cdot\nabla_{\mathbf{x}}W_\delta dz &= \frac{k^2}{4}\int \hat R_0(\mathbf{q})[W_\delta(\mathbf{p}+\mathbf{q})-W_\delta(\mathbf{p})]\frac{d\mathbf{q}}{(2\pi)^d}dz\\
&+\frac{ik}{2}\int e^{i\mathbf{q}\cdot\mathbf{x}/\delta}\left[W_\delta\left(\mathbf{p}-\frac{\mathbf{q}}{2}\right)-W_\delta\left(\mathbf{p}+\frac{\mathbf{q}}{2}\right)\right]\frac{d\mathbf{q}}{(2\pi)^d}d\hat{\mathbf{B}}(z,\mathbf{q}).
\end{aligned}\tag{5.1}$$

Our objective is to analyze the role of the Brownian term in (5.1), and show that in the limit $\delta \to 0$ the rapid oscillatory phase in the $\mathbf{q}$ integral makes it small, so that W_δ converges to the solution of (3.5) in the "locally weak" sense of Theorem 3.4.

The proof is based on the integral formulation of the transport equation and the Picard iteration. In order to develop this argument it is convenient to introduce the function $u(z,\mathbf{x},\mathbf{p}) = W_\delta(z,\mathbf{x},\mathbf{p})\exp(\Sigma z)$ with the total scattering cross-section

$$\Sigma = \frac{k^2}{4}\int \hat R_0(\mathbf{q})\frac{d\mathbf{q}}{(2\pi)^d}.$$

Then equation (5.1) becomes

$$du+\frac{\mathbf{p}}{k}\cdot\nabla_{\mathbf{x}}udz=\frac{k^2}{4}\int\hat{R}_0(\mathbf{q})u(\mathbf{p}+\mathbf{q})\frac{d\mathbf{q}}{(2\pi)^d}dz+\frac{ik}{2}\int\left[u\left(\mathbf{p}-\frac{\mathbf{q}}{2}\right)-u\left(\mathbf{p}+\frac{\mathbf{q}}{2}\right)\right]\frac{e^{i\mathbf{q}\cdot\frac{\mathbf{x}}{\delta}}d\mathbf{q}}{(2\pi)^d}d\hat{\mathbf{B}}(z,\mathbf{q}). \tag{5.2}$$

This, in turn, can be re-written as an integral equation that will be the starting point of our analysis

$$\begin{aligned}u(z,\mathbf{x},\mathbf{p})&=W_0\left(\mathbf{x}-\frac{z\mathbf{p}}{k},\mathbf{p}\right)+\frac{k^2}{4}\int_0^z\int\hat{R}_0(\mathbf{q})u\left(s,\mathbf{x}-(z-s)\frac{\mathbf{p}}{k},\mathbf{p}+\mathbf{q}\right)\frac{d\mathbf{q}ds}{(2\pi)^d}\\&+\frac{ik}{2}\int_0^z\int e^{i\mathbf{q}\cdot\frac{(\mathbf{x}-(z-s)\mathbf{p}/k)}{\delta}}\left[u\left(s,\mathbf{x}-(z-s)\frac{\mathbf{p}}{k},\mathbf{p}-\frac{\mathbf{q}}{2}\right)-u\left(s,\mathbf{x}-(z-s)\frac{\mathbf{p}}{k},\mathbf{p}+\frac{\mathbf{q}}{2}\right)\right]\frac{d\mathbf{q}d\hat{\mathbf{B}}(s,\mathbf{q})}{(2\pi)^d}.\end{aligned} \tag{5.3}$$

The first result addresses the existence and uniqueness of solutions of (5.3). Let us fix $Z>0$ and define the space $X=C([0,Z];L^2(\mathbb{R}^{2d}\times\Omega))$ with the norm

$$\|f\|_X=\sup_{0\le z\le Z}\left(\int\mathbb{E}\left\{|f(z,\mathbf{x},\mathbf{p})|^2\right\}d\mathbf{x}d\mathbf{p}\right)^{1/2}.$$

We have the following proposition.

Proposition 5.1. *Assume that*

$$R\equiv\max\left(\|\hat{R}_0\|_{L^1(\mathbb{R}^d)},\|\hat{R}_0\|_\infty\right)<\infty \tag{5.4}$$

Then there exists a unique solution to (5.3) in the space X.

We also introduce the function $\bar{u}$ that satisfies the deterministic part of (5.3), without the random term

$$\bar{u}(z,\mathbf{x},\mathbf{p})=W_0\left(\mathbf{x}-\frac{z\mathbf{p}}{k},\mathbf{p}\right)+\frac{k^2}{4}\int_0^z\int\hat{R}_0(\mathbf{q})\bar{u}\left(s,\mathbf{x}-(z-s)\frac{\mathbf{p}}{k},\mathbf{p}+\mathbf{q}\right)\frac{d\mathbf{q}ds}{(2\pi)^d}, \tag{5.5}$$

with the initial data $u(0,\mathbf{x},\mathbf{p})=\bar{u}(0,\mathbf{x},\mathbf{p})=W_0(\mathbf{x},\mathbf{p})$.

We will show that u converges to $\bar{u}$ in a locally weak sense. More precisely, the following proposition holds.

Proposition 5.2. *Let $\lambda(\mathbf{x},\mathbf{p})\in C_c^\infty(\mathbb{R}^{2d})$ be a smooth deterministic test function of compact support and define the stretched test function as in (3.6)*

$$\lambda_\delta(\mathbf{x},\mathbf{p})=\frac{\lambda(\mathbf{x}/\delta^{a_1},\mathbf{p}/\delta^{a_2})}{\delta^{(a_1+a_2)d}}. \tag{5.6}$$

Then, under the assumption (5.4) there exists a constant $C=C(k,R,\lambda,Z)>0$ so that for $z\le Z$

$$\mathbb{E}\left\{|\langle u-\bar{u},\lambda_\delta\rangle|^2\right\}(z)\le C(k,R,\lambda,Z)\|W_0\|^2_{L^2(\mathbb{R}^{2d})}\times\left\{\begin{array}{ll}\delta^{1-a_1-2a_2}|\log\delta|, & d=1\\ \delta^{2-3(a_1+a_2)}, & d=2\end{array}\right.. \tag{5.7}$$

Theorem 3.4 follows immediately from Proposition 5.2.

5.2. Existence: proof of Proposition 5.1.

The iterative series. The proof of Proposition 5.1 is by an iterative process expanding the solution into a series according to the order of scattering. We introduce the operators $T_1,T_2:X\to X$ by

$$T_1f(z,\mathbf{x},\mathbf{p})=\frac{k^2}{4}\int_0^z\int\hat{R}_0(\mathbf{q})f\left(s,\mathbf{x}-(z-s)\frac{\mathbf{p}}{k},\mathbf{p}+\mathbf{q}\right)\frac{d\mathbf{q}ds}{(2\pi)^d} \tag{5.8}$$

and

$$T_2f=\frac{ik}{2}\int_0^z\int e^{\frac{i\mathbf{q}\cdot(\mathbf{x}-(z-s)\mathbf{p}/k)}{\delta}}\left[f\left(s,\mathbf{x}-(z-s)\frac{\mathbf{p}}{k},\mathbf{p}-\frac{\mathbf{q}}{2}\right)-f\left(s,\mathbf{x}-(z-s)\frac{\mathbf{p}}{k},\mathbf{p}+\frac{\mathbf{q}}{2}\right)\right]\frac{d\mathbf{q}d\hat{\mathbf{B}}(s,\mathbf{q})}{(2\pi)^d}. \tag{5.9}$$

With this notation, equation (5.3) may be re-written as

$$u(z,\mathbf{x},\mathbf{p}) = W_0\left(\mathbf{x}-\frac{z\mathbf{p}}{k},\mathbf{p}\right) + (T_1+T_2)u(z,\mathbf{x},\mathbf{p}) \tag{5.10}$$

We now represent the solution of (5.10) as a series. Let

$$u_0(z,\mathbf{x},\mathbf{p}) = \bar{u}_0(z,\mathbf{x},\mathbf{p}) = W_0\left(\mathbf{x}-\frac{z\mathbf{p}}{k},\mathbf{p}\right)$$

be the solution of the homogeneous transport equation, and set the "up to n-th order" scattering term as

$$u_n(z,\mathbf{x},\mathbf{p}) = W_0\left(\mathbf{x}-\frac{z\mathbf{p}}{k},\mathbf{p}\right) + (T_1+T_2)u_{n-1}(z,\mathbf{x},\mathbf{p}) \tag{5.11}$$

for $n\geq 1$. We also define the pure n-th order scattering contribution as $v_n = u_n - u_{n-1}$, $n\geq 1$, the solution of

$$v_n(z,\mathbf{x},\mathbf{p}) = (T_1+T_2)v_{n-1}(z,\mathbf{x},\mathbf{p}),\quad v_0(z,\mathbf{x},\mathbf{p}) = u_0(z,\mathbf{x},\mathbf{p}). \tag{5.12}$$

Proposition 5.1 follows from the following lemma.

LEMMA 5.3. *Assume that (5.4) holds. Then there exists a constant $C=C(k,R,Z)$ so that for $n\geq 1$*

$$\|v_n\|_X^2 \leq \|W_0\|^2_{L^2(\mathbb{R}^{2d})}\frac{(C(k,R,Z))^n}{n!}. \tag{5.13}$$

Convergence of the iteration process. We now prove Lemma 5.3. This lemma follows from the Cauchy-Schwarz inequality and the Itô-isometry for stochastic flows. Observe that we have

$$\begin{aligned}&\mathbb{E}\left\{\int_0^{z_1}\int_0^{z_2}\int \lambda(s_1,\mathbf{q}_1)\lambda(s_2,\mathbf{q}_2)\,d\hat{\mathbf{B}}(s_1,\mathbf{q}_1)d\hat{\mathbf{B}}(s_2,\mathbf{q}_2)\,d\mathbf{q}_1 d\mathbf{q}_2\right\}\\ &= \int_0^{[z_1,z_2]}\int \mathbb{E}\left\{\lambda(s,\mathbf{q})\lambda(s,-\mathbf{q})\right\}(2\pi)^d\hat{R}_0(\mathbf{q})\,ds d\mathbf{q}\,,\end{aligned} \tag{5.14}$$

with the correlation function R_0 defined in (2.9) and $[z_1,z_2]=\min(z_1,z_2)$. We then find that the following bounds hold for the operators T_1 and T_2, respectively:

$$\begin{aligned}&\mathbb{E}\left\{\|T_1 v_n(z)\|^2_{L^2(\mathbb{R}^{2d})}\right\}\\ &\leq C\mathbb{E}\left\{\int_0^z\int_0^z\int|\hat{R}_0(\mathbf{q})\hat{R}_0(\mathbf{q}')|\left\{\int|v_n(s,\mathbf{x},\mathbf{p})|^2\,d\mathbf{x}d\mathbf{p}\int|v_n(s',\mathbf{x}',\mathbf{p}')|^2\,d\mathbf{x}'d\mathbf{p}'\right\}^{\frac{1}{2}}\frac{d\mathbf{q}d\mathbf{q}'dsds'}{(2\pi)^{2d}}\right\}\\ &\leq Cz\left(\|\hat{R}_0\|_{L^1(\mathbb{R}^d)}\right)^2\mathbb{E}\left\{\int_0^z\int|v_n(s,\mathbf{x},\mathbf{p})|^2\,d\mathbf{x}d\mathbf{p}ds\right\},\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}\left\{\|T_2 v_n(z)\|^2_{L^2(\mathbb{R}^{2d})}\right\}\leq C\int_0^z\int|\hat{R}_0(\mathbf{q})|\left(\int\mathbb{E}\left\{\left|v_n\left(s,\mathbf{x}-(z-s)\frac{\mathbf{p}}{k},\mathbf{p}-\frac{\mathbf{q}}{2}\right)\right.\right.\right.\\ &\left.\left.\left.-v_n\left(s,\mathbf{x}-(z-s)\frac{\mathbf{p}}{k},\mathbf{p}+\frac{\mathbf{q}}{2}\right)\right|^2\right\}d\mathbf{x}d\mathbf{p}\right)d\mathbf{q}ds\leq C\|\hat{R}_0\|_{L^1(\mathbb{R})}\int_0^z\int\mathbb{E}\left\{\left|v_n\left(s,\mathbf{x},\mathbf{p}\right)\right|^2\right\}d\mathbf{x}d\mathbf{p}ds.\end{aligned}$$

Hence we have

$$\mathbb{E}\left\{\|(T_1+T_2)v_n(z)\|^2_{L^2(\mathbb{R}^{2d})}\right\}\leq C(k,R,Z)\int_0^z\mathbb{E}\left\{\|v_n\left(s\right)\|^2_{L^2(\mathbb{R}^{2d})}\right\}ds. \tag{5.15}$$

Using (5.12) and iterating (5.15) we obtain for $n\geq 1$

$$\mathbb{E}\left\{\|v_n(z)\|^2_{L^2(\mathbb{R}^{2d})}\right\}\leq\frac{(C(k,R,Z)z)^n}{n!}\|W_0\|^2_{L^2(\mathbb{R}^{2d})}\,,$$

and the conclusion of Lemma 5.3 follows. □

The proof of Proposition 5.1 is also complete.

5.3. Convergence to the non-random process: proof of Proposition 5.2.

The iterative series for the error. In order to prove Proposition 5.2 we construct an iterative approximation $\bar{u}_n$ to the function u and estimate the n-th order error $u_n - \bar{u}_n$. Recall that

$$\bar{u}(z,\mathbf{x},\mathbf{p}) = W_0\left(\mathbf{x} - \frac{z\mathbf{p}}{k},\mathbf{p}\right) + T_1\bar{u}(z,\mathbf{x},\mathbf{p}). \tag{5.16}$$

Accordingly, we define

$$\bar{u}_n(z,\mathbf{x},\mathbf{p}) = W_0\left(\mathbf{x} - \frac{z\mathbf{p}}{k},\mathbf{p}\right) + T_1\bar{u}_{n-1}(z,\mathbf{x},\mathbf{p}). \tag{5.17}$$

Similarly to Lemma 5.3 we have the same estimate for $\bar{v}_n = \bar{u}_n - \bar{u}_{n-1}$ as for v_n:

$$\|\bar{v}_n\|_X^2 \le \|W_0\|^2_{L^2(\mathbb{R}^{2d})}\frac{(C(k,R,Z))^n}{n!}. \tag{5.18}$$

Proposition 5.2 follows from the following Lemma.

Lemma 5.4. *Let λ_δ be a stretched test function as in (5.6). Then, under the assumption (5.4) there exists a constant $C = C(k,R,\lambda,Z) > 0$ so that for $z \le Z$*

$$\mathbb{E}\left\{|\langle u_n - \bar{u}_n, \lambda_\delta\rangle|^2\right\}(z) \le C(k,R,\lambda,Z)\|W_0\|^2_{L^2(\mathbb{R}^{2d})} \times \left\{\begin{array}{ll}\delta^{1-a_1-2a_2}|\log\delta|, & d=1\\ \delta^{2-3(a_1+a_2)}, & d=2\end{array}\right. . \tag{5.19}$$

Proof of Proposition 5.2. Lemma 5.3 and (5.18) imply that $\forall\delta > 0\ \exists\ N(\delta) > 0$ so that

$$\mathbb{E}\left\{|\langle u - u_n, \lambda_\delta\rangle|^2\right\}(z) \le \delta^2 \quad \text{for all} \;\; n \ge N(\delta)$$

and

$$\mathbb{E}\left\{|\langle \bar{u} - \bar{u}_n, \lambda_\delta\rangle|^2\right\}(z) \le \delta^2 \quad \text{for all} \;\; n \ge N(\delta).$$

The estimate (5.7) now follows by writing $u - \bar{u} = (u - u_n) + (u_n - \bar{u}_n) + (\bar{u}_n - \bar{u})$ and using Lemma 5.4. □

The proof of Lemma 5.4. The difference $u_n - \bar{u}_n$ satisfies

$$u_n - \bar{u}_n = T_1(u_{n-1} - \bar{u}_{n-1}) + T_2 u_{n-1},$$

which can be written as

$$u_n - \bar{u}_n = \sum_{j=0}^{n-1} T_1^{n-1-j} T_2 u_j.$$

In order to prove Lemma 5.4 we observe from the above that

$$|\langle u_n - \bar{u}_n, \lambda_\delta\rangle|^2 = |\sum_{j,l=0}^{n-1}\langle T_1^{n-1-j}T_2u_j,\lambda_\delta\rangle\langle T_1^{n-1-l}T_2u_l,\lambda_\delta\rangle|. \tag{5.20}$$

The individual terms in (5.20) are estimated with the help of the following lemma.

Lemma 5.5. *Let $\theta_i \in X$, $z_i \le Z$, the stretched test function λ_δ be defined as in (5.6) and $R < \infty$ be defined as in (5.4). Then there exist two constants $C_1(k,R,Z)$ and $C_2(k,R,Z,\lambda)$, the second of which depends in addition on the test function λ, such that*

$$\begin{aligned}&\left|\mathbb{E}\left\{\langle [T_1^jT_2\theta_1](z_1),\lambda_\delta\rangle\langle [T_1^lT_2\theta_2](z_2),\lambda_\delta\rangle\right\}\right|\\ &\le \frac{(C_1(k,R,Z)z_1)^j}{j!}\frac{(C_1(k,R,Z)z_2)^l}{l!}C_2(k,R,Z,\lambda)\sup_{m=1,2}\|\theta_m\|_X^2 \times \left\{\begin{array}{ll}\delta^{(1-a_1-2a_2)}|\log\delta|, & d=1\\ \delta^{2-3a_1-3a_2}, & d=2\ .\end{array}\right.\end{aligned} \tag{5.21}$$

It follows from Lemma 5.5 that

$$\left|\mathbb{E}\left\{\sum_{j,l}\langle[T_1^{n-1-j}T_2\theta_j](z_1),\lambda_\delta\rangle\langle[T_1^{n-1-l}T_2\theta_l](z_2),\lambda_\delta\rangle\right\}\right| \tag{5.22}$$
$$\leq e^{C_1(k,R,Z)(z_1+z_2)}\,C_2(k,R,\lambda,Z)\sup_{j\in 1,\cdots,n-1}\|\theta_j\|_X^2\times\left\{\begin{array}{cc}\delta^{(1-a_1-2a_2)}|\log\delta|, & d=1\\ \delta^{2-3a_1-3a_2}, & d=2\end{array}\right. .$$

Using Lemma 5.3 and expression (5.20) we therefore find that

$$\mathbb{E}\left\{|\langle u_n-\bar{u}_n,\lambda_\delta\rangle|^2\right\}(z)\leq C_3(k,R,\lambda,Z)\,\|W_0\|^2_{L^2(\mathbb{R}^{2d})}\times\left\{\begin{array}{cc}\delta^{(1-a_1-2a_2)}|\log\delta|, & d=1\\ \delta^{2-3a_1-3a_2}, & d=2\end{array}\right. .$$

Thus, the conclusion of Lemma 5.4 follows. □

5.4. The Born expansion: the proof of Lemma 5.5. It remains only to prove Lemma 5.5 in order to finish the proof of Theorem 3.4. First, we obtain the bound (5.21) in the case $j=l=0$. In this case we start by rewriting the expression in (5.20) using the Fourier transform and Itô isometry. In the second step the z-integral in T_2 is decomposed into two intervals. In the final boundary layer interval we simply use the Cauchy-Schwarz inequality as well as the smallness of the boundary layer. Outside the final time boundary layer we use the line integration in (5.9) to produce additional averaging, as is typical in the transport theory. The price is in the factors of (z_1-s) and (z_2-s) appearing in the denominator which produce large contributions if the final boundary layer is taken too small. In the last step we optimize with respect to the width of the boundary layer to obtain a bound for the $j=l=0$ term.

Then we present the induction step that gives the bound in Lemma 5.5 for general j and l. The general term can be written in terms of the corresponding expressions with smaller j and l and bounded using an induction argument. A complicating aspect of the induction is the shift of the arguments in the integrals in (5.8), which we handle by introducing a shift operator in the induction.

Bound on Born Term for the Random Scattering. We begin by proving (5.21) in the special case $j=l=0$. Let $\theta_j\in X$, then we need to show that

$$|\mathbb{E}\left\{\langle[T_2\theta_1](z_1),\lambda_\delta\rangle\langle[T_2\theta_2](z_2),\lambda_\delta\rangle\right\}|\leq C(k,R,\lambda,Z)\sup_{m=1,2}\|\theta_m\|_X^2\times\left\{\begin{array}{cc}\delta^{(1-a_1-2a_2)}|\log\delta|, & d=1\\ \delta^{2-3a_1-3a_2}, & d=2\ .\end{array}\right. \tag{5.23}$$

The left side in (5.23) is given explicitly by

$$\begin{aligned}I_{00}&=|\mathbb{E}\left\{\langle[T_2\theta_1](z_1),\lambda_\delta\rangle\langle[T_2\theta_2](z_2),\lambda_\delta\rangle\right\}|\\
&=\frac{k^2}{4(2\pi)^{2d}}\left|\mathbb{E}\left\{\int_0^{z_1}\int_0^{z_2}\int e^{i\mathbf{q}_1\cdot(\mathbf{x}_1-(z_1-s_1)\mathbf{p}_1/k)/\delta+i\mathbf{q}_2\cdot(\mathbf{x}_2-(z_2-s_1)\mathbf{p}_2/k)/\delta}\right.\right.\\
&\times\left[\theta_1\left(s_1,\mathbf{x}_1-(z_1-s_1)\frac{\mathbf{p}_1}{k},\mathbf{p}_1-\frac{\mathbf{q}_1}{2}\right)-\theta_1\left(s_1,\mathbf{x}_1-(z_1-s_1)\frac{\mathbf{p}_1}{k},\mathbf{p}_1+\frac{\mathbf{q}_1}{2}\right)\right]\\
&\times\left[\theta_2\left(s_2,\mathbf{x}_2-(z_2-s_2)\frac{\mathbf{p}_2}{k},\mathbf{p}_2-\frac{\mathbf{q}_2}{2}\right)-\theta_2\left(s,\mathbf{x}_2-(z_2-s_2)\frac{\mathbf{p}_2}{k},\mathbf{p}_2+\frac{\mathbf{q}_2}{2}\right)\right]\\
&\times\left.\left.\lambda_\delta(\mathbf{x}_1,\mathbf{p}_1)\lambda_\delta(\mathbf{x}_2,\mathbf{p}_2)d\hat{\mathbf{B}}(s_1,\mathbf{q}_1)d\hat{\mathbf{B}}(s_2,\mathbf{q}_2)d\mathbf{q}_1d\mathbf{q}_2d\mathbf{x}_1d\mathbf{x}_2d\mathbf{p}_1d\mathbf{p}_2\right\}\right|\ .\end{aligned} \tag{5.24}$$

Using the Itô isometry (5.14) and writing λ_δ in terms of the Fourier transform we find that the above expression for I_{00} becomes

$$\begin{aligned}I_{00}=C(k)&\left|\mathbb{E}\left\{\int_0^{[z_1,z_2]}\int e^{i\mathbf{q}\cdot(\mathbf{x}_1-(z_1-s)\mathbf{p}_1/k)/\delta-i\mathbf{q}\cdot(\mathbf{x}_2-(z_2-s)\mathbf{p}_2/k)/\delta+i[\mathbf{x}_1\cdot\mathbf{r}_1+\mathbf{x}_2\cdot\mathbf{r}_2]+i[\mathbf{p}_1\cdot\mathbf{l}_1+\mathbf{p}_2\cdot\mathbf{l}_2]}\right.\right.\\
&\times\left[\theta_1\left(s,\mathbf{x}_1-(z_1-s)\frac{\mathbf{p}_1}{k},\mathbf{p}_1-\frac{\mathbf{q}}{2}\right)-\theta_1\left(s,\mathbf{x}_1-(z_1-s)\frac{\mathbf{p}_1}{k},\mathbf{p}_1+\frac{\mathbf{q}}{2}\right)\right]\\
&\times\left[\theta_2\left(s,\mathbf{x}_2-(z_2-s)\frac{\mathbf{p}_2}{k},\mathbf{p}_2+\frac{\mathbf{q}}{2}\right)-\theta_2\left(s,\mathbf{x}_2-(z_2-s)\frac{\mathbf{p}_2}{k},\mathbf{p}_2-\frac{\mathbf{q}}{2}\right)\right]\\
&\times\left.\left.\hat{\lambda}(\delta^{a_1}\mathbf{r}_1,\delta^{a_2}\mathbf{l}_1)\hat{\lambda}(\delta^{a_1}\mathbf{r}_2,\delta^{a_2}\mathbf{l}_2)\hat{R}_0(\mathbf{q})d\mathbf{q}ds\Pi_{j=1}^2d\mathbf{x}_jd\mathbf{p}_jd\mathbf{r}_jd\mathbf{l}_j\right\}\right|\end{aligned} \tag{5.25}$$

with $C(k) = k^2/(4(2\pi)^{5d})$. Making a change of variables $\mathbf{x}'_j = \mathbf{x}_j - (z_j - s)\mathbf{p}_j/k$ and taking the Fourier transform in $\mathbf{x}'_1$ and $\mathbf{x}'_2$, we obtain

$$
\begin{aligned}
I_{00} = C(k) \Bigg| \mathbb{E} \Bigg\{ \int_0^{[z_1,z_2]} \int e^{i[((z_1-s)\mathbf{p}_1/k)\cdot\mathbf{r}_1 + ((z_2-s)\mathbf{p}_2/k)\cdot\mathbf{r}_2] + i[\mathbf{p}_1\cdot\mathbf{l}_1 + \mathbf{p}_2\cdot\mathbf{l}_2]} \\
\times \left[\check{\theta}_1\left(s, -\mathbf{r}_1 - \frac{\mathbf{q}}{\delta}, \mathbf{p}_1 - \frac{\mathbf{q}}{2}\right) - \check{\theta}_1\left(s, -\mathbf{r}_1 - \frac{\mathbf{q}}{\delta}, \mathbf{p}_1 + \frac{\mathbf{q}}{2}\right)\right] \\
\times \left[\check{\theta}_2\left(s, -\mathbf{r}_2 + \frac{\mathbf{q}}{\delta}, \mathbf{p}_2 + \frac{\mathbf{q}}{2}\right) - \check{\theta}_2\left(s, -\mathbf{r}_2 + \frac{\mathbf{q}}{\delta}, \mathbf{p}_2 - \frac{\mathbf{q}}{2}\right)\right] \\
\times \hat{\lambda}(\delta^{a_1}\mathbf{r}_1, \delta^{a_2}\mathbf{l}_1)\hat{\lambda}(\delta^{a_1}\mathbf{r}_2, \delta^{a_2}\mathbf{l}_2)\hat{R}_0(\mathbf{q}) d\mathbf{q} ds \Pi_{j=1}^2 d\mathbf{p}_j d\mathbf{r}_j d\mathbf{l}_j \Bigg\} \Bigg| .
\end{aligned}
\tag{5.26}
$$

Expression (5.26) contains four terms that come from the products of θ_j. We consider one of them, take the Fourier transform in $\mathbf{p}_1$ and $\mathbf{p}_2$ and make a change of variables $\mathbf{q}/\delta \mapsto \mathbf{q}$ to find

$$
\begin{aligned}
\left| I_{00}^{(1)} \right| \le \delta^d C(k) \mathbb{E} \Bigg\{ \int_0^{[z_1,z_2]} \int \Big| \hat{\theta}_1\left(s, -\mathbf{r}_1 - \mathbf{q}, -\mathbf{l}_1 - \mathbf{r}_1(z_1-s)/k\right) \\
\times \ \hat{\theta}_2\left(s, -\mathbf{r}_2 + \mathbf{q}, -\mathbf{l}_2 - \mathbf{r}_2(z_2-s)/k\right) \hat{\lambda}(\delta^{a_1}\mathbf{r}_1, \delta^{a_2}\mathbf{l}_1)\hat{\lambda}(\delta^{a_1}\mathbf{r}_2, \delta^{a_2}\mathbf{l}_2)\hat{R}_0(\delta\mathbf{q}) \Big| d\mathbf{q} \Pi_{j=1}^2 d\mathbf{r}_j d\mathbf{l}_j \, ds \Bigg\} .
\end{aligned}
\tag{5.27}
$$

We now decompose the interval $(0, [z_1, z_2])$ in the above integral as

$$
\begin{aligned}
\mathcal{A}_1 &= \{ s \mid \min(|s - z_1|, |s - z_2|) < \delta^p \} , \\
\mathcal{A}_2 &= (0, [z_1, z_2]) \setminus \mathcal{A}_1,
\end{aligned}
\tag{5.28}
$$

for p some positive constant, then $\left| I_{00}^{(1)} \right| \le I_1 + I_2$, where I_j is the integral (5.27) over the time interval $\mathcal{A}_j$.

Making use of the Cauchy-Schwarz inequality in the integration over the random variable and $\mathbf{q}$ we find

$$
\begin{aligned}
I_1 \le \delta^d C(k) \|\hat{R}_0\|_\infty \int_{\mathcal{A}_1} \int \Bigg[\int \mathbb{E} \left\{ \left| \hat{\theta}_1\left(s, \mathbf{q}, -\mathbf{l}_1 - \mathbf{r}_1(z_1-s)/k\right) \right|^2 \right\} d\mathbf{q} \\
\times \int \mathbb{E} \left\{ \left| \hat{\theta}_2\left(s, \mathbf{q}', -\mathbf{l}_2 - \mathbf{r}_2(z_2-s)/k\right) \right|^2 \right\} d\mathbf{q}' \Bigg]^{1/2} \left| \hat{\lambda}(\delta^{a_1}\mathbf{r}_1, \delta^{a_2}\mathbf{l}_1)\hat{\lambda}(\delta^{a_1}\mathbf{r}_2, \delta^{a_2}\mathbf{l}_2) \right| \Pi_{j=1}^2 d\mathbf{r}_j d\mathbf{l}_j \, ds.
\end{aligned}
$$

It then follows after applying the Cauchy-Schwarz inequality with respect to the $\mathbf{l}_1$ and $\mathbf{l}_2$ variables that

$$
\begin{aligned}
I_1 &\le \delta^d C(k,R) \int_{\mathcal{A}_1} \sup_{j=1,2} \mathbb{E} \left\{ \|\theta_j(s)\|^2_{L^2(\mathbb{R}^{2d})} \right\} ds \int \left[\int \left| \hat{\lambda}(\delta^{a_1}\mathbf{r}_1, \delta^{a_2}\mathbf{l}) \right|^2 d\mathbf{l} \int \left| \hat{\lambda}(\delta^{a_1}\mathbf{r}_2, \delta^{a_2}\mathbf{l}) \right|^2 d\mathbf{l} \right]^{\frac{1}{2}} d\mathbf{r}_1 d\mathbf{r}_2 \\
&\le \delta^{d(1-2a_1-a_2)+p} C(k,R,\lambda) \sup_{j=1,2} \|\theta_j\|^2_X.
\end{aligned}
\tag{5.29}
$$

Next, we derive a bound for I_2, the integral (5.27) over the interval $z \in \mathcal{A}_2$. Using the Cauchy-Schwarz inequality as above, with respect to the random variable and the variable $\mathbf{q}$ first, and then with respect to $\mathbf{r}_1$ and $\mathbf{r}_2$, we obtain

$$
\begin{aligned}
I_2 \le \delta^d C(k,R) \int_{\mathcal{A}_2} \int \Bigg[\int \mathbb{E} \left\{ \left| \hat{\theta}_1\left(s, \mathbf{q}, -\mathbf{l}_1 - \mathbf{r}_1(z_1-s)/k\right) \right|^2 \right\} d\mathbf{q} d\mathbf{r}_1 \\
\times \int \mathbb{E} \left\{ \left| \hat{\theta}_2\left(s, \mathbf{q}', -\mathbf{l}_2 - \mathbf{r}_2(z_2-s)/k\right) \right|^2 \right\} d\mathbf{q}' d\mathbf{r}_2 \int \left| \hat{\lambda}(\delta^{a_1}\mathbf{r}'_1, \delta^{a_2}\mathbf{l}_1) \right|^2 d\mathbf{r}'_1 \int \left| \hat{\lambda}(\delta^{a_1}\mathbf{r}'_2, \delta^{a_2}\mathbf{l}_2) \right|^2 d\mathbf{r}'_2 \Bigg]^{\frac{1}{2}} d\mathbf{l}_1 d\mathbf{l}_2 \, ds.
\end{aligned}
$$

After a change of variables the above integral becomes

$$
\begin{aligned}
I_2 &\le \delta^d C(k,R) \int_{\mathcal{A}_2} \sup_{j=1,2} \mathbb{E} \left\{ \|\theta_j(s)\|^2_{L^2(\mathbb{R}^{2d})} \right\} \left(\frac{k^2}{(z_1-s)(z_2-s)} \right)^{d/2} ds \\
&\quad \times \int \left[\int \left| \hat{\lambda}(\delta^{a_1}\mathbf{r}, \delta^{a_2}\mathbf{l}_1) \right|^2 d\mathbf{r} \int \left| \hat{\lambda}(\delta^{a_1}\mathbf{r}', \delta^{a_2}\mathbf{l}_2) \right|^2 d\mathbf{r}' \right]^{1/2} d\mathbf{l}_1 d\mathbf{l}_2 \\
&\le \delta^{d(1-a_1-2a_2)} C(k,R,\lambda) \sup_{j=1,2} \|\theta_j\|^2_X \int_{\mathcal{A}_2} \left[\left(\frac{k}{z_1-s} \right)^d + \left(\frac{k}{z_2-s} \right)^d \right] ds.
\end{aligned}
\tag{5.30}
$$

We derive from this the following bound for I_2:

$$I_2 \le \delta^{d(1-a_1-2a_2)} C(k,\lambda,R) \sup_{j=1,2} \|\theta_j\|_X^2 \times \begin{cases} |\log \delta^p|, & d=1 \\ \delta^{-p}, & d=2 \end{cases}. \tag{5.31}$$

Using (5.29) and (5.31) we then arrive at the following bound for $I_{00}^{(1)}$ in (5.27):

$$|I_{00}^{(1)}| \le C(k,R,\lambda) \sup_{j=1,2} \|\theta_j\|_X^2 \times \begin{cases} \delta^{(1-2a_1-a_2)+p} + \delta^{(1-a_1-2a_2)}|\log \delta^p|, & d=1 \\ \delta^{2(1-2a_1-a_2)+p} + \delta^{2(1-a_1-2a_2)-p}, & d=2 \end{cases}.$$

We choose now $p = a_1 - a_2$ to obtain at estimate

$$|I_{00}^{(1)}| \le C(k,R,\lambda,Z) \sup_{j=1,2} \|\theta_j\|_X^2 \times \begin{cases} \delta^{(1-a_1-2a_2)}|\log \delta|, & d=1 \\ \delta^{2-3a_1-3a_2}, & d=2 \end{cases}. \tag{5.32}$$

The other terms contributing to I_{00} in (5.26) can be bounded analogously and we can conclude that (5.23) holds. This proves the bound (5.5) when $j = l = 0$.

The estimate of I_{00} in (5.30) is not optimal when $z_1 \neq z_2$ as far s-integration is concerned. While this leads to sub-optimal estimates in the higher order terms with $j, l > 0$, this step does not seem to over-estimate the term I_{00} with $z_1 = z_2$, which does have a contribution to the overall error in Proposition 5.2. A more careful analysis would reveal the relative size of the error produced by various orders of scattering – we do not pursue this avenue here.

Bound on Higher Order Scattering Terms. In this section we treat the general case $j > 0$, $l > 0$ in Lemma 5.5 by induction. In order to account for the shifts of the arguments in various integrals we introduce the shift operator $T_{\mathbf{c}} : X \to X$ defined by

$$[T_{\mathbf{c}}\theta](z,\mathbf{x},\mathbf{p}) = \theta(z, \mathbf{x} + (\mathbf{c}_1 + \mathbf{c}_2 z)\mathbf{p} + (\mathbf{c}_3 + \mathbf{c}_4 z), \mathbf{p} + \mathbf{c}_5), \tag{5.33}$$

where $\mathbf{c}_1, \mathbf{c}_2 \in \mathbb{R}$ and $\mathbf{c}_3, \mathbf{c}_4, \mathbf{c}_5 \in \mathbb{R}^d$. We will establish inductively the following generalized version of Lemma 5.5:

$$\begin{aligned} &\left|\mathbb{E}\left\{\langle [T_{\mathbf{c}} T_1^j T_2 \theta_1](z_1), \lambda_\delta\rangle \langle [T_{\mathbf{c}'} T_1^l T_2 \theta_2](z_2), \lambda_\delta\rangle\right\}\right| \\ &\quad \le \frac{(C_1(k,R,Z)z_1)^j}{j!} \frac{(C_1(k,R,Z)z_2)^l}{l!} C_2(k,R,\lambda,Z) \sup_{m=1,2} \|\theta_m\|_X^2 \times \begin{cases} \delta^{(1-a_1-2a_2)}|\log \delta|, & d=1 \\ \delta^{2-3a_1-3a_2}, & d=2 \end{cases}. \end{aligned} \tag{5.34}$$

The constants in (5.34) are independent of the shift $\mathbf{c}$. We first assume that (5.34) holds for $0 \le j \le \widehat{j}$ and $0 \le l \le \hat{l}$. Note that

$$\begin{aligned} &\left|\mathbb{E}\left\{\langle [T_{\mathbf{c}} T_1^{\widehat{j}+1} T_2 \theta_1](z_1), \lambda_\delta\rangle \langle [T_{\mathbf{c}'} T_1^{\hat{l}} T_2 \theta_2](z_2), \lambda_\delta\rangle\right\}\right| \\ &= \left|\mathbb{E}\left\{\frac{k^2}{4(2\pi)^d}\int_0^{z_1} \hat{R}_0(\mathbf{q}) \langle [T_{\widetilde{\mathbf{c}}(z_1,\mathbf{q})} T_1^{\widehat{j}} T_2 \theta_1](s), \lambda_\delta\rangle \, d\mathbf{q} ds \, \langle [T_{\mathbf{c}'} T_1^{\hat{l}} T_2 \theta_2](z_2), \lambda_\delta\rangle\right\}\right|, \end{aligned}$$

with the vector $\widetilde{\mathbf{c}}$ which has the components

$$\begin{aligned} &\widetilde{\mathbf{c}}(z_1,\mathbf{q})_1 = \mathbf{c}_1 + (\mathbf{c}_2 - 1/k) z_1, \quad \widetilde{\mathbf{c}}(z_1,\mathbf{q})_2 = 1/k, \\ &\widetilde{\mathbf{c}}(z_1,\mathbf{q})_3 = \mathbf{c}_3 + \mathbf{c}_4 z_1 - z_1 \mathbf{c}_5/k, \quad \widetilde{\mathbf{c}}(z_1,\mathbf{q})_4 = \mathbf{c}_5/k, \quad \widetilde{\mathbf{c}}(z_1,\mathbf{q})_5 = \mathbf{c}_5 + \mathbf{q}. \end{aligned}$$

Therefore, it follows from the induction hypothesis that

$$\begin{aligned} &\left|\mathbb{E}\left\{\langle [T_{\mathbf{c}} T_1^{\widehat{j}+1} T_2 \theta_1](z_1), \lambda_\delta\rangle \langle [T_{\mathbf{c}'} T_1^{\hat{l}} T_2 \theta_2](z_2), \lambda_\delta\rangle\right\}\right| \le \frac{k^2}{4(2\pi)^d}\int_0^{z_1} |\hat{R}_0(\mathbf{q})| \frac{(C_1(k,R,Z)s)^{\widehat{j}}}{\widehat{j}!} d\mathbf{q} ds \\ &\quad \times \frac{(C_1(k,R,Z)z_2)^{\hat{l}}}{\hat{l}!} C_2(k,R,\lambda,Z) \sup_{m=1,2} \|\theta_m\|_X^2 \times \begin{cases} \delta^{(1-a_1-2a_2)}|\log \delta|, & d=1 \\ \delta^{2-3a_1-3a_2}, & d=2 \end{cases} \\ &\le \frac{(C_1(k,R,Z)z_1)^{(\widehat{j}+1)}}{(\widehat{j}+1)!} \frac{(C_1(k,R,Z)z_2)^{\hat{l}}}{\hat{l}!} C_2(k,R,\lambda,Z) \sup_{m=1,2} \|\theta_m\|_X^2 \times \begin{cases} \delta^{(1-a_1-2a_2)}|\log \delta|, & d=1 \\ \delta^{2-3a_1-3a_2}, & d=2, \end{cases} \end{aligned}$$

if we take

$$C_1(k,R,Z) \geq \frac{k^2 \int |\hat{R}_0(\mathbf{q})|\, d\mathbf{q}}{4(2\pi)^d}.$$

We conclude that (5.34) holds for $0 \leq j \leq \widehat{j}+1$ and $0 \leq l \leq \hat{l}$.

To complete the induction argument we must finally show that (5.34) is valid for $j = l = 0$. This can be accomplished by a generalization of the argument leading to the bound (5.23) derived in the case without shift and we summarize this step below. We need to estimate

$$\begin{aligned}
\tilde{I}_{00} &= |\mathbb{E}\left\{\langle [T_{\mathbf{c}}T_2\theta_1](z_1),\lambda_\delta\rangle\langle [T_{\mathbf{c}'}T_2\theta_2](z_2),\lambda_\delta\rangle\right\}| \\
&= \frac{k^2}{4(2\pi)^{2d}}\left|\mathbb{E}\left\{\int_0^{z_1}\int_0^{z_2}\int e^{i\mathbf{q}_1\cdot(\tilde{\mathbf{x}}_1-(z_1-s_1)\tilde{\mathbf{p}}_1/k)/\delta+i\mathbf{q}_2\cdot(\tilde{\mathbf{x}}_2-(z_2-s_1)\tilde{\mathbf{p}}_2/k)/\delta}\right.\right. \\
&\quad\times\left[\theta_1\left(s_1,\tilde{\mathbf{x}}_1-(z_1-s_1)\frac{\tilde{\mathbf{p}}_1}{k},\tilde{\mathbf{p}}_1-\frac{\mathbf{q}_1}{2}\right)-\theta_1\left(s_1,\tilde{\mathbf{x}}_1-(z_1-s_1)\frac{\tilde{\mathbf{p}}_1}{k},\tilde{\mathbf{p}}_1+\frac{\mathbf{q}_1}{2}\right)\right] \\
&\quad\times\left[\theta_2\left(s_2,\tilde{\mathbf{x}}_2-(z_2-s_2)\frac{\tilde{\mathbf{p}}_2}{k},\tilde{\mathbf{p}}_2-\frac{\mathbf{q}_2}{2}\right)-\theta_2\left(s,\tilde{\mathbf{x}}_2-(z_2-s_2)\frac{\tilde{\mathbf{p}}_2}{k},\tilde{\mathbf{p}}_2+\frac{\mathbf{q}_2}{2}\right)\right] \\
&\quad\times\left.\left.\lambda_\delta(\mathbf{x}_1,\mathbf{p}_1)\lambda_\delta(\mathbf{x}_2,\mathbf{p}_2)d\hat{\mathbf{B}}(s_1,\mathbf{q}_1)d\hat{\mathbf{B}}(s_2,\mathbf{q}_2)d\mathbf{q}_1d\mathbf{q}_2d\mathbf{x}_1d\mathbf{x}_2d\mathbf{p}_1d\mathbf{p}_2\right\}\right|,
\end{aligned}$$

where

$$\begin{aligned}
&\tilde{\mathbf{x}}_1 = \mathbf{x}_1+(\mathbf{c}_1+\mathbf{c}_2z_1)\mathbf{p}_1+(\mathbf{c}_3+\mathbf{c}_4z_1), \quad \tilde{\mathbf{p}}_1=\mathbf{p}_1+\mathbf{c}_5\,, \\
&\tilde{\mathbf{x}}_2 = \mathbf{x}_2+(\mathbf{c}'_1+\mathbf{c}'_2z_2)\mathbf{p}_2+(\mathbf{c}'_3+\mathbf{c}'_4z_2), \quad \tilde{\mathbf{p}}_2=\mathbf{p}_2+\mathbf{c}'_5.
\end{aligned} \tag{5.35}$$

Using the same transformations as in the passage from (5.24) to (5.26) we obtain the following generalization of (5.26)

$$\begin{aligned}
\tilde{I}_{00} = C(k)&\left|\mathbb{E}\left\{\int_0^{[z_1,z_2]}\int e^{i[((z_1-s)\tilde{\mathbf{p}}_1/k)\cdot\mathbf{r}_1+((z_2-s)\tilde{\mathbf{p}}_2/k)\cdot\mathbf{r}_2]+i[\mathbf{p}_1\cdot\mathbf{l}_1+\mathbf{p}_2\cdot\mathbf{l}_2]}\right.\right. \\
&\times e^{-i[(\mathbf{c}_1+\mathbf{c}_2z_1)\mathbf{p}_1+(\mathbf{c}_3+\mathbf{c}_4z_1)]\cdot\mathbf{r}_1-i[(\mathbf{c}'_1+\mathbf{c}'_2z_2)\mathbf{p}_2+(\mathbf{c}'_3+\mathbf{c}'_4z_2)]\cdot\mathbf{r}_2} \\
&\times\left[\breve{\theta}_1\left(s,-\mathbf{r}_1-\frac{\mathbf{q}}{\delta},\tilde{\mathbf{p}}_1-\frac{\mathbf{q}}{2}\right)-\breve{\theta}_1\left(s,-\mathbf{r}_1-\frac{\mathbf{q}}{\delta},\tilde{\mathbf{p}}_1+\frac{\mathbf{q}}{2}\right)\right] \\
&\times\left[\breve{\theta}_2\left(s,-\mathbf{r}_2+\frac{\mathbf{q}}{\delta},\tilde{\mathbf{p}}_2+\frac{\mathbf{q}}{2}\right)-\breve{\theta}_2\left(s,-\mathbf{r}_2+\frac{\mathbf{q}}{\delta},\tilde{\mathbf{p}}_2-\frac{\mathbf{q}}{2}\right)\right] \\
&\times\left.\left.\hat{\lambda}(\delta^{a_1}\mathbf{r}_1,\delta^{a_2}\mathbf{l}_1)\hat{\lambda}(\delta^{a_1}\mathbf{r}_2,\delta^{a_2}\mathbf{l}_2)\hat{R}_0(\mathbf{q})d\mathbf{q}ds\Pi_{j=1}^2d\mathbf{p}_jd\mathbf{r}_jd\mathbf{l}_j\right\}\right|.
\end{aligned} \tag{5.36}$$

Considering the term similar to that in (5.27) and taking Fourier transform in $\mathbf{p}_1$ and $\mathbf{p}_2$ we obtain

$$\begin{aligned}
\left|\tilde{I}_{00}^{(1)}\right| &\leq C(k)\left|\mathbb{E}\left\{\int_0^{[z_1,z_2]}\int e^{i[\mathbf{q}\cdot((z_1-s)\mathbf{r}_1/k+\mathbf{l}_1-(z_2-s)\mathbf{r}_2/k-\mathbf{l}_2)/2]}\right.\right. \\
&\times e^{-i[(\mathbf{c}_1+\mathbf{c}_2z_1)\mathbf{q}/2+(\mathbf{c}_3+\mathbf{c}_4z_1)]\cdot\mathbf{r}_1+i[(\mathbf{c}'_1+\mathbf{c}'_2z_2)\mathbf{q}/2-(\mathbf{c}'_3+\mathbf{c}'_4z_2)]\cdot\mathbf{r}_2+i\mathbf{c}_5\cdot((\mathbf{c}_1+\mathbf{c}_2z_1)\mathbf{r}_1-\mathbf{l}_1)+i\mathbf{c}'_5\cdot((\mathbf{c}'_1+\mathbf{c}'_2z_2)\mathbf{r}_2-\mathbf{l}_2)} \\
&\times\hat{\theta}_1\left(s,-\mathbf{r}_1-\frac{\mathbf{q}}{\delta},-\mathbf{l}_1-\mathbf{r}_1h_1(s)/k\right)\hat{\theta}_2\left(s,-\mathbf{r}_2+\frac{\mathbf{q}}{\delta},-\mathbf{l}_2-\mathbf{r}_2h_2(s)/k\right) \\
&\times\left.\left.\hat{\lambda}(\delta^{a_1}\mathbf{r}_1,\delta^{a_2}\mathbf{l}_1)\hat{\lambda}(\delta^{a_1}\mathbf{r}_2,\delta^{a_2}\mathbf{l}_2)\hat{R}_0(\mathbf{q})d\mathbf{q}\Pi_{j=1}^2d\mathbf{r}_jd\mathbf{l}_j\,ds\right\}\right|,
\end{aligned}$$

with

$$h_1(s) = (z_1-s)-k(\mathbf{c}_1+\mathbf{c}_2z_1), \quad h_2(s)=(z_2-s)-k(\mathbf{c}'_1+\mathbf{c}'_2z_2). \tag{5.37}$$

After the change of variable $\mathbf{q}/\delta \mapsto \mathbf{q}$ we find

$$\begin{aligned}
\left|I_{00}^{(1)}\right| &\leq \delta^dC(k)\mathbb{E}\left\{\int_0^{[z_1,z_2]}\int\left|\hat{\theta}_1\left(s,-\mathbf{r}_1-\mathbf{q},-\mathbf{l}_1-\mathbf{r}_1h_1(s)/k\right)\right.\right. \\
&\times\left.\left.\hat{\theta}_2\left(s,-\mathbf{r}_2+\mathbf{q},-\mathbf{l}_2-\mathbf{r}_2h_2(s)/k\right)\hat{\lambda}(\delta^{a_1}\mathbf{r}_1,\delta^{a_2}\mathbf{l}_1)\hat{\lambda}(\delta^{a_1}\mathbf{r}_2,\delta^{a_2}\mathbf{l}_2)\hat{R}_0(\delta\mathbf{q})\right|d\mathbf{q}\Pi_{j=1}^2d\mathbf{r}_jd\mathbf{l}_j\,ds\right\}.
\end{aligned}$$

In the above integral we decompose the interval $(0,[z_1,z_2])$ as

$$\tilde{\mathcal{A}}_1 = \{s \mid \min(|h_1(s)|,|h_2(s)|) < \delta^p\}\,, \qquad \tilde{\mathcal{A}}_2 = (0,[z_1,z_2]) \setminus \tilde{\mathcal{A}}_1, \tag{5.38}$$

for p some positive constant, which is a slight modification of the final boundary layer defined in (5.28). The argument following (5.27) can now be repeated verbatim with these slightly modified integration subintervals to give the generalized version of (5.32):

$$|\tilde{I}_{00}^{(1)}| \le \bar{C}(k,R,\lambda,Z) \sup_{j=1,2} \|\theta_j\|_X^2 \times \begin{cases} \delta^{(1-a_1-2a_2)}|\log\delta|, & d=1 \\ \delta^{2-3a_1-3a_2}, & d=2 \end{cases}.$$

The other terms contributing to $\tilde{I}_{00}$ in (5.36) can again be bounded analogously. This concludes the inductive proof of (5.34) and hence also that of Lemma 5.5. □

REFERENCES

[1] F. Bailly, J.F. Clouet and J.P. Fouque, Parabolic and white noise approximation for waves in random media, SIAM Jour. Appl. Math. **56**, 1996, 1445-1470.
[2] G. Bal, On the self-averaging of wave energy in random media. Multiscale Model. Simul. **2**, 2004, 398–420.
[3] G.Bal and L. Ryzhik, Time reversal and refocusing in random media, SIAM Jour. Appl. Math., **63**, 2003, 1475–1498.
[4] G. Bal, T. Komorowski and L. Ryzhik, Self-averaging of Wigner transforms in random media. Comm. Math. Phys. **242**, 2003, 81–135
[5] G.Bal, G. Papanicolaou and L. Ryzhik, Self-averaging in time reversal for the parabolic wave equation. Stoch. Dyn. **2**, 2002, 507–531.
[6] G.Bal, G. Papanicolaou and L. Ryzhik, Radiative transport limit for the random Schrödinger equation, Nonlinearity, **15**, 2002, 513-529.
[7] P. Billingsley, Convergence of probability measures, 2nd edition, New York, Wiley, 1999.
[8] G. Blankenship and G. C. Papanicolaou, Stability and Control of Stochastic Systems with Wide-Band Noise Disturbances, SIAM J. Appl. Math., **34**, 1978, 437-476.
[9] P. Blomgren, G. Papanicolaou, and H. Zhao, Super-Resolution in Time-Reversal Acoustics, J. Acoust. Soc. Am., **111**, 2002, 230-248.
[10] L. Borcea, G. Papanicolaou and C. Tsogka, Interferometric array imaging in clutter, Inverse Problems, **21**, 2005, 1419–1460.
[11] L. Borcea, G. Papanicolaou and C. Tsogka, Adaptive interferometric imaging in clutter and optimal illumination, Inverse Problems, **22**, 2006, 1405–1436.
[12] L. Borcea, G. Papanicolaou and C. Tsogka, Asymptotics for the space-time Wigner transform with applications to imaging, to appear, 2007.
[13] R. Bouc and E. Pardoux, Asymptotic analysis of PDEs with wide-band noise disturbances and expansion of the moments, Stochastic Analysis and Applications, **2**, 1984, 369-422.
[14] D. Dawson and G. Papanicolaou, A random wave process, Appl. Math. Optim., **12**, 1984, 97–114.
[15] D. A. Dawson, Measure-valued Markov processes, in Ecole d' Ete de Probabilite de Saint-Flour XXI—1991. [Saint-Flour Summer School on Probability Theory XXI—1991] edited by P. L. Hennequin, Lecture Notes in Mathematics, 1541, Springer-Verlag, Berlin, 1993.
[16] A. Fannjiang, White-noise and geometrical optics limits of Wigner-Moyal equation for wave beams in turbulent media. Comm. Math. Phys. **254**, 2005, 289–322.
[17] A. Fannjiang, White-noise and geometrical optics limits of Wigner-Moyal equation for beam waves in turbulent media. II. Two-frequency formulation. J. Stat. Phys. **120**, 2005, 543–586.
[18] J.P. Fouque, La convergence en loi pour les processus a valeurs dans un espace nucleaire, Ann. Inst. H. Poincare Probab. Statist., **20**, 1984, 225-245.
[19] J.P. Fouque, G.C. Papanicolaou and Y. Samuelides, Forward and Markov Approximation: The Strong Intensity Fluctuations Regime Revisited, Waves in Random Media, **8**, 1998, 303-314.
[20] I.A. Ibragimov and Yu. V. Linnik, *Independent and Stationary Sequences of Random Variables*, Wolters-Noordhoff, Groningen, 1971.
[21] H. Kunita, *Stochastic flows and stochastic differential equations*. Cambridge Studies in Advanced Mathematics, 24. Cambridge University Press, Cambridge, 1997.
[22] H. Kushner, Approximation and weak convergence methods for random processes, with applications to stochastic systems theory, MIT Press Series in Signal Processing, Optimization, and Control, MIT Press, 1984.
[23] I. Mitoma, Tightness of probabilities on $C([0,1];\mathcal{S}')$ and $D([0,1];\mathcal{S}')$, Ann. Probab., **11**, 1983, 989-999.
[24] B. Nair and B. White, High-frequency wave propagation in random media- a unified approach, SIAM J. Appl. Math., **51**, 1991, 374-411.
[25] G. Papanicolaou, L. Ryzhik and K. Solna, Statistical stability in time reversal, SIAM J. on Appl. Math., **64**, 2004, 1133–1155.
[26] L. Ryzhik, G. Papanicolaou, and J. B. Keller. Transport equations for elastic and other waves in random media, Wave Motion, **24**, 327–370, 1996.
[27] F. Tappert, The parabolic approximation method, Lecture notes in physics, vol. 70, *Wave propagation and underwater acoustics*, Springer-Verlag, 1977.

[28] V. I. Tatarskii, A. Ishimaru and V. U. Zavorotny, editors, Wave Propagation in Random Media (Scintillation), SPIE and IOP, 1993.
[29] B, J. Uscinski, Wave Propagation and Scattering, Oxford Clarendon Press, 1986.