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SELF-AVOIDING WALK ON A HIERARCHICAL LATTICE IN FOUR DIMENSIONS

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We define a Lévy process on a d-dimensional hierarchical lattice. By construction the Green's function for this process decays as $|x|^{2-d}$. For $d = 4$, we prove that the introduction of a sufficiently weak self-avoidance interaction does not change this decay provided the mass ≡ "killing" rate is chosen in a special way, so that the process is critical.

1. Introduction. Self-avoiding walk is of particular interest in four dimensions because it is believed to be the borderline between simple Brownian behavior in dimension $d > 4$ and complex behavior in $d < 4$. It has been argued [Brézin, Le Guillou and Zinn-Justin (1976), De Gennes (1972) and Duplantier (1986)] that a self-avoiding walk of length T on a simple cubic lattice in four dimensions will have an end-to-end distance which is asymptotic to a constant times $T^{1/2}(\log T)^{1/8}$ as $T \to \infty$.

In order to learn about this problem, we begin an analysis of an analogous problem with the simple cubic lattice replaced by a hierarchical lattice. As should be evident in Figure 1, the hierarchical lattice behaves simply under a rescaling of distances. This feature makes it especially suited to studying long-distance or long-time asymptotics via successive rescalings (the "renormalization group" method). Hierarchical models were introduced by Dyson (1969) for lattice spin systems in statistical mechanics. They have since proven a useful guide to the more complicated "real" models [Benfatto, Cassandro, Gallavotti, Nicolo, Olivieri, Presutti and Scacciatelli (1978) and Gawedzki and Kupiainen (1982, 1986)].

In Figure 1 we illustrate how *hierarchical distance*, d_H, is defined on a one-dimensional lattice:

$$d_H(x, y) = 2^{\mathrm{scale}(x, y)} = 2^4.$$

(The distance from a point to itself is 0.)

If we define $|x|_H \equiv d_H(0, x)$, then $d_H(x, y) = |x - y|_H$. Here the minus is not the minus that comes from the usual way of regarding the one-dimen-

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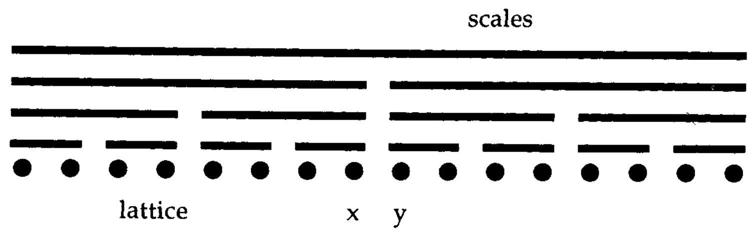

FIG. 1.

sional lattice as the additive group isomorphic to the integers. Instead the lattice points are labeled by elements of $\mathscr{G} \equiv \oplus_{k=0}^{\infty} \mathbb{Z}_2$ (see Figure 2). We call this a one-dimensional lattice because the number of points inside the ball $\{x\colon |x| \le 2^N\}$ is 2^N.

We can replace $\mathbb{Z}_2$ by $\mathbb{Z}_n$ with $n = L^d$, where L is any integer greater than 1. If we define

$$|x|_H \equiv \begin{cases} L^{\text{scale of } x}, & \text{if } x \ne 0, \\ 0, & \text{if } x = 0, \end{cases} \tag{1.1}$$

then we will obtain a lattice $\mathscr{G}$ which is d-dimensional in the sense that the ball $\{x\colon |x|_H \le L^N\}$ contains $O(L^{dN})$ points.

Lévy process. We obtain self-avoiding walk on the hierarchical lattice as a perturbation of a Lévy process (or continuous-time Markov jump process) on $\mathscr{G}$. [Lévy processes on hierarchical lattices were considered in another context in Knapp (1988) and Köhler, Knapp and Blumen (1988), using a different set of allowed transitions.] This is analogous to perturbing about simple random walk to obtain self-avoiding walk in $\mathbb{Z}^4$. A nearest-neighbor random walk on $\mathscr{G}$ would be very dull; it would never get out of the copy of $\mathbb{Z}_n$ in which it starts, but we can create analogues to Euclidean invariant processes by allowing walks to have long-range jumps from x to y with a probability which is a function of $|x - y|_H$. Let $E_x(\cdot)$ denote the expectation associated with the Lévy

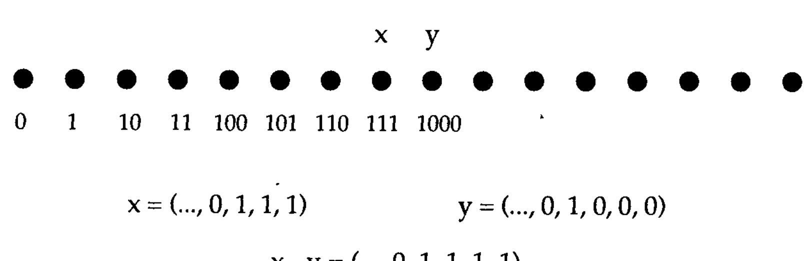

FIG. 2.

process with initial condition $x \in \mathscr{G}$. A sample path will be denoted $\omega(t)$, $0 \le t \le \infty$. The Green's function for the process is defined to be

$$G(x,y) = \int dT\, E_x(\mathbf{1}_{\{\omega(T)=y\}}).$$

In Section 2 we prove the following result.

LEMMA. *The Green's function for a Lévy process on* $\mathscr{G}$ *such that*

$$\text{Prob}(\textit{jump from } x \textit{ to } y) = \text{Const} \cdot |x-y|_H^{-d-2} \tag{1.2}$$

is

$$G(x,y) = \frac{\text{Const}}{|x-y|_H^{d-2}} \quad \textit{for } x \ne y. \tag{1.3}$$

$G(x,y) = G(x-y)$ can be thought of as the hierarchical version of the inverse Laplacian in $\mathbb{Z}^d$. We will be dropping the H subscript from this point onward because we are not going to have any occasion to use the Euclidean norm.

Self-avoidance. We make the process tend to avoid itself by modifying the expectation E for a Lévy process which starts at the origin and satisfies (1.2). Given a path $\omega(t)$, $0 \le t \le T$, for the Lévy process, we define the local time spent at x (up to time T) by

$$\tau(x) \equiv \int_0^T ds\, \mathbf{1}_{\{\omega(s)=x\}}. \tag{1.4}$$

The time spent inside some set $\Lambda \subset \mathscr{G}$ is therefore

$$\tau(\Lambda) \equiv \int_\Lambda dx\, \tau(x), \tag{1.5}$$

where dx is the counting measure which is also the Haar measure on the lattice $\mathscr{G}$. A measure of self-intersection inside a set Λ is

$$\tau^2(\Lambda) \equiv \int_\Lambda dx\, \tau^2(x) = \int\int ds\, dt\, \mathbf{1}_{\{\omega(s)=\omega(t)\in\Lambda\}}. \tag{1.6}$$

The expectation E_λ^T for a self-avoiding walk depends on the strength $\lambda \ge 0$ of the self-repulsion and the time T for which the walk tries to avoid itself. It is given by

$$E_\lambda^T(\cdot) \equiv \frac{E(e^{-\lambda\tau^2(\mathscr{G})}\,\cdot)}{E(e^{-\lambda\tau^2(\mathscr{G})})}. \tag{1.7}$$

Our objective is to prove that the end-to-end distance $E_\lambda^T(|\omega(T)|)$ for this walk is asymptotic to a constant times $T^{1/2}(\log T)^{1/8}$. However, we fall short of this in this paper and instead we investigate the Green's function which we define for $\lambda \geq 0$ by

$$(1.8) \qquad U^a(x) \equiv \lim_{\Lambda \to \mathscr{S}} \int_0^\infty dT\, E\left(e^{-\lambda\tau^2(\Lambda) - a\tau(\Lambda)} \mathbf{1}_{\{\omega(T)=x\}}\right).$$

A more natural object would be obtained by moving the limit inside the integrals to obtain

$$(1.9) \qquad \tilde{U}^a(x) \equiv \int_0^\infty dT\, e^{-aT} E\left(e^{-\lambda\tau^2(\mathscr{S})} \mathbf{1}_{\{\omega(T)=x\}}\right).$$

We have used $T = \tau(\mathscr{S})$. By Fatou's lemma $\tilde{U}^a \leq U^a$. We intend to show that U and $\tilde{U}$ are equal in another paper. In this paper we prove the following theorem.

THEOREM. *Let $d = 4$. For each sufficiently small $\lambda \geq 0$, there exists $a_c = a_c(\lambda) \leq 0$ such that U^a exists for $a \geq a_c$ and*

$$U^{a_c}(x) \sim \frac{\text{Const}(\lambda)}{|x|^2} \quad \text{as } x \to \infty.$$

The absence of logarithmic corrections in the Green's function is expected [Gawedzki and Kupiainen (1982)]. These logarithmic corrections will make their appearance when the numerator and denominator of (1.7) are reconstructed from the Green's function (1.8) or (1.9) by inverting the Laplace transform in T. In other words, U^a as a function of a will have a singularity at $a = a_c$. The logarithms are expected to be in this singularity.

Method of proof. There is a relation between local time and the square of a Gaussian field [Symanzik (1969), Brydges, Fröhlich and Spencer (1983) and Dynkin (1983)]. In McKane (1980), Parisi and Sourlas (1979, 1980) and Le Jan (1987, 1988), a more complete version of this isomorphism was introduced. It involves the unfamiliar concept of Grassman integration but this turns out to be very manageable [e.g., see Luttinger (1983), Brydges and Munoz-Maya (1991), Campanino and Klein (1986) and Klein, Landau and Fernando-Perez (1984) for some applications].

We begin in Section 2 by describing in detail the hierarchical Lévy processes.

In Section 3 we give a review of Grassman integration and in Theorem 3.3 we prove the formula by which McKane and Parisi and Sourlas pass from Green's functions for Markov processes to Gaussian fields:

$$U^a(x) = \lim_{\Lambda \to \mathscr{S}} \int d\mu_G(\Phi) e^{-\lambda(\Phi^2)^2(\Lambda) - a\Phi^2(\Lambda)} \bar{\psi}_0 \psi_x.$$

Essentially it says that (for any Markov process) τ equals $\Phi^2 \equiv \varphi\bar{\varphi} + \psi\bar{\psi}$, where φ is a collection of (complex valued) Gaussian random variables and ψ is the Grassman analogue to Gaussian random variables.

In Sections 4 and 5 we show how the Wilson renormalization group is set up in this language of φ and ψ. We split the covariance G into $G' + \Gamma$, where G' is chosen so that, under a rescaling $\mathscr{R}$ of lengths by a factor L, G' scales back into G. Γ is called a fluctuation covariance because it relates to fluctuations of Φ about its average on a ball of L^{dN} points. We obtain

$$\int d\mu_G(\Phi) e^{-\lambda(\Phi^2)^2(\Lambda) - a\Phi^2(\Lambda)} \bar{\psi}_0 \psi_x$$

$$= \int d\mu_{G'}(\Phi') \int d\mu_\Gamma(\zeta) e^{-\lambda(\Phi^2)^2(\Lambda) - a\Phi^2(\Lambda)} \bar{\psi}_0 \psi_x$$

$$= \int d\mu_G(\Phi) \mathscr{R}\left(\int d\mu_\Gamma(\zeta) e^{-\lambda(\Phi^2)^2(\Lambda) - a\Phi^2(\Lambda)} \bar{\psi}_0 \psi_x \right),$$

where on the right-hand side $\Phi = \Phi' + \zeta$ and ψ and $\bar{\psi}$ are the Grassman components of $\Phi' + \zeta$. In the hierarchical model the fluctuation covariance has finite range. Hence the hierarchical random walk has the simplifying feature that the *renormalization group map*

$$F \to \tilde{T}(F) \equiv \mathscr{R}\left(\int d\mu_\Gamma(\zeta) F(\Phi + \zeta) \right) \equiv \mathscr{R}\mu_\Gamma * F$$

preserves locality: If $F(\Phi^2) = \prod_x f(\Phi^2(x))$, then the action of the renormalization group on F descends to an action on f. $F \to \tilde{T}(F)$ is the same as another map T acting on f. The computation at the level of f is a finite-dimensional problem. This is the essential simplification of the renormalization group method.

After proving several analytical lemmas in Section 6, we proceed to the main part of the paper in Section 7. There we prove that under the renormalization group, $f \to T(f) \to T^2(f) \to \cdots$, f tends to 1, that is, the self-avoidance becomes weaker. In Section 8 we prove that the long-distance behavior of the critical Green's function $U^{a_c}(x)$ is as if there were no self-avoidance. The effect of the self-avoidance appears only as a shift in the diffusion rate, that is, the constant in the theorem above.

Our discussion in Section 7 is an extension of the procedure in Brydges and Yau (1990) and is not the same as the previous approaches to hierarchical models, although there are related ideas in Koch and Wittwer (1986).

We could use the method of this paper to prove results for hierarchical walk with higher degree polynomial or even some nonpolynomial interactions in τ. We also believe that the procedure will extend to self-avoiding walk on a simple cubic lattice.

The Gaussian processes we have just discussed are indexed by a group with an invariant metric which is an ultrametric: $|x + y| \le \max(|x|, |y|)$. There are Gaussian processes indexed by fields carrying an ultrametric, in particular, local fields (e.g., the P-adic numbers). These have been studied by Evans (1988a, b, 1989a, b).

2. The hierarchical random walk. Here we construct the hierarchical Lévy process. The main results are summarized in Proposition 2.3. We also define the "fluctuation covariances," which will be appearing in the rest of the paper.

Fix an integer $L \geq 2$. The points of the d-dimensional hierarchical lattice are elements of the group $\mathscr{G} = \oplus_{k=0}^{\infty} \mathbb{Z}_n$, $n = L^d$. An element x in $\mathscr{G}$ is an infinite sequence

$$x \equiv (\ldots, x_2, x_1, x_0), \qquad x_i \in \text{ additive group } \mathbb{Z}_n,$$

where only finitely many x_i are nonzero.

We define subgroups

$$\begin{aligned} &\{0\} = \mathscr{G}_0 \subset \mathscr{G}_1 \subset \cdots \subset \mathscr{G}, \\ &\mathscr{G}_k = \{x \in \mathscr{G}: x_i = 0, i \geq k\}. \end{aligned} \tag{2.1}$$

We define a norm $|\cdot|$, which depends on L and d, by

$$|x| = \begin{cases} 0, & \text{if } x = 0, \\ L^p, & \text{where } p = \inf\{k: x \in \mathscr{G}_k\} \text{ if } x \neq 0. \end{cases} \tag{2.2}$$

We define dx to be the Haar measure which is also the counting measure on $\mathscr{G}$, implying that

$$\int_{|x| \leq L} dx\, 1 = L^d. \tag{2.3}$$

Thus in this metric the subgroups $\mathscr{G}_k$ are balls $|x| \leq L^k$ containing $L^{dk} \equiv n^k$ points.

The Fourier transform on $\mathscr{G}$. Let $\mathscr{H}$ be the dual of $\mathscr{G}$, $\mathscr{H} = \times_{k=0}^{\infty} \hat{\mathbb{Z}}_n$. An element $\xi \in \mathscr{H}$ is an *infinite* sequence

$$\xi = (\xi_0, \xi_1, \ldots), \qquad \xi_k \in \hat{\mathbb{Z}}_n.$$

The pairing $\langle \mathscr{G}, \mathscr{H} \rangle \to$ unit circle is given by

$$\langle x, \xi \rangle = \prod_{k=0}^{\infty} \langle x_k, \xi_k \rangle. \tag{2.4}$$

The elements of $\hat{\mathbb{Z}}_n$ are $\{\chi_0, \chi_1, \ldots, \chi_{n-1}\}$, where

$$\chi_m(l) \equiv \langle \chi_m, l \rangle = \exp\left(\frac{2\pi i}{n} ml\right), \qquad l \in \mathbb{Z}_n. \tag{2.5}$$

We will write m instead of χ_m (i.e., identify the multiplicative group $\hat{\mathbb{Z}}_n$ with $\mathbb{Z}_n$).

Define $\mathscr{H}_k$ to be the annihilator of $\mathscr{G}_k$, that is,

$$\begin{aligned} \mathscr{H}_k &\equiv \{\xi \in \mathscr{H}: \langle x, \xi \rangle = 1\ \forall\, x \in \mathscr{G}_k\} \\ &= \begin{cases} \{(\xi_0, \xi_1, \ldots): \xi_i = 0 \text{ if } i < k\}, & \text{if } k > 0, \\ \mathscr{H}, & \text{if } k = 0. \end{cases} \end{aligned} \tag{2.6}$$

On $\mathscr{H}$ we define the metric $|\cdot|$,

$$|\xi| = \begin{cases} 0, & \text{if } \xi = 0, \\ L^{-p},\ p = \sup\{k : k \in \mathscr{H}_k\}, & \text{if } \xi \neq 0. \end{cases} \tag{2.7}$$

We define the Fourier transform of a function $f\colon \mathscr{G} \to \mathbb{C}$ by

$$\hat{f}(\xi) = \int_{\mathscr{G}} dx\, f(x) \langle x, \xi \rangle. \tag{2.8}$$

LEMMA 2.1 [Evans (1988a, b, 1989a, b)]. *For $k = 0, 1, \dots,$ let $\mathbf{1}_{\mathscr{G}_k}(x) = 1$ if $x \in \mathscr{G}_k$, 0 otherwise. Then*

$$\hat{\mathbf{1}}_{\mathscr{G}_k} = n^k \mathbf{1}_{\mathscr{H}_k}, \qquad n = L^d. \tag{2.9}$$

PROOF. If $\xi \in \mathscr{H}_k$, then

$$\int_{\mathscr{G}_k} dx \langle x, \xi \rangle = \int_{\mathscr{G}_k} dx\, 1 = n^k.$$

If $\xi \notin \mathscr{H}_k$, then there exists $y \in \mathscr{G}_k$ such that $\langle y, \xi \rangle \neq 1$.

$$\int_{\mathscr{G}_k} dx \langle x, \xi \rangle = \int_{\mathscr{G}_k} dx \langle x + y, \xi \rangle$$

$$= \langle y, \xi \rangle \int_{\mathscr{G}_k} \langle x, \xi \rangle\, dx.$$

Therefore $\int_{\mathscr{G}_k} dx \langle x, \xi \rangle = 0$. □

Lévy processes on $\mathscr{G}$. The Green's function (potential) for simple random walk on a simple cubic lattice $\mathbb{Z}^d$, $d > 2$, has asymptotic behavior Const $\cdot$ $|x|^{2-d}_{\text{Euclidean}}$ as $|x| \to \infty$. We will look for a Lévy process on $\mathscr{G}$ which has the Green's function

$$G(x) = \begin{cases} \dfrac{1 - L^{-d}}{1 - L^{-2}}, & \text{if } x = 0, \\ \dfrac{1}{|x|^{d-2}}, & \text{if } x \neq 0. \end{cases}$$

Following methods in Evans (1988a, b, 1989a, b), we identify the infinitesimal generator by inverting G, using the Fourier transform on the group $\mathscr{G}$.

Suppose X_t is a Lévy process ($\equiv$ continuous-time random walk) on $\mathscr{G}$ with law $P(t, x - y)$:

$$P(X_t = y | X_0 = x) = P(t, x - y). \tag{2.10}$$

We suppose that the process has a probability $r\,dt$ of making a jump in time $[t, t + dt]$ and, given that it jumps, the probability of jumping from x to y is $q(x - y)$.

LEMMA 2.2 (Lévy–Hinčin formula).

$$\int dx\, P(t, x) \langle x, \xi \rangle = e^{-t\psi(\xi)},$$

where

$$\psi(\xi) = r\int_{\mathscr{G}} dx\, q(x)[1 - \langle x, \xi\rangle].$$

This says that $\psi(\xi)$ is the Fourier transform of the infinitesimal generator of the semigroup $P(t, x)$.

PROOF. If we condition on N, the number of steps, then $X(t)$, the position at time t, $= X_1 + \cdots + X_N$, where each step X_i is an independent random variable with density $q(x)$. N is Poisson distributed with mean rt; therefore

$$\begin{aligned} E(\langle \xi, X(t)\rangle) &= \sum_{n=0}^{\infty} \frac{(rt)^n}{n!} e^{-rt} E\left\langle \xi, \sum_{i=1}^{n} X_i \right\rangle \\ &= \sum_{n=0}^{\infty} \frac{(rt)^n}{n!} e^{-rt} \prod_{i=1}^{n} \left(\int dx\, q(x) \langle \xi, x\rangle \right) \\ &= \exp\left(rt \int dx\, q(x) \langle x, \xi\rangle - rt \right). \end{aligned}$$ □

We define the Green's function ($\equiv \beta$-potential) for $\beta > 0$ by

(2.11) $$U^\beta(x, y) \equiv U^\beta(x - y) = \int_0^\beta dt\, e^{-\beta t} P(t, x, y),$$

and deduce that

(2.12) $$\hat{U}^\beta(\xi) = (\beta + \psi(\xi))^{-1}.$$

We now specialize by assuming that

$$q(x) = \begin{cases} \dfrac{(L^{\alpha - d} - 1)}{1 - L^{-d}} |x|^{-\alpha}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

where α is sufficiently large that $\int q(x)\, dx < \infty$. This implies $\alpha > d$. The coefficient is chosen so that $\int q(x)\, dx = 1$. We will now compute ψ and U^β for this process. The results are summarized in Proposition 2.3 below.

For this choice of q,

$$\psi(\xi) = \text{Const} \cdot \sum_{k=1}^{\infty} L^{-\alpha k} \int_{\mathscr{G}_k - \mathscr{G}_{k-1}} dx (1 - \langle x, \xi\rangle),$$

and by Lemma 2.1,

$$\begin{aligned} \int_{\mathscr{G}_k - \mathscr{G}_{k-1}} dx(1 - \langle x, \xi\rangle) &= \int_{\mathscr{G}_k} dx(1 - \langle x, \xi\rangle) - \int_{\mathscr{G}_{k-1}} dx(1 - \langle x, \xi\rangle) \\ &= n^k(1 - \mathbf{1}_{\mathscr{H}_k}) - n^{k-1}(1 - \mathbf{1}_{\mathscr{H}_{k-1}}). \end{aligned}$$

Therefore, if $\xi \in \mathscr{H}_j - \mathscr{H}_{j+1}$, $j = 0, 1, \ldots,$ all terms with $k \leq j$ in the summand in $\psi(\xi)$ vanish and

$$\psi(\xi) = \text{Const} \cdot \left[L^{-\alpha(j+1)} n^{j+1} + \sum_{k=j+2}^{\infty} L^{-\alpha k}(n^k - n^{k-1}) \right].$$

Recalling that $n = L^d$, $|\xi| = L^{-j}$, the last expression

$$= \text{Const} \cdot \left[L^{(d-\alpha)(j+1)} + \frac{L^{(d-\alpha)(j+2)}(1 - L^{-d})}{1 - L^{d-\alpha}} \right]$$

$$= \gamma |\xi|^{\alpha - d},$$

and we can compute $\gamma = r(1 - L^{-\alpha})/(1 - L^{-d})$. From this and (2.12) we learn that

$$\hat{U}^{\beta}(\xi) = \frac{1}{\beta + \gamma |\xi|^{\alpha - d}} = \sum_{k=0}^{\infty} \left(\beta + \gamma L^{-(\alpha-d)k}\right)^{-1} \left(\mathbf{1}_{\mathscr{H}_k}(\xi) - \mathbf{1}_{\mathscr{H}_{k+1}}(\xi)\right). \tag{2.13}$$

By Lemma 2.1,

$$U^{\beta}(x) = \sum_{k=0}^{\infty} \left(\beta L^{kd} + \gamma L^{-(\alpha-2d)k}\right)^{-1} \left(\mathbf{1}_{\mathscr{G}_k}(x) - \frac{1}{n} \mathbf{1}_{\mathscr{G}_{k+1}}(x)\right),$$

and provided $\alpha < 2d$,

$$\lim_{\beta \downarrow 0} U^{\beta}(x) = \frac{1}{\gamma} \sum_{k=0}^{\infty} L^{(\alpha-2d)k} \left(\mathbf{1}_{\mathscr{G}_k} - \frac{1}{n} \mathbf{1}_{\mathscr{G}_{k+1}}\right)(x).$$

Since $\mathbf{1}_{\mathscr{G}_k}(x) = 0$ unless $L^k \geq |x|$, and $= 1$ otherwise, we can evaluate the sums and

$$\lim_{\beta \downarrow 0} U^{\beta}(x) = \frac{1}{\gamma} \left\{ \frac{1 - L^{-d}}{1 - L^{\alpha - 2d}} \mathbf{1}_{\{x=0\}} + \frac{1 - L^{d-\alpha}}{1 - L^{\alpha-2d}} |x|^{\alpha - 2d} \mathbf{1}_{\{x \neq 0\}} \right\}.$$

PROPOSITION 2.3. *Let α satisfy $d < \alpha < 2d$. A Lévy process on $\mathscr{G}$ with jumps governed by a measure $rq(x)\,dx = r(L^{\alpha-d} - 1)/(1 - L^{-d})|x|^{-\alpha}\,dx$ satisfies*

$$E(\langle X_t, \xi \rangle) = e^{-t\psi(\xi)},$$

$$\psi(\xi) = \gamma |\xi|^{\alpha - d},$$

$$\hat{U}^{\beta}(\xi) = \frac{1}{\beta + \psi(\xi)},$$

$$U^{\beta}(x) = \sum_{k=0}^{\infty} \left(\beta L^{kd} + \gamma L^{-(\alpha-2d)k}\right)^{-1} \left(\mathbf{1}_{\mathscr{G}_k}(x) - \frac{1}{n} \mathbf{1}_{\mathscr{G}_{k+1}}(x)\right),$$

$$U^{\beta=0}(x) = \begin{cases} \dfrac{1}{\gamma} \dfrac{1 - L^{-d}}{1 - L^{\alpha-2d}}, & \text{if } x = 0, \\ \rho |x|^{\alpha - 2d}, & \text{if } x \neq 0, \end{cases}$$

where

$$\rho = \frac{1}{\gamma}\frac{1 - L^{d-\alpha}}{1 - L^{(\alpha-2d)}},$$

$$\gamma = \frac{r(1 - L^{-\alpha})}{1 - L^{-d}}.$$

These potentials will serve as covariances for Gaussian processes. The following definition prepares for this.

DEFINITION 2.4. Given any integer $L \geq 2$ and $\beta \geq 0$, let $G(\beta, x) \equiv U^{\beta}(x)$ be the β-potential for the Lévy process with $\alpha = d + 2$ and jumping rate r chosen so that $\rho = 1$. Let $x \mapsto x/L$ be the $\mathscr{G}$-homomorphism defined by

$$(\ldots, x_2, x_1, x_0)/L = (\ldots, x_3, x_2, x_1).$$

Define the *fluctuation covariance* $\Gamma(\beta, x)$ by

$$\Gamma(\beta, x) \equiv (\beta + \gamma)^{-1}\left(\mathbf{1}_{\mathscr{G}_0}(x) - \frac{1}{n}\mathbf{1}_{\mathscr{G}_1}(x)\right), \tag{2.14}$$

where $\gamma = (1 - L^{-2})/(1 - L^{2-d})$.

The Green's function $G(\beta, x)$ has an expansion in terms of scaled copies of Γ:

$$G(\beta, x) \equiv \sum_{k=0}^{\infty} L^{(2-d)k}\Gamma(\beta L^{2k}, x/L^k), \tag{2.15}$$

$$G(\beta = 0, x) \equiv \begin{cases} \dfrac{1 - L^{-d}}{1 - L^{-2}}, & \text{if } x = 0,\\[2mm] \dfrac{1}{|x|^{d-2}}, & \text{if } x \neq 0.\end{cases} \tag{2.16}$$

Note that

$$G(\beta, x) = L^{2-d}G(\beta L^2, x/L) + \Gamma(\beta, x). \tag{2.17}$$

Γ is a rank 2 positive semidefinite function of x. Its Fourier transform is the first term in (2.13),

$$\hat{\Gamma}(\beta, \xi) = \frac{1}{\beta + \gamma}(\mathbf{1}_{\mathscr{H}_0} - \mathbf{1}_{\mathscr{H}_1})(\xi). \tag{2.18}$$

From Definition 2.4, we see that

$$\Gamma(\beta, x - y) = 0 \quad \text{if } |x - y| > L, \tag{2.19}$$

which means that fields in different *blocks* of radius L are independent wrt $d\mu_{\Gamma}$. Since balls of radius L are the same as balls of diameter L are the same as $\mathscr{G}_1$ cosets, we call them *blocks*. If we set

$$G'(\beta, x) = L^{2-d}G(\beta L^2, x/L), \tag{2.20}$$

then

$$G'(\beta, x) = G'(\beta, y) \quad \text{if } |x - y| \le L, \tag{2.21}$$

so that $G'(x - y)$ is constant on blocks. It is a singular covariance; any $f(x)$ with $\int_{|x| \le L} dx\, f(x) = 0$ is in its kernel.

In the remainder of the paper we need G, Γ only at $\beta = 0$ and so we put $G(x) = G(\beta = 0, x)$ and $\Gamma(x) = \Gamma(\beta = 0, x)$.

3. Grassman algebras, Berezin integration. The main result of this section is Theorem 3.3, in which the isomorphism which transforms self-avoiding walk into a lattice field theory is described. This theorem involves Grassman integration and we begin by a review of this topic. The standard reference is Berezin (1966).

Let $\mathbb{G}$ be the algebra over the ring of complex-valued C^∞ functions on $\mathbb{R}^{2n}$ with a $2n$-tuple $(\bar{\psi}_1, \psi_1, \dots, \bar{\psi}_n, \psi_n)$ of generators satisfying the anticommutation relations

$$\psi_i^\# \psi_j^\# + \psi_j^\# \psi_i^\# = 0, \qquad i, j = 1, \dots, n,\ \psi_i^\# = \psi_i \text{ or } \bar{\psi}_i.$$

Thus the elements of $\mathbb{G}$ can be uniquely expressed in the form

$$g = \sum_\alpha g^{(\alpha)}(\varphi) \psi^\alpha,$$

where α is a multi-index with $2n$ components $(\bar{\alpha}_1, \alpha_1, \dots, \bar{\alpha}_n, \alpha_n)$ which take values 0 or 1. ψ^α denotes the product of those generators for which the corresponding α_i or $\bar{\alpha}_i$ is 1. The product is in the order

$$\psi^\alpha \equiv \bar{\psi}_1^{\bar{\alpha}_1} \psi_1^{\alpha_1} \cdots \bar{\psi}_n^{\bar{\alpha}_n} \psi_n^{\alpha_n}.$$

For each α, $g^{(\alpha)}(\varphi) \equiv g^{(\alpha)}(\varphi, \bar{\varphi})$ is a C^∞ function on $\mathbb{R}^{2n}$, but we will use the complex variables $(\bar{\varphi}_1, \varphi_1, \dots, \bar{\varphi}_n, \varphi_n)$ to denote a point in $\mathbb{R}^{2n}$. We will refer to φ and ψ as fields. Throughout this paper we will use the words *Bosonic* and *Fermionic* in a loose way to distinguish constructions that involve φ and ψ, respectively.

The Berezin integrals. Let α be a multi-index. We define the Berezin integral $\int d^\alpha \psi$ to be the map from $\mathbb{G} \to \mathbb{G}$ which is linear over C^∞ and satisfies

$$\int d^\alpha \psi (\psi^\alpha \psi^\beta) = \left(1/\sqrt{\pi}\right)^{|\alpha|} \psi^\beta \tag{3.1}$$

whenever $\psi^\alpha \psi^\beta \ne 0$. In case α is the top index $(1, 1, \dots, 1, 1)$, we write $d^\alpha \psi = d\psi$. We also define the combined "Fermionic and Bosonic" integration, $\int d\Phi$, by

$$\int d\Phi\, g = \int d\varphi \int d\psi\, g,$$

where $d\varphi = d^n(\operatorname{Re} \varphi)\, d^n(\operatorname{Im} \varphi)$.

The definition of $\int d\psi$ is basis dependent; however, the following change of variables formula holds: Let $\bar{\psi}', \psi'$ be new generators obtained by

$$\begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \psi' \\ \bar{\psi}' \end{pmatrix} \equiv A \begin{pmatrix} \psi' \\ \bar{\psi}' \end{pmatrix},$$

where a, b, c, d are $n \times n$ matrices and $\det A \neq 0$. Then

$$\int d\psi' \, g = \det A \int d\psi \, g \tag{3.2}$$

(the opposite to ordinary integration). Note that if we simultaneously transform $\varphi, \bar{\varphi}$ to $\varphi', \bar{\varphi}'$ by the same linear transformation, then

$$\int d\Phi' \, g = \int d\Phi \, g. \tag{3.3}$$

Functions evaluated on $\mathbb{G}$. The algebra $\mathbb{G}$ contains a subalgebra $\mathbb{G}_{\text{even}}$ consisting of all elements of $\mathbb{G}$ which are even polynomials in ψ. Any element g of $\mathbb{G}$ can be split into its degree-zero part $[g]_0 \equiv g^{(0,\dots,0)}$ and the remaining part $[g]_{>0} \equiv g - [g]_0$.

Let $t \equiv (t_1, \dots, t_N)$ and let $G(t)$ be a C^∞ function. Then G has a power series expansion $G(t+s) \sim \Sigma(1/\beta!)G^{(\beta)}(t)s^\beta$. Let $g \equiv (g_1, \dots, g_N)$ be any N-tuple of elements of $\mathbb{G}_{\text{even}}$. We define an element $G(g) \in \mathbb{G}_{\text{even}}$ by

$$G(g) = \sum_\beta \frac{1}{\beta!} G^{(\beta)}([g]_0)[g]_{>0}^\beta,$$

where $[g]_0 \equiv ([g_1]_0, \dots, [g_N]_0)$, $[g]_{>0}^\beta \equiv ([g_1]_{>0}^{\beta_1} \cdots [g_N]_{>0}^{\beta_N})$. Note that this series terminates after finitely many terms. Also the order of the product $[g]_{>0}^\beta$ is immaterial because each component of g is in the commutative algebra $\mathbb{G}_{\text{even}}$. The map $G \to G(g)$ from C^∞ to $\mathbb{G}_{\text{even}}$ is an algebra homomorphism. It also respects composition of functions: $(G \circ F)(g) = G(F(g))$ holds whenever G and F can be composed as C^∞ functions.

We will frequently use the following example of this construction. Let

$$\mathbf{\Phi}^2 \equiv (\Phi_1^2, \dots, \Phi_n^2),$$

$$\Phi_i^2 \equiv \varphi_i \bar{\varphi}_i + \psi_i \bar{\psi}_i.$$

Given $G(t_1, \dots, t_n)$, a C^∞ function on $\mathbb{R}^n$, we define

$$\begin{aligned} G(\Phi^2) &\equiv G(\Phi_1^2, \dots, \phi_n^2) \\ &\equiv \sum \frac{1}{\beta!} G^{(\beta)}(\varphi\bar{\varphi})(\psi\bar{\psi})\beta, \end{aligned}$$

where $\beta = (\beta_1, \dots, \beta_n)$, $\varphi\bar{\varphi} = (\varphi_1\bar{\varphi}_1, \dots, \varphi_n\bar{\varphi}_n)$, $\psi\bar{\psi} = (\psi_1\bar{\psi}_1, \dots, \psi_n\bar{\psi}_n)$ and $G^{(\beta)}$ denotes the β multiderivative of G.

Differentiation. We define $\partial/\partial\psi_i$: $\mathbb{G} \to \mathbb{G}$ by the rules:

(i) $$\frac{\partial}{\partial\psi_i}\psi_j = \delta_{ij}, \qquad \frac{\partial}{\partial\psi_i}\bar{\psi}_j = 0;$$

(ii) $$\frac{\partial}{\partial\psi_i}\psi^\alpha\psi^\beta = \left(\frac{\partial}{\partial\psi_i}\psi^\alpha\right)\psi^\beta + (-1)^{|\alpha|}\psi^\alpha\left(\frac{\partial}{\partial\psi_i}\psi^\beta\right);$$

(iii) linearity over $C^\infty(\mathbb{R}^{2N})$.

$\partial/\partial\bar{\psi}_i$ is defined analogously. It follows from these rules that

$$\frac{\partial}{\partial\psi_i^\#}\frac{\partial}{\partial\psi_j^\#} = -\frac{\partial}{\partial\psi_j^\#}\frac{\partial}{\partial\psi_i^\#}.$$

In fact, $\partial/\partial\psi_i$ is, apart from a constant, the same as integration with respect to a single generator.

Gaussian integrals. Let A be a complex $n \times n$ matrix. Then, by our definitions,

$$\int d\psi\, e^{-\psi A\bar{\psi}} = \pi^{-n}\det A,$$

(3.4)

$$\int d\psi\, e^{-\psi A\bar{\psi}}\bar{\psi}_{i_1}\psi_{j_1}\cdots\bar{\psi}_{i_p}\psi_{j_p} = \pi^{-n}\det A_{\mathbf{j},\mathbf{i}},$$

where $\mathbf{i} = i_1 < i_2 < \cdots < i_p$, $\mathbf{j} = j_1 < \cdots < j_p$, and $A_{\mathbf{j},\mathbf{i}}$ is the signed minor of A obtained by deleting rows $\mathbf{j}$ and columns $\mathbf{i}$ from A. See Brydges and Munoz-Maya (1991) for more details.

Given a nonsingular $n \times n$ matrix $C = A^{-1}$, we define

$$\int d\mu(\psi)(\cdot) = \frac{\int d\psi\, e^{-\psi A\bar{\psi}}(\cdot)}{\int d\psi\, e^{-\psi A\bar{\psi}}}. \tag{3.5}$$

We find from (3.4) and (a generalized) Cramér's rule that

$$\int d\mu(\psi)\bar{\psi}_i\psi_j = C_{ij},$$

(3.6)

$$\int d\mu(\psi)g = \left[\exp\left[\sum_{i,j}\frac{\partial}{\partial\psi_j}C_{ij}\frac{\partial}{\partial\bar{\psi}_i}\right]g\right]_{\psi=0}.$$

Setting $\psi = 0$ means projecting on the degree-zero part of the Grassman element. The exponential is defined by its power series, which terminates after finitely many terms.

Suppose C is singular: then the right-hand side of (3.6) is defined, but the formula for $d\mu(\psi)$ is not. Since the right-hand side of (3.6) can serve as a definition of $d\mu(\psi)$, we only outline how to extend the formula for $d\mu$ to cover this case.

EXAMPLE. Suppose C is diagonal with $C_{11} = 0$ and $C_{ii} \neq 0$ for $i \neq 1$. In this case, let

$$d\mu(\psi) = d\psi\, \bar{\psi}_1 \psi_1\, e^{-\psi A \bar{\psi}}/\text{Normalization},$$

with A the diagonal matrix with $A_{11} = 0$ and $A_{ii} = C_{ii}^{-1}$, $i \neq 1$. It is easy to verify that (3.6) holds.

More generally, we can have more than one diagonal entry of C vanish and achieve (3.6) by inserting further factors of $\bar{\psi}_i \psi_i$ corresponding to zero diagonal entries.

The general case may be reduced to the case where C is diagonal by writing the singular value decomposition

$$C = U(\text{Diagonal})V^{-1},$$

with U, V unitary matrices. We use the linear transformation $\psi' = V\psi$, $\bar{\psi}' = U\bar{\psi}$, and (3.2) to transform to an integral with C diagonal.

The Fourier transform. There is an analogue to the formula for the Fourier transform of a Gaussian integral. Introduce a copy of $\mathbb{G}$ with $\psi, \bar{\psi}$ replaced by $u_1, \ldots, u_n\, \bar{u}_1, \ldots, \bar{u}_n$. $\exp(\psi \bar{u} + u \bar{\psi})$ is an element of $\mathbb{G} \wedge \mathbb{G}$. We claim that

$$\int d\mu(\psi) \exp(\psi \bar{u} + u \bar{\psi}) = \exp(u C \bar{u}). \tag{3.7}$$

PROOF. Let C^t be the transpose of C. By (3.6) and algebraic manipulation,

$$\begin{aligned}
\int d\mu(\psi) e^{\psi \bar{u} + u \bar{\psi}} &= \exp\left(\frac{\partial}{\partial \psi} C^t \frac{\partial}{\partial \bar{\psi}} \right) e^{\psi \bar{u} + u \bar{\psi}} \Big|_{\psi = 0} \\
&= e^{\psi \bar{u} + u \bar{\psi}} e^{-\psi \bar{u} - u \bar{\psi}} \exp\left(\frac{\partial}{\partial \psi} C^t \frac{\partial}{\partial \bar{\psi}} \right) e^{\psi \bar{u} + u \bar{\psi}} 1 \Big|_{\psi = 0} \\
&= e^{\psi \bar{u} + u \bar{\psi}} \exp\left(\left[e^{-\psi \bar{u}} \frac{\partial}{\partial \psi} e^{\psi \bar{u}} \right] C^t \left[e^{-u \bar{\psi}} \frac{\partial}{\partial \bar{\psi}} e^{u \bar{\psi}} \right] \right) 1 \Big|_{\psi = 0} \\
&= \exp\left(\left[\frac{\partial}{\partial \psi} + \bar{u} \right] C^t \left[\frac{\partial}{\partial \bar{\psi}} - u \right] \right) 1 \\
&= \exp(-\bar{u} C^t u) = \exp(u C \bar{u}). \qquad \square
\end{aligned}$$

The combined Fermion–Boson Gaussian measure. Let A be any $n \times n$ matrix with real part $A + \bar{A}^t$ positive definite. Let $C = A^{-1}$. We define the Gaussian measure

$$d\mu_C(\Phi) = d\Phi\, e^{-\Phi A \bar{\Phi}}, \tag{3.8}$$

where $\Phi A \bar{\Phi} = \varphi A \bar{\varphi} + \psi A \bar{\psi}$. The omission of any normalization factor is deliberate, since even as it stands, $\int d\mu(\Phi) 1 = 1$.

From the definitions, it follows that

$$\int d\mu_C(\Phi)\bar{\varphi}_i\varphi_j = \int d\mu_C(\Phi)\bar{\psi}_i\psi_j = C_{ij}, \tag{3.9}$$

and ψ, φ covariances vanish.

These formulas are extended to allow C to be singular by including δ functions in the φ, $\bar{\varphi}$ integrals and suitable polynomials in ψ, $\bar{\psi}$ (which act as δ functions) in the Fermionic integral; see the example above concerning C singular.

Convolution of Gaussian measures. Let $g \equiv g(\Phi) = \sum g^{(\alpha)}(\varphi)\psi^\alpha$ be in $\mathbb{G}$. We consider a new copy $\mathbb{G}'$ of $\mathbb{G}$ and let the corresponding two fields be Φ, Φ'. g defines an element $g(\Phi + \Phi') \in \mathbb{G} \wedge \mathbb{G}'$ by

$$g(\Phi + \Phi') \equiv \sum g^{(\alpha)}(\varphi + \varphi')(\psi + \psi')^\alpha,$$

where

$$(\psi + \psi')^\alpha \equiv \left(\bar{\psi}_1 + \bar{\psi}_1'\right)^{\bar{\alpha}_1}(\psi_1 + \psi_1')^{\alpha_1} \cdots \left(\bar{\psi}_n + \bar{\psi}_n'\right)^{\bar{\alpha}_n}(\psi_n + \psi_n')^{\alpha_n}.$$

Convolution $\mu_C *$ by a Gaussian measure is the map from $\mathbb{G} \to \mathbb{G}$ given by

$$g(\Phi) \to \int d\mu_C(\Phi')g(\Phi + \Phi').$$

LEMMA 3.1. *Suppose C, C_1, C_2 are $N \times N$ matrices with positive semi-definite real parts and $C = C_1 + C_2$. Let $g \in \mathbb{G}$. Then using the notation above,*

$$\int d\mu_C(\Phi)g(\Phi) = \int d\mu_{C_1}(\Phi)(\mu_{C_2} * g)(\Phi).$$

PROOF. The formula is linear in g so it is enough to consider a monomial $g^{(\alpha)}(\varphi)\psi^\alpha$. In this case the formula is implied by

$$\int d\mu_C(\varphi)g^{(\alpha)}(\varphi) = \int d\mu_{C_1}(\varphi)\mu_{C_2} * g^{(\alpha)}(\varphi),$$

$$\int d\mu_C(\psi)\psi^\alpha = \int d\mu_{C_1}(\psi)\int d\mu_{C_2}(\psi')(\psi + \psi')^\alpha.$$

The first formula is a standard fact for Gaussian integrals. The second formula is a Grassman analogue which follows from (3.6). □

Markov chains and Gaussian integrals. Let A be a real $n \times n$ matrix which is the generator of a Markov process with n states, that is, $\exp(-tA)$ is the semigroup of transition probabilities for the process whose histories $\omega(t)$ are functions ω: $\mathbb{R}^+ \to \{1, \dots, n\}$. On the space of histories Ω there are defined conditional expectations satisfying

$$\begin{aligned} E(\omega(t) = j|\omega(s) = i) &= (e^{-(t-s)A})_{ij}, \\ E(\cdot|\omega(s), s \le t) &= E(\cdot|\omega(t)), \end{aligned} \tag{3.10}$$

where $t \ge s$.

PROPOSITION 3.2 (Feynman and Kac). *Let* $v\colon \{1,\dots,n\} \to \mathbb{R}, \mathbb{C}$.

$$E\Big(e^{-\int_s^t v(\omega(\tau))\,d\tau} 1_{\{\omega(t)=j\}} | \omega(s) = i\Big) = \big(e^{-(t-s)(A+V)}\big)_{ij},$$

where V *is the diagonal matrix* $V_{ij} = v(i)\delta_{ij}$.

PROOF. Let $Q(s,t,i,j)$ denote the left-hand side. Then computations show (i) Q is a semigroup and (ii) its generator is $A + V$. □

THEOREM 3.3 [McKane (1980) and Parisi and Sourlas (1979, 1980)]. *Let* $F \in C^\infty(\mathbb{R}_+^n)$ *decay exponentially*, $|F(t)| \le \text{Const} \cdot \exp(-b\Sigma t_i)$ *for some* $b > 0$. *Then*

$$\int_0^\infty dt\, E\big(F(\boldsymbol{\tau}^t) 1_{\{\omega(t)=j\}} | \omega(0) = i\big) = \int d\mu_C(\Phi) F(\Phi^2) \begin{pmatrix} \bar{\psi}_i \psi_j \\ or \\ \bar{\varphi}_i \varphi_j \end{pmatrix},$$

where $C = A^{-1}$, $\boldsymbol{\tau}^t = (\tau_1^t,\dots,\tau_n^t)$, *and*

$$\tau_i^t = \int_0^t ds\, 1_{\{\omega(s)=i\}}.$$

REMARK. If M is a monomial in $\varphi, \bar{\varphi}, \psi, \bar{\psi}$ with p factors of φ, p factors of $\bar{\varphi}$, q factors of ψ and q factors of $\bar{\psi}$, then $\int d\mu_C\, FM$ can be expressed in terms of an expectation of F evaluated on local times for $p + q$ independent random walks summed at each state.

PROOF [See also page 119 of Brydges, Fröhlich and Sokal (1983)]. Since both sides are linear in F and

$$F(\mathbf{u}) = (2\pi)^{-n/2} \int_\Gamma d^n v\, \hat{F}(\mathbf{v}) e^{i\mathbf{vu}},$$

where Γ is the contour

$$\Gamma = \{v \in \mathbb{C}^n\colon \operatorname{Im} v_i = b/2 > 0, \text{all } i\},$$

it suffices to prove Theorem 3.3 for the special case

$$F(\mathbf{u}) = e^{i\mathbf{vu}}, \qquad \operatorname{Im} v_i > 0.$$

(The contour is chosen to enable the use of Fubini's theorem for interchanging the v and φ integrals.) In this case we note that

$$\begin{aligned} F(\boldsymbol{\tau}^t) &= \exp\Big(i \sum_{j=1} v_j \tau_j^t\Big) \\ &= \exp\Big(i \sum_{j=1} v_j \int_0^t 1_{\{\omega(s)=j\}}\, ds\Big) \\ &= \exp\Big(i \int_0^t ds\, v_{\omega(s)}\Big). \end{aligned}$$

Therefore, by the Feynman–Kac formula, Proposition 3.2,

$$\int_0^\infty dt\, E\big(F(\tau^t)1_{\{\omega(t)=j\}}|\omega(0)=i\big) = \int_0^t dt\, \exp(-t[A+V])_{ij} = (A+V)^{-1}_{ij}$$

$$= \int d\Phi\, e^{-\Phi(A+V)\bar{\Phi}}\bar{\psi}_i\psi_j \quad [\text{by (3.8) and (3.9)}]$$

$$= \int d\mu_C(\Phi)F(\Phi^2)\bar{\psi}_i\psi_j. \qquad \square$$

Processes with infinitely many states. Let ω be a transient Markov process with countably infinite state space. In this case $\mathbb{G}$ is the inductive limit (i.e., "union") of all algebras $\mathbb{G}_\Lambda$, where Λ is any finite subset of state space. Let C_{ij} be an infinite matrix such that when i and j are restricted to a finite set Λ, then the resulting finite matrix $C_{\Lambda,ij}$ has positive semidefinite real part. We define $d\mu_C$ by

$$\int d\mu_C\, g = \lim_{\Lambda\uparrow}\int d\mu_{C_\Lambda}\, g.$$

The limit exists because $g \in \mathbb{G}$ means that $g \in \mathbb{G}_{\Lambda'}$ for some Λ'. As soon as Λ contains Λ', the integral no longer depends on Λ.

PROPOSITION 3.4. *Let ω be a transient Markov process with countably infinite state space. Suppose F depends on local times at only finitely many states. Then the conclusion of Theorem* 3.3 *holds.*

PROOF. Let F depend on local times only in the infinite set Λ'. Let $\hat{\omega}$, $\hat{E}$ be the process killed on its first exit from some subset Λ that contains Λ'. We apply Theorem 3.3 to this finite state process $\hat{\omega}$ and take the limit as Λ grows. The left-hand side becomes the expectation for ω by monotone convergence. Convergence of the Gaussian integral on the right-hand side follows from

$$\hat{C}_{\Lambda',ij} \equiv \int_0^\infty dt\, \hat{E}\big(1_{\{\omega(t)=j\}}|\omega(0)=i\big), \qquad i,j\in\Lambda',$$

$$\to C_{\Lambda',ij} \equiv \int_0^\infty dt\, E\big(1_{\{\omega(t)=j\}}|\omega(0)=i\big), \qquad i,j\in\Lambda'$$

(which is a consequence of transience and the monotone convergence theorem). $\square$

4. The renormalization group. In this section we define the renormalization group and establish its properties in Theorem 4.2.

Let F be dependent on only a finite number of fields, by which we mean that F is in the Grassman algebra which is built from the fields at finitely many points in $\mathscr{S}$.

By Lemma 3.1 and (2.17),

$$\int d\mu_G(\Phi)F(\Phi) = \int d\mu_{G'}(\Phi')(\mu_\Gamma * F)(\Phi'), \tag{4.1}$$

where

$$(\mu_\Gamma * F)(\Phi') \equiv \int d\mu_\Gamma(\zeta)F(\Phi' + \zeta),$$

and

$$G'(x) \equiv L^{2-d}G\left(\frac{x}{L}\right). \tag{4.2}$$

As noted at the end of Section 2, G' is singular; it is constant on blocks $x + \mathscr{S}_1$. Therefore, as a convolution operator, G' annihilates any function f whose integral over blocks $\int_{\mathscr{S}_1} dx f(x+y)$ vanishes. This means that if $|x-y| \le L$, then $\varphi(x) = \varphi(y)$ almost surely $d\mu_{G'}$ and $\int d\mu_{G'}$ annihilates anything in the ideal, in the Grassman algebra $\mathbb{G}$, generated by $\psi(x) - \psi(y), \bar{\psi}(x) - \bar{\psi}(y)$. Thus, in the integrand in the right-hand side of (4.1), we can eliminate fields using

$$\Phi'(x) = \Phi'(y) \quad \text{if } |x-y| \le L. \tag{4.3}$$

In this sense $\Phi'(x)$ is block-wise constant. This means we can shrink blocks to points by the rescaling operation which appears in the following.

DEFINITIONS 4.1. Let $\mathbb{G}_N$ be the Grassman algebra built from fields $\Phi(x)$, x in $\mathscr{S}_N$ for some N. Let $x \mapsto Lx$ be the $\mathscr{S}$ homomorphism defined by $L(\dots, x_1, x_0) = (\dots, x_1, x_0, 0)$.

1. $\mathscr{R}$: $\mathbb{G}_N \to \mathbb{G}_{N-1}$ is an algebra homomorphism defined by the action of $\mathscr{R}$ on the generators and the coefficient ring $C^\infty(\mathbb{R}^{2N})$:

$$\mathscr{R}(\psi_z^\#) \equiv L^{-(d-2)/2}\psi_x^\#, \qquad \forall\, z \in Lx + \mathscr{S}_1, \qquad \psi^\# = \psi \text{ or } \bar{\psi},$$

$$(\mathscr{R}f)(\varphi) \equiv f(\mathscr{R}\varphi), \qquad f \in C^\infty(\mathbb{R}^{2N}), \tag{4.4}$$

$$\mathscr{R}(\varphi_z) \equiv L^{-(d-2)/2}\varphi_x, \qquad \forall\, z \in Lx + \mathscr{S}_1.$$

2. A renormalization group step is the map $\tilde{T}$: $\mathbb{G}_N \to \mathbb{G}_{N-1}$,

$$F \mapsto \tilde{T}(F) \equiv \mathscr{R}(\mu_\Gamma * F).$$

3. For each $x \in \mathscr{S}$, let f_x be a function of Φ_x only. Define new functions $(Tf)_x$ by

$$(Tf)_x(\Phi_x) \equiv \tilde{T}\left(\prod_{y\in\mathscr{S}_1} f_{Lx+y}\right)(\Phi_x)$$

$$= \int d\mu_\Gamma(\zeta) \prod_{y\in\mathscr{S}_1} f_{Lx+y}\left(L^{-(d-2)/2}\Phi_x + \zeta_{Lx+y}\right).$$

THEOREM 4.2. *If $F = \sum f^{(\alpha)}(\varphi)\psi^{\alpha}$ is an element of the Grassman algebra $\mathbb{G}_N$ and its coefficients $f^{(\alpha)}(\varphi)$ have Gaussian decay at ∞ in φ, then*:

(i) $\int d\mu_G\, F = \int d\mu_G\, \tilde{T}(F)$.

(ii) *If $F = \Pi_{x \in \mathscr{I}_N} f_x$, then*

$$\tilde{T}(F) = \prod_{x \in \mathscr{I}_{N-1}} (Tf)_x.$$

(iii) *If for each z in a block $Lx + \mathscr{I}_1$, $f_z(\Phi)$ is a polynomial in $\varphi_z, \overline{\varphi}_z$ with coefficients which are functions of Φ_z^2,*

$$f_z(\Phi) = \sum_{\alpha, \overline{\alpha}} c_{z,\alpha,\overline{\alpha}}\big(\Phi_z^2\big)\varphi_z^{\alpha}\overline{\varphi}_z^{\overline{\alpha}},$$

then $(Tf)_x$ is a polynomial of the same form with $(\alpha, \overline{\alpha})$ degrees equal to that of $F = \Pi_{z \in Lx + \mathscr{I}_1} f_z$. $(Tf)_x$ is even (odd) if and only if F is even (odd).

(iv) $(Tf)(0) = f(0)$.

REMARKS. (i) follows immediately from (4.1) and (4.2). It is a standard property for renormalization groups.

(ii) is a very special and simplifying feature of hierarchical models. It means that for these models one can descend from studying trajectories of $\tilde{T}$ on a huge space of functions on the whole lattice and need only study functions of a few variables and their trajectories under T. The proof of (ii) is immediate from the independence of disjoint blocks implied by (3.21).

(iii) is also special to the hierarchical model. It is telling us that the renormalization group preserves the property that the Grassman integral represents a random walk problem.

PROOF OF THEOREM 4.2 [Parts (iii) and (iv)]. We give computational proofs. One could also use arguments based on supersymmetry as in Klein, Landau and Fernando-Perez (1984).

Fix x, let $F = \Pi_{y \in \mathscr{I}_1} f_{Lx+y}$, then $(Tf)_x = \tilde{T}(F)$. By hypothesis, $F = \sum F_{\alpha,\overline{\alpha}}(\Phi^2)\varphi^{\alpha}\overline{\varphi}^{\overline{\alpha}}$, where $F_{\alpha,\overline{\alpha}} \in C^{\infty}$ has Gaussian decay. [For part (iv), $F_{\alpha,\overline{\alpha}} = 0$ for α or $\overline{\alpha} \neq 0$.] Thus

$$F = \sum \int dv\, \hat{F}_{\alpha,\overline{\alpha}}(v)\exp\Big(i\sum_x v_x\Phi_x^2\Big)\varphi^{\alpha}\overline{\varphi}^{\overline{\alpha}},$$

with contour of integration for v_x, $x \in \mathscr{I}_N$, chosen so that $\operatorname{Im} v_x \geq \delta$, $\delta > 0$, $\forall\, x$. The choice of contour allows us to interchange $\tilde{T}$ and $\int dv$, so this formula reduces the proof to the special case

$$F(\Phi) = \exp\big(i\sum v_x\Phi_x^2\big)\varphi^{\alpha}\overline{\varphi}^{\overline{\alpha}}.$$

If we instead consider

$$F(k, \Phi) = \exp\Big(\sum \big(iv_x\Phi_x^2 - k_x\overline{\varphi}_x - \overline{k}_x\varphi_x\big)\Big),$$

then we can recover $F(\Phi)$ by differentiating $F(k, \Phi)$ with respect to $k, \overline{k}$ at 0.

k and $\bar{k}$ are complex conjugate vectors. To avoid confusion over the meaning of conjugation, we will give the proof for v a vector whose components are pure imaginary so that iv is real and negative. The general case is recovered by analytic continuation.

Γ is positive semidefinite. To avoid complicating our formulas by delta functions, we imagine that Γ is replaced by $\Gamma + \varepsilon I$, $\varepsilon > 0$, and recover the result for Γ by letting $\varepsilon \downarrow 0$ at the end.

Let $V = (V_{xy}) = -iv_x \delta_{xy}$ and define $\eta, \bar{\eta}$ so that $\Sigma(k_x \bar{\varphi}_x + \varphi_x \bar{k}_x) \equiv \eta\bar{\Phi} + \Phi\bar{\eta}$. Then

$$F(k, \Phi) = e^{-\Phi V \bar{\Phi} - \eta\bar{\Phi} - \Phi\bar{\eta}}$$
$$= e^{-(\Phi + V^{-1}\eta)V(\bar{\Phi} + V^{-1}\bar{\eta})} e^{\eta V^{-1}\bar{\eta}}.$$

By (3.8) followed by (3.7),

$$\mu_\Gamma * (e^{-\Phi V \bar{\Phi}}) = \int d\mu_\Gamma(\zeta) e^{-(\Phi + \zeta)V(\bar{\Phi} + \bar{\zeta})}$$
$$= \int d\mu_{\Gamma U}(\zeta) e^{-\zeta(V\bar{\Phi}) - (V\Phi)\bar{\zeta}} e^{-\Phi V \bar{\Phi}}$$
$$= \exp\{\Phi[V\Gamma UV - V]\bar{\Phi}\} = \exp\{-\Phi UV\bar{\Phi}\},$$

where $U = (1 + V\Gamma)^{-1}$ so that $\Gamma U = (V + \Gamma^{-1})^{-1}$. We replace Φ by $\Phi + V^{-1}\eta$ and use $UV = VU^t$, $V^{-1} - V^{-1}U = \Gamma U$ to obtain

$$\mu_\Gamma * F(k, \Phi) = \exp\{-\Phi UV\Phi - \eta(U^t\bar{\Phi}) - (U^t\Phi)\bar{\eta} + \eta\Gamma U\bar{\eta}\}.$$

Part (iv) is proven by taking $\eta, \bar{\eta} = 0$ and noting that $F(k = 0, \Phi = 0) = 1 = (\tilde{T}f)(k = 0, \Phi = 0)$. For part (iii), we convert $F(k, \Phi)$ into $F(\Phi)$ by differentiating with respect to η and $\bar{\eta}$ and setting $\eta = \bar{\eta} = 0$. Under this operation, $\mu_\Gamma * F(k, \Phi)$ becomes $\exp\{-\Phi UV\bar{\Phi}\} \times$ polynomial in $(\varphi, \bar{\varphi})$. Under $\mathscr{R}$, $\Phi(z) \to L^{-(d-2)/2}\Phi'(x)$ for all $z \in Lx + \mathscr{G}_1$, so it becomes a polynomial in $\varphi', \bar{\varphi}'$ whose coefficients are functions of $\Phi'^2(x)$ as claimed. The other assertions in (iii) are clear. □

5. Weakly self-avoiding walk on $\mathscr{G}$. The natural measure for self-intersections at site x for walks living for time T is

$$\frac{1}{2}\left(\tau_x^T\right)^2 = \int_{0 \le s_1 \le s_2 \le T} ds_1\, ds_2\, \mathbf{1}_{\{\omega(s_1) = \omega(s_2) = x\}}. \tag{5.1}$$

We will use Theorem 3.3 to study the expectation, with respect to the hierarchical random walk introduced in Section 2, of

$$I_N(\boldsymbol{\tau}) = \prod_{x \in \mathscr{G}_N} e^{-\lambda\tau_x^2 - a\tau_x}. \tag{5.2}$$

The a is introduced because the $\exp(-\lambda\tau^2)$ will kill the process at a rate that turns out to be exponential. This will be compensated by a delicate choice of a.

In the Φ representation obtained from Theorem 3.3, this becomes

$$I_N(\Phi^2) = \prod_{x \in \mathscr{I}_N} g(\Phi_x), \tag{5.3}$$

where

$$\begin{aligned} g(\Phi) &= \exp(-v(\Phi)), \\ v(\Phi) &= \lambda :\Phi^4:_G + \mu^2 :\Phi^2:_G, \end{aligned} \tag{5.4}$$

$\Phi^4 = (\Phi^2)^2$, $\mu^2 = a + 2\lambda G(0)$. We have Wick-ordered with respect to the covariance G. Wick ordering is defined, for $P = P(\Phi)$ a polynomial, by

$$:P(\Phi):_G \equiv \exp(-\Delta_G)P, \tag{5.5}$$

where $\exp(-\Delta_G)$ is defined on polynomials by its power series and

$$\begin{aligned} \Delta_G &= \int dx\, dy \frac{\partial}{\partial \Phi(x)} G(x-y) \frac{\partial}{\partial \overline{\Phi}(y)} \\ &\equiv \int dx \int dy\, G(x-y) \left\{ \frac{\partial}{\partial \varphi(x)} \frac{\partial}{\partial \overline{\varphi}(y)} + \frac{\partial}{\partial \psi(x)} \frac{\partial}{\partial \overline{\psi}(y)} \right\}. \end{aligned} \tag{5.6}$$

We will use the following standard properties of normal ordering which all follow easily from (5.5) and $\mu *$ (Polynomial) $= \exp(\Delta)$(Polynomial).

LEMMA 5.1. *Let B, A and $B - A$ be covariances. Then*:

(i) $\mu_A * :P:_B = :P:_{B-A}$.
(ii) *Suppose P is a monomial of degree p in Φ and $\alpha \in \mathbb{R}$, then*

$$:P:_A(\alpha\Phi) = \alpha^P :P:_{\alpha^{-2}A}(\Phi).$$

(iii) $(\partial/\partial\Phi):P_A: = :\partial P/\partial\Phi:_A$.
(iv) *If $\overline{\alpha} \neq \beta$, then* $\int d\mu_A :\Phi^\alpha:_A :\Phi^\beta:_A = 0$.

EXAMPLE 5.2. By parts (i) and (ii) and (2.17),

$$\mathscr{R}\mu_\Gamma * :\Phi^\alpha:_G = L^{-|\alpha|} :\Phi^\alpha:_G$$

in $d = 4$ dimensions.

It happens that $:\Phi^2: = \Phi^2$ because Boson and Fermion contributions to $\Delta\Phi^2$ cancel. More generally, $\Delta^m :\Phi^{2m}: = 0$ for $m = 1, 2, \dots$.

6. Analyticity for Fermi and Bose fields. Analyticity methods have proven very useful in controlling renormalization group transformations in Bosonic models [Gawedzki and Kupiainen (1985, 1986)]. We set up a framework which includes Fermi fields. We summarize the results in Definition 6.0, Lemmas 6.1–6.4 and Corollary 6.5. These results will be used in Sections 7 and 8 to study the action of the renormalization group.

To begin with, we drop the Fermions and consider functions of Bose fields only. Let $\gamma, h > 0$. For $g(\varphi)$ a complex C^∞ function of $(\varphi_1, \dots, \varphi_p) \in \mathbb{C}^p$, we define

$$(6.1)\qquad \begin{aligned} |g|_w &= \sup_\varphi \left| g(\varphi) \prod_{i=1}^{p} w(\varphi_i)^{-1} \right|, \\ |g|_{w,h} &= \sum \frac{h^\alpha}{\alpha!} |g^{(\alpha)}|_w, \qquad h^\alpha \equiv h^{|\alpha|}. \end{aligned}$$

Here w is a positive weight function and $g^{(\alpha)}$ is the αth derivative of g with respect to φ and $\overline{\varphi}$. α is a multi-index. The $|\cdot|_{w,h}$ norm behaves very like a weighted supremum over a polystrip, with w the weighting factor and h the half-width of a strip about the real axis. For example,

$$(6.2)\qquad |g_1 g_2|_{w_1 w_2, h} \le |g_1|_{w_1, h} |g_2|_{w_2, h},$$

$$(6.3)\qquad |g^{(\alpha)}|_{w,h} \le \frac{\alpha!}{(h' - h)^\alpha} |g|_{w,h'} \quad \text{for } h' > h,$$

$$(6.4)\qquad |\mu_C * g|_{w',h} \le |g|_{w,h},$$

where w' satisfies

$$(6.5)\qquad \mu_C * w \le w'.$$

For proofs, see Appendix A.

Now suppose g belongs to a Grassman algebra $\mathbb{G}$ so that

$$g = \sum_\beta g^{(\beta)}(\varphi)\psi^\beta, \qquad \left(= \sum_\beta \frac{1}{\beta!} g^{(\beta)}(\varphi)\psi^\beta \right).$$

If we set

$$(6.6)\qquad |g|_{w,h} = \sum_\beta h^\beta |g^{(\beta)}|_{w,h},$$

where the $|\cdot|_{w,h}$ on the right-hand side is the norm on Bosonic functions that we have just defined, then we can substitute in the definition of the Bosonic norm to see that $|g|_{w,h}$ can also be defined as follows.

Definition 6.0.

$$(6.7)\qquad \begin{aligned} |g|_{w,h} &= \sum_{\beta,\alpha} \frac{h^\beta}{\beta!} \frac{h^\alpha}{\alpha!} |g^{(\beta,\alpha)}|_w \\ &\equiv \sum_\gamma \frac{h^\gamma}{\gamma!} |g^{(\gamma)}|_w, \end{aligned}$$

where $\gamma = (\beta, \alpha)$ is a multi-index with components of β being 0 or 1, $g^{(\beta,\alpha)}$ is the αth derivative of $g^{(\beta)}$ with respect to φ and $\overline{\varphi}$ and $h^\gamma \equiv h^{|\gamma|}$.

While this unifies the notations, the Fermionic variables are fundamentally different in that there is nothing to take a supremum over.

It is easy to prove for $g_1, g_2 \in \mathbb{G}$ that (6.2) holds. Note that

$$\begin{aligned}\left|\left(\frac{\partial}{\partial\psi}\right)^{\alpha} g\right|_{w,h} &= \left|\sum_{\beta} g^{(\beta)}\left(\frac{\partial}{\partial\psi}\right)^{\alpha}\psi^{\beta}\right|_{w,h}\\ &= \left|\sum_{\beta} g^{(\alpha+\beta)}(\pm\psi^{\beta})\right|_{w,h}\\ &= \sum_{\beta} |g^{(\alpha+\beta)}|_{w,h} h^{\beta+\alpha}h^{-\alpha}\\ &\le h^{-\alpha}|g|_{w,h},\end{aligned} \tag{6.8}$$

so (6.3) generalizes to

$$\left|\left(\frac{\partial}{\partial\Phi}\right)^{\alpha} g\right|_{w,h} \le \frac{\alpha!}{(h'-h)^{\alpha}}|g|_{w,h'} \tag{6.9}$$

if $h' > h$.

We will now consider convolution by a Fermionic Gaussian measure μ_C. A slight generalization of (3.6) is

$$\mu * g = \exp\left(\sum_{i,j}\frac{\partial}{\partial\psi_j}C_{ij}\frac{\partial}{\partial\bar{\psi}_i}\right)g = \prod\left(1+\frac{\partial}{\partial\psi_j}C_{ij}\frac{\partial}{\partial\bar{\psi}_i}\right)g. \tag{6.10}$$

Consequently, by (6.8),

$$\begin{aligned}|\mu * g|_{w,h} &\le \prod\left(1+h^{-2}|C_{ij}|\right)|g|_{w,h}\\ &\le \exp\left(\sum_{i,j} h^{-1}|C_{ij}|h^{-1}\right)|g|_{w,h}.\end{aligned} \tag{6.11}$$

Assuming $\sum_j |C_{ij}| < \infty$, we conclude that convolution worsens the bound by a factor exponential in the number of sites.

We combine (6.11) and (6.4) to conclude that if μ_C is a mixed Fermionic and Bosonic integral, then

$$|\mu_C * g|_{w',h} \le \exp\left(\sum h^{-1}|C_{ij}|h^{-1}\right)|g|_{w,h}, \tag{6.12}$$

where w' satisfies $\mu_C * w \le w'$. For a proof, see Appendix A.

Until now we have considered φ and $\bar{\varphi}$ to be complex conjugates; thus

$$\varphi_i = \varphi_i^{(1)} + i\varphi_i^{(2)}, \qquad \bar{\varphi}_i = \varphi_i^{(1)} - i\varphi_i^{(2)},$$

with $\varphi^{(1)}, \varphi^{(2)}$ real. Keeping the same definitions of $\varphi_i, \bar{\varphi}_i$ in terms of $\varphi_i^{(j)}$, we now let $\varphi_i^{(j)}$ be complex. We set

$$\begin{aligned}D_h(\varphi) = \Big\{&(\varphi'_i, \bar{\varphi}'_i, u'_i, \bar{u}'_i) \in \mathbb{C}^4 \colon \forall\, i = 1,\dots,p,\\ &\forall\, j = 1,2,\ |\varphi_i'^{(j)} - \varphi_i^{(j)}|, |u'_i|, |\bar{u}'_i| \le h\Big\},\\ S_h = &\bigcup_{\varphi^{(j)}\ \text{real}} D_h(\varphi).\end{aligned}$$

Given F analytic on $D_h(\varphi)$, let

$$|F|_h(\varphi) = \sup_{\varphi' \in D_h(\varphi)} |F(\varphi')|,$$

and let $F(\Phi)$ denote any element of the Grassman algebra obtained by substituting $u = \psi$, $\bar{u} = \bar{\psi}$ in the power series

$$F(\varphi, \bar{\varphi}, u, \bar{u}) = \sum \frac{1}{\bar{\alpha}!\alpha!} (\partial_{\bar{u}, u}^{\bar{\alpha}, \alpha} F)(\bar{\varphi}, \varphi, 0, 0) \bar{u}^{\bar{\alpha}} u^{\alpha},$$

with any convention for the order of ψ's in the product ($\alpha! = \bar{\alpha}! = 1$).

LEMMA 6.1 (Comparison of norms). *Suppose F is analytic in S_{Ah} and $g = F(\Phi)$, then for $A > 1$,*

$$|g|_{w,h} \leq \left(\frac{1}{1 - A^{-1}} \right)^{4p} \|F\|_{w,Ah},$$

where

$$\|F\|_{w,h} \equiv \sup_{\varphi \in S_n} |F|_h(\varphi) w^{-1}(\varphi).$$

PROOF. By

$$\frac{\partial}{\partial \varphi} = \frac{1}{2} \left(\frac{\partial}{\partial \varphi^{(1)}} - i \frac{\partial}{\partial \varphi^{(2)}} \right)$$

and the Cauchy formula,

$$|F^{(\alpha)}(\varphi, \bar{\varphi}, 0, 0)| \leq \frac{\alpha!}{(Ah)^{\alpha}} |F|_{Ah}(\varphi),$$

where φ and $\bar{\varphi}$ are complex conjugates. Consequently,

$$|g|_{w,h} \leq \left(\sum_{\alpha} A^{-\alpha} \right) \sup_{\varphi} \left(|F|_{Ah}(\varphi) w(\varphi)^{-1} \right),$$

and $\sum_{\alpha} A^{-\alpha} = (1 - A^{-1})^{-4p}$. □

We now specialize this machinery to the cases we will need in the rest of the paper. Let X be a finite subset of the infinite lattice $\mathscr{S}$, and let $\mathbb{G}_X$ be the Grassman algebra generated by fields $\Phi(x)$, $x \in X$. We will write g_X for an element in $\mathbb{G}_X$.

Let us define

$$|g_X|_{a,h} = |g_X|_{w,h},$$

where $w(X, \varphi) = \exp(-a \int_X dx |\varphi(x)|^2)$. We shall henceforth use only the new notation.

LEMMA 6.2 (Properties of $|\cdot|_{a,h}$). *Suppose $h \geq 0$, $g_X \in \mathbb{G}_X$.*

(i) *If $X \subset \mathscr{S}_1$, then $|\mathscr{R}g_X|_{a,h} = |g_X|_{L^2a/|X|, h/L}$.*

(ii) $|g_X|_{a,h} \leq |g_X|_{a',h'}$, $a' \geq a$, $h' \geq h$.

(iii) $|\mu_C * g_X|_{a,h} \leq \exp[\int_x dx \int_y dy |C(x,y)| h^{-2}] |g_X|_{\tilde{a},h}$, $\tilde{a} = a(1 - a\|C\|)^{-1}$, *where $\|C\|$ is the norm of the covariance $C(x,y)$ regarded as an operator on $L^2(X, dx)$ and $a \geq 0$.*

(iv) $|(\partial/\partial\Phi)^\alpha g_X|_{a,h} \leq \alpha!(h' - h)^{-\alpha} |g_X|_{a,h'}$.

(v) *If $X \cap Y = \varnothing$,*

$$|g_X g_Y|_{a,h} \leq |g_X|_{a,h} |g_Y|_{a,h}.$$

(vi) $|g_X g_Y|_{a+b,h} \leq |g_X|_{a,h} |g_Y|_{b,h}$.

The proof of each part in Lemma 6.2 is an easy consequence of the definitions and the work we have already done. It is important to note that in part (i) the a changes to $a/|X|$ in the norm because $\mathscr{R}g$ depends only on one variable, whereas g depends on $|X|$ variables.

We will use the next lemma to show that parts of the interaction become small under the rescaling operation $\mathscr{R}$.

LEMMA 6.3. *Given $g_X \in \mathbb{G}_X$ and $t \in \mathbb{R}$, let $g_X(t\Phi)$ be defined by replacing $\varphi, \bar{\varphi}, \psi, \bar{\psi}$ by $t\varphi, t\bar{\varphi}, t\psi, t\bar{\psi}$ in g_X. Let a such that $2 \geq a > 0$ be given. Suppose that $(d^j/dt^j) g_X(t\Phi)|_{t=0} = 0$ for $j = 0, \dots, k-1$ with $k \geq 1$. Then there exists $C > 0$ such that for $h > 0$ and $l \geq 2$,*

$$|g_X|_{-al^2/h^2, h/l} \leq k! \left(C \frac{|X|}{l\sqrt{a}} \right)^k |g_X|_{0,h}.$$

PROOF.

$$g_X(\Phi) = \int_0^1 dt \frac{(1-t)^{k-1}}{(k-1)!} \sum_{\alpha: |\alpha| = k} g^{(\alpha)}(t\Phi) \Phi^\alpha,$$

where the Fermion fields have to be ordered correctly. By Lemma 6.2(vi),

$$|g_X|_{-al^2/h^2, h/l} \leq \frac{|X|^k}{k!} \sup_{t, |\alpha| = k} |g_X^{(\alpha)}(t\Phi)|_{0, h/l} |\Phi^\alpha|_{-al^2/h^2, h/l},$$

and by Lemma 6.2(iv),

$$\leq |X|^k \left(\frac{l}{h} \frac{1}{l-1} \right)^k |g_X|_{0,h} k! O\left(\frac{h}{l\sqrt{a}} \right)^k,$$

which proves the lemma. □

LEMMA 6.4 (Convolution $\cong$ identity). *Let C be a covariance and let*

$$\Delta = \Delta_C = \int dx\, dy\, C(x,y) \left(\frac{\partial}{\partial\varphi(x)} \frac{\partial}{\partial\bar{\varphi}(y)} + \frac{\partial}{\partial\psi(x)} \frac{\partial}{\partial\bar{\psi}(y)} \right),$$

then for $k = 1, 2, \dots$,

$$\mu_C * g = \sum_{j=0}^{k-1} \frac{1}{j!}(\Delta^j g) + \int_0^1 dt \frac{(1-t)^{k-1}}{(k-1)!} \mu_{tC} * (\Delta^k g).$$

PROOF. Replace C by tC and apply Taylor's theorem. □

The importance of the lemma resides in the fact that $|\Delta^k g|_{a,h}$ is smaller than g by Lemma 6.2(iv), if $|g|_{a,h'} < \infty$ and h' is large.

The next corollary says that if a function of fields has derivatives that vanish at the origin, then convolution almost preserves this feature.

COROLLARY 6.5. *Suppose that* $(d^j/dt^j) g_X(t\Phi)|_{t=0} = 0$ *for* $j = 0, \dots, k$ *with* $k \geq 0$. (*Refer to Lemma* 6.3.) *Then for* $j \leq k$,

$$\left| \frac{d^j}{dt^j}(\mu_C * g_X)\bigg|_{t=0} \right|_{0,h} \leq \frac{l!}{h^l} |2C|^{(l-j)/2} e^{h^{-2}|C|} |g_X|_{0,h},$$

where l *is the smallest integer with* $l > k$ *and* $l - j$ *even and*

$$|C| = \int_X dx \int_X dy |C(x,y)|.$$

PROOF. Apply Lemma 6.4 to the jth t-derivative of $g_X(t\Phi)$ with the k in Lemma 6.4 replaced by $(l-j)/2$. Then estimate the error term using Lemma 6.2(iii) and (iv), with $a = 0$ and $h = 0$. □

7. Interaction after renormalization. We set the dimension $d = 4$ and study the trajectory of $I_N(\Phi^2)$ in (5.3) under repeated application of the renormalization transformation (Definition 4.1). The main result, Theorem 7.2, shows precisely how the self-avoidance interaction tends to 0 with successive transformations.

We ignore, for the time being, the factors ψ and $\bar{\psi}$ in Theorem 3.3; that is, we only consider blocks not containing such factors. From Theorem 4.2, we see that the renormalization group amounts to considering the trajectory of the function g in (5.4) under repeated application of T defined in Definition 4.1.

In this form the model resembles the hierarchical model for φ^4 field theory considered by Gawedzki and Kupiainen (1982, 1986). Here, of course, we have Fermions as well as Bosons and also the fluctuation covariance Γ has a particular form required for a random walk interpretation.

Initially, we have

$$\begin{aligned} g(\Phi) &= \exp(-v(\Phi)), \\ v(\Phi) &= \lambda {:}\Phi^4{:}_G + \mu^2 {:}\Phi^2{:}_G, \end{aligned} \tag{7.1}$$

where the Green's function G was defined in Definition 2.4. The colons denote normal ordering which was defined in Section 5.

After applying T one or more times, we will have the following representation:

$$g(\Phi) = e^{-v(\Phi)}\left(1 + \eta{:}\Phi^6{:}_G\right) + r(\Phi), \tag{7.2}$$

where v has the same form as in (7.1) but the parameters λ, μ are changed from their starting values. The last term is a remainder which is small in an appropriate sense.

Fix two numbers $c_0, c_1 > 0$. The following assumption will be verified for $T^{np}g$ for a sequence λ_n tending to 0.

Inductive assumption $A(\lambda, \mu^2)$. The function $g(\Phi)$ can be represented as in (7.2). The parameter η satisfies $|\eta| \le c_0\lambda^2$. The remainder term $r(\Phi)$ can be represented as $R(\Phi^2)$ and it satisfies the following estimates with $h = \lambda^{-1/4}$:

$$|r|_{\sqrt{\lambda}, h} \le c_1 \lambda,$$

$$\left(\frac{d}{dt}\right)^j r(t\Phi)\bigg|_{t=0} = 0, \qquad 0 \le j < 8, \tag{7.3}$$

where $r(t\Phi)$ is defined by replacing $\varphi, \bar{\varphi}, \psi, \bar{\psi}$ by $t\varphi, t\bar{\varphi}, t\psi, t\bar{\psi}$ in r. Equivalently,

$$r^{(\alpha)}(0) \equiv \left(\frac{\partial}{\partial\Phi}\right)^{\alpha} r(\Phi)\bigg|_{\Phi=0} = 0, \qquad |\alpha| < \infty.$$

The $\Phi = 0$ subscript means set $\varphi = 0$ and project onto the zero-degree part of the Grassman algebra. Derivatives with respect to ψ and $\bar{\psi}$ were defined in Section 3. Any ordering convention for the derivatives can be chosen.

Our starting g satisfies $A(\lambda, \mu^2)$ with $r = 0$. We wish to show that μ can be chosen so that $A(\lambda_n, \mu_n^2)$ holds for $T^n g$ and so that λ_n decreases as $(\lambda^{-1} + \beta_2 n)^{-1}$, where β_2 is a calculable constant.

Some remarks on the form of the inductive assumption are in order. The important parameter is λ, and we wish to get the change in λ correct up to errors $O(\lambda^3)$. Since there is an $O(\lambda^2)\phi^4$ term implicit in $\eta{:}\phi^6{:}$, we keep track of this term as well. One can see that the remainder should have norm $O(\lambda)$ because the leading term in perturbation theory is $O(\lambda^3)\Phi^8$ and $\Phi \sim h = \lambda^{-1/4}$.

We will study the transformation $g \to Tg$ when L is large, or alternatively, if L is not large, we study $g \to T^p g$ with p chosen so that L^p is large.

Recall from Definition 4.1 that $Tg = \mathscr{R}\mu_\Gamma * \Pi g$. $T^p g$ has a similar structure:

$$T^p g = \mathscr{R}^p \mu_{\Gamma_p} * \prod_{x \in \mathscr{G}_p} g(\Phi(x)),$$

$$\Gamma_p(x) = \sum_{k=0}^{p-1} L^{-2k}\Gamma(x/L^k).$$

From this formula for Γ_p,

$$|\Gamma_p|_1 \equiv \int dx |\Gamma_p(x)| \leq cL^{2p},$$

$$|\Gamma_p|_\infty \equiv \sup_x \Gamma_p(x) \leq c.$$

PROPOSITION 7.1. *Let L^p be sufficiently large, and let $c_0 = c_0(L^p)$, $c_1 = c_1(L^p)$ appearing in $A(\lambda, \mu^2)$ be sufficiently large. Let λ be sufficiently small, depending on L^p, and let $|\mu^2| \leq c_0\lambda^2$. Then [g satisfies $A(\lambda, \mu^2)$] $\Rightarrow$ [$T^p g$ satisfies $A(\lambda', \mu'^2)$], where*

$$\lambda' = \lambda - \beta_2\lambda^2 + O(\lambda^3),$$

$$\mu'^2 = L^{2p}\mu^2 - \gamma_2\lambda^2 + O(\lambda^3).$$

Here, $\beta_2, \gamma_2 > 0$ depend on L, p as does $O(\lambda^3)$. The new coefficients λ' and μ'^2 depend on λ and μ^2 continuously.

PROOF. We give the proof for the special case in which $p = 1$ and L is sufficiently large. The general case is obtained by replacing $\mathscr{R}$ by $\mathscr{R}^p$, L by L^p, $n = L^d$ by L^{pd}, and so on. Continuity is left to the reader to verify.

We will write μ for μ_Γ. We set $h = \lambda^{-1/4}$, $h' = \lambda'^{-1/4}$. We do not yet know what λ' is. We prove estimates under the assumptions that $\lambda \geq \lambda' \geq \frac{9}{10}\lambda$, which will be justified a posteriori by the equations for λ', μ' in Proposition 7.1.

We will use $\tilde{r} = \tilde{r}(\Phi_0)$ to denote a function of the field $\Phi_{x=0}$ at the origin, which for λ sufficiently small, satisfies

$$|\tilde{r}|_{\sqrt{\lambda'}, h'} \leq O(L^{-4})c_1(L)\lambda,$$

$$|\tilde{r}^{(\alpha)}(0)| \leq c^{(\alpha)}(L)\lambda^3 \quad \text{for } |\alpha| < 8,$$

for L-dependent constants $c^{(\alpha)}(L)$. Here $c_1 = c_1(L)$ is the constant appearing in the induction hypothesis. We allow $\tilde{r}$ to change from one equation to the next. It will be used to accumulate errors which will eventually be shown to be harmless. Note that our choice of λ allows us to bound any constant depending on L by $\lambda^{-\varepsilon}$, $\varepsilon > 0$.

We begin by using Lemma 6.1 to estimate $|e^{-v}|_{2\sqrt{\lambda'}, h'}$. We need an estimate on the real part of the following analytic function of four complex variables:

$$v_c \equiv \lambda\left(\varphi^{(1)2} + \varphi^{(2)2} + u\bar{u}\right)^2 + O(\lambda^2)\left(\varphi^{(1)2} + \varphi^{(2)2} + u\bar{u}\right)$$

on the domain $|\operatorname{Im}\varphi^{(1)}|, |\operatorname{Im}\varphi^{(2)}|, |u|, |\bar{u}| \leq Ah$ with $A = 2$. It is easy to show that there is a constant C such that

$$\operatorname{Re} v_c \geq \lambda\left\{\left(\operatorname{Re}\varphi^{(1)}\right)^4 + \left(\operatorname{Re}\varphi^{(2)}\right)^4\right\}$$
$$- C\lambda^{1/2}\left\{\left(\operatorname{Re}\varphi^{(1)}\right)^2 + \left(\operatorname{Re}\varphi^{(2)}\right)^2\right\} - C.$$

Therefore, for any $a > 0$, there is a constant C_a such that, using the notation of Lemma 6.1,

$$|e^{-v_c}|_{h'}(\varphi) \le C_a e^{-a\sqrt{\lambda'}|\varphi|^2}.$$

Consequently, by Lemma 6.1, for any $a, b > 0$ there exist $c_{a,b}$ and $c_{a,b,q}$ such that

$$(7.4)\qquad \begin{aligned} &|e^{-bv}|_{a\sqrt{\lambda'},\,h'} \le c_{a,b},\\ &|e^{-bv}\Phi^q|_{a\sqrt{\lambda'},\,h'} \le c_{a,b,q}\lambda^{-q/4}, \qquad q = 0,1,2,\dots, \end{aligned}$$

where Φ^q denotes any monomial of degree q in the components of Φ.

Now we will analyze

$$Tg = \mathscr{R}(\mu * g^{\mathscr{G}_1}),$$

where

$$g^X = \prod_{x\in X} g(\Phi_x),$$
$$g(\Phi_x) = e^{-v(\Phi_x)}(1 + \eta{:}\Phi^6_{x:G}) + r(\Phi_x).$$

We substitute for g in Tg and expand the product grouping the resulting terms into cases (i), (ii) and (iii).

(i) Two or more factors of r. Let

$$S_{\ge 2}(\Phi) = \sum_{X\subset\mathscr{G}_1,\,|X|\ge 2} r^X g_0^{\mathscr{G}_1 - X},$$

where $g_0 = g$ with r set to 0. We will show that each factor of r is $O(\lambda)$, the factors g_0 or r at each $x \in \mathscr{G}_1$ contribute one $\exp(-\sqrt{\lambda}\,|\varphi|^2)$ per site so that the total decay for the block variable $\Phi'(0)$ is $\exp(-n\sqrt{\lambda}\,|\varphi'|^2)$, and so $TS_{\ge 2}$ is small enough to be put into $\tilde r$.

By Lemma 6.2,

$$\begin{aligned} |\mathscr{R}(\mu_\Gamma * S_{\ge 2})|_{\sqrt{\lambda'},\,h'} &= |\mu_\Gamma * S_{\ge 2}|_{\sqrt{\lambda'}L^{-2},\,h'/L}\\ &\le 2|S_{\ge 2}|_{\sqrt{\lambda},\,h} \le 2\sum_{X,\,|X|\ge 2} |r|^X_{\sqrt{\lambda},\,h}|g_0|^{\mathscr{G}_1 - X}_{\sqrt{\lambda},\,h} \le O(\lambda^2 c^n), \end{aligned}$$

by (7.4) and the inductive assumption on r. Thus $\mathscr{R}(\mu_\Gamma * S_{\ge 2})$ obeys $\tilde r$-estimates. (The bound on the derivatives follows from Corollary 6.5 and the vanishing of derivatives of r at the origin.)

(ii) One factor of r. Let

$$(7.5)\qquad S_1 = \int_{\mathscr{G}_1} dx\, r(\Phi_x) g_0^{\mathscr{G}_1 - \{x\}},$$

where we have used the same notation as in case (i). By Lemma 6.4 with $k = 1$, $\mu_\Gamma * S_1 = S_1 + \tilde r$. The bounds on $\tilde r$ are obtained by the same steps as in case (i), together with Lemma 6.2(iv) to estimate $\Delta(rg_0^{\mathscr{G}_1 - \{x\}})$. [Each derivative contributes $O(h^{-1}) = \lambda^{1/4}$ and r contributes $c_1(L)\lambda$.]

Now we consider the size of $\mathscr{R}S_1$. It depends on a single field $\Phi = \Phi(0)$:

$$\mathscr{R}S_1 = e^{-(n-1)v(\Phi/L)}\left(1 + \eta{:}\left(\frac{\Phi}{L}\right)^6{:}\right)^{n-1} n\mathscr{R}r \equiv A(\Phi)n\mathscr{R}r.$$

We will show this is small by exploiting $r(\Phi/L) = O(L^{-8})c_1(L)\lambda$. By Lemma 6.2(vi),

$$|\mathscr{R}S_1|_{\sqrt{\lambda'},h'} \le |A|_{2\sqrt{\lambda'},h'} n|\mathscr{R}r|_{-\sqrt{\lambda'},h'}.$$

By (7.4), the norm of A is bounded by a constant. By Lemma 6.2(i),

$$|\mathscr{R}r|_{-\sqrt{\lambda'},h'} = |r|_{-L^2\sqrt{\lambda'},h'/L} = |r|_{-l^2/h^2,h/l},$$

with $l = Lh/h'$.

Therefore, by Lemma 6.3 (with $l = Lh/h'$, $|X| = 1$),

$$|n\mathscr{R}r|_{-\sqrt{\lambda'},h'} \le nO(L^{-8})|r|_{\sqrt{\lambda},h} \le O(L^{-4})|r|_{\sqrt{\lambda},h} \le O(L^{-4})c_1(L)\lambda,$$

by the inductive assumptions on r. Hence $c_1(L)$ carries over from the induction hypothesis; the choice of $c_1(L)$ takes place below. Thus $\mathscr{R}(\mu_\Gamma * S_1)$ obeys estimates of type $\tilde{r}$.

(iii) No $r(\Phi)$ factors. Here we will extract the lowest-order perturbative effects. We compute to second order in λ, by using Lemma 6.4 with $k = 3$. Let

$$S_0 = e^{-\int_{\mathscr{G}_1} dx\, v}\left(1 + \eta{:}\Phi^6{:}_G\right)^{\mathscr{G}_1}. \tag{7.6}$$

Then $\mu_\Gamma * S_0 = (1 + \Delta + \frac{1}{2}\Delta^2)S_0 + \tilde{r}$, by (7.4), Lemma 6.2, hypotheses on η, μ^2 and Corollary 6.5.

Next, we claim that

$$\begin{aligned}\mathscr{R}\left(1 + \Delta + \tfrac{1}{2}\Delta^2\right)S_0 &= \mathscr{R}\left(1 + \Delta + \tfrac{1}{2}\Delta^2\right)\left(e^{-\int dx\, v}\right)\\ &\quad\times\left(1 + \int dx\,\eta{:}\Phi^6{:}_G\right) + \tilde{r},\end{aligned} \tag{7.7}$$

where the Laplacians act only on $\exp(-\int v)$. This follows by expanding $(1 + \eta{:}\Phi^6{:})^{\Phi_1}$ in S_0 and estimating all the terms with two or more $:\Phi^6:$ factors using Lemma 6.2. The remaining terms have one or no $:\Phi^6:$ factors. For terms with one $:\Phi^6:$,

$$\mathscr{R}\left(1 + \Delta + \tfrac{1}{2}\Delta^2\right){:}\Phi^6{:}_G = \mathscr{R}\left(\mu * {:}\Phi^6{:}_G\right) = L^{-6}{:}\Phi^6{:}_G, \tag{7.8}$$

by Example 5.2 and $\Delta^3\Phi^6 = 0$.

There are also terms in which one $\partial/\partial\Phi$ in a Laplacian acts on $\exp(-\int dx\, v)$ and the other $\partial/\partial\Phi$ acts on the $:\Phi^6:$. These are of type $\tilde{r}$ by Lemmas 6.2, 7.4 and the inductive assumptions on λ, μ, ν. Let us, for example, consider the most delicate of these terms which is

$$S \equiv \eta e^{-nv(\Phi/L)}\mathscr{R}\left\{\int_{\mathscr{G}_1} dx \int_{\mathscr{G}_1} dy \frac{\partial v}{\partial\Phi(x)}\Gamma(x,y){:}\Phi^4\Phi{:}(y)\right\},$$

up to a constant.

Then by (7.4),

$$|S|_{\sqrt{\lambda'}, h'} \le C\eta n |\Gamma|_1 L^{-8} \lambda \lambda^{-2} = O(L^{-2})\lambda.$$

The rescaling produces the L^{-8} and $n|\Gamma|_1$ is an estimate for $\int dx\,dy\,\Gamma(x,y)$. This implies the first of the $\tilde{r}$-estimates, for large enough $c_1(L)$. The second $\tilde{r}$-estimate for S is also easy to obtain. [In fact, $S = 0$ because $\int \Gamma(x-y)\,dy$ vanishes! All tree graphs vanish but we prefer not to take advantage of this special feature.]

CONCLUSION FROM (i), (ii) AND (iii).

$$(Tg)(\Phi) = e^{-nv(\Phi/L)}\big(P(\Phi) + L^{-2}\eta{:}\Phi^6{:}\big) + \tilde{r}(\Phi), \tag{7.9}$$

where P is a sixth-order polynomial obtained by applying $1 + \Delta + \frac{1}{2}\Delta^2$ to $\exp(-\int dx\, v)$, rescaling $(\mathscr{R})$ and dropping terms of degree greater than 6 (which are absorbed into $\tilde{r}$). It has the form

$$P(\Phi) = 1 + O(\lambda)\Phi^2 + O(\lambda^2)\Phi^4 + O(\lambda^2)\Phi^6. \tag{7.10}$$

The constant term is 1 because $\Delta{:}\Phi^2{:} = \Delta^2{:}\Phi^4{:} = 0$. The coefficients have dependence on L.

RENORMALIZATION. Our task is now to take terms up to degree Φ^6 in $\tilde{r}$ and P and cancel them by a shift in λ, μ^2, η up to $O(\Phi^8)$ terms which will be combined with the rest of $\tilde{r}$. This puts Tg back into the form of the inductive assumption $A(\lambda, \mu'^2)$.

As a first step we write

$$\begin{aligned}(Tg)(\Phi) &= e^{-nv(\Phi/L)}\big(P + L^{-2}\eta{:}\Phi^6{:} + f(\Phi)\big)\\ &= e^{-nv(\Phi/L)}\big(P + L^{-2}\eta{:}\Phi^6{:} + f^{(\le 6)}(\Phi)\big) + \tilde{r}_1(\Phi),\end{aligned} \tag{7.11}$$

where $f(\Phi) = \tilde{r}(\Phi)\exp(+nv(\Phi/L))$ and $f^{(\le 6)}$ is its Taylor series through sixth order. $\tilde{r}_1 = (f - f^{(\le 6)})\exp(-nv(\Phi/L))$. We have

$$\begin{aligned}|\tilde{r}_1|_{\sqrt{\lambda'}, h'} &= |\tilde{r} - f^{(\le 6)}e^{-nv(\Phi/L)}|_{\sqrt{\lambda'}, h'}\\ &\le |\tilde{r}|_{\sqrt{\lambda'}, h'} + |f^{(\le 6)}e^{-nv(\Phi/L)}|_{\sqrt{\lambda'}, h'}\\ &\le |\tilde{r}|_{\sqrt{\lambda'}, h'} + O(\lambda^{3-6/4}),\end{aligned} \tag{7.12}$$

so $\tilde{r}_1$ obeys $\tilde{r}$-estimates. By construction $\tilde{r}_1^{(\alpha)}(0) = 0$ for $|\alpha| < 8$.

By matching coefficients of Φ^2, Φ^4, Φ^6,

$$\begin{aligned}\tilde{P} &\equiv P + L^{-2}\eta{:}\Phi^6{:} + f^{(\le 6)}\\ &= \left(1 + A\Phi^2 + \frac{1}{2}A^2\Phi^4 + \frac{1}{3!}A^3\Phi^6\right)\\ &\quad\times(1 + B\Phi^4)(1 + C{:}\Phi^6{:}) \mod \Phi^8,\end{aligned} \tag{7.13}$$

where $A = O(\lambda)$, $B = O(\lambda^2)$, $C = L^{-2}\eta + O(\lambda^2)$. The constant term is 1 by

Theorem 4.2(iv), and because $:\Phi^6: = 0$ at $\Phi = 0$ and $\tilde{r}_1(0) = 0$. Therefore

$$(7.14) \qquad (Tg)(\Phi) = e^{-nv(\Phi/L)} e^{A\Phi^2} e^{B\Phi^4}(1 + C{:}\Phi^6{:}) + r'(\Phi),$$

where $\tilde{r}_1$ has changed to r' to absorb $O(L^{-2}\lambda^3)\Phi^8\exp(-nv(\Phi/L))$ terms.

We claim this representation satisfies $A(\lambda', \mu'^2)$ if L is chosen sufficiently large. First, it is clear that the derivatives $r^{(\alpha)}(0)$, $|\alpha| \le 7$, vanish. Second, to prove the estimate for $\eta' = C$, note that η' receives a contribution: (a) From $\eta{:}\Phi^6{:}$ which by (7.7) and the equation after (7.7) is $(\eta/L^6)n = \eta L^{-2}$, so this contribution is less than $\frac{1}{3}c_0\lambda^2$ for $L \gg 1$. (b) From P in (7.9). This arises by applying $(1 + \Delta + \Delta^2/2)$ to $\exp(-\int dx\, v)$ and in particular is independent of r so we may choose c_0 depending on L so that this contribution is less than $\frac{1}{3}c_0\lambda^2$. This is how c_0 is determined in Proposition 7.1. (c) From $f^{(\le 6)}$ and cross-terms between P and $f^{(\le 6)}$ in (7.13). These terms are $O(\lambda^{>2})$ and are smaller than $\frac{1}{6}c_0\lambda^2$ by choice of λ. The total contribution from (a), (b) and (c) is therefore such that $\eta' \le c_0\lambda^2$.

Third, to prove the estimate $|r|_{\sqrt{\lambda'}, h'} \le c_1\lambda$, note that $|r'|_{\sqrt{\lambda'}, h'}$ has contributions: (a) From $\tilde{r}_1$ in (7.11). These are less than $\frac{1}{2}c_1\lambda$ by taking L large in $\tilde{r}$-estimates. (b) From mod Φ^8 contributions produced in steps (7.13) and (7.14), coming solely from P and η. These are less than $\frac{1}{3}c_1\lambda$ by choosing c_1 depending on L. Here and in $\tilde{r}$-estimates on type (iii) terms above, r is not involved in the estimates so there is no circularity in the choice of $c_1(L)$. (c) Contributions from $f^{(\le 6)}$ and cross-terms with $f^{(\le 6)}$ in (7.13). These are $O(\lambda^{>1})$ in norm and are less than $\frac{1}{3}c_1\lambda$ by choice of λ. Altogether, we have $|r'|_{\sqrt{\lambda'}, h'} \le c_1\lambda$.

Next, we claim that

$$(7.15) \qquad nv(\Phi/L) - A\Phi^2 - B\Phi^4 = \lambda'{:}\Phi^4{:}_G + \mu'^2{:}\Phi^2{:}_G,$$

with $\lambda' = \lambda - \beta_2\lambda^2 + O(\lambda^3)$, $\mu'^2 = L^2\mu^2 - \gamma_2\lambda^2 + O(\lambda^3)$, $\beta_2 > 0$, $\gamma_2 > 0$. To do this, we compute A to second order in λ. To first order we find, starting from (7.7), that

$$A\Phi^2 = -\mathscr{R}\Delta\left(\int dx\, v(\Phi(x))\right),$$

so that

$$nv(\Phi/L) - A\Phi^2 = \mathscr{R}e^{\Delta}\left(\int dx\, v\right) = \lambda{:}\Phi^4{:}_G + \mu^2L^2{:}\Phi^2{:}_G,$$

by Example 5.2. Thus the $O(\lambda)$ part of $A\Phi^2$ adjusts the normal ordering in v. The remaining parts of A and B are $O(\lambda^2)$. The coefficients of λ^2 are determined completely by perturbation theory; that is, they depend only on integrals over $\mathscr{I}_1$ of products of Γ's. Derivatives of e^{-v} alternate in sign, and this leads to $\beta_2 > 0$. Beyond λ^2 there are corrections bounded by L dependent constants times λ^3. These come from the Taylor coefficients of the $\tilde{r}$ errors.

Finally, $r(\Phi)$ is a function $R(\Phi^2)$ by Theorem 4.2. □

THEOREM 7.2 (Asymptotic freedom). *Let $L \geq 2$ be given. Let p be a sufficiently large integer, and λ_0 be sufficiently small. Then there exists $\mu_0^2(\lambda_0) = O(\lambda_0^2)$ such that if*

$$g(\Phi) = \exp\{-\lambda_0{:}\Phi^4{:}_G - \mu_0^2{:}\Phi^2{:}_G\},$$

then $T^{np}g$ satisfies $A(\lambda_{np}, \mu_{np}^2)$ with $\lambda_n = (\lambda_0^{-1} + \beta_2 n + O(\log n))^{-1}$ for all n. Here $O(\cdot)$ may depend on L^p.

DEFINITION 7.3. Let $\mu_c^2(\lambda) = \mu_0^2(\lambda)$ as given in Theorem 7.2. Then μ_c^2 is called the critical mass.

PROOF. We use the well-known method of Bleher and Sinai (1973) to find μ_c^2. Again, for simplicity, let us discuss the case $p = 1$.

Let J_n be the interval $c_0\lambda_n^2[-1, 1]$ and let $I_0 = J_0$. Let us suppose, as an inductive hypothesis, that we have constructed an interval I_{-n} such that when μ_0^2 sweeps I_{-n} then μ_n^2 sweeps the interval J_n. To construct $I_{-(n+1)}$, note that Proposition 7.1 says that as μ_n^2 sweeps J_n, μ_{n+1}^2 sweeps $L^2 J_n - \gamma_2\lambda_n^2 + O(\lambda_n^3)$. For L large enough or for c_0 larger than γ_2, this interval contains J_{n+1}. Hence we define $I_{-(n+1)}$ as the preimage of J_{n+1} under the mapping $\mu_0^2 \to \mu_{n+1}^2$.

The intervals I_{-n} decrease as nested sets to a single point which we define to be μ_c^2. Only at this point will the bound $|\mu_n^2| \leq c_0\lambda_n^2$ be satisfied for all n so that Proposition 7.1 can be applied arbitrarily many times. The recursion $\lambda_{n+1} = \lambda_n - \beta_2\lambda_n^2 + O(\lambda_n^3)$ can be rewritten as $\lambda_{n+1}^{-1} = \lambda_n^{-1} + \beta_2 - O(\lambda_n)$. This can easily be solved to obtain $\lambda_n = (\lambda_0^{-1} + \beta_2 n + O(\log n))^{-1}$. □

8. The Green's function. We will now study the Green's function for walks from site x to site y in $\mathscr{G}$. Our main result is Theorem 8.1.

THEOREM 8.1. *Fix any integer $L \geq 2$. Let $\lambda > 0$ be sufficiently small. Then there exists $a_c = a_c(\lambda) < 0$ such that*

$$U^a(x,y) \equiv \lim_{N\to\infty} \int_0^\infty dt\, E_x\biggl(\exp\biggl[-2\lambda\int_{0\leq u<v\leq T} du\, dv\, \mathbf{1}_{\{\omega_u=\omega_v\in\mathscr{G}_N\}} - a\int_{0\leq u\leq T} du\, \mathbf{1}_{\{\omega_u\in\mathscr{G}_N\}}\biggr]\mathbf{1}_{\{\omega_T=y\}}\biggr)$$

exists for all $a \geq a_c$. At $a = a_c$,

$$U^{a_c}(x,y) = (1 + O(\lambda))G(x-y)(1 + \varepsilon(x-y)),$$

where $|\varepsilon(x)| \leq O(\lambda)(1 + \lambda\log(1 + |x|))^{-1}$. Here $O(\lambda)$ may depend on L and

$$a_c = \mu_c^2 - 2\lambda G(0),$$

where μ_c^2 was defined in Definition 7.3.

We prove this theorem at the end of the section.

By Theorem 3.3 and the remark after it,

$$(8.1)\quad \int_0^\infty dt\, E_x\left\{ \prod_{z\in\mathscr{G}_N} e^{-\lambda\tau_z^2 - a\tau_z} 1_{\{\omega(t)=y\}} \right\} = \langle \overline{\varphi}_x \varphi_y \rangle_N = \int d\mu_G\, dg f^{\mathscr{G}_N} \overline{\varphi}_x \varphi_y,$$

where $g^X(\Phi) = \Pi_{x\in X} g(\Phi_x)$, and

$$g(\Phi) = e^{-v(\Phi)}(1 + \eta :\Phi^6:) + r(\Phi),$$

with $\eta = 0$, $r = 0$, $v(\Phi) = \lambda :(\Phi^2)^2: + \mu^2 :\Phi^2:$, $\mu^2 = a - 2\lambda G(0)$. In order to analyze (8.1) properly, one needs to study a renormalization group transformation S which acts on an observable f (a function of fields Φ_y, $y \in X \subset x + \mathscr{G}_1$) according to

$$(8.2)\quad \begin{aligned} S(f)T(g) &= \mathscr{R}\mu_\Gamma * (f g^{\mathscr{G}_1}) \\ &= \mathscr{R}\mu_\Gamma * (f g^X g^{\mathscr{G}_1 - X}). \end{aligned}$$

The second line shows that we are actually defining a transformation $fg^X \mapsto S(f)T(g)$ which is a function of the block field $\Phi(x/L)$. Although we will refer to $S(f)$, all our arguments involve only products $S(f)T(g)$ (or can easily be reworded this way). Thus we do not assume $T(g)$ is an invertible element of $\mathbb{G}$.

Initially, we have $f(\Phi) = \varphi(x)$. We should also consider cases in which φ is replaced by $\overline{\varphi}$, ψ or $\overline{\psi}$, but there are no essential differences.

We will begin by formulating an inductive assumption which characterizes functions which "look like" φ. Eventually, the renormalization group transformations will rescale the x and y so that $\overline{\varphi}_x$ and φ_y are located in the same block and after that a different inductive assumption which characterizes functions which "look like" Const + Const $\overline{\varphi}\varphi$ will be used.

Fix a constant $c_2 > 0$.

ASSUMPTION $B(g, a, b)$. The function $f = f(\Phi)$ of a field at a single point, $\Phi = \Phi(x)$, satisfies

$$f(\Phi)g(\Phi) = \{a\varphi + b\lambda :\varphi\Phi^2:_G\} g(\Phi) + s(\Phi),$$

where $s(\Phi)$ is a function of the form $\varphi F(\Phi^2)$, and

$$s^{(\alpha)}(0) = 0 \quad \text{for } |\alpha| \le 4,$$
$$|s|_{\sqrt{\lambda}, h} \le c_2 |a| \lambda^{3/4},$$

and b satisfies $|b| \le c_2 |a|$.

The same basic principles guide the formulation of this assumption. In this case, we wish to follow the coefficient of φ carefully since its flow determines the rate of decrease of the Green's function. The cubic term is isolated so as not to obscure the convergence of $L^{pn} a_n$ as $n \to \infty$.

PROPOSITION 8.2. *Let L^p be sufficiently large and let $c_2 = c_2(L^p)$ appearing in $B(g, a, b)$ be sufficiently large. Let λ be sufficiently small, depending on L^p, and let $|\mu^2| \le c_0 \lambda^2$, where c_0 is the constant appearing in the inductive* *assumption $A(\lambda,\mu)$ of Section 7. Let g satisfy $A(\lambda,\mu)$ and let f satisfy $B(g,a,b)$. Then $S^p f$ satisfies $B(T^p g, a', b')$ with*

$$a' = L^{-p}a + O(\lambda^2 a),$$

$$b' = L^{-3p}\left\{b - 2\int dx\,\Gamma_p(x)a\right\} + O(\lambda a).$$

Here $O(\lambda^2 a)$ and $O(\lambda a)$ depend on L^p.

PROOF. Without loss of generality, we take $f = f(\Phi(0))$. We write out the proof for the special case $p = 1$ and L large. As in Section 7, the general case is the same argument with $\mathscr{R}$ replaced by $\mathscr{R}^p$, Γ by Γ_p, $\mathscr{G}_1$ by $\mathscr{G}_p$, $n = |\mathscr{G}_1| = L^d$ by $|\mathscr{G}_p| = L^{pd}$ and L by L^p.

The claim that s has the functional form $\varphi F(\Phi^2)$ follows from Theorem 4.2.

We use $\tilde{s}$ to denote any function of Φ at a single point in $\mathscr{G}$ such that

$$|\tilde{s}|_{\sqrt{\lambda'},h'} \le O(L^{-5})c_2(L)|a|\lambda^{3/4},$$

$$|\tilde{s}^{(\alpha)}(0)| \le c^{(\alpha)}(L)|a|\lambda^2 \quad \text{if } |\alpha| \le 4.$$

We will call these *$\tilde{s}$-estimates.*

By Lemma 6.4,

$$S(f)T(g) = \mathscr{R}(\mu_\Gamma * fg^{\mathscr{G}_1}) \tag{8.3}$$
$$= \mathscr{R}E(fg^{\mathscr{G}_1}) + \text{remainder},$$

where $E = 1 + \Delta + \frac{1}{2}\Delta^2 + (1/3!)\Delta^3$. We claim that the remainder obeys $\tilde{s}$-estimates. By Lemma 6.2,

$$\begin{aligned} |\mathscr{R}(\mu_{t\Gamma} * \Delta^4 fg^{\mathscr{G}_1})|_{\sqrt{\lambda'},h'} &\le |\mu_{t\Gamma} * \Delta^4 fg^{\mathscr{G}_1}|_{\sqrt{\lambda'}L^{-2},h'/L} \\ &\le 2|\Delta^4 fg^{\mathscr{G}_1}|_{2\sqrt{\lambda'}L^{-2},h'/L} \\ &\le O(h^{-8})|fg^{\mathscr{G}_1}|_{\sqrt{\lambda'},h} \\ &\le |a|O(\lambda^{7/4}), \end{aligned} \tag{8.4}$$

by using $B(g,a,b)$, $A(\lambda,\mu)$, Lemma 6.2 and (7.4). By Lemma 6.4, we see that the remainder is bounded by $O(\lambda^{7/4})|a|$. By Lemma 6.2, the derivatives are $O(\lambda^2)$ since the zeroth derivative vanishes at $\Phi = 0$. For λ small depending on L, the remainder obeys $\tilde{s}$-estimates. The claim is proven.

By $B(g,a,b)$,

$$\mathscr{R}E(fg^{\mathscr{G}_1}) = \mathscr{R}E\{(a\varphi_0 + b\lambda{:}\varphi_0\Phi_0^2{:})g^{\mathscr{G}_1}\} + \mathscr{R}E(sg^{\mathscr{G}_1-\{0\}}). \tag{8.5}$$

We claim the last term is an $\tilde{s}$.

By estimates similar to (8.4), we see that each Δ in E contributes $\lambda^{1/2}$ and s is already $O(a\lambda^{3/4})$, so E can be replaced by the identity, up to $\tilde{s}$ terms:

$$\mathscr{R}E(sg^{\mathscr{G}_1-\{0\}}) = \mathscr{R}(sg^{\mathscr{G}_1-\{0\}}) + \tilde{s}.$$

Furthermore, by $A(\lambda, \mu)$ and Lemma 6.2,

$$\mathscr{R}sg^{\mathscr{G}_1-\{0\}} = \mathscr{R}s(e^{-v})^{\mathscr{G}_1-\{0\}} + \tilde{s},$$

$$|\mathscr{R}s(e^{-v})^{\mathscr{G}_1-\{0\}}|_{\sqrt{\lambda'},h'} \le |\mathscr{R}s|_{-\sqrt{\lambda'},h'}|\mathscr{R}(e^{-v})^{\mathscr{G}_1-\{0\}}|_{2\sqrt{\lambda'},h'}.$$

By Lemma 6.2, $|\mathscr{R}s|_{-\sqrt{\lambda'},h'} = |s|_{-l^2h^2,h/l}$ with $l = Lh/h'$. We apply Lemma 6.3, (7.4) and $B(g, a, b)$ to get that the last expression is

$$\le cL^{-5}|s|_{0,h} \le O(L^{-5})c_2(L)|a|\lambda^{3/4}.$$

We have proven the first $\tilde{s}$-estimate. Here $c_2(L)$ carries over from the induction hypothesis; the choice of $c_2(L)$ takes place below. Since $s^{(\alpha)}(0) = 0$ for $|\alpha| \le 4$, we have the second $\tilde{s}$-estimate also. We have proven the claim that the second term in (8.5) is $\tilde{s}$.

So far, we have proven that

$$S(f)T(g) = \mathscr{R}E\{Pg^{\mathscr{G}_1}\} + \tilde{s}, \tag{8.6}$$

where $P = (a\varphi + b\lambda{:}\varphi\Phi^2{:})(x)$. To organize the results of expanding E using the Leibniz rule, we will use $\doteq$ to denote equality mod (eighth-order $\partial/\partial\Phi$ derivatives), for example, $E \doteq \exp(\Delta)$. Also we write $\Delta = \Delta_{PP} + 2\Delta_{Pg} + \Delta_{gg}$, where Δ_{PP} means that both $\partial/\partial\Phi$'s in Δ act on P, and so forth. Then, in the above equation we may write

$$\begin{aligned} E &\doteq \exp(\Delta) \doteq \exp(\Delta_{PP} + 2\Delta_{Pg} + \Delta_{gg}) \\ &= \exp(2\Delta_{Pg})\exp(\Delta_{PP})\exp(\Delta_{gg}) \\ &\doteq \sum_{j=0}^{3} \frac{2^j}{j!}\Delta_{Pg} E_P E_g. \end{aligned}$$

Although we have manipulated formal power series in the intermediate equalities, the left- and right-hand sides are finite series and therefore agree up to finitely many eighth- or higher-order derivatives. When such derivatives act on $Pg^{\mathscr{G}_1}$, the results can be put in $\tilde{s}$ by estimates similar to (8.4).

Since P is a cubic polynomial $EP = \exp(\Delta)P$ and, by Lemma 5.1, $\exp(\Delta)$ changes the normal ordering in P from G to $G - \Gamma$. Let

$$Q = a\varphi + b\lambda{:}\varphi\Phi^2{:}_{G-\Gamma}.$$

Then we have proven:

$$S(f)T(g) = \mathscr{R}\sum_{j=0}^{3} \frac{2^j}{j!}\Delta_{Qg}^j\{QEg^{\mathscr{G}_1}\} + \tilde{s}. \tag{8.7}$$

By using the explicit form for Q and the representation (7.2) for g, we find that the $j = 2, 3$ terms in the sum are $\tilde{s}$-terms.

$\mathscr{R}$ acts on a Wick power as follows (Lemma 5.1):

$$R{:}\Phi^m{:}_{G-\Gamma} = L^{-m}{:}\Phi^m{:}_G.$$

Therefore $\mathscr{R}Q = P_L$, where

$$P_L = aL^{-1}\varphi + b\lambda L^{-3}{:}\varphi\Phi^2{:}_G.$$

Also, $P_L \mathscr{R} E g^{\mathscr{I}_1} = P_L \mathscr{R} \mu_\Gamma * g^{\mathscr{I}_1} + \tilde{s} = P_L Tg + \tilde{s}$, and

$$S(f)T(g) = P_L(Tg) + 2\mathscr{R}\Delta_{Qg}(QEg^{\mathscr{I}_1}) + \tilde{s}. \tag{8.8}$$

Consider the term with Δ_{Qg}. Up to changes in $\tilde{s}$ in each equality,

$$\mathscr{R}\Delta_{Qg}\{QE(g^{\mathscr{I}_1})\} = -\mathscr{R}\int dx\, \Gamma(x) Q_{\Phi_0} E\{g^{\mathscr{I}_1} v_{\Phi_x}\} + \tilde{s},$$

where Φ subscripts indicate $\partial/\partial\Phi$ derivatives,

$$= \left(\int dx\, \Gamma(x)\right)(\mathscr{R}Q_\Phi)(\mathscr{R}Ev_\Phi)(Tg) + \tilde{s}$$

$$= \left(\int \Gamma\right) a\{2\lambda L^{-3}{:}\Phi^2\varphi{:}_G + \mu^2 L^{-1}\varphi\}(Tg) + \tilde{s}.$$

This term is actually 0, since $\int\Gamma = 0$, but we have written it into Proposition 8.1 as part of the shift in b. In the above analysis of $\mathscr{R}E\{Pg^{\mathscr{I}_1}\}$, various terms have been put in $\tilde{s}$ either because they are $O(\lambda^{>3/4})|a|$ or $O(L^{-5})c_2(L)|a|\lambda^{3/4}$. Here a one-time choice of $c_2(L)$ is made (note that s is not involved so there is no circularity in this choice).

Let

$$\tilde{P} = aL^{-1}\varphi + L^{-3}\lambda\left(b - \frac{1}{2}\int \Gamma a\right){:}\varphi\Phi^2{:}_G. \tag{8.9}$$

Then we have proven

$$S(f)T(g) = \tilde{P}(Tg) + \tilde{s}. \tag{8.10}$$

Now we let q be the terms through order 3 in the Taylor series for $\tilde{s}/(Tg)$ about $\Phi = 0$. Note that Tg is invertible near $\varphi = 0$ by Proposition 7.1. Then

$$S(f)T(g) = (\tilde{P} + q)(Tg) + \tilde{s} - q(Tg),$$

and here the coefficients of q are $O(\lambda^2 a)$ because $\tilde{s}^{(\alpha)}(0) = O(\lambda^2 a)$, $|\alpha| \le 4$. Since $S(f)T(g)$ and $P(Tg)$ have the functional form $\varphi f(\Phi^2)$, $\tilde{s}$ must also by (8.10), and so must q. Therefore q renormalizes a, b by $O(\lambda^2 a)$. Let $s' = \tilde{s} - q(Tg)$. Then

$$|s'|_{\sqrt{\lambda'}, h'} \le |\tilde{s}|_{\sqrt{\lambda'}, h'} + |q|_{-\sqrt{\lambda'}, h'} |(Tg)|_{2\sqrt{\lambda'}, h'}$$

$$\le O(L^{-5})c_2(L)\lambda^{3/4} + c|q|_{-\sqrt{\lambda'}, h'},$$

by $\tilde{s}$-estimates and (7.4). Since $q^{(\alpha)} = (\tilde{s}/(Tg))^{(\alpha)}(0)$, the second $\tilde{s}$-estimate and $A(\lambda, \mu)$ imply

$$|q^{(\alpha)}(0)| \le c(L)|a|\lambda^2,$$

so that $|q|_{-\sqrt{\lambda'},h'} \le c(L)|a|\lambda^{5/4}$. Therefore

$$|s'|_{\sqrt{\lambda'},h'} \le c_2(L)|a'|\lambda^{3/4},$$

where a' is the coefficient of φ in $\tilde{P} + q$. The rescaling caused the original s to shrink by $O(L^{-5})$, leaving plenty of room for new contributions to s'.

Finally, $s' = (Tg)(\tilde{s}(Tg)^{-1} - q)$ so by the definition of q, s' satisfies $s'^{(\alpha)}(0) = 0$, if $|\alpha| \le 3$, and if $|\alpha| = 4$, because s is an odd function.

In summary, we have proven that

$$S(f)T(g) = (\tilde{P} + q)(Tg) + s'$$

satisfies the inductive assumption $B(Tg, a', b')$, where

$$\tilde{P} + q = a'\varphi - b'\lambda' {:}\varphi\Phi^2{:},$$
$$a' = L^{-1}a + O(\lambda^2 a),$$
$$b' = L^{-3}\left(b - \frac{1}{2}\int \Gamma a\right) + O(\lambda a).$$

The change in λ induces a small shift in b which is absorbed in $O(\lambda a)$. □

If we are able to do k iterations, where $L^{kp} = |u - v|$, then two observables $S^{kp}f_u$ and $S^{kp}f_v$ will be in the same block, $u/L^{kp} + \mathscr{G}_p$. We write

$$h_0(\Phi) = (S^{kp}f_u)(S^{kp}f_v)$$

and make the following inductive assumption (which we will prove is valid for h_0).

Assumption $C(g, \mathbf{d})$. Let u, v be two not necessarily distinct points with $|u - v| \le L^p$. The observable h can be represented in the form

$$h(\Phi)g^{\{u,v\}} = \left(d^{(0)} + d^{(2)}{:}\,\varphi(u)\bar{\varphi}(v){:}_G\right)g^{\{u,v\}} + t,$$
$$g^{\{u,v\}} \equiv \prod_{w \in \{u,v\}} g(\Phi_w).$$

Here t has the functional form $\varphi_u\bar{\varphi}_v F(\Phi_u^2, \Phi_v^2)$ with $F(0,0) = 0$. It satisfies $t^{(\alpha)}(0) = 0$ for $|\alpha| < 4$ and

$$|t|_{\sqrt{\lambda},h} \le d^{(4)}.$$

Now let us suppose that $f(\Phi_x) = \varphi_x$ and $\bar{f}(\Phi_y) = \bar{\varphi}_y$, where $|x - y| = L^{k+1}$. From Proposition 8.2 iterated k times, and the flow of λ given by Theorem 7.2, we find

$$S^{kp}f = a_{kp}\varphi + b_{kp}\lambda_{kp}{:}\varphi\Phi^2{:} + s_{kp}g_{kp}^{-1}, \tag{8.11}$$

$$a_{kp} = L^{-kp}\prod_{j=1}^{k}\left(1 + O(\lambda_{pj}^2)\right) = L^{-kp}(1 + O(\lambda_0))(1 + O(\lambda_{kp})), \tag{8.12}$$

$$|b_{kp}| \le O(a_{kp}),$$
$$|s_{kp}|_{\sqrt{\lambda_{kp}},h_{kp}} \le O\left(a_{kp}\lambda_{kp}^{3/4}\right). \tag{8.13}$$

Therefore h_0 satisfies $C(g_{kp}, \mathbf{d})$ with

$$
\begin{aligned}
d^{(0)} &= G(x-y)(1+O(\lambda_0))(1+O(\lambda_{kp})),\\
d^{(2)} &= O(d^{(0)}), \qquad\qquad (8.14)\\
d^{(4)} &= O(d^{(0)}).
\end{aligned}
$$

The term $d^{(0)}$ comes from normal-ordering $a_{kp}^2\varphi\overline{\varphi}$. The leading contribution to $d^{(4)}$ comes from $a_{kp}b_{kp}\lambda\varphi\overline{\varphi}\Phi^2$. In (8.12)–(8.14) and below, $O(\cdot)$ may depend on L^p, unless the dependence on L^p is written explicitly, as in $O(L^{-4p})$.

PROPOSITION 8.3. *Assume the same hypotheses on L, p, λ, g as in Proposition 8.2. Then $[h$ satisfies $C(g, \mathbf{d})] \Rightarrow [S^p(h)$ satisfies $C(T^p g, \mathbf{d}')]$ with*

$$
\begin{aligned}
d'^{(0)} &= d^{(0)} + O(d^{(2)}\lambda^2) + O(d^{(4)}\lambda),\\
d'^{(2)} &= L^{-2p}d^{(2)}(1+O(\lambda)) + O(d^{(4)}\lambda),\\
d'^{(4)} &= O(L^{-4p})d^{(4)} + O(\lambda d^{(2)}),\\
u', v' &= u/L^p, v/L^P.
\end{aligned}
$$

PROOF. With the basic method established in the previous proposition, we indicate only the main points. The coefficient $d'^{(0)}$ receives its shift from Laplacians which contract the $d^{(2)}{:}\varphi\overline{\varphi}{:}$ and t terms to g. The normal ordering is responsible for the vanishing of $O(\lambda)$ contributions, and $d'^{(2)}$ scales down by L^{-2p} because $\mathscr{R}^p\mu_{\Gamma_p} * {:}\varphi\overline{\varphi}{:} = L^{-2p}{:}\varphi\overline{\varphi}{:}$. In addition, it is shifted as indicated above by convolution acting on t and Laplacian contractions to g. The constant $d'^{(4)}$ scales down by $O(L^{-4p})$ because $t^{(\alpha)}(0) = 0$ for $|\alpha| < 4$. It also receives "new" contributions of order $\lambda d^{(2)}$. □

PROOF OF THEOREM 8.1. We give the proof for the case where $p = 1$. The general case requires more notation but no new ideas.

Consider

$$I_N(x,y) \equiv \int d\mu_G(\Phi) g^{\mathscr{G}_N}\overline{\varphi}_x\varphi_y.$$

We claim that the limit as $N \to \infty$ exists pointwise in x, y when $\mu^2 \ge \mu_c^2$.

By hypothesis we can apply Theorem 7.2 and conclude that there is a global trajectory $g \to Tg \to T^2 g \to \cdots$ which starts with $\mu_0^2 = \mu_c^2$ and $\lambda_0 \to \lambda_1 \to \cdots$ with $\lambda_n^{-1} = \lambda^{-1} + \beta_2 n + O(\log n)$. Given x, y, let k be the smallest integer such that $|(x-y)/L^k| \le L$. By applying Proposition 8.2 and Theorem 4.2 k times, we find that

$$I(x,y) \equiv \lim_{N\to\infty} I_N(x,y) = \lim_{N\to\infty} \int d\mu_G(\Phi) g_k^{\mathscr{G}_{N-k}} h_0,$$

where h_0 is as in (8.14) and $g_k \equiv T^k g$. Here h_0 is independent of N for N large enough so that $\mathscr{G}_N$ contains x, y. We apply Proposition 8.3 $N - k$ times and evaluate the resulting h at $\Phi = 0$ to get $I(x, y)$. This picks out the final $d^{(0)}$ which we now asymptotically evaluate using Propositions 8.2 and 8.3.

From (8.14) the initial $d^{(0)}$ for Proposition 8.3 is

$$G(x-y)(1+O(\lambda_0))(1+O(\lambda_k)). \tag{8.15}$$

A similar factor gives the k-dependent part of the first $L^{2kp}d^{(0)}$. The subsequent flow of $d^{(0)}, d^{(2)}$ is given by Proposition 8.3. After n steps, $d^{(2)}$ is down by $L^{-2np}\prod_{j=1}^{\infty}(1 + O(\lambda_{j+k}))$. Therefore $d^{(2)}$ (and also $d^{(4)}$) are decreasing exponentially in n so that $d^{(0)}$ stabilizes quickly close to its starting value and the k-dependence is still of the form (8.15). Since $k = O(\log(x - y))$, we find that in the limit as $N \to \infty$,

$$I(x,y) = G(x-y)[1+O(\lambda)][1+\varepsilon(x-y)],$$

with ε as in the statement of Theorem 8.1.

By Theorem 3.3 and the remarks below it and the definition of local time τ,

$$U^a(x,y) = \lim_{N\to\infty} \int_0^{\infty} dT\, E_x\left(e^{-\lambda\tau^2(\mathscr{G}_N) - a\tau(\mathscr{G}_N)} \mathbf{1}_{\{\omega(T)=y\}}\right)$$

$$= \lim_{N\to\infty} \int d\mu_G(\Phi) g^{\mathscr{G}_N} \bar{\varphi}_x \varphi_y = I(x,y).$$

Therefore the $a = a_c$ conclusion of Theorem 8.1 is proven. We obtain the existence of $U^a(x, y)$ for $a \geq a_c$ by the Lebesgue dominated convergence theorem applied to the random walk expression for U^a. □

APPENDIX

PROOF OF (6.2).

$$|g_1 g_2|_{w_1 w_2, h} = \sum_{\alpha} \frac{h^{\alpha}}{\alpha!} |(g_1 g_2)^{(\alpha)}|_{w_1 w_2}$$

$$= \sum_{\alpha} \frac{h^{\alpha}}{\alpha!} \sum_{\beta,\gamma:\, \beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} |g_1^{(\beta)} g_2^{(\gamma)}|_{w_1 w_2}$$

$$\leq \sum_{\beta,\gamma} \frac{h^{\beta+\gamma}}{\beta!\gamma!} |g_1^{(\beta)}|_{w_1} |g_2^{(\gamma)}|_{w_2}$$

$$= |g_1|_{w_1, h} |g_2|_{w_2, h}.$$

□

PROOF OF (6.3).

$$
\begin{aligned}
|g^{(\alpha)}|_{w,h} &= \sum_\beta \frac{h^\beta}{\beta!}|g^{(\alpha+\beta)}|_w \\
&= \frac{\alpha!}{(h'-h)^\alpha}\sum_\beta \frac{(\alpha+\beta)!}{\alpha!\beta!}h^\beta(h'-h)^\alpha\frac{1}{(\alpha+\beta)!}|g^{(\alpha+\beta)}|_w \\
&\le \frac{\alpha!}{(h'-h)^\alpha}\sum_\beta \frac{h'^{\alpha+\beta}}{(\alpha+\beta)!}|g^{(\alpha+\beta)}|_w \\
&\le \frac{\alpha!}{(h'-h)^\alpha}|g|_{w,h'}. \qquad \square
\end{aligned}
$$

PROOF OF (6.4).

$$
\begin{aligned}
|(\mu_C * g)^{(\alpha)}| &= |\mu_C * g^{(\alpha)}| \\
&= |\mu_C * (ww^{-1}g^{(\alpha)})| \\
&\le |\mu_C * w||g^{(\alpha)}|_w \\
&\le w'|g^{(\alpha)}|_w.
\end{aligned}
$$

Therefore

$$|\mu_C * g^{(\alpha)}|_{w'} \le |g^{(\alpha)}|_w,$$

which implies $|\mu_C * g|_{w',h} \le |g|_{w,h}$. □

PROOF OF (6.12). Let $g = \sum_\beta g^\beta(\varphi)\psi^\beta$. Then if μ_C is a Fermionic ⊗ Bosonic Gaussian measure, we have

$$\mu_C * g = \sum_\beta (\mu_C * g^\beta)(\varphi)\mu_C * \psi^\beta,$$

and by (6.2), (6.4) and (6.11),

$$
\begin{aligned}
|\mu_C * g|_{w,h} &\le \sum_\beta |\mu_C * g^\beta|_{w,h}|\mu_C * \psi^\beta|_{1,h} \\
&\le \sum_\beta |g^\beta|_{w',h}\exp\left(h^{-2}\sum|C_{ij}|\right)|\psi^\beta|_{1,h} \\
&= \exp\left(h^{-2}\sum|C_{ij}|\right)\sum_\beta |g^\beta|_{w',h}h^\beta \\
&= \exp\left(h^{-2}\sum|C_{ij}|\right)|g|_{w',h}. \qquad \square
\end{aligned}
$$

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