Self-duality of Coble's quartic hypersurface and applications

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Abstract

The moduli space $\mathcal{M}_0$ of semi-stable rank 2 vector bundles with fixed trivial determinant over a non-hyperelliptic curve C of genus 3 is isomorphic to a quartic hypersurface in $\mathbb{P}^7$ (Coble's quartic). We show that $\mathcal{M}_0$ is self-dual and that its polar map associates to a stable bundle $E \in \mathcal{M}_0$ a bundle F which is characterized by $\dim H^0(C, E \otimes F) = 4$. The projective space $\mathbb{P}H^0(C, E \otimes F)$ is equipped with a net of quadrics Π and it is shown that the map which associates to $E \in \mathcal{M}_0$ the isomorphism class of the plane quartic Hessian curve of Π is a dominant map to the moduli space of genus 3 curves.

1 Introduction

In his book [C] A.B. Coble constructs for any non-hyperelliptic curve C of genus 3 a quartic hypersurface in $\mathbb{P}^7$ which is singular along the Kummer variety $\mathcal{K}_0 \subset \mathbb{P}^7$ of C. It is shown in [NR] that this hypersurface is isomorphic to the moduli space $\mathcal{M}_0$ of semi-stable rank 2 vector bundles with fixed trivial determinant. For many reasons Coble's quartic hypersurface may be viewed as a genus-3-analogue of a Kummer surface,i.e. a quartic surface $S \subset \mathbb{P}^3$ with 16 nodes. For example the restriction of $\mathcal{M}_0$ to an eigenspace $\mathbb{P}^3_\alpha \subset \mathbb{P}^7$ for the action of a 2-torsion point $\alpha \in JC[2]$ is isomorphic to a Kummer surface (of the corresponding Prym variety). It is classically known (see e.g. [GH]) that a Kummer surface $S \subset \mathbb{P}^3$ is self-dual.

In this paper we show that this property also holds for the Coble quartic $\mathcal{M}_0$ (Theorem 3.1). The rational polar map $\mathcal{D} : \mathbb{P}^7 \longrightarrow (\mathbb{P}^7)^*$ maps $\mathcal{M}_0$ birationally to $\mathcal{M}_\omega \subset (\mathbb{P}^7)^*$, where $\mathcal{M}_\omega$ ($\cong \mathcal{M}_0$) is the moduli space parametrizing vector bundles with fixed canonical determinant. More precisely we show that the embedded tangent space at a stable bundle E to $\mathcal{M}_0$ corresponds to a semi-stable bundle $\mathcal{D}(E) = F \in \mathcal{M}_\omega$, which is characterized by the condition $\dim H^0(C, E \otimes F) = 4$ (its maximum). We also show that $\mathcal{D}$ resolves to a morphism $\widetilde{\mathcal{D}}$ by two successive blowing-ups, and that $\mathcal{D}$ contracts the trisecant scroll of $\mathcal{K}_0$ to the Kummer variety $\mathcal{K}_\omega \subset \mathcal{M}_\omega$.

The condition which relates E to its "tangent space bundle" F, namely $\dim H^0(C, E \otimes F) = 4$, leads to many geometric properties. First we observe that $\mathbb{P}H^0(C, E \otimes F)$ is naturally equipped with a net of quadrics Π whose base points (Cayley octad) correspond bijectively to the 8 line subbundles of maximal degree of E (and of F). The Hessian curve $\mathrm{Hess}(E)$ of the net of quadrics $\Pi \cong |\omega|^*$ is a plane quartic curve, which is everywhere tangent (Proposition 4.7) to the canonical curve $C \subset |\omega|^*$,i.e. $\mathrm{Hess}(E) \cap C = 2\Delta(E)$ for some divisor $\Delta(E) \in |\omega^2|$. Since these constructions are $JC[2]$-invariant, we introduce the quotient $\mathcal{N} = \mathcal{M}_0 / JC[2]$ parametrizing $\mathbb{P}\mathrm{SL}_2$-bundles over C and we show (Proposition 4.13) that the map $\mathcal{N} \xrightarrow{\Delta} |\omega^2|$, $E \longmapsto \Delta(E)$ is the restriction of the projection from the projective space $\mathcal{N} \subset |\overline{\mathcal{L}}|^* = \mathbb{P}^{13}$ ($\overline{\mathcal{L}}$ is the ample generator of $\mathrm{Pic}(\mathcal{N})$) with center of projection given by the linear span of the Kummer variety $\mathcal{K}_0 \subset \mathcal{N}$ ($\mathcal{K}_0$ parametrizes decomposable $\mathbb{P}\mathrm{SL}_2$-bundles).

We also show (Corollary 4.16) that the Hessian map $\mathcal{N} \to \mathcal{R}$, $E \mapsto \mathrm{Hess}(E)$ is finite of degree 72, where $\mathcal{R}$ is the rational space parametrizing plane quartics everywhere tangent to $C \subset |\omega|^* = \mathbb{P}^2$. Considering the isomorphism class of $\mathrm{Hess}(E)$, we deduce that the map $\mathrm{Hess} : \mathcal{N} \to \mathcal{M}_3$ is dominant, where $\mathcal{M}_3$ is the moduli space of smooth genus 3 curves. We actually prove that some Galois-covers $\widetilde{\mathcal{N}} \to \mathcal{N}$ and $\mathcal{P}_C \to \mathcal{R}$ are birational (Proposition 4.15). In particular we endow the space $\widetilde{\mathcal{N}}$, parametrizing $\mathbb{P}\mathrm{SL}_2$-bundles E with an ordered set of 8 line subbundles of E of maximal degree, with an action of the Weyl group $W(E_7)$ such that the action of the central element $w_0 \in W(E_7)$ coincides with the polar map $\mathcal{D}$.

We hope that these results will be useful for dealing with several open problems, e.g. rationality of the moduli spaces $\mathcal{M}_0$ and $\mathcal{N}$.

I would like to thank S. Ramanan for some inspiring discussions on Coble's quartic.

2 The geometry of Coble's quartic

In this section we briefly recall some known results related to Coble's quartic hypersurface, which can be found in the literature, e.g. [DO], [L2], [NR], [OPP]. We refer to [B1], [B2] for the results on the geometry of the moduli of rank 2 vector bundles.

2.1 Coble's quartic as moduli of vector bundles

Let C be a smooth non-hyperelliptic curve of genus 3 with canonical line bundle ω. Let $\mathrm{Pic}^d(C)$ be the Picard variety parametrizing degree d line bundles over C and $JC := \mathrm{Pic}^0(C)$ be the Jacobian variety. We denote by $\mathcal{K}_0$ the Kummer variety of JC and by $\mathcal{K}_\omega$ the quotient of $\mathrm{Pic}^2(C)$ by the involution $\xi \mapsto \omega\xi^{-1}$. Let $\Theta \subset \mathrm{Pic}^2(C)$ be the Riemann Theta divisor and let $\Theta_0 \subset JC$ be a symmetric Theta divisor, i.e. a translate of Θ by a theta-characteristic. We also recall that the two linear systems $|2\Theta|$ and $|2\Theta_0|$ are canonically dual to each other via Wirtinger duality ([Mu2] p. 335),i.e. we have an isomorphism $|2\Theta|^* \cong |2\Theta_0|$.

Let $\mathcal{M}_0$ (resp. $\mathcal{M}_\omega$) denote the moduli space of semi-stable rank 2 vector bundles over C with fixed trivial (resp. canonical) determinant. The singular locus of $\mathcal{M}_0$ is isomorphic to $\mathcal{K}_0$ and points in $\mathcal{K}_0$ correspond to bundles E whose S-equivalence class $[E]$ contains a decomposable bundle of the form $M \oplus M^{-1}$ for $M \in JC$. We have natural morphisms

$$\mathcal{M}_0 \xrightarrow{D} |2\Theta| = \mathbb{P}^7, \qquad \mathcal{M}_\omega \xrightarrow{D} |2\Theta_0| = |2\Theta|^*,$$

which send a stable bundle $E \in \mathcal{M}_0$ to the divisor $D(E)$ whose support equals the set $\{L \in \mathrm{Pic}^2(C) \mid \dim H^0(C, E \otimes L) > 0\}$ (if $E \in \mathcal{M}_\omega$, replace $\mathrm{Pic}^2(C)$ by JC). On the semi-stable boundary $\mathcal{K}_0$ (resp. $\mathcal{K}_\omega$) the morphism D restricts to the Kummer map. The moduli spaces $\mathcal{M}_0$ and $\mathcal{M}_\omega$ are isomorphic, although non-canonically (consider tensor product with a theta-characteristic). It is known that the Picard group $\mathrm{Pic}(\mathcal{M}_0)$ is $\mathbb{Z}$ and that $|\mathcal{L}|^* = |2\Theta|$, where $\mathcal{L}$ is the ample generator of $\mathrm{Pic}(\mathcal{M}_0)$.

The main theorem of [NR] asserts that D embeds $\mathcal{M}_0$ as a quartic hypersurface in $|2\Theta| = \mathbb{P}^7$, which was originally described by A.B. Coble [C] (section 33(6)). Coble's quartic is characterized by a uniqueness property: it is the unique (Heisenberg-invariant) quartic which is singular along the Kummer variety $\mathcal{K}_0$ (see [L2] Proposition 5).

We recall that Coble's quartic hypersurfaces $\mathcal{M}_0 \subset |2\Theta|$ and $\mathcal{M}_\omega \subset |2\Theta_0|$ contain some distinguished points. First ([C] section 48(4), [L1], [OPP]) there exists a unique stable bundle $A_0 \in \mathcal{M}_\omega$ such that $\dim H^0(C, A_0) = 3$ (its maximal dimension). We define for any theta-characteristic κ and for any 2-torsion point $\alpha \in JC[2]$ the stable bundles, called *exceptional* bundles

$$A_\kappa := A_0 \otimes \kappa^{-1} \in \mathcal{M}_0 \qquad \text{and} \qquad A_\alpha := A_0 \otimes \alpha \in \mathcal{M}_\omega. \tag{2.1}$$

2.2 Global and local equations of Coble's quartic

Let F_4 be the Coble quartic,i.e. the equation of $\mathcal{M}_0 \subset |2\Theta| = \mathbb{P}^7$. Then the eight partials $C_i = \frac{\partial F_4}{\partial X_i}$ for $1 \leq i \leq 8$ (X_i are coordinates for $|2\Theta|$) define the Kummer variety $\mathcal{K}_0$ scheme-theoretically ([L2] Theorem IV.6). We also need the following results ([L2] Theorem 6 bis).

(i) The étale local equation (in affine space $\mathbb{A}^7$) of Coble's quartic at the point $[\mathcal{O} \oplus \mathcal{O}]$ is $T^2 = \det [T_{ij}]$, with coordinates T, T_{ij} with $T_{ij} = T_{ji}$ and $1 \leq i, j \leq 3$.

(ii) The étale local equation at the point $[M \oplus M^{-1}]$ with $M^2 \neq \mathcal{O}$ is a rank 4 quadric $\det[T_{ij}] = 0$, where T_{ij}, $1 \leq i, j \leq 2$ are four coordinates on $\mathbb{A}^7$.

Hence any point $[M \oplus M^{-1}] \in \mathcal{K}_0$ has multiplicity 2 on $\mathcal{M}_0$.

2.3 Extension spaces

Given $L \in \mathrm{Pic}^1(C)$ we introduce the 3-dimensional space $\mathbb{P}_0(L) := |\omega L^2|^* = \mathbb{P}\mathrm{Ext}^1(L, L^{-1})$. A point $e \in \mathbb{P}_0(L)$ corresponds to an isomorphism class of extensions

$$0 \longrightarrow L^{-1} \longrightarrow E \longrightarrow L \longrightarrow 0 \qquad (e) \tag{2.2}$$

and the composite of the classifying map $\mathbb{P}_0(L) \to \mathcal{M}_0$ followed by the embedding $D : \mathcal{M}_0 \to |2\Theta|$ is linear and injective ([B2] Lemme 3.6). It is shown that a point $e \in \mathbb{P}_0(L)$ represents a stable bundle precisely away from $\varphi(C)$, where φ is the map induced by the linear system $|\omega L^2|$. A point $e = \varphi(p)$ for $p \in C$ is represented by the decomposable bundle $L(-p) \oplus L^{-1}(p)$.

We also introduce the projective spaces $\mathbb{P}_\omega(L) := |\omega^2 L^{-2}|^* = \mathbb{P}\mathrm{Ext}^1(\omega L^{-1}, L)$. A point $f \in \mathbb{P}_\omega(L)$ corresponds to an extension

$$0 \longrightarrow L \longrightarrow F \longrightarrow \omega L^{-1} \longrightarrow 0 \qquad (f) \tag{2.3}$$

Similarly, we have an injective classifying map $\mathbb{P}_\omega(L) \to \mathcal{M}_\omega$. Although we will not use this fact, we observe that $\mathbb{P}_0(L) = \mathbb{P}_\omega(\kappa L^{-1})$ for any theta-characteristic κ.

It is well-known that the Kummer variety $\mathcal{K}_0 \subset |2\Theta|$ admits a 4-dimensional family of trisecant lines (e.g. [Mu3]). It follows from [OPP] Theorem 1.4 and Theorem 2.1 that any trisecant line to $\mathcal{K}_0$ is contained in some space $\mathbb{P}_0(L)$, where it is a trisecant to the curve $\varphi(C) \subset \mathbb{P}_0(L)$. We denote by $\mathcal{T}_0$ the trisecant scroll, which is a divisor in $\mathcal{M}_0$. Similarly we define $\mathcal{T}_\omega \subset \mathcal{M}_\omega$.

The main tool for the proof of the self-duality is that $\mathcal{M}_0$ (resp. $\mathcal{M}_\omega$) can be covered by the projective spaces $\mathbb{P}_0(L)$ (resp. $\mathbb{P}_\omega(L)$). This is expressed by the following result [NR](see also [OP2]): there exist rank 4 vector bundles $\mathcal{U}_0$ and $\mathcal{U}_\omega$ over $\mathrm{Pic}^1(C)$ such that $\forall L \in \mathrm{Pic}^1(C)$, $(\mathbb{P}\mathcal{U}_0)_L \cong \mathbb{P}_0(L)$, $(\mathbb{P}\mathcal{U}_\omega)_L \cong \mathbb{P}_\omega(L)$ and their associated classifying morphisms ψ_0 and ψ_ω,

$$\begin{array}{ccc} \mathbb{P}\mathcal{U}_0 & \xrightarrow{\psi_0} & \mathcal{M}_0 \subset |2\Theta| \\ \downarrow & & \\ \mathrm{Pic}^1(C) & & \end{array} \qquad \begin{array}{ccc} \mathbb{P}\mathcal{U}_\omega & \xrightarrow{\psi_\omega} & \mathcal{M}_\omega \subset |2\Theta_0| \\ \downarrow & & \\ \mathrm{Pic}^1(C) & & \end{array}$$

are surjective (Nagata's theorem) and of degree 8 (see section 4.1).

2.4 Tangent spaces to Theta-divisors

Following [B2] section 2, we associate to any $[F] \in \mathcal{M}_\omega \subset |2\Theta_0|$ the divisor $\Delta(F) \subset \mathcal{M}_0 \subset |2\Theta|$ which has the properties

1. $\mathrm{supp}\, \Delta(F) = \{[E] \in \mathcal{M}_0 \mid \dim H^0(C, E \otimes F) > 0\}$,
2. $\Delta(F) \in |\mathcal{L}| \cong |2\Theta|^*$ is mapped to $[F]$ under the canonical duality $|2\Theta|^* \cong |2\Theta_0|$.

Symmetrically, we associate to any $E \in \mathcal{M}_0$ the divisor $\Delta(E) \subset \mathcal{M}_\omega$ with the analogous properties.

For any E, F with $[E] \in \mathcal{M}_0$ and $[F] \in \mathcal{M}_\omega$, the rank 4 vector bundle $E \otimes F = \mathcal{H}\mathrm{om}(E, F)$ is equipped with a ω-valued non-degenerate quadratic form (given by the determinant of local sections), hence, by Mumford's parity theorem [Mu1], the parity of $\dim H^0(C, E \otimes F)$ is constant under degeneration. Considering e.g. a degeneration of either E or F to a decomposable bundle, we obtain that $\dim H^0(C, E \otimes F)$ is even. The divisor $\Delta(F)$ is defined as the Pfaffian divisor associated to a family $\mathcal{E} \otimes F$ of orthogonal bundles [LS] and satisfies the equality

$$2\Delta(F) = \mathrm{detdiv}(\mathcal{E} \otimes F),$$

where $\mathrm{detdiv}(\mathcal{E} \otimes F)$ is the determinant divisor of the family $\mathcal{E} \otimes F$. Thus for any stable bundle $E \in \mathcal{M}_0$, we have

$$\mathrm{mult}_{[E]} \Delta(F) = \frac{1}{2} \mathrm{mult}_{[E]} \mathrm{detdiv}(\mathcal{E} \otimes F) \geq \frac{1}{2} \dim H^0(C, E \otimes F).$$

The last inequality is Corollaire II.3 [L1].

2.1 Lemma. *Suppose that E is stable and that* $\dim H^0(C, E \otimes F) \geq 4$. *Then $\Delta(F) \subset \mathcal{M}_0$ is singular at E and the embedded tangent space $\mathbb{T}_E \mathcal{M}_0 \in |2\Theta|^* \cong |2\Theta_0|$ corresponds to the point $[F] \in |2\Theta_0|$.*

Proof. The first assertion is an immediate consequence of the previous inequality. To show the second, it is enough to observe that, since E is a singular point of the divisor $\Delta(F)$, we have equality between the Zariski tangent spaces $T_E \Delta(F) = T_E \mathcal{M}_0$ and $T_E \Delta(F)$ coincides with the hyperplane cutting out the divisor $\Delta(F)$, which corresponds to the point $[F]$ by property (2). □

We will also need the dual version.

2.2 Lemma. *Suppose that F is stable and that* $\dim H^0(C, E \otimes F) \geq 4$. *Then $\Delta(E) \subset \mathcal{M}_\omega$ is singular at F and the embedded tangent space $\mathbb{T}_F \mathcal{M}_\omega \in |2\Theta_0|^* \cong |2\Theta|$ corresponds to the point $[E] \in |2\Theta|$.*

3 Self-duality

3.1 Statement of the main theorem

Let $\mathcal{D}$ be the rational map defined by the polars of Coble's quartic F_4,i.e. the eight cubics C_i,

$$\begin{array}{ccc} \mathcal{D} : |2\Theta| & \longrightarrow & |2\Theta|^* \cong |2\Theta_0| \\ \cup & & \cup \\ \mathcal{M}_0 & & \mathcal{M}_\omega \end{array}$$

Note that $\mathcal{D}$ is defined away from $\mathcal{K}_0$. Geometrically, $\mathcal{D}$ maps a stable bundle $E \in \mathcal{M}_0$ to the hyperplane defined by the embedded tangent space $\mathbb{T}_E\mathcal{M}_0$ at the smooth point E. The main theorem of this paper is the following

3.1 Theorem (Self-duality). *The moduli space $\mathcal{M}_0$ is birationally mapped by $\mathcal{D}$ to $\mathcal{M}_\omega$,i.e. $\mathcal{M}_\omega$ is the dual hypersurface of $\mathcal{M}_0$. More precisely, we have*

1. *$\mathcal{D}$ restricts to an isomorphism $\mathcal{M}_0 \setminus \mathcal{T}_0 \xrightarrow{\sim} \mathcal{M}_\omega \setminus \mathcal{T}_\omega$.*
2. *$\mathcal{D}$ contracts the divisor $\mathcal{T}_0$ ($\in |\mathcal{L}^8|$) to $\mathcal{K}_\omega$.*
3. *For any stable $E \in \mathcal{M}_0$, the moduli point $\mathcal{D}(E) \in \mathcal{M}_\omega$ can be represented by a semi-stable bundle F, which satisfies $\dim H^0(C, E \otimes F) \geq 4$. Moreover, if $E \in \mathcal{M}_0 \setminus \mathcal{T}_0$ then there exists a unique stable bundle $F = \mathcal{D}(E)$ for which $\dim H^0(C, E \otimes F)$ has its maximal value 4.*
4. *$\mathcal{D}$ resolves to a morphism $\widetilde{\mathcal{D}}$ from a blowing-up $\widetilde{\mathcal{M}_0}$,*

$$\begin{array}{ccccc} \mathcal{E} & \subset & \widetilde{\mathcal{M}_0} & & \\ \downarrow & & \downarrow & & \\ \widetilde{\mathcal{K}_0} & \subset & \mathcal{B}l_s(\mathcal{M}_0) & \searrow^{\widetilde{\mathcal{D}}} & \\ \downarrow & & \downarrow & & \\ \mathcal{K}_0 & \subset & \mathcal{M}_0 & \xrightarrow{\mathcal{D}} & \mathcal{M}_\omega \end{array}$$

where $\widetilde{\mathcal{M}_0}$ is obtained by two successive blowing-ups: first we blow-up the singular points of $\mathcal{K}_0$ and secondly we blow-up $\mathcal{B}l_s(\mathcal{M}_0)$ along the smooth proper transform $\widetilde{\mathcal{K}_0}$ of $\mathcal{K}_0$. The exceptional divisor $\mathcal{E}$ is mapped by $\widetilde{\mathcal{D}}$ onto the divisor $\mathcal{T}_\omega$.

3.2 Restriction of $\mathcal{D}$ to the extension spaces

The strategy of the proof is to restrict $\mathcal{D}$ to the extension spaces $\mathbb{P}_0(L)$. We start by defining a map

$$\mathcal{D}_L : \mathbb{P}_0(L) \longrightarrow \mathcal{M}_\omega$$

as follows: consider a point $e \in \mathbb{P}_0(L)$ (2.2) and denote by $W_e \subset H^0(C, \omega L^2)$ the corresponding 3-dimensional linear subspace of divisors. If we suppose that $e \notin \varphi(C)$, then the evaluation map $\mathcal{O}_C \otimes W_e \xrightarrow{ev} \omega L^2$ is surjective and we define $F_e = \mathcal{D}_L(e)$ to be the rank 2 vector bundle such that $\ker(ev) \cong (F_e L)^*$,i.e. we have an exact sequence

$$0 \longrightarrow (F_e L)^* \longrightarrow \mathcal{O}_C \otimes W_e \xrightarrow{ev} \omega L^2 \longrightarrow 0. \tag{3.1}$$

If there is no ambiguity, we will drop the subscript e.

3.2 Lemma. *Suppose that $e \notin \varphi(C)$. Then*

1. *The bundle F_e has canonical determinant, is semi-stable and F_eL is generated by global sections.*

2. *There exists a nonzero map $L \to F_e$, hence $[F_e]$ defines a point in $\mathbb{P}_\omega(L)$.*

3. *We have* $\dim H^0(C, E \otimes F_e) \geq 4$, *where E is the stable bundle associated to e* (2.2).

Proof. (1) The first assertion is immediately deduced from the exact sequence (3.1). We take the dual of (3.1)

$$0 \longrightarrow \omega^{-1}L^{-2} \longrightarrow \mathcal{O}_C \otimes W^* \longrightarrow FL \longrightarrow 0. \tag{3.2}$$

Taking global sections leads to the inclusion $W^* \subset H^0(FL)$, which proves the last assertion. Let us check semi-stability: suppose that there exists a line subbundle M which destabilizes FL (assume M saturated),i.e. $0 \to M \to FL \to \omega L^2 M^{-1} \to 0$. Then $\deg M \geq 4$, which implies that $\deg \omega L^2 M^{-1} \leq 2$. Hence $\dim H^0(\omega L^2 M^{-1}) \leq 1$, so the subspace $H^0(M) \subset H^0(FL)$ has codimension ≤ 1, which contradicts that FL is globally generated.

(2) Since $\det F = \omega$, we have $(FL)^* = FL^{-1}\omega^{-1}$. Taking global sections of the exact sequence (3.1) tensored with ω leads to

$$0 \longrightarrow H^0(FL^{-1}) \longrightarrow H^0(\omega) \otimes W \longrightarrow H^0(\omega^2 L^2) \longrightarrow \cdots$$

Now we observe that $\dim H^0(\omega) \otimes W = 9$ and $\dim H^0(\omega^2 L^2) = 8$ (Riemann-Roch), which implies that $\dim H^0(FL^{-1}) \geq 1$.

(3) We tensor the exact sequence (2.2) defined by e with F and take global sections

$$0 \longrightarrow H^0(FL^{-1}) \longrightarrow H^0(E \otimes F) \longrightarrow H^0(FL) \xrightarrow{\cup e} H^1(FL^{-1}) \longrightarrow \cdots$$

The coboundary map is the cup-product with the extension class $e \in H^1(L^{-2})$ and, since $\det F = \omega$, the coboundary map $\cup e$ is skew-symmetric (by Serre-duality $H^1(FL^{-1}) = H^0(FL)^*$). Hence the linear map $\epsilon \mapsto \cup\epsilon$ factorizes as follows

$$H^0(\omega L^2)^* \longrightarrow \Lambda^2 H^0(FL)^* \subset \mathcal{H}\mathrm{om}(H^0(FL), H^1(FL)), \tag{3.3}$$

and its dual map $\Lambda^2 H^0(FL) \xrightarrow{\mu} H^0(\omega L^2)$ coincides with exterior product of global sections (see e.g. [L1]). On the other hand, it is easy to check that the image under μ of the subspace $\Lambda^2 W^* \subset \Lambda^2 H^0(FL)$ equals $W \subset H^0(\omega L^2)$ and μ restricts to the canonical isomorphism $\Lambda^2 W^* = W$. Therefore the linear map $\cup e$ is zero on $W^* \subset H^0(FL)$, from which we deduce that $\dim H^0(E \otimes F) = \dim H^0(FL^{-1}) + \dim \ker (\cup e) \geq 4$. □

It follows that the map $\mathcal{D}_L$ factorizes

$$\mathcal{D}_L : \mathbb{P}_0(L) \longrightarrow \mathbb{P}_\omega(L) \subset \mathcal{M}_\omega. \tag{3.4}$$

Moreover, by Lemma 3.2(3) and Lemma 2.1 the point $\mathcal{D}_L(e)$ corresponds to the embedded tangent space at $e \in \mathbb{P}_0(L)$, hence $\mathcal{D}_L$ is the restriction of $\mathcal{D}$ to $\mathbb{P}_0(L)$. In particular, $\mathcal{D}_L$ is given by a linear system of cubics through $\varphi(C)$.

We recall from section 2.3 that the restriction of the trisecant scroll $\mathcal{T}_0$ to $\mathbb{P}_0(L)$ is the surface, denoted by $\mathcal{T}_0(L)$, ruled out by the trisecants to $\varphi(C) \subset \mathbb{P}_0(L)$.

3.3 Lemma. *Given a point $e \in \mathbb{P}_0(L)$ such that $e \notin \varphi(C)$. The bundle F_e is stable if and only if $e \notin \mathcal{T}_0$. Moreover*

- *if* $\dim H^0(L^2) = 0$, *then the trisecant $\overline{pqr}$ to $\varphi(C)$ is contracted to the semi-stable point $[L(u) \oplus \omega L^{-1}(-u)] = \varphi(u) \in \mathbb{P}_\omega(L)$ for some point $u \in C$, which satisfies $p+q+r \in |L^2(u)|$.*

- *if* $\dim H^0(L^2) > 0$, *then* $\omega L^{-2} = \mathcal{O}_C(u+v)$ *for some points* $u, v \in C$ *and any trisecant* $\overline{pqr}$ *is contracted to the semi-stable point* $[L(u) \oplus L(v)]$.

Proof. The bundle F fits into an exact sequence $0 \to L \to F \to \omega L^{-1} \to 0$. Suppose that F has a line subbundle M of degree 2 and consider the composite map $\alpha : M \to F \to \omega L^{-1}$.

First we consider the case $\alpha = 0$: then $M = L(u) \hookrightarrow F$ for some $u \in C$, or equivalently $\dim H^0(FL^{-1}(-u)) > 0$. We tensor (3.1) with $\omega(-u)$ and take global sections

$$0 \longrightarrow H^0(FL^{-1}(-u)) \longrightarrow H^0(\omega(-u)) \otimes W \xrightarrow{m} H^0(\omega^2 L^2(-u)) \longrightarrow \cdots$$

The second map m is the multiplication map of global sections. Since $W \subset H^0(\omega L^2)$, let us consider for a moment the extended multiplication map $\tilde{m} : H^0(\omega(-u)) \otimes H^0(\omega L^2) \longrightarrow H^0(\omega^2 L^2(-u))$. By the base-point-free-pencil-trick applied to the pencil $|\omega(-u)|$, we have $\ker \tilde{m} = H^0(L^2(u))$ and a tensor in $\ker \tilde{m}$ is of the form $s \otimes t\alpha - t \otimes s\alpha$, with $\{s,t\}$ a basis of $H^0(\omega(-u))$ and $\alpha \in H^0(L^2(u))$. We denote by $p+q+r$ the zero divisor of α. Then we see that $\ker m \neq \{0\}$ if and only if W contains the linear space spanned by $t\alpha$ and $s\alpha$. Dually, this means that $e \in \overline{pqr}$, the trisecant through the points p, q, r. Conversely, any $e \in \overline{pqr}$ is mapped by $\mathcal{D}_L$ to $[L(u) \oplus \omega L^{-1}(-u)]$.

Secondly we consider the case $\alpha \neq 0$: then $M = \omega L^{-1}(-u) \hookrightarrow F$ for some $u \in C$, or equivalently $\dim H^0(F\omega^{-1}L(u)) > 0$. As in the first case we take global sections of (3.1) tensored with $L^2(u)$ and we obtain that $H^0(F\omega^{-1}L(u))$ is the kernel of the multiplication map $H^0(L^2(u)) \otimes W \xrightarrow{m} H^0(\omega L^4(u))$. Then $\ker \tilde{m} \neq \{0\}$ implies that $\dim H^0(L^2(u)) = 2$. Hence $L^2(u) = \omega(-v)$ for some point $v \in C$,i.e. $\omega L^{-2} = \mathcal{O}_C(u+v)$, which implies that $\dim H^0(\omega L^{-2}) = \dim H^0(L^2) > 0$. Furthermore the multiplication map becomes $H^0(\omega(-v)) \otimes W \xrightarrow{m} H^0(\omega^2 L^2(-v))$. We can now conclude exactly as in the first case, with the additional observation that any trisecant $\overline{pqr}$ is contracted to the point $[L(v) \oplus \omega L^{-1}(-v)] = [L(v) \oplus L(u)]$. □

Now we are going to construct along the same lines an inverse map to $\mathcal{D}_L$ (3.4)

$$\mathcal{D}'_L : \mathbb{P}_\omega(L) \longrightarrow \mathbb{P}_0(L).$$

Given an extension class $f \in \mathbb{P}_\omega(L)$ such that $f \notin \varphi(C)$, we denote by $W_f \subset H^0(C, \omega^2 L^{-2})$ the corresponding 3-dimensional linear space of divisors and we define $E_f = \mathcal{D}'_L(f)$ to be the rank 2 vector bundle which fits in the exact sequence

$$0 \longrightarrow E_f \omega^{-1} L \longrightarrow W_f \otimes \mathcal{O}_C \xrightarrow{ev} \omega^2 L^{-2} \longrightarrow 0.$$

Exactly as in Lemma 3.2 we show that E_f has the following properties.

3.4 Lemma. *Suppose that* $f \notin \varphi(C)$. *Then*

1. *The bundle* E_f *has trivial determinant, is semi-stable and* $E_f \omega L^{-1}$ *is generated by global sections.*
2. *There exists a nonzero map* $L^{-1} \to E_f$, *hence* $[E_f]$ *defines a point in* $\mathbb{P}_0(L)$.
3. *We have* $\dim H^0(C, E_f \otimes F) \geq 4$, *where* F *is the stable bundle associated to* f *(2.3).*

Similarly the analogue of Lemma 3.3 holds for the bundle E_f.

3.5 Lemma. *The map* $\mathcal{D}'_L$ *is the birational inverse of* $\mathcal{D}_L$,*i.e.*

$$\mathcal{D}'_L \circ \mathcal{D}_L = Id_{\mathbb{P}_0(L)} \qquad \textit{and} \qquad \mathcal{D}_L \circ \mathcal{D}'_L = Id_{\mathbb{P}_\omega(L)}.$$

Proof. Start with $e \in \mathbb{P}_0(L)$ with $e \notin \mathcal{T}_0(L)$. Then (Lemma 3.3) $\mathcal{D}_L(e) = F_e$ is stable and (Lemma 3.2(3)) $\dim H^0(C, E \otimes F_e) \geq 4$. Now the stable bundle F_e determines an extension class $f \in \mathbb{P}_\omega(L)$ with $f \notin \varphi(C)$. Let us denote $E_f = \mathcal{D}'_L(f)$. We know (Lemma 3.4(3)) that $\dim H^0(C, E_f \otimes F_e) \geq 4$ and since F is stable we deduce from Lemma 2.2 that the embedded tangent space $\mathbb{T}_F\mathcal{M}_\omega$ corresponds to $[E]$ and $[E_f]$. Hence $[E] = [E_f]$ and since E is stable, we have $E = E_f$. □

We deduce that $\mathcal{D}_L$ restricts to an isomorphism $\mathbb{P}_0(L) \setminus \mathcal{T}_0(L) \xrightarrow{\sim} \mathbb{P}_\omega(L) \setminus \mathcal{T}_\omega(L)$. Since $\mathcal{M}_0$ is covered by the spaces $\mathbb{P}_0(L)$ and since $\mathcal{D}$ restricts to $\mathcal{D}_L$ on $\mathbb{P}_0(L)$, we obtain that $\mathcal{D}$ restricts to a birational bijective morphism from $\mathcal{M}_0 \setminus \mathcal{T}_0$ to $\mathcal{M}_\omega \setminus \mathcal{T}_\omega$. Hence by Zariski's Main Theorem $\mathcal{D}$ is an isomorphism on these open sets, which proves part (1) of Theorem 3.1. Lemma 3.3 implies part (2). As for part (3), we choose a $\mathbb{P}_0(L)$ containing $E \in \mathcal{M}_0$. This determines a point $e \in \mathbb{P}_0(L)$ and we consider $F := F_e = \mathcal{D}_L(e)$. By Lemma 3.2(3) and Lemma 2.1 $\mathcal{D}_L(e) = \mathcal{D}(e)$ - which shows that this construction does not depend on the choice of L. Moreover if $e \notin \mathcal{T}_0$, then F is stable and characterized by the property $\dim H^0(C, E \otimes F) \geq 4$. One easily shows that $\dim H^0(C, E \otimes F) \geq 6$ cannot occur if $e \notin \mathcal{T}_0$ (see also Remark 3.4(2)).

3.3 Blowing-up

Even if part (4) of Theorem 3.1 is a straight-forward consequence of the results obtained in [L2], we give the complete proof for the convenience of the reader. First we consider the blowing-up $\mathcal{B}l_s(\mathbb{P}^7)$ of $\mathbb{P}^7 = |2\Theta|$ along the 64 singular points of $\mathcal{K}_0$. Because of the invariance of $\mathcal{K}_0$ and $\mathcal{M}_0$ under the Heisenberg group, it is enough to consider the blowing-up at the "origin" $O := [\mathcal{O} \oplus \mathcal{O}]$. We denote by $\widetilde{\mathcal{K}_0}$ (resp. $\mathcal{B}l_s(\mathcal{M}_0)$) the proper transform of $\mathcal{K}_0$ (resp. $\mathcal{M}_0$), and by $\mathbb{P}(T_O\mathbb{P}^7) \subset \mathcal{B}l_s(\mathbb{P}^7)$ the exceptional divisor (over O).

By [L2] Remark 5 the Zariski tangent spaces $T_O\mathcal{K}_0$ and $T_O\mathcal{M}_0$ at the origin O to $\mathcal{K}_0$ and $\mathcal{M}_0$ satisfy the relations

$$\mathrm{Sym}^2 H^0(\omega)^* \cong T_O\mathcal{K}_0 \subset T_O\mathcal{M}_0 = T_O\mathbb{P}^7 \qquad \text{and} \qquad T_O\mathcal{M}_0/T_O\mathcal{K}_0 \cong \Lambda^3 H^0(\omega)^*.$$

Moreover with the notation of section 2.2 the equation of the hyperplane $T_O\mathcal{K}_0 \subset T_O\mathcal{M}_0$ is $T = 0$ and T_{ij} are coordinates on $\mathrm{Sym}^2 H^0(\omega)^*$. We deduce from the local equation of $\mathcal{M}_0$ at the origin O (section 2.2(ii)) that $\widetilde{\mathcal{K}_0} \cap \mathbb{P}\mathrm{Sym}^2 H^0(\omega)^*$ is the Veronese surface $S := \mathrm{Ver} H^0(\omega)^*$ and that $\widetilde{\mathcal{K}_0}$ is smooth. Moreover the linear system spanned by the proper transforms of the cubics C_i is given by the six quadrics $Q_{ij} := \frac{\partial}{\partial T_{ij}} (\det [T_{ij}])$ vanishing on S.

Given a smooth point $x = [M \oplus M^{-1}] \in \mathcal{K}_0$ with $M^2 \neq \mathcal{O}$, the Zariski tangent spaces $T_x\mathcal{K}_0$ and $T_x\mathcal{M}_0$ satisfy the relations

$$H^0(\omega)^* \cong T_x\mathcal{K}_0 \subset T_x\mathcal{M}_0 = T_x\mathbb{P}^7 \qquad \text{and} \qquad T_x\mathcal{M}_0/T_x\mathcal{K}_0 \cong H^0(\omega M^2)^* \otimes H^0(\omega M^{-2})^*.$$

The tangent space $T_x\mathcal{K}_0 \subset T_x\mathcal{M}_0$ is cut out by the four equations $T_{ij} = 0$, where the T_{ij} are natural coordinates on $H^0(\omega M^2)^* \otimes H^0(\omega M^{-2})^*$. Let $\widetilde{\mathcal{E}}$ be the exceptional divisor of the blowing-up of $\mathcal{B}l_s(\mathbb{P}^7)$ along the smooth variety $\widetilde{\mathcal{K}_0}$ and let $\mathcal{E}$ be its restriction to the proper transform $\widetilde{\mathcal{M}_0}$. We denote by $\widetilde{\mathcal{E}}_x$ and $\mathcal{E}_x$ the fibres of $\widetilde{\mathcal{E}}$ and $\mathcal{E}$ over a point $x \in \mathcal{K}_0$. Then for a smooth point x, it follows from the local equation at x (section 2.2 (ii)) that $\mathcal{E}_x$ is the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 = |\omega M^2|^* \times |\omega M^{-2}|^* \hookrightarrow \mathbb{P}H^0(\omega M^2)^* \otimes H^0(\omega M^{-2})^* = \widetilde{\mathcal{E}}_x$ and the linear system spanned by the proper transforms of the cubics C_i is given by the four linear forms T_{ij}.

At a singular point (we take $x = O$), it follows from the preceeding discussion that $\mathcal{E}_O$ is the exceptional divisor of the blowing-up of $\mathbb{P}\mathrm{Sym}^2 H^0(\omega)^*$ along the Veronese surface S,i.e. the projectivized normal bundle over S. It is a well-known fact (duality of conics) that the rational map given by the quadrics Q_{ij} resolves by blowing-up S.

It remains to show that $\widetilde{\mathcal{D}}$ maps $\mathcal{E}$ onto the trisecant scroll $\mathcal{T}_\omega$. Since $\mathcal{E}$ is irreducible, it will be enough to check this on an open subset of $\mathcal{E}$. We consider again the extension spaces $\mathbb{P}_0(L) \subset \mathcal{M}_0$. For simplicity we choose L such that

(1) $\mathbb{P}_0(L)$ does not contain a singular point of $\mathcal{K}_0$,

(2) the morphism $\varphi : C \longrightarrow \mathbb{P}_0(L)$ is an embedding, or equivalently $\dim H^0(L^2) = 0$.

Let $\widetilde{\mathbb{P}_0(L)}$ be the blowing-up of $\mathbb{P}_0(L)$ along the curve C, with exceptional divisor $\mathcal{E}_L$. Because of assumptions (1) and (2), we have an embedding $\widetilde{\mathbb{P}_0(L)} \hookrightarrow \widetilde{\mathcal{M}_0}$, $\mathcal{E}$ restricts to $\mathcal{E}_L$, and $\mathcal{E}_L$ is the projectivized normal bundle N of the embedded curve $C \subset \mathbb{P}_0(L)$. We have the following commutative diagram

$$\begin{array}{ccccc} \mathbb{P}(N) = \mathcal{E}_L & \subset & \widetilde{\mathbb{P}_0(L)} & & \\ \downarrow^{\pi} & & \downarrow & \searrow^{\widetilde{\mathcal{D}}_L} & \\ C & \subset & \mathbb{P}_0(L) & \xrightarrow{\mathcal{D}_L} & \mathbb{P}_\omega(L) \end{array}$$

In order to study the image $\widetilde{\mathcal{D}}_L(\mathcal{E}_L)$ we introduce, for a point $u \in C$, the rank 2 bundle E_u which is defined by the exact sequence

$$0 \longrightarrow E_u^* \longrightarrow \mathcal{O}_C \otimes H^0(\omega L^2(-u)) \xrightarrow{ev} \omega L^2(-u) \longrightarrow 0.$$

Note that $H^0(\omega L^2(-u))$ corresponds to the hyperplane defined by $u \in C \subset \mathbb{P}_0(L)$. Then exactly as in Lemma 3.2(1) we show that $\det E_u = \omega L^2(-u)$, E_u is stable and globally generated with $H^0(E_u) \cong H^0(\omega L^2(-u))^*$. We introduce the Hecke line $\mathcal{H}_u$ defined as the set of bundles which are (negative) elementary transformations of $E_u L^{-1}(u)$ at the point u,i.e. the set of bundles which fit into the exact sequence

$$0 \longrightarrow F \longrightarrow E_u L^{-1}(u) \longrightarrow \mathbb{C}_u \longrightarrow 0. \qquad (3.5)$$

Since E_u is stable, any F is semi-stable (and $\det F = \omega$) and we have a linear map (see [B2]) $\mathbb{P}^1 \cong \mathcal{H}_u \to \mathcal{M}_\omega$.

3.6 Lemma. *Given a point $u \in C$, the fibre $\mathbb{P}(N_u) = \mathcal{E}_{L,u}$ is mapped by $\widetilde{\mathcal{D}}_L$ to the Hecke line $\mathcal{H}_u \subset \mathbb{P}_\omega(L)$. Moreover $\mathcal{H}_u$ coincides with the trisecant line $\overline{pqr}$ to $C \subset \mathbb{P}_\omega(L)$, with $p + q + r \in |\omega L^{-2}(u)|$.*

Proof. The Zariski tangent space $T_u\mathbb{P}_0(L)$ at the point u is identified with $H^0(\omega L^2(-u))^* \cong H^0(E_u)$. Under this identification the tangent space T_uC corresponds to the subspace $H^0(E_u(-u))$. Hence we obtain a canonical isomorphism of $\mathbb{P}(N_u)$ with the projectivized fibre over the point u of the bundle E_u,i.e. the Hecke line $\mathcal{H}_u$. It is straight-forward to check that $\widetilde{\mathcal{D}}_L$ restricts to the isomorphism $\mathbb{P}(N_u) \cong \mathcal{H}_u$. To show the last assertion, it is enough to observe that the Hecke line $\mathcal{H}_u$ intersects the curve $C \subset \mathbb{P}_\omega(L)$ at a point p if and only if $\dim H^0(E_u L^{-1}(u - p)) > 0$ and to continue as in the proof of Lemma 3.3. □

Since the union of those $\mathcal{E}_L$ such that L satisfies assumptions (1) and (2) form an open subset of $\mathcal{E}$, we conclude that $\widetilde{\mathcal{D}}(\mathcal{E}) = \mathcal{T}_\omega$. This completes the proof of Theorem 3.1.

3.4 Some remarks

(1) The divisor $\mathcal{T}_\omega \in |\mathcal{L}^8|$. This is seen as follows: it suffices to restrict $\mathcal{T}_\omega$ to a general $\mathbb{P}_\omega(L) \subset \mathcal{M}_\omega$ and to compute the degree of the trisecant scroll $\mathcal{T}_\omega(L) \subset \mathbb{P}_\omega(L)$. By Lemma 3.6 $\mathcal{T}_\omega(L)$ is the image of $\mathcal{E}_L = \mathbb{P}(N)$ under the morphism $\widetilde{\mathcal{D}}_L$. The hyperplane bundle over $\mathbb{P}_\omega(L)$ pulls-back under $\widetilde{\mathcal{D}}_L$ to $\mathcal{O}_{\mathbb{P}}(1) \otimes \pi^* (\omega^3 L^6)$ over the ruled surface $\mathbb{P}(N)$. Since $\widetilde{\mathcal{D}}_{L|\mathcal{E}_L}$ is birational, we obtain that $\deg \mathcal{T}_\omega(L) = \deg \pi_* \mathcal{O}_{\mathbb{P}}(1) \otimes \omega^3 L^6 = \deg N^* \omega^3 L^6 = 8$.

(2) Using the same methods as before, one can show a refinement of Theorem 3.1(3). Consider E stable with $E \in \mathcal{M}_0$ and F semi-stable with $[F] \in \mathcal{M}_\omega$.

- The only pairs (E, F) for which $\dim H^0(C, E \otimes F) = 6$ are the 64 exceptional pairs $E = A_\kappa$ and $F = \kappa \oplus \kappa$ for a theta-characteristic κ (2.1). We note that $\mathcal{D}(A_\kappa) = [\kappa \oplus \kappa]$.
- Suppose $\mathcal{D}(E) = [M \oplus \omega M^{-1}]$ for some M and $E \neq A_\kappa$,i.e. $M^2 \neq \omega$. Then there are exactly three semi-stable bundles F such that $\mathcal{D}(E) = [F]$ and $\dim H^0(C, E \otimes F) = 4$, namely
  (1) the decomposable bundle $F = M \oplus \omega M^{-1}$ (note that $\dim H^0(EM) = 2$).
  (2) two indecomposable bundles with extension classes in $\mathrm{Ext}^1(M, \omega M^{-1}) = H^0(M^2)^*$ and $\mathrm{Ext}^1(\omega M^{-1}, M) = H^0(\omega^2 M^{-2})^*$ defined by the images of the exterior product maps

$$\Lambda^2 H^0(EM) \longrightarrow H^0(M^2) \qquad \text{and} \qquad \Lambda^2 H^0(E\omega M^{-1}) \longrightarrow H^0(\omega^2 M^{-2}).$$

(3) As a corollary of Lemma 3.6 we obtain that the morphism $\widetilde{\mathcal{D}}$ maps the exceptional divisor $\widetilde{\mathcal{E}}$ onto the dual hypersurface $\mathcal{K}_0^*$ of the Kummer variety $\mathcal{K}_0$ (more precisely $\widetilde{\mathcal{D}}$ maps $\widetilde{\mathcal{E}}_x = \mathbb{P}^3$ isomorphically to the subsystem of divisors singular at $x \in \mathcal{K}_0^{sm}$) and that the hypersurface $\widetilde{\mathcal{D}}(\widetilde{\mathcal{E}}) = \mathcal{K}_0^*$ intersects (set-theoretically) $\mathcal{M}_\omega$ along the trisecant scroll $\mathcal{T}_\omega$. It is worthwhile to figure out the relationship with other distinguished hypersurfaces in $|2\Theta|$, e.g. the octic G_8 defined by the equation $\mathcal{D}^{-1}(F_4) = F_4 \cdot G_8$ and the Hessian H_{16} of Coble's quartic F_4.

4 Applications

4.1 The 8 maximal line subbundles of $E \in \mathcal{M}_0$

In this section we recall the results of [LN] (see also [OPP], [OP2]) on line subbundles of stable bundles $E \in \mathcal{M}_0$ and $F \in \mathcal{M}_\omega$. We introduce the closed subsets $\mathbf{M}_0(E)$ and $\mathbf{M}_\omega(F)$ of $\mathrm{Pic}^1(C)$ parametrizing line subbundles of maximal degree of E and F,

$$\mathbf{M}_0(E) := \{L \in \mathrm{Pic}^1(C) \mid L^{-1} \hookrightarrow E\} \qquad \text{and} \qquad \mathbf{M}_\omega(F) := \{L \in \mathrm{Pic}^1(C) \mid L \hookrightarrow F\}.$$

The next lemma follows from [LN] section 5 and Nagata's theorem. For simplicity we assume that C is not bi-elliptic.

4.1 Lemma. *The subsets $\mathbf{M}_0(E)$ and $\mathbf{M}_\omega(F)$ are non-empty and 0-dimensional, unless E and F are exceptional (see (2.1)). In these cases we have*

$$\mathbf{M}_0(A_\kappa) = \{\kappa(-p) \mid p \in C\} \cong C \qquad \text{and} \qquad \mathbf{M}_\omega(A_\alpha) = \{\alpha(p) \mid p \in C\} \cong C.$$

Note that $A_\kappa \in \mathcal{T}_0$ and $A_\alpha \in \mathcal{T}_\omega$ (see [OPP] Theorem 5.3) and that in the bi-elliptic case, we additionally have a $JC[2]$-orbit in $\mathcal{M}_0$ (resp. $\mathcal{M}_\omega$) of bundles E (resp. F) with 1-dimensional $\mathbf{M}_0(E)$ (resp. $\mathbf{M}_0(F)$).

Since $\mathbf{M}_0(E)$ is non-empty, any stable $E \in \mathcal{M}_0$ lies in at least one extension space $\mathbb{P}_0(L)$ for some $L \in \mathrm{Pic}^1(C)$ with extension class $e \notin \varphi(C)$. Now [LN] Proposition 2.4 says that there exists a bijection between the sets of

(1) effective divisors $p+q$ on C such that e lies on the secant line $\overline{pq}$

(2) line bundles $M \in \mathrm{Pic}^1(C)$ such that $M^{-1} \hookrightarrow E$ and $M \neq L$.

The two data are related by the equation

$$L \otimes M = \mathcal{O}_C(p+q). \tag{4.1}$$

Let us count secant lines to $\varphi(C)$ through a *general* point $e \in \mathbb{P}_0(L)$: composing φ with the projection from e maps C birationally to a plane nodal sextic S. By the genus formula, we obtain that the number of nodes of S (= number of secants) equals 7. Hence, for E general, the cardinal $|\mathbf{M}_0(E)|$ of the finite set $\mathbf{M}_0(E)$ is 8. We write

$$\mathbf{M}_0(E) = \{L_1, \cdots, L_8\}.$$

From now on we assume that E is sufficiently general in order to have $|\mathbf{M}_0(E)| = 8$. Since $E \in \mathbb{P}_0(L_i)$ for $1 \leq i \leq 8$, we deduce from relation (4.1) that

$$1 \leq i < j \leq 8, \qquad L_i \otimes L_j = \mathcal{O}_C(D_{ij}), \tag{4.2}$$

where D_{ij} is an effective degree two divisor on C.

4.2 Lemma. *The 8 line bundles L_i satisfy the relation $\bigotimes_{i=1}^8 L_i = \omega^2$.*

Proof. We represent E as a point $e \in \mathbb{P}_0(L_8)$ and assume that the plane sextic curve $S \subset \mathbb{P}^2$ obtained by projection with center e has 7 nodes as singularities. It will be enough to prove the equality for such a bundle E. Then $C \xrightarrow{\pi} S$ is the normalization of S and, by the adjunction formula, we have $\omega = \pi^*\mathcal{O}_S(3) \otimes \mathcal{O}_C(-\Delta)$, where Δ is the divisor lying over the 7 nodes of S,i.e. $\Delta = \sum_{i=1}^7 D_{i8}$. Hence

$$\omega = \omega^3 L_8^6 \left(- \sum_{i=1}^{7} D_{i8} \right) = \omega^3 L_8^{-1} \otimes \bigotimes_{i=1}^{7} (L_8(-D_{i8})) = \omega^3 \otimes \bigotimes_{i=1}^{8} L_i^{-1},$$

where we used relations (4.2). □

4.3 Remark. Conversely, suppose we are given 8 line bundles L_i which satisfy the 28 relations (4.2). Then there exists a unique stable bundle $E \in \mathcal{M}_0$ such that $\mathbf{M}_0(E) = \{L_1, \ldots, L_8\}$. This is seen as follows: take e.g. L_8 and consider any two secant lines $\overline{D}_{i8}$ and $\overline{D}_{j8}$ ($i < j < 8$) in $\mathbb{P}_0(L_8)$. Then relations (4.2) imply that these two lines intersect in a point e. It is straight-forward to check that the bundle E associated to e does not depend on the choices we made.

4.2 Nets of quadrics

We consider $E \in \mathcal{M}_0$ and we assume that $E \notin \mathcal{T}_0$ and $|\mathbf{M}_0(E)| = 8$. Then $F = \mathcal{D}(E)$ is stable and dim $H^0(C, E \otimes F) = 4$. We recall that the rank 4 vector bundle $E \otimes F$ is equipped with a non-degenerate quadratic form (we note that $E = E^*$)

$$\det : E \otimes F = \mathcal{H}\mathrm{om}(E, F) \longrightarrow \omega.$$

Taking global sections on both sides endows the projective space $\mathbb{P}^3 := \mathbb{P}H^0(C, \mathcal{H}\mathrm{om}(E, F))$ with a net $\Pi = |\omega|^*$ of quadrics. We denote by $Q_x \subset \mathbb{P}^3$ the quadric associated to $x \in \Pi$ and, identifying C with its canonical embedding $C \subset |\omega|^* = \Pi$, we see that (the cone over) the quadric Q_p for $p \in C$ corresponds to the sections

$$Q_p := \{\phi \in H^0(C, \mathcal{H}\mathrm{om}(E, F)) \mid E_p \xrightarrow{\phi_p} F_p \text{ not surjective}\}, \tag{4.3}$$

where E_p, F_p denote the fibres of E, F over $p \in C$. It follows from Lemma 3.2(2) that $\mathbf{M}_0(E) = \mathbf{M}_\omega(F)$, or equivalently any line bundle $L_i \in \mathbf{M}_0(E)$ fits into a sequence of maps

$$x_i : E \longrightarrow L_i \longrightarrow F.$$

We denote by $x_i \in \mathbb{P}^3$ the composite map (defined up to a scalar).

4.4 Lemma. *The base locus of the net of quadrics* Π *consists of the* 8 *distinct points* $x_i \in \mathbb{P}^3$*.*

Proof. A base point x corresponds to a vector bundle map $x : E \to F$ such that $\mathrm{rk}\, x \leq 1$ (since $x \in Q_p$, $\forall p$). Hence there exists a line bundle L such that $E \to L \to F$ and since E and F are stable, of slope 0 and 2, we obtain that $\deg L = 1$ and $L \in \mathbf{M}_0(E) = \mathbf{M}_\omega(F)$. □

The set of base points $\overline{x} = \{x_1, \ldots, x_8\}$ of a net of quadrics in $\mathbb{P}^3$ is *self-associated* (for the definition of (self-)association of point sets we refer to [DO] chapter 3) and called a *Cayley octad.* We recall ([DO] chapter 3 example 6) that ordered Cayley octads $\overline{x} = \{x_1, \ldots, x_8\}$ are in 1-to-1 correspondence with ordered point sets $\overline{y} = \{y_1, \ldots, y_7\}$ in $\mathbb{P}^2$ (note that we consider here general ordered point sets up to projective equivalence). The correspondence goes as follows: starting from $\overline{x}$ we consider the projection with center x_8, $\mathbb{P}^3 \xrightarrow{pr_{x_8}} \mathbb{P}^2$, and define $\overline{y}$ to be the projection of the remaining seven points. Conversely, given $\overline{y}$ in $\mathbb{P}^2$, we obtain by association 7 points $x_1, \ldots, x_7$ in $\mathbb{P}^3$. The missing 8-th point x_8 of $\overline{x}$ is the additional base point of the net of quadrics through the 7 points $x_1, \ldots, x_7$.

Consider a general $E \in \mathcal{M}_0$ and choose a line subbundle $L_8 \in \mathbf{M}_0(E)$. We denote by x_8 the corresponding base point of the net Π. We consider the two (different) projections onto $\mathbb{P}^2$

(1) projection with center x_8 of $\mathbb{P}^3 = \mathbb{P}H^0(C, \mathcal{H}\mathrm{om}(E, F)) \xrightarrow{pr_{x_8}} \mathbb{P}^2$. Let $\overline{y} = \{y_1, \ldots, y_7\} \subset \mathbb{P}^2$ be the projection of the 7 base points $x_1, \ldots, x_7$.

(2) projection with center e of $\mathbb{P}_0(L_8) \xrightarrow{pr_e} \mathbb{P}^2$. Let $\overline{z} = \{z_1, \ldots, z_7\} \subset \mathbb{P}^2$ be the images of the 7 secant lines to $\varphi(C)$ through e. Note that $z_1, \ldots, z_7$ are the 7 nodes of the plane sextic S.

4.5 Lemma. *The two target* $\mathbb{P}^2$*'s of the projections (1) and (2) are canonically isomorphic (to* $\mathbb{P}W_e^*$*) and the two point sets* $\overline{y}$ *and* $\overline{z}$ *coincide.*

Proof. First we recall from the proof of Lemma 3.2 that we have an exact sequence

$$0 \longrightarrow H^0(FL_8^{-1}) \xrightarrow{i} H^0(E \otimes F) \xrightarrow{\pi} H^0(FL_8) \longrightarrow 0,$$

and that $H^0(FL_8) \cong W_e^*$ and $\dim H^0(FL_8^{-1}) = 1$. Moreover it is easily seen that $\mathbb{P}(\mathrm{im} i) = x_8 \in \mathbb{P}^3$, hence the projectivized map π identifies with pr_{x_8}. The images $pr_{x_8}(x_i)$ for $1 \leq i \leq 7$ are given by the sections $s_i \in H^0(FL_8)$ vanishing at the divisor D_{i8} (since $L_i L_8 = \mathcal{O}_C(D_{i8}) \hookrightarrow FL_8$). It remains to check that the section $s_i \in H^0(FL_8) \cong W_e^*$ correponds to the 2-dimensional subspace $H^0(\omega L^2(-D_{i8})) \subset W_e \subset H^0(\omega L^2)$, which is standard. □

We introduce the non-empty open subset $\mathcal{M}_0^{reg} \subset \mathcal{M}_0$ of stable bundles E which satisfy $E \notin \mathcal{T}_0$, $|\mathbf{M}_0(E)| = 8$ and for any $L \in \mathbf{M}_0(E)$ the point set $\overline{z} \subset \mathbb{P}^2$ is such that no three points in $\overline{z}$ are collinear.

4.3 The Hessian construction

It is classical (see e.g. [DO] chapter 9) to associate to a net of quadrics Π on $\mathbb{P}^3$ its Hessian curve parametrizing singular quadrics,i.e.

$$\mathrm{Hess}(E) := \{x \in \Pi = |\omega|^* \mid Q_x \text{ singular}\}.$$

Note that C and $\mathrm{Hess}(E)$ lie in the same projective plane.

4.6 Lemma. *We suppose that $E \in \mathcal{M}_0^{reg}$. Then the curve* $\mathrm{Hess}(E)$ *is a smooth plane quartic.*

Proof. It follows from [DO] chapter 9 Lemma 5 that $\mathrm{Hess}(E)$ is smooth if and only if every 4 points of $\overline{x} = \{x_1, \ldots, x_8\}$ span $\mathbb{P}^3$. Projecting from one of the x_i's and using Lemma 4.5 we see that this condition holds for $E \in \mathcal{M}_0^{reg}$. □

First we determine for which bundles $E \in \mathcal{M}_0^{reg}$ the Hessian curve $\mathrm{Hess}(E)$ equals the base curve C. We need to recall some facts about nets of quadrics and Cayley octads [DO]. The net Π determines an even theta-characteristic θ over the smooth curve $\mathrm{Hess}(E)$, such that the Steinerian embedding

$$\mathrm{Hess}(E) \xrightarrow{\mathrm{St}} \mathbb{P}^3 = |\omega\theta|^*, \qquad x \longmapsto \mathrm{Sing}(Q_x),$$

is given by the complete linear system $|\omega\theta|$. The image $\mathrm{St}(E)$ is called the *Steinerian* curve. Given two distinct base points $x_i, x_j \in \mathbb{P}^3$ of the net Π, the pencil Λ_{ij} of quadrics of the net Π which contain the line $\overline{x_i x_j}$ is a bitangent to the curve $\mathrm{Hess}(E)$. In this way we obtain all the $28 = \binom{8}{2}$ bitangents to $\mathrm{Hess}(E)$. Let u, v be the two intersection points of the bitangent Λ_{ij} with $\mathrm{Hess}(E)$. Then the secant line to the Steinerian curve $\mathrm{St}(E)$ determined by $\mathrm{St}(u)$ and $\mathrm{St}(v)$ coincides with $\overline{x_i x_j}$.

Conversely, given a smooth plane quartic $X \subset \mathbb{P}^2$ with an even theta characteristic θ, taking the symmetric resolution over $\mathbb{P}^2$ of the sheaf θ supported at the curve X gives a net of quadrics Π whose Hessian curve equals X. Thus the correspondence between nets of quadrics Π and the data (X, θ) is 1-to-1.

This correspondence allows us to construct some more distinguished bundles in $\mathcal{M}_0$. We consider a triple (θ, L, x) consisting of an even theta-characteristic θ over C, a square-root $L \in \mathrm{Pic}^1(C)$,i.e. $L^2 = \theta$, and a base point x of the net of quadrics Π associated to (C, θ). We denote by

$$A(\theta, L, x) \in \mathcal{M}_0 \tag{4.4}$$

the stable bundle defined by the point $x \in \mathbb{P}_0(L) = |\omega\theta|^*$. Since C is smooth, we have $A(\theta, L, x) \in \mathcal{M}_0^{reg}$. These bundles will be called *Aronhold* bundles (see Remark 4.12). We leave it to the reader to deduce the following characterization: E is an Aronhold bundle if and only if the 28 line bundles $L_i L_j$ $(1 \leq i < j \leq 8)$ are the odd theta-characteristics, with $L_i \in \mathbf{M}_0(E)$.

4.7 Proposition. *Given a bundle $E \in \mathcal{M}_0^{reg}$. Then*

1. *We have* $\mathrm{Hess}(E) = C$ *if and only if E is an Aronhold bundle.*
2. *Assuming* $\mathrm{Hess}(E) \neq C$, *the curves C and* $\mathrm{Hess}(E)$ *are everywhere tangent. More precisely, the scheme-theoretical intersection $C \cap \mathrm{Hess}(E)$ is non-reduced of the form $2\Delta(E)$, with $\Delta(E) \in |\omega^2|$.*

Proof. We deduce from (4.3) that the intersection $C \cap \mathrm{Hess}(E)$ corresponds (set-theoretically) to the sets of points where the evaluation map of global sections

$$\mathcal{O}_C \otimes H^0(C, \mathcal{H}\mathrm{om}(E,F)) \xrightarrow{ev} \mathcal{H}\mathrm{om}(E,F) \qquad (4.5)$$

is not surjective.

Let us suppose that $C = \mathrm{Hess}(E)$. Then ev is not generically surjective ($\mathrm{rk}\, ev \leq 3$). We choose a line subbundle $L_8 \in \mathbf{M}_0(E)$ and we consider (as in Lemma 4.5) the exact sequence

$$\begin{array}{ccccccccc} 0 \longrightarrow & H^0(FL_8^{-1}) & \longrightarrow & H^0(\mathcal{H}\mathrm{om}(E,F)) & \longrightarrow & H^0(FL_8) & \longrightarrow 0 \\ & \downarrow^{\cong} & & \downarrow^{ev} & & \downarrow^{ev'} & \\ 0 \longrightarrow & \mathcal{O}_C & \longrightarrow & \mathcal{H}\mathrm{om}(E,F) & \longrightarrow & \mathcal{E} & \longrightarrow 0 \end{array}$$

where the vertical arrows are evaluation maps. Note that $\mathcal{O}_C \hookrightarrow FL_8^{-1} \hookrightarrow \mathcal{H}\mathrm{om}(E,F)$ corresponds to the section of $H^0(FL_8^{-1})$. We denote by $\mathcal{E}$ the rank 3 quotient. Then $ev' : H^0(FL_8) \longrightarrow \mathcal{E}$ is not generically surjective either. But $\mathcal{E}$ has a quotient $E \rightarrow FL_8$ with kernel ωL_8^{-2}. Now since $H^0(FL_8) \xrightarrow{ev} FL_8$ is surjective, we obtain a direct sum decomposition $\mathcal{E} = \omega L_8^{-2} \oplus FL_8$. Furthermore since $E \otimes F$ is poly-stable (semi-stable and orthogonal) and of slope 2, we obtain that ωL_8^{-2} is an orthogonal direct summand. Hence $\omega L_8^{-2} = \theta$ for some theta-characteristic θ. Now we can do this reasoning for any line bundle $L_i \in \mathbf{M}_0(E)$, establishing that all ωL_i^{-2} are theta-characteristics contained in $\mathcal{H}\mathrm{om}(E,F)$. Projecting to FL_8 shows that $L_i^2 = L_8^2 = \theta$ for all i and therefore the 28 line bundles L_iL_j are the odd theta-characteristics. It follows that E is an Aronhold bundle.

Assuming $C \neq \mathrm{Hess}(E)$, the evaluation map (4.5) is injective

$$0 \longrightarrow \mathcal{O}_C \otimes H^0(C, \mathcal{H}\mathrm{om}(E,F)) \xrightarrow{ev} \mathcal{H}\mathrm{om}(E,F) \longrightarrow \mathbb{C}_{\Delta(E)} \longrightarrow 0.$$

The cokernel is a sky-scraper sheaf supported at a divisor $\Delta(E)$. Since $\det \mathcal{H}\mathrm{om}(E,F) = \omega^2$, we have $\Delta(E) \in |\omega^2|$. This shows that set-theoretically we have $C \cap \mathrm{Hess}(E) = \Delta(E)$. Let us determine the local equation of $\mathrm{Hess}(E)$ at a point $p \in \Delta(E)$. We denote by m the multiplicity of $\Delta(E)$ at the point p. Then, since there is no section of $\mathcal{H}\mathrm{om}(E,F)$ vanishing twice at p (stability of E and F), we have $\dim H^0(\mathcal{H}\mathrm{om}(E,F)(-p)) = m$. We choose a basis $\phi_1, \ldots, \phi_m$ of sections of the subspace $H^0(\mathcal{H}\mathrm{om}(E,F)(-p)) \subset H^0(\mathcal{H}\mathrm{om}(E,F))$ and complete it (if necessary) by $\phi_{m+1}, \ldots, \phi_4$. Let z be a local coordinate in an analytic neighbourhood centered at the point p. With these notations the quadrics Q_z of the net can be written

$$Q_z(\lambda_1, \ldots, \lambda_4) = \det \left(\sum_{i=1}^{4} \lambda_i \phi_i(z) \right),$$

where the $\phi_i(z)$ are a basis of the fibre $\mathcal{H}\mathrm{om}(E,F)_z$ for $z \neq 0$. By construction we have for $1 \leq i \leq m$, $\phi_i(z) = z\psi_i(z)$, and the local equation of $\mathrm{Hess}(E)$ is the determinant of the symmetric 4×4 matrix

$$\mathrm{Hess}(E)(z) = \det \left[B(\phi_i(z), \phi_j(z)) \right]_{1 \leq i,j \leq 4},$$

where B is the polarization of the determinant. We obtain that $\mathrm{Hess}(E)(z)$ is of the form $z^{2m}R(z)$. Hence $\mathrm{mult}_p(\mathrm{Hess}(E)) \geq 2m$, proving the statement. $\square$

4.8 Definition. *We call the divisor $\Delta(E)$ the discriminant divisor of E and the rational map $\Delta : \mathcal{M}_0 \longrightarrow |\omega^2|$ the discriminant map.*

In the sequel of this paper we will show that the bundle E and its Hessian curve $\mathrm{Hess}(E)$ are in bijective correspondence (modulo some discrete structure, which will be defined in section 4.5.2). A first property is the following: given $E \in \mathcal{M}_0^{reg}$, we associate to the 28 degree two effective divisors D_{ij} (4.2) on the curve C their corresponding secant lines $\overline{D}_{ij} \subset |\omega|^*$.

4.9 Proposition. *The secant line $\overline{D}_{ij}$ to the curve C coincides with the bitangent Λ_{ij} to the smooth quartic curve* $\mathrm{Hess}(E)$.

Proof. Since the bitangent Λ_{ij} to $\mathrm{Hess}(E)$ corresponds to the pencil of quadrics in Π containing the line $\overline{x_i x_j}$, it will be enough to show that Q_a and Q_b belong to Λ_{ij}, for $D_{ij} = a + b$, with $a, b \in C$. Consider the vector bundle map $\pi_i \oplus \pi_j : E \longrightarrow L_i \oplus L_j$, where π_i and π_j are the natural projection maps. Since $L_i L_j = \mathcal{O}(D_{ij})$, the map $\pi_i \oplus \pi_j$ has cokernel $\mathbb{C}_a \oplus \mathbb{C}_b$, which is equivalent to saying that the two linear forms $\pi_{i,a} : E_a \to L_{i,a}$ and $\pi_{j,a} : E_a \to L_{j,a}$ are proportional (same for b). This implies that any map $\phi \in \overline{x_i x_j}$ factorizes at the point a through $\pi_{i,a} = \pi_{j,a}$, hence $\det \phi_a = 0$. This means that $\overline{x_i x_j} \subset Q_a$,i.e. $Q_a \in \Lambda_{ij}$ (same for b). $\square$

4.4 Moduli of $\mathbb{P}\mathrm{SL}_2$-bundles and the discriminant map Δ

The finite group $JC[2]$ of 2-torsion points of JC acts by tensor product on $\mathcal{M}_0$ and $\mathcal{M}_\omega$. Since Coble's quartic is Heisenberg-invariant, it is easily seen that the polar map $\mathcal{D} : \mathcal{M}_0 \longrightarrow \mathcal{M}_\omega$ is $JC[2]$-equivariant,i.e. $\mathcal{D}(E \otimes \alpha) = \mathcal{D}(E) \otimes \alpha$, $\forall \alpha \in JC[2]$. This implies that the constructions we made in sections 4.2 and 4.3, namely the projective space $\mathbb{P}^3 = \mathbb{P}H^0(\mathcal{H}\mathrm{om}(E, F))$, the net of quadrics Π, its Hessian curve $\mathrm{Hess}(E)$ and discriminant divisor $\Delta(E)$, only depend on the class of E modulo $JC[2]$, which we denote by $\overline{E}$. It is therefore useful to introduce the quotient $\mathcal{N} = \mathcal{M}_0/JC[2]$, which can be identified with the moduli space of semi-stable $\mathbb{P}\mathrm{SL}_2$-vector bundles with fixed trivial determinant. We observe that $\mathcal{N}$ is canonically isomorphic to the quotient $\mathcal{M}_\omega/JC[2]$. Therefore the $JC[2]$-invariant polar map $\mathcal{D}$ descends to a birational involution

$$\overline{\mathcal{D}} : \mathcal{N} \longrightarrow \mathcal{N}. \tag{4.6}$$

We recall [BLS] that the generator $\overline{\mathcal{L}}$ of $\mathrm{Pic}(\mathcal{N}) = \mathbb{Z}$ pulls-back under the quotient map $q : \mathcal{M}_0 \longrightarrow \mathcal{N}$ to $q^*\overline{\mathcal{L}} = \mathcal{L}^4$ and that global sections $H^0(\mathcal{N}, \overline{\mathcal{L}}^k)$ correspond to $JC[2]$-invariant sections of $H^0(\mathcal{M}_0, \mathcal{L}^{4k})$.

The Kummer variety $\mathcal{K}_0$ is contained in the singular locus of $\mathcal{N}$: since the composite map $JC \xrightarrow{i} \mathcal{M}_0 \xrightarrow{q} \mathcal{N}$, with $i(L) = [L \oplus L^{-1}]$, is $JC[2]$-invariant, it factorizes $JC \xrightarrow{[2]} JC \xrightarrow{\bar{i}} \mathcal{N}$, and the image $\bar{i}(JC) \cong \mathcal{K}_0 \subset \mathcal{N}$.

We also recall from [OP1] that we have a morphism

$$\Gamma : \mathcal{N} \longrightarrow |3\Theta|_+ = \mathbb{P}^{13}, \qquad \overline{E} \longmapsto \Gamma(\overline{E}) = \{L \in \mathrm{Pic}^2(C) \mid \dim H^0(C, \mathrm{Sym}^2(E) \otimes L) > 0\},$$

which is well-defined since $\Gamma(\overline{E})$ only depends on $\overline{E}$. The subscript $+$ denotes invariant (w.r.t. $\xi \longmapsto \omega\xi^{-1}$) theta-functions. When restricted to $\mathcal{K}_0$ the morphism Γ is the Kummer map,i.e we have a commutative diagram

$$\begin{array}{ccc} \mathcal{K}_0 & \xrightarrow{Kum} & |2\Theta| = \mathbb{P}^7 \\ \downarrow & & \downarrow{+\Theta} \\ \mathcal{N} & \xrightarrow{\Gamma} & |3\Theta|_+ = \mathbb{P}^{13} \end{array}$$

The main result of [OP1] is

4.10 Proposition. *The morphism $\Gamma : \mathcal{N} \longrightarrow |3\Theta|_+$ is given by the complete linear system $|\overline{\mathcal{L}}|$,i.e. there exists an isomorphism $|\overline{\mathcal{L}}|^* \cong |3\Theta|_+$.*

4.11 Remark. Using the same methods as in [NR], one can show that $\Gamma : \mathcal{N} \longrightarrow |3\Theta|_+$ is an embedding. We do not use that result.

Since the open subset $\mathcal{M}_0^{reg}$ is $JC[2]$-invariant, we obtain that $\mathcal{M}_0^{reg} = q^{-1}(\mathcal{N}^{reg})$. By passing to the quotient $\mathcal{N}$, the Aronhold bundles (4.4) determine $36 \cdot 8 = 288$ distinct points $A(\theta, x) := \overline{A(\theta, L, x)} \in \mathcal{N}^{reg}$, the exceptional bundles (2.1) determine one point in $\mathcal{N}$, denoted by A_0, and we obtain a (rational) discriminant map (4.8)

$$\Delta : \mathcal{N} \longrightarrow |\omega^2|$$

defined on the open subset $\mathcal{N}^{reg} \setminus \{A(\theta, x)\}$. We also note that the 28 line bundles L_iL_j for $L_i \in \mathbf{M}_0(E)$ only depend on $\overline{E}$.

4.12 Remark. The 288 points $A(\theta, x)$ are in 1-to-1 correspondence with unordered Aronhold sets (see [DO] page 167),i.e. sets of seven odd theta-characteristics θ_i $(1 \leq i \leq 7)$ such that $\theta_i + \theta_j - \theta_k$ is even $\forall i, j, k$. The seven θ_i are cut out on the Steinerian curve by the seven lines $\overline{xx_i}$, where x, x_i are the base points of Π.

The main result of this section is

4.13 Proposition. *We have a canonical isomorphism $|3\Theta_{|\Theta}|_+ \cong |\omega^2|$, which makes the right diagram commute*

$$\begin{array}{ccccc} \mathcal{K}_0 & \subset & \mathcal{N} & \xrightarrow{\Delta} & |\omega^2| \\ \cap & & \downarrow{\Gamma} & & \downarrow{\cong} \\ |2\Theta| & \xrightarrow{+\Theta} & |3\Theta|_+ & \xrightarrow{res_\Theta} & |3\Theta_{|\Theta}|_+. \end{array}$$

In other words, considering $\mathcal{N}$ (via Γ) as a subvariety in $|3\Theta|_+$, the discriminant map Δ identifies with the projection with center $|2\Theta| = \mathrm{Span}(\mathcal{K}_0)$, or equivalently, with the restriction map of $|3\Theta|_+$ to the Theta divisor $\Theta \subset \mathrm{Pic}^2(C)$.

Proof. First we show that the discriminant map Δ is given by a linear subsystem of $|\overline{\mathcal{L}}|$ ($\cong |3\Theta|_+^*$). Consider a line bundle $L \in \mathrm{Pic}^1(C)$ and the composite map

$$\psi_L : \mathbb{P}^3 := \mathbb{P}_0(L) \longrightarrow \mathcal{M}_0 \xrightarrow{q} \mathcal{N} \xrightarrow{\Delta} |\omega^2|.$$

Then it will be enough to show that $\psi_L^*(H) \in |\mathcal{O}_{\mathbb{P}^3}(4)|$ (since $q^*\overline{\mathcal{L}} = \mathcal{L}^4$) for a hyperplane H in $|\omega^2|$. We denote by p (resp.q) the projection of $\mathbb{P}^3 \times C$ onto C (resp.$\mathbb{P}^3$). There exists a universal extension bundle $\mathbb{E}$ over $\mathbb{P}^3 \times C$

$$0 \longrightarrow p^*L^{-1} \longrightarrow \mathbb{E} \longrightarrow p^*L \otimes q^*\mathcal{O}_{\mathbb{P}^3}(-1) \longrightarrow 0 \tag{4.7}$$

such that $\forall e \in \mathbb{P}_0(L)$ the vector bundle $\mathbb{E}_{|\{e\}\times C}$ corresponds to the extension class e. We denote by $\mathbb{W} \hookrightarrow \mathcal{O}_{\mathbb{P}^3} \otimes H^0(\omega L^2)$ the universal rank 3 subbundle over $\mathbb{P}^3$ and we define the family $\mathbb{F}$ over $U \times C$ by the exact sequence

$$0 \longrightarrow (\mathbb{F} \otimes p^*L)^* \longrightarrow q^*\mathbb{W} \xrightarrow{ev} p^*(\omega L^2) \longrightarrow 0, \tag{4.8}$$

where U is the open subset $\mathbb{P}^3 \setminus C$. We have $\mathbb{F}_{|\{e\}\times C} \cong F_e$ (see (3.1)). Note that $\mathrm{Pic}(U) = \mathrm{Pic}(\mathbb{P}^3)$. It follows immediately from (4.7) and (4.8) that $\det \mathbb{E} = q^*\mathcal{O}(-1)$, $\det \mathbb{F} = q^*\mathcal{O}(1) \otimes p^*\omega$, and that $\det \mathbb{E} \otimes \mathbb{F} = p^*\omega^2$. After removing (if necessary) the point A_0 from U (see Remark 3.4(2)), we obtain that $\forall e \in U$, $\dim H^0(C, \mathbb{E} \otimes \mathbb{F}_{|\{e\}\times C}) = 4$, hence by the base change theorems, the direct image sheaves $q_*(\mathbb{E} \otimes \mathbb{F})$ and $R^1q_*(\mathbb{E} \otimes \mathbb{F})$ are locally free over U. Suppose that the hyperplane H consists of divisors in $|\omega^2|$ containing a point $p \in C$. Then $\psi_L^*(H)$ is given by the determinant of the evaluation map over U (see (4.5))

$$q_*(\mathbb{E} \otimes \mathbb{F}) \xrightarrow{ev} \mathbb{E} \otimes \mathbb{F}_{|U \times \{p\}}.$$

Since $\det\,(\mathbb{E} \otimes \mathbb{F}_{|U\times\{p\}}) = \mathcal{O}_U$, the result will follow from the equality $\det q_*(\mathbb{E} \otimes \mathbb{F}) = \mathcal{O}_U(-4)$, which we prove by using some properties of the determinant line bundles [KM]. Given any family of bundles $\mathcal{F}$ over $U \times C$, we denote the determinant line bundle associated to the family $\mathcal{F}$ by $\det Rq_*(\mathcal{F})$. First we observe that by relative duality [K] we have

$$q_*(\mathbb{E} \otimes \mathbb{F}) \xrightarrow{\sim} \left(R^1q_*(\mathbb{E} \otimes \mathbb{F})\right)^*,$$

hence $\det Rq_*(\mathbb{E} \otimes \mathbb{F}) = (\det q_*(\mathbb{E} \otimes \mathbb{F}))^{\otimes 2}$. Next we tensor (4.7) with $\mathbb{F}$

$$0 \longrightarrow \mathbb{F} \otimes p^*L^{-1} \longrightarrow \mathbb{E} \otimes \mathbb{F} \longrightarrow \mathbb{F} \otimes p^*L \otimes q^*\mathcal{O}(-1) \longrightarrow 0.$$

Since $\det Rq_*$ is multiplicative, we obtain

$$\det Rq_*(\mathbb{E} \otimes \mathbb{F}) \cong \det Rq_*(\mathbb{F} \otimes p^*L^{-1}) \otimes \det Rq_*(\mathbb{F} \otimes p^*L \otimes q^*\mathcal{O}(-1)).$$

Again by relative duality we have $\det Rq_*(\mathbb{F} \otimes p^*L^{-1}) \cong \det Rq_*(\mathbb{F} \otimes p^*L \otimes q^*\mathcal{O}(-1))$, hence (since $\mathrm{Pic}(U) = \mathbb{Z}$) we can divide by 2 to obtain

$$\det q_*(\mathbb{E} \otimes \mathbb{F}) \cong \det Rq_*(\mathbb{F} \otimes p^*L \otimes q^*\mathcal{O}(-1)) \cong \det Rq_*(\mathbb{F} \otimes p^*L) \otimes \mathcal{O}(-2).$$

The last equation holds since $\chi(F_eL) = 2$. Finally, we apply the functor $\det Rq_*$ to the dual of (4.8)

$$\begin{aligned} \det Rq_*(\mathbb{F} \otimes p^*L) &\cong \det Rq_*(q^*\mathbb{W}^*) \otimes \det Rq_*(p^*\omega L^2)^{-1} \\ &\cong (\det \mathbb{W}^*)^{\otimes \chi(\mathcal{O})} \cong \mathcal{O}(-2). \end{aligned}$$

which proves $\det q_*(\mathbb{E} \otimes \mathbb{F}) = \mathcal{O}(-4)$.

We also deduce from this construction that the exceptional locus of the rational discriminant map Δ is the union of the Kummer variety $\mathcal{K}_0$, the exceptional bundle A_0, and the 288 Aronhold bundles $A(\theta, x)$. Therefore the map Δ is given by the composite of Γ with a projection map $\pi : |\overline{\mathcal{L}}|^* \cong |3\Theta|_+ \longrightarrow |\omega^2|$, whose center of projection $\ker \pi$ contains $\mathrm{Span}(\mathcal{K}_0) = |2\Theta|$. In order to show that $\ker \pi = |2\Theta|$, it suffices (for dimensional reasons) to show that Δ is dominant:

Consider a general divisor $\delta = a_1 + \cdots + a_8 \in |\omega^2|$ and choose $M \in \mathrm{Pic}^2(C)$ such that $a_1 + \cdots + a_4 \in |M^2|$, or equivalently $a_5 + \cdots + a_8 \in |\omega^2 M^{-2}|$. Using Lemma 3.3 we can find a stable $E \in \mathcal{T}_0$ such that $[\mathcal{D}(E)] = [M \oplus \omega M^{-1}]$. We easily deduce from Remark 3.4(2) that $\Delta(E) = \delta$.

Finally, we deduce from the natural exact sequence associated to the divisor $\Theta \subset \mathrm{Pic}^2(C)$

$$0 \longrightarrow H^0(\mathrm{Pic}^2(C), 2\Theta) \xrightarrow{+\Theta} H^0(\mathrm{Pic}^2(C), 3\Theta)_+ \xrightarrow{res_\Theta} H^0(\Theta, 3\Theta_{|\Theta})_+ \longrightarrow 0$$

that the projectivized restriction map res_Θ identifies with the projection π. □

4.14 Remark. Geometrically the assertion on the exceptional locus of Δ given in the proof means that (we map $\mathcal{N}$ via Γ into $|3\Theta|_+$)

$$\mathcal{N} \cap |2\Theta| = \mathcal{K}_0 \cup \{A_0\} \cup \{A(\theta, x)\},$$

or equivalently, that the 3θ-divisors $\Gamma(A_0)$ and $\Gamma(A(\theta, x))$ are reducible of the form

$$\Gamma(A_0) = \Theta + \Gamma^{res}(A_0), \qquad \Gamma(A(\theta, x)) = \Theta + \Gamma^{res}(A(\theta, x)),$$

where the residual divisors $\Gamma^{res}(A_0)$ and $\Gamma^{res}(A(\theta, x))$ lie in $|2\Theta|$. This can be checked directly.

(1) exceptional bundle A_0: since $\Theta \cong \mathrm{Sym}^2 C$, the inclusion $\Theta \subset \Gamma(A_0)$ is equivalent to $\dim H^0(C, \mathrm{Sym}^2(A_0) \otimes \omega^{-1}(p+q)) > 0$, $\forall p, q \in C$ (here we take $A_0 \in \mathcal{M}_\omega$ see (2.1)), or $\dim H^0(C, \mathrm{Sym}^2(A_0)(-u-v)) > 0$, $\forall u, v \in C$. But this follows immediately from $\dim H^0(C, A_0) = 3$, which implies that $\forall u$ there exists a nonzero section $s_u \in H^0(C, A_0(-u))$. Taking the symmetric product, we obtain $s_u \cdot s_v \in H^0(C, \mathrm{Sym}^2(A_0)(-u-v))$.

(2) Aronhold bundles $A(\theta, x)$: similarly we have to show that $\dim H^0(C, \mathrm{Sym}^2(A) \otimes \omega(-p-q)) > 0$, $\forall p, q \in C$ (take $A = A(\theta, L, x) \in \mathcal{M}_0$). Since $\mathbf{M}_0(A)$ is invariant under the involution $L_i \mapsto \theta L_i^{-1}$, we have $\mathcal{D}(A) = A \otimes \theta$ and $\dim H^0(C, A \otimes A \otimes \theta) = \dim H^0(C, \mathrm{Sym}^2(A) \otimes \theta) = 4$. Hence $\forall p$ there exists a nonzero section $s_p \in H^0(C, \mathrm{End}_0(A) \otimes \theta(-p))$ (note that $\mathrm{End}_0(A) = \mathrm{Sym}^2(A)$) and by taking the End_0-part of the composite section $s_p \circ s_q$, we obtain a nonzero element of $H^0(C, \mathrm{Sym}^2(A) \otimes \omega(-p-q))$.

Moreover it can be shown by standard methods that $\mathrm{Sym}^2(A_0)$ and $\mathrm{Sym}^2(A(\theta, x))$ are stable bundles. It would be interesting to describe explicitly the 2θ-divisors $\Gamma^{res}(A_0)$ and $\Gamma^{res}(A(\theta, x))$, which, we suspect, do not lie on the Coble quartic $\mathcal{M}_0$.

4.5 The action of the Weyl group $W(E_7)$

The aim of this section is to show that the Hessian map (section 4.3), which associates to a $\mathbb{P}\mathrm{SL}_2$-bundle $\overline{E} \in \mathcal{N}^{reg}$ the isomorphism class of the smooth curve $\mathrm{Hess}(\overline{E}) \in \mathcal{M}_3$, is dominant.

4.5.1 Some group theory related to genus 3 curves

We recall here (see e.g. [A], [DO], [Ma]) the main results on root lattices and Weyl groups. Let $\Gamma \subset \mathbb{P}^2$ be a smooth plane quartic and let V be its associated degree 2 del Pezzo surface,i.e. the degree 2 cover $\pi : V \to \mathbb{P}^2$ branched along the curve Γ. We choose an isomorphism (called geometric marking of V) of the Picard group $\mathrm{Pic}(V)$,

$$\varphi : \mathrm{Pic}(V) \xrightarrow{\sim} H_7 = \bigoplus_{i=0}^{7} \mathbb{Z} e_i, \tag{4.9}$$

with the hyperbolic lattice H_7, such that φ is orthogonal for the intersection form on $\mathrm{Pic}(V)$ and the quadratic form on H_7 defined by $e_0^2 = 1$; $e_i^2 = -1$, $(i \neq 0)$; $e_i \cdot e_j = 0$, $(i \neq j)$. The anti-canonical class $-k$ of V equals $3e_0 - \sum_{i=1}^{7} e_i$. We put $e_8 := \sum_{i=1}^{7} e_i - 2e_0 = e_0 + k$. Then the 63 positive roots of H_7 are of two types

$$(1)\ \alpha_{ij} = e_i - e_j,\ 1 \leq i < j \leq 8, \qquad (2)\ \alpha_{ijk} = e_0 - e_i - e_j - e_k,\ 1 \leq i < j < k \leq 7. \tag{4.10}$$

The 28 roots of type (1) correspond to the 28 positive roots of the Lie algebra $\mathfrak{sl}_8$ seen as a subalgebra of the exceptional Lie algebra $\mathfrak{e}_7$. Similarly the 56 exceptional lines of H_7 are of two types

$$(1)\ l_{ij} = e_i + e_j - e_8, \qquad (2)\ l'_{ij} = e_0 - e_i - e_j, \qquad 1 \leq i < j \leq 8. \tag{4.11}$$

The Weyl group $W(\mathrm{SL}_8)$ equals the symmetric group Σ_8 and is generated by the reflections s_{ij} associated to the roots α_{ij} of type (1). The action of the reflection s_{ij} on the exceptional lines l_{pq} and l'_{pq} is given by applying the transposition (ij) to the indices pq. The Weyl group $W(E_7)$ is generated by the reflections s_{ij} and s_{ijk} (associated to α_{ijk}) and the reflection s_{ijk} acts on the exceptional lines as follows

- if $|\{i,j,k,8\} \cap \{p,q\}| = 1$, then $s_{ijk}(l_{pq}) = l_{pq}$,
- if $|\{i,j,k,8\} \cap \{p,q\}| = 0$ or 2, then $s_{ijk}(l_{pq}) = l'_{st}$ such that $\{p,q,s,t\}$ equals $\{i,j,k,8\}$ or its complement in $\{1,\ldots,8\}$.

Let us consider the restriction map $\mathrm{Pic}(V) \xrightarrow{res} \mathrm{Pic}(\Gamma)$ to the ramification divisor $\Gamma \subset V$. Then we have the beautiful fact (see [DO] Lemma 8 page 190) that res maps bijectively the 63 positive roots $\{\alpha_{ij}, \alpha_{ijk}\}$ (4.10) to the 63 nonzero 2-torsion points $J\Gamma[2] \setminus \{0\}$, thus endowing the Jacobian $J\Gamma$ with a level-2-structure,i.e. a symplectic isomorphism $\psi : J\Gamma[2] \cong \mathbb{F}_2^3 \times \mathbb{F}_2^3$ (for the details, see [DO] chapter 9). We also observe that the partition of $J\Gamma[2]$ into the two sets $\{res(\alpha_{ij})\}$ (28 points) and $\{res(\alpha_{ijk}), 0\}$ (36 points) corresponds to the partition into odd and even points (w.r.t. the level-2-structure ψ). Moreover the images of the 56 exceptional lines (4.11) are the 28 odd theta-characteristics on Γ, which we denote by $res(l_{ij}) = res(l'_{ij}) = \theta_{ij}$. Moreover $\pi(l_{ij}) = \pi(l'_{ij}) = \Lambda_{ij}$, where Λ_{ij} is the bitangent to Γ corresponding to θ_{ij}.

Two geometric markings φ, φ' (4.9) differ by an element $g \in O(H_7) = W(E_7)$ and their induced level-2-structures ψ, ψ' differ by $\bar{g} \in \mathrm{Sp}(6, \mathbb{F}_2)$. The restriction map $W(E_7) \to \mathrm{Sp}(6, \mathbb{F}_2)$, $g \mapsto \bar{g}$ is surjective with kernel $\mathbb{Z}/2 = \langle w_0 \rangle = \mathrm{Center}(W(E_7))$. The element $w_0 \in W(E_7)$ acts as -1 on the root lattice, leaves k invariant $w_0(k) = k$ and exchanges the exceptional lines $w_0(l_{ij}) = l'_{ij}$.

We also note that $w_0 \notin \Sigma_8 \subset W(E_7)$ and that the injective composite map $\Sigma_8 \to W(E_7) \to \mathrm{Sp}(6, \mathbb{F}_2)$ identifies Σ_8 with the stabilizer of an even theta-characteristic.

4.5.2 Two moduli spaces with $W(E_7)$-action

We introduce the Σ_8-Galois cover $\widetilde{\mathcal{M}_0} \to \mathcal{M}_0^{reg}$ parametrizing stable bundles $E \in \mathcal{M}_0^{reg}$ with an order on the 8 line subbundles $\mathbf{M}_0(E) = \{L_1, \ldots, L_8\}$. The group $JC[2]$ acts on $\widetilde{\mathcal{M}_0}$ and we denote the quotient $\widetilde{\mathcal{M}_0}/JC[2]$ by $\widetilde{\mathcal{N}}$, which is a Σ_8-Galois cover $\widetilde{\mathcal{N}} \to \mathcal{N}^{reg}$. The polar map $\overline{\mathcal{D}} : \mathcal{N} \to \mathcal{N}$ (4.6) lifts to a Σ_8-equivariant birational involution $\widetilde{\mathcal{D}} : \widetilde{\mathcal{N}} \to \widetilde{\mathcal{N}}$.

We also consider the moduli space $\mathcal{P}_C$ parametrizing pairs (Γ, φ), with $\Gamma \subset |\omega|^* = \mathbb{P}^2$ a smooth plane quartic curve which satisfies $\Gamma \cap C = 2\Delta$ and $\Delta \in |\omega^2|$, and φ a geometric marking (4.9) for the Del Pezzo surface V associated to Γ. Then the forgetful map $(\Gamma, \varphi) \mapsto \Gamma$ realizes $\mathcal{P}_C$ as a $W(E_7)$-Galois cover of the space $\mathcal{R}$ of smooth quartic curves Γ satisfying the above intersection property. Since the general fibre $f^{-1}(\Delta)$ of the projection map $\mathcal{R} \xrightarrow{f} |\omega^2|$ corresponds to the pencil of curves spanned by the curve C and the double conic Q^2 defined by $Q \cap C = \Delta$, we see that $\mathcal{R}$ is an open subset of a $\mathbb{P}^1$-bundle over $|\omega^2|$, hence rational.

4.15 Proposition. *The Hessian map (section 4.3) induces a birational map*

$$\widetilde{\mathrm{Hess}} : \widetilde{\mathcal{N}} \longrightarrow \mathcal{P}_C,$$

which endows $\widetilde{\mathcal{N}}$ with a $W(E_7)$-action. The action of w_0 corresponds to the polar map $\widetilde{\mathcal{D}}$.

Proof. Let $\overline{E} \in \widetilde{\mathcal{N}}$ be represented by $E \in \mathcal{M}_0^{reg}$ and by an ordered set $\mathbf{M}_0(E) = \{L_1, \ldots, L_8\}$. In order to construct the data (Γ, φ), we consider the Del Pezzo surface $V \xrightarrow{\pi} \mathbb{P}^2$ associated to the Hessian curve $\Gamma = \mathrm{Hess}(E) \subset |\omega|^* = \mathbb{P}^2$. Since $\Gamma \cap C = 2\Delta(E)$, the preimage $\pi^{-1}(C) \subset V$ splits into two irreducible components $C_1 \cup C_2$, with $C_1 = C_2 = C$. More generally, it can be shown that the preimage $\pi^{-1}(C \times \mathcal{R}) \subset \mathcal{V}$ has two irreducible components, where $\mathcal{V} \to \mathcal{R}$ is the family of Del Pezzo's parametrized by $\mathcal{R}$. This allows us to choose uniformly a component C_1. Then by Proposition 4.9 the secant line $\overline{D}_{ij}$ coincides with a bitangent to Γ. Therefore the preimage $\pi^{-1}(\overline{D}_{ij})$ splits into two exceptional lines and we denote by l_{ij} the line which cuts out the divisor D_{ij} on the curve $C_1 = C$. Then the other line l'_{ij} cuts out the divisor D'_{ij} on C_1 with $D_{ij} + D'_{ij} \in |\omega|$. Now it is immediate to check that the classes $e_i = l_{i8}$ for $1 \leq i \leq 7$ and $e_0 = e_i + e_j - l_{ij} - k$ determine a geometric marking (4.9).

Conversely, given V and a geometric marking φ, we choose a line bundle $L_8 \in \mathrm{Pic}^1(C)$ such that $\omega L_8^2 = e_{0|C=C_1}$. Next we define L_i for $1 \leq i \leq 7$ by $L_i L_8 = e_{i|C=C_1}$. Then one verifies that $l_{ij|C=C_1} = L_i L_j$ and therefore by Remark 4.3 there exists a bundle $E \in \mathcal{M}_0$ such that $\mathbf{M}_0(E) = \{L_1, \ldots, L_8\}$. Since L_8 is defined up to $JC[2]$, this construction gives an element of $\widetilde{\mathcal{N}}$.

Since the element $\overline{E} \in \widetilde{\mathcal{N}}$ is determined by the 28 line bundles $L_i L_j$, it will be enough to describe the action of $\widetilde{\mathcal{D}}$ and $w_0 \in W(E_7)$ on the $L_i L_j$'s. Suppose $\widetilde{\mathcal{D}}(\overline{E}) = \overline{F}$ with $\mathbf{M}_0(F) = \{M_1, \ldots, M_8\}$, then it follows from the equality $\mathbf{M}_\omega(F) = \mathbf{M}_0(E)$ (assuming $F = \mathcal{D}(E)$) that $M_i M_j = \omega L_i^{-1} L_j^{-1}$. On the other hand we have $w_0(l_{ij}) = l'_{ij}$ and $l_{ij} + l'_{ij} = -k$. Restricting to $C = C_1$ $(-k_{|C} = \omega)$, we obtain that $w_0 = \widetilde{\mathcal{D}}$. □

4.16 Corollary. *The morphism* $\mathrm{Hess} : \mathcal{N}^{reg} \longrightarrow \mathcal{R}$, $\overline{E} \longmapsto \mathrm{Hess}(\overline{E})$ *is finite of degree* 72 *and, if C is general, the map*

$$\mathcal{N}^{reg} \longrightarrow \mathcal{M}_3, \qquad \overline{E} \longmapsto \textit{iso class}(\mathrm{Hess}(\overline{E}))$$

is dominant.

Proof. The first assertion follows from $|W(E_7)/\Sigma_8| = 72$. For the second it suffices to show that the forgetful map $\mathcal{R} \to \mathcal{M}_3$ is dominant: let $[C] \in |\mathcal{O}_{\mathbb{P}^2}(4)| = \mathbb{P}^{14}$ denote the quartic equation of C. Projection with center $[C]$ maps $|\mathcal{O}_{\mathbb{P}^2}(4)| \longrightarrow |\omega^4|$. We immediately see that $\mathcal{R}$ equals the cone with vertex $[C]$ over the Veronese variety $\mathrm{Ver}|\omega^2| \hookrightarrow |\omega^4|$. If C is general, one can show (e.g. by computing the differential of the natural map $\mathbb{P}\mathrm{GL}_3 \times \mathcal{R} \longrightarrow |\mathcal{O}_{\mathbb{P}^2}(4)|$) that the $\mathbb{P}\mathrm{GL}_3$-orbit of the cone $\mathcal{R}$ (note that $\dim \mathcal{R} = 6$) in $|\mathcal{O}_{\mathbb{P}^2}(4)| = \mathbb{P}^{14}$ is dense and since $\mathcal{M}_3 = |\mathcal{O}_{\mathbb{P}^2}(4)|/\mathbb{P}\mathrm{GL}_3$, we obtain the result.

□

4.17 Remark. The action of the reflection $s_{ijk} \in W(E_7)$ on $\widetilde{\mathcal{N}}$ is easily deduced from its action on the exceptional lines l_{pq} and l'_{pq} (see section 4.5.1). Representing an element $\overline{E} \in \widetilde{\mathcal{N}}$ by $e \in |\omega L_8^2|^*$, it is easily checked that the restriction of s_{ijk} to $|\omega L_8^2|^*$ is given by the linear system of quadrics on $|\omega L_8^2|^*$ passing through the 6 points $D_{ijk} = D_{i8} + D_{j8} + D_{k8}$. In this way we can construct the $72 = 2(1 + \binom{7}{3})$ bundles in the fibre of $\mathrm{Hess} : \mathcal{N}^{reg} \to \mathcal{R}$.

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