Tôhoku Math. J.
49 (1997), 259–275

SELF-DUALITY OF METRICS OF TYPE (2, 2) ON FOUR-DIMENSIONAL MANIFOLDS

HIROYUKI KAMADA[1] AND YOSHINORI MACHIDA[2]

(Received November 22, 1995, revised December 26, 1995)

Abstract. We study self-duality of pseudo-Riemannian metrics of type (2, 2). We show correspondences between self-duality of Riemannian metrics and that of pseudo-Riemannian metrics of type (2, 2) under appropriate conditions. Moreover we give global constructions for four-dimensional manifolds with self-dual metrics of type (2, 2).

1. Introduction. The twistor theory, constructed by Penrose for four-dimensional *complex* space-times, is the double fibering correspondence between these space-times and the complex three-dimensional manifolds called twistor spaces. Penrose and his colleagues investigated conformally invariant equations on these space-times (e.g., massless field equations, self-dual Yang-Mills equations) by transforming them into more tractable objects in algebraic and complex analytic geometry of the twistor spaces (e.g., cohomology, holomorphic vector bundles). When one generalizes the twistor theory constructed for the flat space-times to that for curved space-times, it is then essential to deform complex structures on the corresponding twistor spaces under appropriate conditions. This leads to the fact that the curved space-times under consideration enjoy self-duality (or half conformal flatness) of the Weyl conformal curvature tensor, and vice versa. See Penrose-Ward [P-W], Ward-Wells [W-W].

Subsequently, from another point of view, Atiyah, Hitchin and Singer [A-H-S] constructed the twistor theory for *real* four-dimensional Riemannian manifolds with *positive definite metrics* ((4, 0)-*metrics*). Their construction can also be carried out under self-duality of the conformal tensor.

Motivated by these results, we will study in this paper self-duality for (2, 2)-metrics, as a first step to constructing the twistor theory for *real* four-dimensional pseudo-Riemannian manifolds with *neutral metrics* ((2, 2)-*metrics*). For the twistor theory for (2, 2)-metrics, see also [M-S]. More precisely, as our standpoint, we will investigate the relation between self-duality for (4, 0)-metrics and self-duality for (2, 2)-metrics in detail. In particular, we will show, for self-dual, Einstein and Kählerian cases, the exact correspondence between Bianchi type IX (4, 0)-metrics and Bianchi type VIII (2, 2)-

* 1991 *Mathematics Subject Classification*. Primary 53C50; Secondary 53B30, 53C25, 83C60, 83C20.

[1] Partly supported by the Grants-in-Aid for Scientific as well as Co-operative Research, The Ministry of Education, Science and Culture, Japan.

[2] Partly supported by the Grants-in-Aid for Scientific Research, The Ministry of Education, Science and Culture, Japan.

metrics. Furthermore, we will also give global constructions for four typical examples.

In mathematical physics, (2, 2)-metrics have appeared in various context. For instance, it has been known that, from the compatibility of quantization, the target spaces of N = 2 superstrings must be four-dimensional as a critical dimension and have Kähler, Ricci-flat (2, 2)-metrics. See Ooguri and Vafa [O-V]. In the effective theory, an N = 2 superstring is regarded as only one massless scalar field, which is identified with the Kähler potential. It is also well-known that two-dimensional nonlinear principal sigma models (harmonic maps) are obtained from the reduction of self-dual Yang-Mills fields (gauge fields) on four-dimensional spaces with (2, 2)-metrics (cf. Hitchin [H], Ward [Wa]).

However, compared with those for Riemannian metrics ((4, 0)-metrics) and for Lorentzian metrics ((3, 1)-metrics), the study for (2, 2)-metrics are not well explored. It is therefore worth while studying (2, 2)-metrics which are not well understood.

This paper is organized as follows: In §2, for later convenience, we review several conditions for four-dimensional manifolds to admit (2, 2)-metrics and examine some examples.

In §3, we recall the decomposition of the curvature tensor of pseudo-Riemannian manifolds with (2, 2)-metrics and define the notion of (conformal) self-duality of (2, 2)-metrics. Also, we study self-duality of (2, 2)-metrics on (2, 2)-Kähler surfaces.

In §4, we show that a (2, 2)-metric of Bianchi type VIII is self-dual (resp. anti-self-dual, Einstein, Kähler) if and only if the (4, 0)-metric of Bianchi type IX with the same coefficients is anti-self-dual (resp. self-dual, Einstein, Kähler). To show these correspondences, we first recall polar coordinates of the pseudo-Euclidean space $\boldsymbol{R}_2^4$ of signature (2, 2), and then introduce Bianchi type VIII (2, 2)-metrics and Bianchi type IX (4, 0)-metrics. We also obtain, for self-duality and Kählerian property, the exact correspondence between Gibbons-Hawking type (2, 2)- and (4, 0)-metrics.

§5 is devoted to the study of removable singularities of Bianchi type VIII (2, 2)-metrics, which are called nut and bolt singularities. We define four typical Bianchi type VIII (2, 2)-metrics (which are the counterparts of those Bianchi type IX (4, 0)-metrics):

(1) Fubini-Study type (anti-self-dual, Einstein and Kähler)
(2) Eguchi-Hanson type (self-dual, Ricci-flat and Kähler)
(3) Taub-NUT type (anti-self-dual, Ricci-flat and non-Kähler)
(4) LeBrun type (self-dual, scalar-flat and Kähler).

Furthermore, since these metrics have only nut or bolt singularities, we can construct topologically manifolds with globally defined (anti-)self-dual (2, 2)-metrics by removing the singularities and extending the metrics. We also show that these examples with suitable orientations can be constructed, at least locally, as a one-parameter family of anti-self-dual (2, 2)-metrics. Finally, we reconstruct these manifolds by modifying the indefinite complex hyperbolic space $H_1^2(\boldsymbol{C})$.

In this paper, we will use the following convention and notation (cf. Wolf [Wo]). Let $\boldsymbol{R}_s^n$ denote the vector space of real n-tuples $x = (x_0, \ldots, x_{n-1})$ with the inner product

$$(g_0)^n_s(x, y) = -\sum_{\alpha=0}^{s-1} x_\alpha y_\alpha + \sum_{i=s}^{n-1} x_i y_i .$$

Then $\boldsymbol{R}^n_s$ can be regarded as a pseudo-Riemannian manifold of signature $(n-s, s)$, which is often called the *pseudo-Euclidean space* of signature $(n-s, s)$. Moreover we define

$$S^n_s = \{x \in \boldsymbol{R}^{n+1}_s \mid (g_0)^{n+1}_s(x, x) = 1\} , \quad H^n_s = \{x \in \boldsymbol{R}^{n+1}_{s+1} \mid (g_0)^{n+1}_{s+1}(x, x) = -1\} .$$

It is well-known that S^n_s and H^n_s are complete pseudo-Riemannian manifolds of signature $(n-s, s)$ and of constant curvature $+1$ and -1, respectively. S^n_s is called the *pseudo-Riemannian sphere* and H^n_s the *pseudo-Riemannian hyperbolic space*.

The authors would like to thank Professor Seiki Nishikawa for his helpful suggestions.

2. Existence of (2, 2)-metrics. We begin by recalling several conditions for four-dimensional manifolds to admit (2, 2)-metrics and examine some examples.

It is well-known that a four-dimensional manifold admits a (2, 2)-metric if and only if there exists a two-plane field, that is, a two-dimensional tangential distribution, on it (see Steenrod [S]). Concerning the existence of two-plane fields, the following conditions have been known. Given a compact oriented four-dimensional manifold M, Hirzebruch and Hopf [H-H] obtained a necessary and sufficient condition for M to admit an orientable two-plane field in terms of the Euler characteristic $\chi(M)$ and the Hirzebruch signature $\tau(M)$. Comparing their condition with that of Wu [Wu] concerning the existence of almost complex structures, we can see that M admits an orientable two-plane field if and only if it has two almost complex structures, one being compatible with the fixed orientation and the other being compatible with the reversed orientation (see Matsushita [M]). If M has two such almost complex structures, then it admits a (2, 2)-metric. Furthermore, if M is simply connected, the converse is also true. As an application, we see that the four-dimensional sphere S^4 admits no (2, 2)-metrics, since there are no almost complex structures on S^4.

For a compact complex surface M, Matsushita [M] proved that the existence of an orientable two-plane field on M is equivalent to $\chi(M) \equiv 0 \pmod 2$. Thus we see that $P^2(\boldsymbol{C})$, the complex projective space of complex dimension two with canonical orientation, admits no (2, 2)-metrics, since $\chi(P^2(\boldsymbol{C}))$ is three. In general, it is known that the n-times connected sum of $P^2(\boldsymbol{C})$'s, $nP^2(\boldsymbol{C}) = P^2(\boldsymbol{C}) \sharp \cdots \sharp P^2(\boldsymbol{C})$, admits no (2, 2)-metrics (see [M]).

On the other hand, there exist many four-dimensional manifolds which admit (2, 2)-metrics. For example, the following manifolds admit (2, 2)-metrics: the total space of a fibre bundle on a surface with two-dimensional fibres; $P^2(\boldsymbol{C}) \sharp \overline{P^2(\boldsymbol{C})}$, where $\overline{P^2(\boldsymbol{C})}$ is the $P^2(\boldsymbol{C})$ with the orientation reversed; $K3$ surfaces.

3. Self-duality of (2, 2)-metrics. We recall briefly some basic facts about four-dimensional pseudo-Riemannian geometry of (2, 2)-metrics (cf. Besse [B]).

Let (M, g) be an oriented four-dimensional pseudo-Riemannian manifold with a (2, 2)-metric g. As in the Riemannian case, the Hodge star operator $*$ of (M, g) satisfies $*^2 = \mathrm{Id}$ on $\bigwedge^2 T^*M$, the bundle of 2-forms. Thus $\bigwedge^2 T^*M$ decomposes as $\bigwedge^2 T^*M = \bigwedge_+ \oplus \bigwedge_-$, where $\bigwedge_\pm = \{\alpha \in \bigwedge^2 T^*M \mid *\alpha = \pm\alpha\}$. This corresponds to the Lie algebra decomposition $\mathfrak{o}(2, 2) = \mathfrak{sl}(2, \boldsymbol{R}) \oplus \mathfrak{sl}(2, \boldsymbol{R})$. Denote respectively by R, Z, s and W, the curvature tensor, the traceless Ricci tensor, the scalar curvature and the Weyl conformal tensor of (M, g). Considering them as endomorphisms on $\bigwedge^2 T^*M$, we see that $*W = W*$ and $*Z = -Z*$. Thus we have the following decomposition:

$$R = (W_+ \oplus W_-) \oplus Z \oplus (s/12)\mathrm{Id}\,, \tag{3.1}$$

where $W_\pm = W|_{\bigwedge_\pm}$ are endomorphisms on $\bigwedge_\pm$, respectively. These being understood we recall the following:

DEFINITION. A (2, 2)-metric g is said to be *self-dual* (resp. *anti-self-dual*) if $W_- \equiv 0$ (resp. $W_+ \equiv 0$). A (2, 2)-metric g is said to be *Einstein* if $Z \equiv 0$.

We note that the notion of (anti-)self-duality of (2, 2)-metrics is conformally invariant.

REMARK. Recalling the Gauss-Bonnet formula for a compact oriented four-dimensional pseudo-Riemannian manifold (M, g) with (2, 2)-metric g (see Avez [A], Chern [C]), we can write the Euler characteristic $\chi(M)$ in terms of the components of the curvature tensor as

$$\chi(M) = -\frac{1}{8\pi^2}\int_M \left\{\|W_+\|^2 + \|W_-\|^2 - \|Z\|^2 + \frac{s^2}{24}\right\} v_g\,, \tag{3.2}$$

$\|\cdot\|^2$ being the squared norm. We can also write the Hirzebruch signature $\tau(M)$ as

$$\tau(M) = \frac{1}{12\pi^2}\int_M \{\|W_+\|^2 - \|W_-\|^2\} v_g\,. \tag{3.3}$$

Since the squared norm $\|\cdot\|^2$ is indefinite, $\|T\|^2 = 0$ no longer implies $T \equiv 0$ in general. So we are led to the following question: *Does there exist a manifold which admits a* (2, 2)*-metric but does not admit any* (*anti-*)*self dual* (2, 2)*-metric?* In particular, we are interested in whether or not the following manifolds admit (anti-)self-dual (2, 2)-metrics: (1) $P^2(\boldsymbol{C}) \# \overline{P^2(\boldsymbol{C})}$, (2) $S^1 \times S^3$, (3) $K3$ surfaces, etc.

We next consider an almost complex structure J on (M, g). If J is compatible with (2, 2)-metric g, then (M, g, J) is called an *almost* (2, 2)*-Hermitian manifold*. If J is also parallel with respect to the Levi-Civita connection of (M, g), then (M, g, J) is called a (2, 2)*-Kähler surface*. Note that J is integrable if (M, g, J) is (2, 2)-Kähler. In the following, we assume that J is compatible with the orientation of (M, g).

As in the Riemannian case (cf. Derdziński [D], Itoh [I]), we obtain the following:

PROPOSITION 3.1. *Let (M, g, J) be a $(2, 2)$-Kähler surface. Then (M, g) is self-dual if and only if (M, g) is scalar-flat.*

PROPOSITION 3.2. *Suppose that (M, g, J) is a compact $(2, 2)$-Kähler surface. If g is an anti-self-dual $(2, 2)$-metric, then $\tau(M)$ is nonpositive. Furthermore, $\tau(M)=0$ if and only if (M, g) is scalar-flat and conformally flat.*

PROPOSITION 3.3. *Suppose that (M, g, J) is a compact $(2, 2)$-Kähler surface. If g is an anti-self-dual, Einstein $(2, 2)$-metric, then $\chi(M)$ is nonpositive. Furthermore, $\chi(M)=0$ if and only if (M, g) is flat.*

4. Bianchi type VIII (2, 2)-metrics. A four-dimensional pseudo-Riemannian manifold (M, g) is said to be *spatially homogeneous* if a three-dimensional Lie group acts isometrically on (M, g) and simply transitively on each orbit. According to the classification of three-dimensional Lie groups, spatially homogeneous pseudo-Riemannian manifolds are classified into nine types called Bianchi types. In this section, we introduce Bianchi type IX (4, 0)-metrics and Bianchi type VIII (2, 2)-metrics.

4.1. Polar coordinates of $\boldsymbol{R}_2^4$. We recall the standard (2, 2)-metric of $\boldsymbol{R}_2^4=(\boldsymbol{R}^4, g_0)$:

$$g_0=-dx_0^2-dx_1^2+dx_2^2+dx_3^2\,, \tag{4.1.1}$$

where (x_0, x_1, x_2, x_3) denote the standard coordinates of $\boldsymbol{R}^4$. For simplicity, we set

$$\boldsymbol{R}_{2\pm}^4=\{(x_0,\dots,x_3)\mid \pm(x_0^2+x_1^2-x_2^2-x_3^2)>0\}\,,\quad \mathcal{N}=\{(x_0,\dots,x_3)\mid x_0^2+x_1^2=x_2^2+x_3^2\}\,.$$

$\mathcal{N}$ is often called the nullcone of $\boldsymbol{R}_2^4$. Then $\boldsymbol{R}^4$ decomposes as $\boldsymbol{R}^4=\boldsymbol{R}_{2-}^4\coprod\mathcal{N}\coprod\boldsymbol{R}_{2+}^4$ (disjoint union).

Note that $\boldsymbol{R}_{2+}^4$ is identified with $\boldsymbol{R}_+\times SL(2,\boldsymbol{R})$ by the isomorphism $x=(x_0,\dots, x_3)\mapsto(r,\Theta)$, where

$$r=(x_0^2+x_1^2-x_2^2-x_3^2)^{1/2}\quad\text{and}\quad \Theta=\frac{1}{r}\begin{pmatrix} x_0-x_2 & -x_1+x_3\\ x_1+x_3 & x_0+x_2\end{pmatrix}.$$

We call (r,Θ) polar coordinates of $\boldsymbol{R}_{2+}^4$. We define an orthonormal basis $\{e_0, e_1, e_2, e_3\}$ for $\boldsymbol{R}\oplus\mathfrak{sl}(2,\boldsymbol{R})$ as follows:

$$e_0=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad e_1=\begin{pmatrix}0&-1\\1&0\end{pmatrix},\quad e_2=\begin{pmatrix}-1&0\\0&1\end{pmatrix},\quad e_3=\begin{pmatrix}0&1\\1&0\end{pmatrix}, \tag{4.1.2}$$

which forms an orthonormal frame field for $\boldsymbol{R}_{2+}^4\cong\boldsymbol{R}_+\times SL(2,\boldsymbol{R})$ such that

$$g_0(e_0,e_0)=g_0(e_1,e_1)=-1\,,\quad g_0(e_2,e_2)=g_0(e_3,e_3)=1\,, \tag{4.1.3}$$

$$[e_2,e_3]=2e_1\,,\quad [e_3,e_1]=-2e_2\,,\quad [e_1,e_2]=-2e_3\,,\quad [e_0,e_A]=O\ \ (0\le A\le 3)\,. \tag{4.1.4}$$

Note that $\{e_1, e_2, e_3\}$ in (4.1.2) defines an orthonormal frame field for $SL(2,\boldsymbol{R})$ satisfying (4.1.3) and (4.1.4). We denote by $\{\tau_1,\tau_2,\tau_3\}$ the dual coframe field consisting of left-invariant 1-forms on $SL(2, \boldsymbol{R})$, which satisfies the following structure equations:

(4.1.5) $$d\tau_1 = -2\tau_2 \wedge \tau_3\,, \quad d\tau_2 = 2\tau_3 \wedge \tau_1\,, \quad d\tau_3 = 2\tau_1 \wedge \tau_2\,.$$

Then the standard metric g_0 on $\boldsymbol{R}^4_{2+}$ can be expressed as

(4.1.6) $$g_0 = -dr^2 + r^2(-\tau_1^2 + \tau_2^2 + \tau_3^2)\,,$$

where r is the radial function in polar coordinates of $\boldsymbol{R}^4_{2+}$. We note that $SL(2, \boldsymbol{R})$ with the (2, 1)-metric $-\tau_1^2 + \tau_2^2 + \tau_3^2$ is isometric to $H^3_1 \subset \boldsymbol{R}^4_2$.

We also comment on the standard metric of $\boldsymbol{R}^4_{2-}$ and its representation in polar coordinates. For $(x_0, \dots, x_3)$ of $\boldsymbol{R}^4_{2-}$, we put $\rho = (-x_0^2 - x_1^2 + x_2^2 + x_3^2)^{1/2}$ (>0). Let σ be a 2×2 matrix such that $\det(\sigma) = -1$. Then σ gives an identification $\boldsymbol{R}^4_{2-}$ with $\boldsymbol{R}_+ \times SL(2, \boldsymbol{R})$, and hence the standard metric g_0 is expressed as

$$g_0 = d\rho^2 - \rho^2(-\tau_1^2 + \tau_2^2 + \tau_3^2)\,.$$

4.2. Bianchi type VIII and IX metrics. Let r be the coordinate function of $\boldsymbol{R}_+$, and f, a, b, c smooth (non-zero) functions of r.

A (2, 2)-metric g, defined on an open subset of $\boldsymbol{R}^4_{2+} = \boldsymbol{R}_+ \times SL(2, \boldsymbol{R})$, is of Bianchi type VIII if g is expressed in polar coordinates as

(4.2.1) $$g = -f(r)^2 dr^2 - a(r)^2\tau_1^2 + b(r)^2\tau_2^2 + c(r)^2\tau_3^2\,,$$

where $\{\tau_1, \tau_2, \tau_3\}$ is the orthonormal coframe field of left-invariant 1-forms on $SL(2, \boldsymbol{R}) = H^2_1$ satisfying the structure equations (4.1.5).

A (4, 0)-metric h, defined on an open subset of $\boldsymbol{R}^4 \setminus \{0\} = \boldsymbol{R}_+ \times SU(2)$, is of Bianchi type IX if h is expressed in polar coordinates as

$(\overline{4.2.1})$ $$h = f(r)^2 dr^2 + a(r)^2\sigma_1^2 + b(r)^2\sigma_2^2 + c(r)^2\sigma_3^2\,,$$

where $\{\sigma_1, \sigma_2, \sigma_3\}$ is the orthonormal coframe field of left-invariant 1-forms on $SU(2) = S^3$ satisfying the structure equations: $d\sigma_1 = 2\sigma_2 \wedge \sigma_3$, $d\sigma_2 = 2\sigma_3 \wedge \sigma_1$, $d\sigma_3 = 2\sigma_1 \wedge \sigma_2$.

Comparing the components of the curvature operators of g and h, we have the following:

THEOREM 4.1. *Let g and h be a* (2, 2)*-metric and a* (4, 0)*-metric defined respectively by* (4.2.1) *and* $(\overline{4.2.1})$. *Define almost complex structures J and $\bar{J}$ respectively by $J(fdr) = a\tau_1$, $J(b\tau_2) = c\tau_3$ and $\bar{J}(fdr) = a\sigma_1$, $\bar{J}(b\sigma_2) = c\sigma_3$. Then the following hold*:

(1) *g is self-dual* (*resp. anti-self-dual*) *if and only if h is anti-self-dual* (*resp. self-dual*).

(2) *g is Einstein if and only if so is h.*

(3) (g, J) *is Kähler if and only if so is* $(h, \bar{J})$.

PROOF. For convenience, we set $A = \dot{a}/f$, $B = \dot{b}/f$, $C = \dot{c}/f$ ($\dot{}$ means d/dr),

$$L=\frac{-a^2+b^2+c^2}{bc}, \qquad M=\frac{a^2-b^2+c^2}{ca}, \qquad N=\frac{a^2+b^2-c^2}{ab};$$

$$e^0=fdr, \quad e^1=a\tau_1, \quad e^2=b\tau_2, \quad e^3=c\tau_3;$$

$$\bar{e}^0=fdr, \quad \bar{e}^1=a\sigma_1, \quad \bar{e}^2=b\sigma_2, \quad \bar{e}^3=c\sigma_3;$$

$$\lambda^1_\pm=e^0\wedge e^1\pm e^2\wedge e^3, \quad \lambda^2_\pm=e^0\wedge e^2\pm e^1\wedge e^3, \quad \lambda^3_\pm=e^0\wedge e^3\pm e^2\wedge e^1;$$

$$\bar{\lambda}^1_\pm=\bar{e}^0\wedge \bar{e}^1\pm \bar{e}^2\wedge \bar{e}^3, \quad \bar{\lambda}^2_\pm=\bar{e}^0\wedge \bar{e}^2\pm \bar{e}^3\wedge \bar{e}^1, \quad \bar{\lambda}^3_\pm=\bar{e}^0\wedge \bar{e}^3\pm \bar{e}^1\wedge \bar{e}^2.$$

The connection forms $\omega=(\omega^I{}_J)$ and $\bar{\omega}=(\bar{\omega}^I{}_J)$ with respect to $\{e^I\}$ and $\{\bar{e}^I\}$ are then determined respectively by

$$\begin{cases} \omega^0{}_1=-(A/a)e^1, & \omega^2{}_3=\ \ (L/a)e^1, \\ \omega^0{}_2=\ \ (B/b)e^2, & \omega^1{}_3=\ \ (M/b)e^2, \\ \omega^0{}_3=\ \ (C/c)e^3, & \omega^2{}_1=-(N/c)e^3, \end{cases} \quad \text{and} \quad \begin{cases} \bar{\omega}^0{}_1=-(A/a)e^1, & \bar{\omega}^2{}_3=(L/a)e^1, \\ \bar{\omega}^0{}_2=-(B/b)e^2, & \bar{\omega}^3{}_1=(M/b)e^2, \\ \bar{\omega}^0{}_3=-(C/c)e^3, & \bar{\omega}^1{}_2=(N/c)e^3. \end{cases}$$

Note that ω is an $\mathfrak{o}(2,2)$-valued 1-form and that $\bar{\omega}$ is an $\mathfrak{o}(4)$-valued 1-form.

Now recall that (g, J) is Kähler if and only if $\omega^0{}_2-\omega^1{}_3=\omega^0{}_3+\omega^2{}_1=0$. Similarly, $(h, \bar{J})$ is Kähler if and only if $\bar{\omega}^0{}_2+\bar{\omega}^3{}_1=\bar{\omega}^0{}_3+\bar{\omega}^1{}_2=0$. These are equivalent to $B=M$, $C=N$. Hence we see (3).

The components W, Z of R are expressed as

$$W(\lambda^I_\pm)=W_{I\pm}\lambda^I_\pm, \quad Z(\lambda^I_+)=Z_I\lambda^I_- \quad (I=1,2,3).$$

Then $W_{I\pm}$, Z_I and the scalar curvature s are explicitly given by

$$W_{1\pm}=\frac{1}{2fabc}\{bc(A\pm 2L)^{\cdot}+fa(BC-2L+MN)\}-s/12,$$

$$Z_1=\frac{1}{2fabc}\{\dot{A}bc-fa(BC-2L+MN)\},$$

$$s=\frac{2}{fabc}\{\dot{A}bc+\dot{B}ca+\dot{C}ab+fa(BC-MN)+fb(CA-NL)+fc(AB-LM)\},$$

and we obtain the other components $W_{I\pm}$, Z_I $(I=2,3)$ by changing $(a, b, c; A, B, C; L, M, N)$ with $(b, c, a; B, C, A; M, N, L)$ and $(c, a, b; C, A, B; N, L, M)$.

For a quantity Υ defined for (2, 2)-metric g, let $\bar{\Upsilon}$ denote the corresponding quantity defined for (4, 0)-metric h. Then the components $\bar{W}$, $\bar{Z}$ of $\bar{R}$ are expressed as

$$\bar{W}(\bar{\lambda}^I_\pm)=\bar{W}_{I\pm}\bar{\lambda}^I_\pm, \quad \bar{Z}(\bar{\lambda}^I_+)=\bar{Z}_I\bar{\lambda}^I_- \quad (I=1,2,3).$$

It then follows that

$$\bar{W}_{I\pm}=-W_{I\mp}, \quad \bar{Z}_I=-Z_I \quad (I=1,2,3) \quad \text{and} \quad \bar{s}=-s.$$

Hence (1) and (2) are verified. □

REMARK. Let g be a $(2,2)$-metric defined by (4.2.1) such that $b^2=c^2$. In the following four cases, g is generically expressed as follows (cf. Dancer-Strachan [D-S], Tod [T]):

(1) g is Ricci-flat and self-dual:

(4.2.2) $$g=-\frac{da^2}{(1+\lambda a^2)^4}-a^2\tau_1^2+\frac{a^2}{(1+\lambda a^2)^2}(\tau_2^2+\tau_3^2) \quad \text{if } g \text{ is non-Kähler}.$$

(4.2.3) $$g=-\frac{db^2}{1+\lambda/b^4}-b^2(1+\lambda/b^4)\tau_1^2+b^2(\tau_2^2+\tau_3^2) \quad \text{if } g \text{ is Kähler}.$$

(2) g is Einstein and Kähler:

(4.2.4) $$g=-\frac{db^2}{1+(s/24)b^2+\mu/b^4}-b^2(1+(s/24)b^2+\mu/b^4)\tau_1^2+b^2(\tau_2^2+\tau_3^2),$$

where s is the scalar curvature (constant). Note that the right hand side of (4.2.4) coincides with that of (4.2.3) if g is Ricci-flat.

(3) g is anti-self-dual and Kähler:

(4.2.5) $$g=-\frac{db^2}{1+\mu_1 b^2+\mu_2 b^4}-b^2(1+\mu_1 b^2+\mu_2 b^4)\tau_1^2+b^2(\tau_2^2+\tau_3^2).$$

(4) g is self-dual and Kähler:

(4.2.6) $$g=-\frac{db^2}{1+\nu_1/b^2+\nu_2/b^4}-b^2(1+\nu_1/b^2+\nu_2/b^4)\tau_1^2+b^2(\tau_2^2+\tau_3^2).$$

Here λ, μ, μ_1, μ_2, ν_1, ν_2 are constants.

We note the following relation between anti-self-dual Kähler $(2,2)$-metrics (Case (3)) and self-dual Kähler $(2,2)$-metrics (Case (4)). Suppose that g is a self-dual Kähler $(2,2)$-metric expressed by (4.2.6). By changing b with $1/b$, we have

$$g=b^{-4}\left\{-\frac{db^2}{1+\nu_1 b^2+\nu_2 b^4}-b^2(1+\nu_1 b^2+\nu_2 b^4)\tau_1^2+b^2(\tau_2^2+\tau_3^2)\right\}$$

and hence see that $b^4 g$ is nothing but an anti-self-dual Kähler metric in (4.2.5).

4.3. Gibbons-Hawking type metrics. We recall that a Gibbons-Hawking type $(4,0)$-metric h on an open subset of $\boldsymbol{R}\times\boldsymbol{R}^3$ (resp. $\boldsymbol{R}\times H^3$) is a Ricci-flat Kähler metric (resp. conformally equivalent to a scalar-flat Kähler metric) defined by using a $U(1)$-monopole on $\boldsymbol{R}^3$ (resp. H^3) (see Gibbons-Hawking [G-H], LeBrun [L2]). Thus h is anti-self-dual.

To show an analogous result for $(2,2)$-metrics, we first introduce Gibbons-Hawking type $(2,2)$-metrics. Let M_1^3 be $\boldsymbol{R}_1^3$ or S_1^3, g_0 the standard $(2,1)$-metric of M_1^3 and (V,A) a $U(1)$-monopole on an open subset of M_1^3 (i.e., a pair of a non-zero function V and a 1-form A such that $dV=*_0 dA$, where $*_0$ is the Hodge star operator of M_1^3). We assume, for simplicity, that V is a positive function. We now define a Gibbons-Hawking type (2, 2)-metric g_{GH} by

$$g_{GH}=-V^{-1}(dt+A)^2+Vg_0\,, \tag{4.3.1}$$

where t is the coordinate function of $\boldsymbol{R}$. Let $\{e_0^1, e_0^2, e_0^3\}$ be an orthonormal coframe field of M_1^3 such that $g_0(e_0^1, e_0^1)=-1$, $g_0(e_0^2, e_0^2)=g_0(e_0^3, e_0^3)=1$. We define an almost complex structure J by $J(dt+A)=Ve_0^1$, $Je_0^2=e_0^3$. We can show the following:

THEOREM 4.2. *Let g_{GH} be a Gibbons-Hawking* (2, 2)*-metric defined by* (4.3.1), *and J the almost complex structure defined as above. Then the following hold*:

(1) *If $M_1^3=\boldsymbol{R}_1^3$, then g_{GH} is a Ricci-flat and self-dual* (2, 2)*-Kähler metric.*

(2) *If $M_1^3=S_1^3$, then g_{GH} is locally conformally equivalent to a scalar-flat* (2, 2)*-Kähler metric and hence is self-dual.*

PROOF. By an argument similar to that in the second case $M_1^3=S_1^3$, the assertion in the first case $M_1^3=\boldsymbol{R}_1^3$ can also be verified. Thus we only prove the case $M_1^3=S_1^3$.

Recall that $S_1^3=\{X=(X_1, X_2, X_3, X_4)\in \boldsymbol{R}_1^4 \mid -X_1^2+X_2^2+X_3^2+X_4^2=1\}$. For $X\in S_1^3\setminus\{X_1+X_4=0\}$, we define a local coordinate system $x=(x^1, x^2, x^3)$ by

$$x^1=1/(X_1+X_4)\,,\quad x^2=X_2/(X_1+X_4)\,,\quad x^3=X_3/(X_1+X_4)\,.$$

It is then immediate that $(x^1)^2 g_0=-(dx^1)^2+(dx^2)^2+(dx^3)^2$.

We will show that $(x^1)^2 g_{GH}$ is a scalar-flat (2, 2)-Kähler metric. We define $\{e^0, \dots, e^3\}$ by

$$e^0=x^1V^{-1/2}(dt+A)\,,\quad e^I=V^{1/2}dx^I \quad (I=1, 2, 3)\,,$$

forming an orthonormal coframe field with respect to $(x^1)^2 g_{GH}$. Then the connection form $\omega=(\omega^I{}_J)$, which is an $\mathfrak{o}(2, 2)$-valued 1-form, is given by

$$\omega^0{}_1=\frac{-1}{2V\sqrt{V}}\left\{\left(\frac{\partial V}{\partial x^1}-\frac{2V}{x^1}\right)e^0+\frac{\partial V}{\partial x^3}e^2-\frac{\partial V}{\partial x^2}e^3\right\},$$

$$\omega^2{}_3=\frac{1}{2V\sqrt{V}}\left\{\frac{\partial V}{\partial x^1}e^0+\frac{\partial V}{\partial x^3}e^2-\frac{\partial V}{\partial x^2}e^3\right\}.$$

$$\omega^0{}_2=\frac{-1}{2V\sqrt{V}}\left\{\frac{\partial V}{\partial x^2}e^0-\frac{\partial V}{\partial x^3}e^1-\frac{\partial V}{\partial x^1}e^3\right\},$$

$$\omega^1{}_3=\frac{-1}{2V\sqrt{V}}\left\{\frac{\partial V}{\partial x^2}e^0-\frac{\partial V}{\partial x^3}e^1-\frac{\partial V}{\partial x^1}e^3\right\},$$

$$\omega^0{}_3=\frac{-1}{2V\sqrt{V}}\left\{\frac{\partial V}{\partial x^3}e^0+\frac{\partial V}{\partial x^2}e^1+\frac{\partial V}{\partial x^1}e^2\right\},$$

$$\omega^1{}_2=\frac{1}{2V\sqrt{V}}\left\{\frac{\partial V}{\partial x^3}e^0+\frac{\partial V}{\partial x^2}e^1+\frac{\partial V}{\partial x^1}e^2\right\},$$

We see that

$$\omega^0{}_1+\omega^2{}_3=V^{-1}(dt+A)\,,\quad \omega^0{}_2-\omega^1{}_3=0\,,\quad \omega^0{}_3+\omega^1{}_2=0\,.$$

Hence $((x^1)^2 g_{\mathrm{GH}}, J)$ is a (2, 2)-Kähler metric. Furthermore, we obtain $W_-+(s/12)\,\mathrm{Id}=0$, since it holds that

$$R(\lambda^1_-)=-\frac{1}{x^1V^2}\left(\frac{\partial V}{\partial x^1}\lambda^1_+ +\frac{\partial V}{\partial x^2}\lambda^2_+ +\frac{\partial V}{\partial x^3}\lambda^3_+\right),\quad R(\lambda^2_-)=R(\lambda^3_-)=0\,,$$

where $\{\lambda^1_\pm, \lambda^2_\pm, \lambda^3_\pm\}$ is defined as in §4.2. Hence we have $W_-=0$ and $s=0$. □

It should be remarked that, for any Gibbons-Hawking type (4, 0)-metric h, LeBrun-Nayatani-Nitta [L-N-N] showed that $\mathrm{Ric}_h=-2h_0$. Analogously, for any Gibbons-Hawking type (2, 2)-metric g, we can show that $\mathrm{Ric}_g=2g_0$.

5. Global constructions of self-dual (2, 2)-metrics. In this section, we construct examples of manifolds with global (anti-)self-dual (2, 2)-metrics of Bianchi type VIII. We first define removable singularities for Bianchi type VIII (2, 2)-metrics, which are called nut and bolt singularities.

5.1. Nut and bolt singularities. For a (2, 2)-metric Bianchi type VIII, we introduce two conditions on removable singularities, which we will call "nuts" and "bolts". (For nuts and bolts for Riemannian metrics of Bianchi type IX, see, e.g., Eguchi-Gilkey-Hanson [E-G-H], Eguchi-Hanson [E-H].) We first recall the relation between the standard (Cartesian) coordinates and polar coordinates of $\boldsymbol{R}^4_{2+}$. The standard coordinates $x=(x_0, x_1, x_2, x_3)$ can be expressed as

$$x_0=r\cosh(\theta/2)\cos(\psi+\varphi)/2\,,\quad x_1=r\cosh(\theta/2)\sin(\psi+\varphi)/2\,,$$
$$x_2=r\sinh(\theta/2)\cos(\psi-\varphi)/2\,,\quad x_3=r\sinh(\theta/2)\sin(\psi-\varphi)/2\,,$$

where $r>0$, $\theta\geq 0$, $0\leq\varphi\leq 2\pi$ and $0\leq\psi\leq 4\pi$. Moreover $\{\tau_1, \tau_2, \tau_3\}$ is given by

$$(\tau_1, \tau_2, \tau_3)=\frac{1}{2}(d\psi+\cosh\theta d\varphi,\ \cos\psi d\theta+\sinh\theta\sin\psi d\varphi,\ \sin\psi d\theta-\sinh\theta\cos\psi d\varphi)\,.$$

Thus we have

$$\begin{pmatrix} dr \\ r\tau_1 \\ r\tau_2 \\ r\tau_3 \end{pmatrix}=\frac{1}{r}\begin{pmatrix} x_0 & x_1 & -x_2 & -x_3 \\ -x_1 & x_0 & x_3 & -x_2 \\ -x_2 & x_3 & x_0 & -x_1 \\ -x_3 & -x_2 & x_1 & x_0 \end{pmatrix}\begin{pmatrix} dx_0 \\ dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}.$$

For a metric $g=-f(r)^2dr^2-a(r)^2\tau_1^2+b(r)^2\tau_2^2+c(r)^2\tau_3^2$, we get the following expression for g by a variable change.

$$g=-d\tau^2-a(\tau)^2\tau_1^2+b(\tau)^2\tau_2^2+c(\tau)^2\tau_3^2\,,$$

where τ is given by $fdr=d\tau$. For simplicity, we assume that a, b, c are finite and nonsingular for finite τ, and have singularities at $\tau=0$.

A metric has a removable *nut singularity* if

$$a^2, b^2, c^2 \to \tau^2 \quad (\text{as } \tau\to 0)\,.$$

In this case, at $\tau=0$ we have simply a coordinate singularity in the flat polar coordinate system on $\boldsymbol{R}_2^4$, whose nullcone is expressed as $\tau^2=0$. By using Cartesian coordinates on $\boldsymbol{R}_2^4$, we can remove the singularity, and can extend g onto a neighbourhood of the nullcone. Near $\tau^2=0$, the manifold is topologically $\boldsymbol{R}_2^4$.

A metric has a removable *bolt singularity* if

$$a^2\to m^2\tau^2\,,\quad b^2=c^2\to \text{finite}\quad (\text{as } \tau\to 0)\,,$$

where m is an integer. Here $b^2=c^2$ implies the canonical metric $(d\theta^2+\sinh^2\theta d\varphi^2)/4$ on H^2 for the $(b^2\tau_2^2+c^2\tau_3^2)$ part of the metric, while at constant (θ, φ), the $(d\tau^2+a^2\tau_1^2)$ part of the metric looks like $d\tau^2+(m^2\tau^2/4)d\psi^2$. If the range of ψ is adjusted to $0\le m\psi/2\le 2\pi$, we can remove the apparent singularity at $\tau^2=0$ by using Cartesian coordinates on $\boldsymbol{R}^2$. The topology of the manifold is locally $\boldsymbol{R}^2\times H^2$ and the $\boldsymbol{R}^2$ shrinks to a point on H^2 as $\tau\to 0$. More precisely, we can regard g as a metric on an $\boldsymbol{R}^2$-bundle over H^2, by considering $(\tau, m\psi/2)$ as polar coordinates of each fiber $\boldsymbol{R}^2$. The $\boldsymbol{R}^2$-bundle is given as follows. Let $K^{1/2}=H_1^3\times_{S^1}\boldsymbol{R}^2\to H^2$ be the $\boldsymbol{R}^2$-bundle induced by the hyperbolic Hopf fibration $H_1^3\to H^2$ which is an S^1-bundle over H^2 analogous to the standard Hopf fibration $S^3\to S^2$. By the adjustment of ψ, we can regard g as a metric on $K^{m/2}=(K^{1/2})^{\otimes m}=H_1^3\times_{S^1}\boldsymbol{R}^2/\boldsymbol{Z}_m$, where $\boldsymbol{Z}_m=\boldsymbol{Z}/m\boldsymbol{Z}$ and $k=m-1\in\boldsymbol{Z}_{\ge 0}$.

EXAMPLE 1. *Pseudo-Euclidean* (2, 2)*-space* $\boldsymbol{R}_2^4=(\boldsymbol{R}^4, g_0)$ (cf. § 4.1):

$$g_0=-dx_0^2-dx_1^2+dx_2^2+dx_3^2\,.$$

Clearly, the space $\boldsymbol{R}_2^4$ is complete and its curvature tensor vanishes. This metric g_0 can be rewritten in polar coordinates as

$$(5.1.1)\qquad g_0=\begin{cases} -(dr^2+r^2\tau_1^2)+r^2(\tau_2^2+\tau_3^2) & (\text{on } \boldsymbol{R}_{2+}^4)\,,\\ d\rho^2+\rho^2\tau_1^2-\rho^2(\tau_2^2+\tau_3^2) & (\text{on } \boldsymbol{R}_{2-}^4)\,.\end{cases}$$

The nullcone $r^2=\rho^2=0$ is an apparent singularity, which is removable.

EXAMPLE 2. *Pseudo-Riemannian hyperbolic* (2, 2)*-space* $H_2^4=(H, g_{H_2^4})$:

$$H=\{x=(t, x_0, x_1, x_2, x_3)\in\boldsymbol{R}_3^5\,|\,t^2+x_0^2+x_1^2-x_2^2-x_3^2=1\}\,,\quad g_{H_2^4}=(g_0)_3^5|_H\,.$$

The space H_2^4 is also complete and has constant curvature -1. Note that $(H, -g_{H_2^4})$

coincides with the pseudo-Riemannian sphere S_2^4. By means of the stereographic projection from the south pole $(-1, 0, 0, 0, 0)$ we regard $g_{H_2^4}$ as a metric on an open subset of $\boldsymbol{R}^4$. Then $g_{H_2^4}$ can be expressed as

$$g_{H_2^4}=\frac{4g_0}{(1+x_0^2+x_1^2-x_2^2-x_3^2)^2}=\begin{cases}\dfrac{4\{-(dr^2+r^2\tau_1^2)+r^2(\tau_2^2+\tau_3^2)\}}{(1+r^2)^2} & (\text{on } \boldsymbol{R}^4_{2+}),\\[2ex] \dfrac{4\{(d\rho^2+\rho^2\tau_1^2)-\rho^2(\tau_2^2+\tau_3^2)\}}{(1-\rho^2)^2} & (\text{on } \rho^2\neq 1).\end{cases}$$

The nullcone $r^2=\rho^2=0$ is a removable apparent singularity. Furthermore, the apparent singularities $r=+\infty$ and $\rho=+\infty$ express the same subset $\{(-1, x_0, x_1, x_2, x_3)\,|\,x_0^2+x_1^2=x_2^2+x_3^2\}$ in $H_2^4\subset\boldsymbol{R}_3^5$. Under the stereographic projection from the north pole, the set above is mapped to the nullcone of $\boldsymbol{R}_2^4$.

5.2. Global constructions of four types. We first recall the following four types of (4, 0)-metrics (see, e.g., [E-G-H], [E-H], LeBrun [L1]):

($\bar{1}$) the Fubini-Study metric on $P^2(\boldsymbol{C})$ (self-dual, Einstein, Kähler),

($\bar{2}$) Eguchi-Hanson metrics on the cotangent bundle T^*S^2 of S^2 (anti-self-dial, Ricci-flat, Kähler),

($\bar{3}$) Taub-NUT metrics on $\boldsymbol{R}^4$ (self-dual, Ricci-flat, non-Kähler),

($\bar{4}$) LeBrun metrics on complex line bundles $K^{(k+1)/2}$ over $P^1(\boldsymbol{C})$ (anti-self-dual, scalar-flat, Kähler), where K is the canonical bundle over $P^1(\boldsymbol{C})$ $(k\in\boldsymbol{Z}_{\geq 0})$.

Each of these metrics is of Bianchi type IX on an open dense subset of the ambient manifold.

By virtue of Theorem 4.1, we can immediately construct Bianchi type VIII (2, 2)-metrics corresponding to the (4, 0)-metrics above. We then give explicit expressions for these (2, 2)-metrics, and construct manifolds on which these metrics are globally defined (cf. §5.1).

(1) Fubini-Study type.

$$g_{\mathrm{FS}}=-\frac{dr^2+r^2\tau_1^2}{(1+r^2)^2}+\frac{r^2(\tau_2^2+\tau_3^2)}{1+r^2}.$$

The metric is anti-self-dual, Einstein and Kähler. Note that we can obtain g_{FS} from (4.2.5) by putting $\mu_1=-1$, $\mu_2=0$, $b^2=r^2/(r^2+1)$. The singularities of g_{FS} are described as follows: $r^2=0$ is a nut singularity, and hence g_{FS} can be extended to $\rho^2=-(x_0^2+x_1^2-x_2^2-x_3^2)<1$, a neighbourhood of the nullcone $\mathcal{N}=\{r^2=0\}$. $r^2=+\infty$ is a bolt singularity, and hence g_{FS} is regarded as a metric on the $\boldsymbol{R}^2$-bundle $K^{1/2}$. Thus g_{FS} is extended to $M_{\mathrm{FS}}=U\cup_{U\cap K^{1/2}=\boldsymbol{R}^4_{2+}}K^{1/2}$, where $U=\boldsymbol{R}^4\setminus\{\rho^2\geq 1\}$.

(2) Eguchi-Hanson type.

$$g_{\mathrm{EH}}=-\frac{dr^2}{1-(a/r)^4}-r^2(1-(a/r)^4)\tau_1^2+r^2(\tau_2^2+\tau_3^2)\qquad(a>0).$$

The metric is self-dual, Ricci-flat and Kähler (cf. [O-V]). Note that we can obtain g_{EH} from (4.2.3) by putting $\lambda=-a^4$ and $b^2=r^2$. The singularity of g_{EH} is given by $r^2=a^2$, which is a bolt singularity, and hence g_{EH} can be extended to the $\boldsymbol{R}^2$-bundle $K^1=T^*H^2$ over $H^2(r^2=a^2$ corresponds to $H^2)$.

We note that g_{EH} is also regarded as a metric globally defined on the cotangent bundle $T^*\Sigma_g$ of a compact Riemann surface Σ_g of genus $g\geq 2$, since g_{EH} is $SL(2, \boldsymbol{R})$-invariant.

(3) Taub-NUT type.

$$g_{\mathrm{TN}}=-(1+r^2)dr^2-\frac{r^2}{1+r^2}\tau_1^2+r^2(1+r^2)(\tau_2^2+\tau_3^2).$$

The metric is self-dual with suitable orientation, Ricci-flat and non-Kähler. Note that we can obtain g_{TN} from (4.2.2) by putting $\lambda=-1$ and $a^2=r^2/(1+r^2)$. The singularity of g_{TN} given by $r^2=0$ is a nut singularity, and hence g_{TN} can be extended to $\rho^2=-(x_0^2+x_1^2-x_2^2-x_3^2)<1$. Thus it is globally defined on $\boldsymbol{R}^4\setminus\{\rho^2\geq 1\}$.

(4) LeBrun type.

$$g_{\mathrm{L}}^k=-\frac{dr^2}{(1-(a/r)^2)(1+k(a/r)^2)}-r^2(1-(a/r)^2)(1+k(a/r)^2)\tau_1^2+r^2(\tau_2^2+\tau_3^2)$$

$(a>0, k\in \boldsymbol{Z}_{\geq 0})$. The metric is self-dual, scalar-flat and Kähler. Note that we can obtain g_{L}^k from (4.2.6) by putting $v_1=(k-1)a^2$, $v_2=-ka^4$ and $b^2=r^2$. The singularity of g_{L}^k is given by $r^2=a^2$, which is a bolt singularity, and g_{L}^k is globally defined on $K^{(k+1)/2}$.

In the following, we verify directly that M_{FS} is nothing but the indefinite complex hyperbolic space $H_1^2(\boldsymbol{C})$, and that g_{FS} is globally defined on $H_1^2(\boldsymbol{C})$. We define M by

$$M=\{(z_0:z_1:z_2)\in P^2(\boldsymbol{C})\,|\,-|z_0|^2-|z_1|^2+|z_2|^2<0\},$$

where $(z_0:z_1:z_2)$ denotes the homogeneous coordinate of $P^2(\boldsymbol{C})$. Note that M can be regarded as the homogeneous space $H_1^2(\boldsymbol{C})=U(2, 1)/(U(1)\times U(1, 1))$. We also define a quadratic form on $\boldsymbol{C}^3\setminus\{0\}$ by

$$\frac{-|dz_0|^2-|dz_1|^2+|dz_2|^2}{|z_0|^2+|z_1|^2-|z_2|^2}+\frac{|z_0\overline{dz_0}+z_1\overline{dz_1}-z_2\overline{dz_2}|^2}{(|z_0|^2+|z_1|^2-|z_2|^2)^2}.$$

This is $\boldsymbol{C}^*$-invariant and hence defines a (2, 2)-metric g on $M=H_1^2(\boldsymbol{C})$. It follows from the condition $-|z_0|^2-|z_1|^2+|z_2|^2<0$ that either $z_0\neq 0$ or $z_1\neq 0$. Since g is symmetric with respect to z_0 and z_1, we may assume $z_0\neq 0$. Setting $\zeta_1=z_1/z_0$, $\zeta_2=z_2/z_0$, we have an expression for g in (ζ_1, ζ_2) as

$$g=\frac{-|d\zeta_1|^2+|d\zeta_2|^2}{1+|\zeta_1|^2-|\zeta_2|^2}+\frac{|\zeta_1\overline{d\zeta_1}-\zeta_2\overline{d\zeta_2}|^2}{(1+|\zeta_1|^2-|\zeta_2|^2)^2}.$$

Setting $r^2=|\zeta_1|^2-|\zeta_2|^2$ if $|\zeta_1|^2-|\zeta_2|^2>0$ and $\rho^2=|\zeta_2|^2-|\zeta_1|^2$ if $-1<|\zeta_1|^2-|\zeta_2|^2$

<0, we obtain the following expression for g in polar coordinates of $\boldsymbol{C}^2=\boldsymbol{R}_2^4$:

$$g=\begin{cases} -\dfrac{dr^2+r^2\tau_1^2}{(1+r^2)^2}+\dfrac{r^2(\tau_2^2+\tau_3^2)}{1+r^2} & (r^2=|\zeta_1|^2-|\zeta_2|^2>0)\,, \\ \dfrac{d\rho^2+\rho^2\tau_1^2}{(1-\rho^2)^2}-\dfrac{\rho^2(\tau_2^2+\tau_3^2)}{1-\rho^2} & (0<\rho^2=|\zeta_2|^2-|\zeta_1|^2<1)\,. \end{cases}$$

Thus g is nothing but the Fubini-Study type (2, 2)-metric g_{FS}. In the expression for g above, the apparent singularity at $r^2=\rho^2=0$ is clearly removable and $g\approx g_0$ near $r^2=\rho^2=0$ (where g_0 is the standard metric of $\boldsymbol{R}_2^4$). The apparent singularity $r^2=+\infty$ is also removable. To see this, recall the expression for g in terms of $(z_0:z_1:z_2)$. Since $r^2=+\infty$ means $z_0=0$, we use the coordinates $(\xi_0=z_0/z_1,\ \xi_2=z_2/z_1)$ near $z_0=0$. In these coordinates, g can be written as

$$g=\frac{-|d\xi_0|^2+|d\xi_2|^2}{1+|\xi_0|^2-|\xi_2|^2}+\frac{|\xi_0\overline{d\xi_0}-\xi_2\overline{d\xi_2}|^2}{(1+|\xi_0|^2-|\xi_2|^2)^2}\,.$$

In particular, we have $g=-|d\xi_0|^2/(1-|\xi_2|^2)+|d\xi_2|^2/(1-|\xi_2|^2)^2$ on $\{(0:z_1:z_2)\}$ $(\xi_0=0)$. Since $|\xi_0|^2-|\xi_2|^2>-1$, we see that $|\xi_2|^2<1$ on $\xi_0=0$. The first term $-|d\xi_0|^2/(1-|\xi_2|^2)$ is the standard (0, 2)-metric of $\boldsymbol{R}^2$ except for the conformal factor $1/(1-|\xi_2|^2)$. The second term $|d\xi_2|^2/(1-|\xi_2|^2)^2$ is exactly the standard metric of H^2 up to the factor 1/4 (the Poincaré metric on $D^2(-4)$). Therefore $r^2=+\infty(\Leftrightarrow z_0=0)$ is removable and locally M is topologically $\boldsymbol{R}^2\times H^2$ near $r^2=+\infty$. Thus we see that $g=g_{\mathrm{FS}}$ is globally defined on $H_1^2(\boldsymbol{C})$. Note that $(M,\,-g)$ coincides with the indefinite complex projective space $P_1^2(\boldsymbol{C})$.

Summarizing these, we have the following:

THEOREM 5.1. *The following hold*:

(1) g_{FS} *is anti-self-dual, Einstein and Kähler, and is globally defined on* $H_1^2(\boldsymbol{C})$.

(2) g_{EH} *is Ricci-flat and Kähler* (*thereby self-dual*), *and is globally defined on* T^*H^2.

(3) g_{TN} *is self-dual, Ricci-flat and non-Kähler, and is globally defined on* $\boldsymbol{R}^4\setminus\{\rho^2\geq 1\}$.

(4) g_{L}^k *is scalar-flat and Kähler* (*thereby self-dual*), *and is globally defined on* $K^{(k+1)/2}$.

5.3. Unified aspects of the four types. Let us comment on some local relations among the four types of (2 ,2)-metrics, which we discussed in §5.2. We also remark that the manifolds on which the four types of (2, 2)-metrics are globally defined can be constructed by modification of the manifold $M=H_1^2(\boldsymbol{C})$.

We define a one-parameter family of (2, 2)-metrics by

$$g_{(t)}=-\frac{1+r^2}{r^4(1+tr^2)}dr^2-\frac{1+tr^2}{r^2(1+r^2)}\tau_1^2+\frac{(1+r^2)}{r^2}(\tau_2^2+\tau_3^2)\,. \tag{5.3.1}$$

Putting

$$f(r)=\frac{1}{r^2}\left(\frac{1+r^2}{1+tr^2}\right)^{1/2}, \quad a(r)=\frac{1}{r}\left(\frac{1+tr^2}{1+r^2}\right)^{1/2}, \quad b(r)=\frac{1}{r^2}(1+r^2)^{1/2}(=c(r))$$

and regarding $\{f(r)dr, a(r)\tau_1, b(r)\tau_2, c(r)\tau_3\}$ as an oriented coframe field, we see that the one-parameter family $g_{(t)}$ has the following properties:

$$W_+\equiv 0; \quad s\equiv 0; \quad Z_1=\frac{2r^4(2-t)}{(1+r^2)^2}, \quad Z_2=Z_3\equiv 0 .$$

Namely, for any t, each $g_{(t)}$ is an anti-self-dual and scalar-flat (2, 2)-metric, and $g_{(t)}$ is Einstein if and only if $t=2$.

If $t=0$, then $g_{(t)}$ is locally conformal to the Taub-NUT type (2, 2)-metric g_{TN}, since

$$g_{(0)}=\frac{1}{r^4}\left\{-(1+r^2)dr^2-\frac{r^2}{1+r^2}\tau_1^2+r^2(1+r^2)(\tau_2^2+\tau_3^2)\right\}.$$

If $t=1$, then $g_{(t)}$ is locally conformal to the Fubini-Study type (2, 2)-metric g_{FS}, since

$$g_{(1)}=\frac{(1+r^2)^2}{r^4}\left\{-\frac{dr^2+r^2\tau_1^2}{(1+r^2)^2}+\frac{r^2(\tau_2^2+\tau_3^2)}{1+r^2}\right\}.$$

If $t=k+1$, then we can regard $g_{(t)}$ as a LeBrun type (2, 2)-metric g_{L}^k, up to constant factor, by replacing r with $a(r^2-a^2)^{-1/2}$. In particular, $g_{(2)}$ locally corresponds conformally to an Eguchi-Hanson type (2, 2)-metric g_{EH}. Therefore we obtain the following:

Proposition 5.2. *Consider the four types in* §5.2 *as* (2, 2)*-metrics defined on* $\boldsymbol{R}^4_{2+}$ *with suitable orientations (if necessary, by replacing the variable r with* $a(r^2-a^2)^{-1/2}$*). Then the conformal classes of the four types can be connected by a one-parameter family* (5.3.1) *consisting of anti-self-dual* (2, 2)*-metrics.*

In §5.2, we investigated the topology of the manifolds on which the four types of (2, 2)-metrics are globally defined. In what follows, we can also obtain these manifolds respectively by taking off subsets from the manifold $M=H_1^2(\boldsymbol{C})$.

Recall that the Fubini-Study type (2, 2)-metric g_{FS} is globally defined on the manifold $M=\{(z_0:z_1:z_2)\,|\,-|z_0|^2-|z_1|^2+|z_2|^2<0\}$. Here we define M_-, M_0, M_+ and M_∞ by

$$M_-=\{(z_0:z_1:z_2)\in M\,|\,|z_0|^2>0,\ |z_1|^2-|z_2|^2<0\},$$
$$M_0=\{(z_0:z_1:z_2)\in M\,|\,|z_0|^2>0,\ |z_1|^2-|z_2|^2=0\},$$
$$M_+=\{(z_0:z_1:z_2)\in M\,|\,|z_0|^2>0,\ |z_1|^2-|z_2|^2>0\},$$
$$M_\infty=\{(0:z_1:z_2)\in M\,|\,|z_1|^2-|z_2|^2>0\}.$$

Then we have $M=M_-\coprod M_0\coprod M_+\coprod M_\infty$. We also define M_t by

$$M_t = \{(z_0 : z_1 : z_2) \in M_+ \mid t^2 = |z_0|^2/(|z_1|^2 - |z_2|^2)\}$$
$$\cong \{(\zeta_1, \zeta_2) \mid |\zeta_1|^2 - |\zeta_2|^2 = 1/t^2\} \cong H_1^3 \quad \text{for } t > 0 .$$

Note that M_0, M_+ and M_∞ can respectively be identified with the nullcone $\mathcal{N}$ (corresponding to the nut singularity of g_{FS}), $\boldsymbol{R}^4_{2+}$ and H^2 (corresponding to the bolt singularity of g_{FS}). Furthermore, we will use, for simplicity, the notation

$$M_{(-1,0]} := M_- \coprod M_0 , \quad M_{[0,\infty)} := M_0 \coprod M_+ , \quad M_{(0,\infty]} := M_+ \coprod M_\infty .$$

The Taub-NUT (2, 2)-metric g_{TN} is defined on $\boldsymbol{R}^4 \setminus \{\rho^2 \geq 1\} \cong M \setminus M_\infty$, which we obtain by taking off the bolt singularity of g_{FS} from the manifold M.

The (2, 2)-metric $g_{(1)}$ corresponding to that of LeBrun type with $k=0$ is globally defined on $M_{(0,\infty]}$, which we obtain by taking off the nut singularity of g_{FS} and M_- from M. Recall that

$$M_{(0,\infty]} = \{(z_0 : z_1 : z_2) \in M \mid -|z_1|^2 + |z_2|^2 < 0\} \cong \{(\xi_0, \xi_2) \in \boldsymbol{C}^2 \mid |\xi_2|^2 < 1\} .$$

Hence the manifold $M_{(0,\infty]}$ is nothing but $\boldsymbol{C} \times H^2$.

We next consider other LeBrun type (2, 2)-metrics ($k > 0$). Note that, since we can identify M_t and M_∞ with H_1^3 and H^2, respectively, we can regard $M_{(0,\infty]} \to M_\infty$ ($(z_0 : z_1 : z_2) \mapsto (0 : z_1 : z_2)$) as the complex line bundle $K^{1/2}$ associated to the hyperbolic Hopf fibration $H_1^3 \to H^2$. We now define an action of $\boldsymbol{Z}_{k+1}$ on M by

$$T_{k+1} : \boldsymbol{Z}_{k+1} \times M \ni (\omega, (z_0 : z_1 : z_2)) \mapsto (\omega z_0 : z_1 : z_2) \in M .$$

Note that the action T_{k+1} fixes M_∞, and $\boldsymbol{Z}_{k+1}$ acts freely on each M_t. Thus we can consider the quotient space $M_{(0,\infty]}/\boldsymbol{Z}_{k+1} \cong (H_1^3 \times_{S^1} \boldsymbol{C})/\boldsymbol{Z}_{k+1}$. From the argument on bolt singularities in §5.1, we obtain $K^{(k+1)/2} = (H_1^3 \times_{S^1} \boldsymbol{C})/\boldsymbol{Z}_{k+1}$ as the entire manifolds for LeBrun type (2, 2)-metrics. Summarizing these, we have the following:

PROPOSITION 5.3. *The following hold*:

(1) g_{FS} *is a* (2, 2)-*metric on* M.
(2) g_{EH} *is a* (2, 2)-*metric on* $M_{(0,\infty]}/\boldsymbol{Z}_2$.
(3) g_{TN} *is a* (2, 2)-*metric on* $M \setminus M_\infty$.
(4) g_{L}^k *is a* (2, 2)-*metric on* $M_{(0,\infty]}/\boldsymbol{Z}_{k+1}$.

REFERENCES

[A-H-S] M. F. ATIYAH, N. J. HITCHIN AND I. M. SINGER, Self-duality in four-dimensional Riemannian geometry, Proc. Roy. Soc. London Ser. A 362 (1978), 425–461.

[A] A. AVEZ, Formule de Gauss-Bonnet-Chern en métrique de signature quelconque, De Revista de la Unión Mat. Argentina 21 (1963), 191–197.

[B] A. L. BESSE, Einstein manifolds, Springer-Verlag, Berlin, Heidelberg, New York, 1987.

[C] S. S. CHERN, Pseudo-Riemannian geometry and the Gauss-Bonnet formula, An. da Acad. Brasileira de Ciências 35 (1963), 17–26.

[D] A. Derdziński, Self-dual Kähler manifolds and Einstein manifolds of dimension 4, Compositio Math. 49 (1983), 405–433.

[D-S] A. S. Dancer and I. A. B. Strachan, Cohomogeneity-one Kähler metrics, in Twistor theory (S. Huggett, ed.), vol. 169, M. Dekker, New York, Basel, Hong Kong, 1995, pp. 9–27.

[E-G-H] T. Eguchi, P. B. Gilkey and A. J. Hanson, Gravitation, gauge theories and differential geometry, Phys. Reports (Review Section of Phys. Lett.) 66 (1980), 213–393.

[E-H] T. Eguchi and A. J. Hanson, Self-dual solutions to Euclidean gravity, Ann. of Phys. 120 (1979), 82–106.

[G-H] G. W. Gibbons and S. W. Hawking, Gravitational multi-instantons, Phys. Lett. B 78 (1978), 430–432.

[H] N. J. Hitchin, Monopoles, minimal surfaces and algebraic curves, Presses de l'Université de Montréal, Montréal, 1978.

[H-H] F. Hirzebruch and H. Hopf, Felder von Flächenelementen in 4-dimensionalen Mannigfaltigkeiten, Math. Ann. 136 (1958), 156–172.

[I] M. Itoh, Self-duality of Kähler surfaces, Compositio Math. 51 (1984), 265–273.

[L1] C. LeBrun, Counter-examples to the generalized positive action conjecture, Comm. Math. Phys. 118 (1988), 591–596.

[L2] C. LeBrun, Explicit self-dual metrics on $\boldsymbol{CP}_2\# \cdots \# \boldsymbol{CP}_2$, J. Differential Geom. 34 (1991), 223–253.

[L-N-N] C. LeBrun, S. Nayatani and T. Nitta, Self-dual manifolds with positive Ricci curvature, preprint.

[M-S] Y. Machida and H. Sato, Conformal structures of signature (2, 2) and totally null plane bundles, in preparation.

[M] Y. Matsushita, Fields of 2-planes and two kinds of almost complex structures on compact 4-dimensional manifolds, Math. Z. 207 (1991), 281–291.

[O-V] H. Ooguri and C. Vafa, Geometry of $N=2$ strings, Nuclear Physics B 361 (1991), 469–518.

[P-W] R. Penrose and R. S. Ward, Twistors for flat and curved space-time, in General relativity and gravitation (A. Held, ed.), vol. 2, Plenum Press, New York, London, 1980, pp. 283–328.

[S] N. Steenrod, Topology of fiber bundles, Princeton Univ. Press, 1958.

[T] K. P. Tod, Cohomogeneity-one metrics with self-dual Weyl tensor, in Twistor theory (S. Huggett, ed.), vol. 169, M. Dekker, New York, Basel, Hong Kong, 1995, pp. 171–184.

[Wa] R. S. Ward, Integrable systems in twistor theory, in Twistors in mathematics and physics (T. N. Bailey and R. J. Baston, eds.), Cambridge Univ. Press, 1990, pp. 246–259.

[W-W] R. S. Ward and R. O. Wells, Twistor geometry and field theory, Cambridge Univ. Press, 1990.

[Wo] J. A. Wolf, Spaces of constant curvature, Publish or Perish, Wilmington, Delaware, 1984.

[Wu] W. T. Wu, Sur les classes caractéristiques des structures fibrées sphériques, Actualités Sci. Indust., vol. 1183, Hermann, Paris, 1952.

Numazu College of Technology
3600 Ooka, Numazu-shi
Shizuoka 410
Japan