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SELF-FOCUSING WITH FOURTH-ORDER DISPERSION*

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Abstract. We analyze self-focusing and singularity formation in the nonlinear Schrödinger equation (NLS) with high-order dispersion $i\psi_t \pm \Delta^q \psi + |\psi|^{2\sigma}\psi = 0$, in the isotropic mixed-dispersion NLS $i\psi_t + \Delta\psi + \epsilon\Delta^2\psi + |\psi|^{2\sigma}\psi = 0$, and in nonisotropic mixed-dispersion NLS equations which model propagation in fiber arrays.

Key words. blowup, critical power, critical exponent, Gagliardo–Nirenberg inequality, optimal constants

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1. Introduction.

The canonical model for propagation of intense laser beams in a bulk medium with Kerr nonlinearity is given by the nonlinear Schrödinger equation (NLS)

$$i\psi_t(x, y, t) + \Delta\psi + |\psi|^2\psi = 0, \qquad \psi(x, y, 0) = \psi_0(x, y), \tag{1}$$

where $\Delta = \partial_{xx} + \partial_{yy}$. This equation is a special case of the NLS in d dimensions and with a general power-law nonlinearity

$$i\psi_t(\mathbf{x}, t) + \Delta\psi + |\psi|^{2\sigma}\psi = 0, \qquad \psi(\mathbf{x}, 0) = \psi_0(\mathbf{x}), \tag{2}$$

where $\mathbf{x} = (x_1, \dots, x_d)$ and $\Delta = \partial_{x_1x_1} + \cdots + \partial_{x_dx_d}$.

Two key questions in NLS theory are (1) Can solutions become singular (blow up, collapse) in finite time? and (2) Do finite-energy waveguide solutions $\psi = e^{i\omega t}Q_\omega(\mathbf{x})$ exist, and are they stable? It is well known that the answer to these questions for the NLS (2) depends on the product σd as follows. When $\sigma d < 2$ (subcritical NLS), all solutions exist globally, and the waveguide solutions are stable. When, however, $\sigma d = 2$ (critical NLS) or $\sigma d > 2$ (supercritical NLS), solutions of the NLS (2) can become singular, and the waveguide solutions are unstable.

The role of small fourth-order dispersion has been considered in a series of papers by Karpman and Shagalov (see [21] and the references therein), who studied the equation

$$i\psi_t(t, \mathbf{x}) + \Delta\psi + |\psi|^{2\sigma}\psi + \epsilon\Delta^2\psi = 0 \tag{3}$$

in the case when $\epsilon < 0$, where Δ^2 is the biharmonic operator. Using a combination of stability analysis and numerical simulations, they showed that (1) when $d\sigma \le 2$, the waveguide solutions of equation (3) are stable for all $\epsilon < 0$, (2) when $2 < d\sigma < 4$, the waveguide solutions are stable for $\epsilon \ll -1$, and (3) when $4 \le d\sigma$, the waveguide solutions are unstable, and the instability can result in a collapse.

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The results of Karpman and Shagalov show that there is a second critical exponent/dimension $d\sigma = 4$. In this study, we show that the reason for this is that $\sigma d = 4$ is the critical exponent/dimension of the *biharmonic NLS*

$$i\psi_t(t, \mathbf{x}) + \epsilon\Delta^2\psi + |\psi|^{2\sigma}\psi = 0.$$

The biharmonic NLS was considered in [20, 31] in the context of stability of solitons in magnetic materials when the effective quasi-particle mass becomes infinite but received little attention otherwise. In section 4, we show that some but not all properties of the NLS (which are briefly reviewed in section 3) remain valid for the biharmonic NLS. In particular, when $\epsilon > 0$, all solutions of the biharmonic NLS exist globally. When $\epsilon < 0$, the critical exponent for singularity formation is $\sigma d = 4$. In addition, when $\sigma d = 4$, a sufficient condition for global existence is that the input power (L^2 norm) be below a threshold value. As in the case of the critical NLS, the calculation of the threshold power reduces to the determination of (new) optimal constants in Sobolev inequalities. The results for the biharmonic NLS are extended to the NLS with any order of dispersion in section 5.

In section 6, we consider the NLS (3) with mixed dispersion. We first show that the sufficient conditions for global existence are the same as for the biharmonic NLS. Thus we might be tempted to conclude that the biharmonic term has a defocusing effect when $\epsilon > 0$ and that it is focusing when $\epsilon < 0$. As is often the case in NLS theory, however, global estimates can lead to misleading conclusions. Indeed, we show that when the NLS is critical ($\sigma d = 2$) and ϵ is small, during the initial stages of propagation the biharmonic term is focusing when $\epsilon > 0$ and is defocusing when $\epsilon < 0$. Thus *the effect of the biharmonic term (focusing or defocusing) depends on whether it is small or large compared with the Laplacian.*

We can use the above conclusions to motivate the results of Karpman and Shagalov as follows. When $\sigma d < 2$, NLS waveguides are stable, and therefore a small defocusing (i.e., $\epsilon < 0$) biharmonic term does not affect stability. When $\sigma d = 2$, NLS waveguides are only mildly unstable, and therefore any amount of defocusing by the biharmonic term is sufficient to stabilize the waveguides. When $\sigma d > 2$, NLS waveguides are "genuinely" unstable, and therefore the defocusing biharmonic term has to be sufficiently large in order to stabilize the waveguides. In other words, $|\epsilon|$ should be sufficiently large so that waveguide propagation is predominantly governed by the corresponding biharmonic equation, which is stable for $\sigma d < 4$.

2. Physical motivation. In this section, we discuss the physical motivation behind (3). We first note that high-order dispersion terms do not arise naturally in the NLS in the sense that they do not correspond to high-order terms in a Taylor expansion. Indeed, let us recall the traditional derivation of the NLS (1) in nonlinear optics from the scalar nonlinear Helmholtz equation (NLH)

$$(4) \qquad (\partial_{xx} + \partial_{yy} + \partial_{zz})E(x, y, z) + k^2 E = 0, \quad k^2 = k_0^2\left(1 + \frac{4n_2}{n_0}|E|^2\right),$$

where E is the electric field, k_0 is the wavenumber, n_0 is the linear index of refraction, and n_2 is the Kerr coefficient. We separate the fast oscillations from the slowly varying amplitude and change to nondimensional variables using[1]

$$(5) \quad \tilde{x} = \frac{x}{r_0}, \quad \tilde{y} = \frac{y}{r_0}, \quad \tilde{t} = \frac{z}{2L_{\mathrm{DF}}}, \quad \psi(\tilde{x}, \tilde{y}, \tilde{t}) = 2r_0 k_0 \sqrt{\frac{n_2}{n_0}} E(x, y, z) e^{-ik_0 z},$$

[1] Note that in nonlinear optics the physical variable z plays the role of "time."

where ψ is the dimensionless electric-field amplitude and $L_{\mathrm{DF}} = k_0 r_0^2$ is the diffraction length. Dropping the tilde signs, the nondimensional NLH is given by

$$\frac{\delta}{4}\psi_{tt}(x,y,t) + i\psi_t + \Delta\psi + |\psi|^2\psi = 0, \tag{6}$$

where $\Delta = \partial_{xx} + \partial_{yy}$ and $\delta = 1/r_0^2 k_0^2 > 0$. Because the input beam width r_0 is typically much larger than its wavelength $\lambda = 2\pi/k_0$, it follows that

$$\delta = \frac{\lambda^2}{\pi^2 r_0^2} \ll 1.$$

In fact, even if the beam width gets to the order of a single wavelength, i.e., $r_0 \approx \lambda$, then $\delta \approx 1/\pi^2 \ll 1$. This observation is the motivation behind the *paraxial approximation*, which amounts to setting $\delta = 0$, in which case the NLH (6) reduces to the NLS (1).

2.1. Nonparaxiality. As we have already said, solutions of the NLS (1) can become singular at a finite time. This indicates that some of the small terms that are neglected in the derivation of the NLS from Maxwell equations become important near the singularity and prevent the collapse. Since the last step in the derivation of the NLS from Maxwell equations is the paraxial approximation, a natural question is whether nonparaxiality arrests the blowup.

At present, there is no definite answer to this question, which is based on rigorous analysis. Numerical simulations [5, 8, 29] and asymptotic analysis [9] suggest that nonparaxiality always arrests the collapse. Because solving the NLH numerically as a true boundary value problem is hard,[2] the standard approach in solving the NLH numerically is to approximate it with an initial value problem (see, e.g., [5, 7, 8, 29]). This can be done by approximating the nonparaxial term ψ_{tt} with[3]

$$\psi_{tt} = -\left[\Delta^2\psi + 4|\psi|^2\Delta\psi + 4\psi\nabla\psi\nabla\psi^* + 2\psi^*\nabla\psi\nabla\psi + |\psi|^4\psi\right] + O(\delta).$$

The resulting perturbed NLS with nonparaxial effects, which is an initial value problem, is given by

$$\begin{aligned} i\psi_t(x,y,t) + \Delta\psi + |\psi|^2\psi - \delta\left[\Delta^2\psi + 4|\psi|^2\Delta\psi + 4\psi\nabla\psi\nabla\psi^*\right. \\ \left. + 2\psi^*\nabla\psi\nabla\psi + |\psi|^4\psi\right] = 0. \end{aligned} \tag{7}$$

In the scalar NLH (4) (hence in (7)) the coupling between the components of the vectorial electric field is neglected. When vectorial effects are included, the model equation is given by the vectorial NLH. It can be shown [12] that, to leading order, the vectorial NLH can be approximated with the scalar equation

$$\begin{aligned} & i\psi_t(x,y,t) + \Delta\psi + |\psi|^2\psi \\ & \underbrace{-\frac{\delta}{4}\left[\Delta^2\psi + 4|\psi|^2\Delta\psi + 4\psi\nabla\psi\nabla\psi^* + 2\psi^*\nabla\psi\nabla\psi + |\psi|^4\psi\right]}_{\text{nonparaxiality}} \\ & \underbrace{+\delta\left[\frac{4+6\gamma}{1+\gamma}|\psi_x|^2\psi + (\psi_x)^2\psi^* + \frac{1+2\gamma}{1+\gamma}\left(|\psi|^2\psi_{xx} + \psi^2\psi^*_{xx}\right)\right]}_{\text{vectorial effects}} = 0, \end{aligned} \tag{8}$$

[2]For a recent numerical study of the scalar NLH as a true boundary value problem, see [16].
[3]For derivation, see Appendix E in [12].

where γ is a constant. For convenience, we have noted the terms that correspond to nonparaxiality and to vectorial coupling. Thus, in (7) and (8), the small biharmonic term with $\epsilon = -\delta < 0$ arises as (part of) the nonparaxial correction to the NLS.

2.2. Fiber arrays. In the last few years, it has been suggested that faster transmission in optical fibers may be achieved by using an array of coupled optical waveguides arranged on a line, in which the pulses undergo two-dimensional self-focusing. The model equation for the nth fiber is given by

$$i\psi_z^n - \beta_2\psi_{tt}^n + 2\gamma|\psi^n|^2\psi^n + \delta(\psi^{n+1} - 2\psi^n + \psi^{n-1}) = 0, \tag{9}$$

where $\psi^n(t,z)$ is the electric field envelope in the nth fiber, δ is the coupling coefficient between neighboring fibers, β_2 is the group velocity dispersion, and γ is the nonlinear coefficient. For theoretical and numerical studies of (9), see, e.g., [1, 2, 3, 4, 24, 33].

When time dispersion is anomalous ($\beta_2 < 0$), the change of variables

$$\tilde{t} = \delta h^2 z, \quad \tilde{\psi}^n = \frac{1}{h}\left(\frac{2\gamma}{\delta}\right)^{1/2}\psi^n, \quad \tilde{y} = \left[h\left(\frac{\delta}{|\beta_2|}\right)^{1/2}\right]t,$$

where h is the distance between fibers, yields (after dropping the tildes)

$$i\psi_t^n(y,t) + \psi_{yy}^n + |\psi^n|^2\psi^n + \frac{\psi^{n+1} - 2\psi^n + \psi^{n-1}}{h^2} = 0. \tag{10}$$

Let $\psi(x = nh, y, t) = \psi^n(y,t)$, and assume that the optical field is slowly varying over a number of fibers in the x direction, i.e., $h \ll 1$. Using Taylor expansion, we have that

$$\frac{\psi^{n+1} - 2\psi^n + \psi^{n-1}}{h^2} = \psi_{xx} + \epsilon\psi_{xxxx} + O(\epsilon^2), \tag{11}$$

where $\epsilon = h^2/12$. Therefore, propagation in fiber arrays can be approximated with the perturbed NLS

$$i\psi_t(x,y,t) + \Delta\psi + |\psi|^2\psi + \epsilon\psi_{xxxx} = 0. \tag{12}$$

Thus the discrete coupling between the fibers gives rise to an anisotropic biharmonic term with $\epsilon > 0$.

3. Review of NLS theory. We now give a short review of the theory of singularity formation in the NLS, which is mostly due to Weinstein [32]. For more information, see also [14, 30]. Let us consider the NLS

$$i\psi_t(t,\mathbf{x}) + \Delta\psi + \mu|\psi|^{2\sigma}\psi = 0, \qquad \mu = \pm 1. \tag{13}$$

Two important conserved quantities of (13) are the power and the Hamiltonian, i.e.,

$$||\psi||_2^2 \equiv ||\psi_0||_2^2, \qquad H(\psi) \equiv H(\psi_0), \tag{14}$$

where

$$H(\psi) = ||\nabla\psi||_2^2 - \frac{\mu}{\sigma+1}||\psi||_{2\sigma+2}^{2\sigma+2},$$

and

$$||f(\mathbf{x})||_p = \left(\int |f(\mathbf{x})|^p \, d\mathbf{x} \right)^{1/p}.$$

Solutions of the NLS (13) satisfy the *variance identity*

$$V_{tt} = 8H - 8\mu \frac{d\sigma - 2}{2\sigma + 2} ||\psi||_{2\sigma+2}^{2\sigma+2},$$

where

$$V = \int |r|^2 |\psi|^2 \, d\mathbf{x} \quad \text{and} \quad r = |\mathbf{x}|.$$

From the variance identity and Hamiltonian conservation, it follows that when $\mu = 1$, $\sigma d \geq 2$, and $H(\psi_0) < 0$, the solution of the NLS (13) becomes singular at a finite time.

From the local existence theory for the NLS (13), it follows that a priori bounds of the H^1 norm imply global existence [19, 22]. Therefore, from (14), it follows that when $\mu = -1$ (*defocusing NLS*), all solutions of (13) exist globally. Global existence theory for the *focusing NLS* (i.e., $\mu = 1$) is based on the *Gagliardo–Nirenberg inequality* [17, 18, 28]

$$||f||_{2\sigma+2}^{2\sigma+2} \leq C_{\sigma,d} ||\nabla f||_2^{\sigma d} ||f||_2^{2+\sigma(2-d)}, \tag{15}$$

where σ satisfies

$$\begin{cases} 0 \leq \sigma, & d \leq 2, \\ 0 \leq \sigma < 2/(d-2), & d > 2, \end{cases} \tag{16}$$

and $C_{\sigma,d}$ is a constant which depends on d and σ. From power conservation, Hamiltonian conservation, and the Gagliardo–Nirenberg inequality, one can obtain the a priori bound

$$||\nabla \psi||_2^2 \leq H(0) + \kappa ||\nabla \psi||_2^{\sigma d},$$

where κ is a positive constant whose value is given by

$$\kappa = \frac{C_{\sigma,d}}{\sigma + 1} ||\psi_0||_2^{2+\sigma(2-d)}.$$

Hence solutions of the NLS (13) with $\mu = 1$ exist globally if $\sigma d < 2$ (subcritical NLS) or if $\sigma d = 2$ (critical NLS) and $||\psi_0||_2^2 < N_c$, where the critical power N_c is given by

$$N_c = \left(\frac{\sigma + 1}{C_{\sigma,d}} \right)^{1/\sigma}. \tag{17}$$

Because solutions of the NLS (13) can only become singular when $\sigma d \geq 2$, *the critical exponent/dimension for singularity formation in the NLS* (13) *is given by* $\sigma d = 2$.

The calculation of the optimal constant $C_{\sigma,d}$ in inequality (15) is based on minimization of the functional

$$J[f] := \frac{||\nabla f||_2^{\sigma d} \, ||f||_2^{2+\sigma(2-d)}}{||f||_{2\sigma+2}^{2\sigma+2}}. \tag{18}$$

By calculating the variational derivative of $J[f]$, Weinstein [32] showed that, in the critical case $\sigma d = 2$,

$$\frac{1}{C_{\sigma,d}} = \frac{1}{\sigma+1}||R||_2^{2\sigma},$$

where R is the radially symmetric ground-state[4] solution of

$$\Delta R(r) - R + R^{4/d+1} = 0, \qquad R'(0) = 0, \quad R(\infty) = 0. \tag{19}$$

Therefore, when $\mu = 1$ and $\sigma d = 2$, the critical power for singularity formation is given by $N_c = ||R||_2^2$. In addition, the ground-state solution of (19) is positive and monotonically decreasing and has zero Hamiltonian ($H(R) = 0$).

When $d = 1$ and $\sigma = 2$, (19) becomes

$$R''(r) - R + R^5 = 0, \qquad R'(0) = 0, \quad R(\infty) = 0.$$

This equation has a unique nontrivial solution $R = 3^{1/4}\text{sech}^{1/2}(2r)$. Therefore, when $d = 1$, the critical power is given by $N_c = \int_{-\infty}^{\infty} R^2(r)\, dr = \sqrt{3}\pi/2$.

When $d \geq 2$, (19) has an infinite number of solutions in H^1. However, (19) has a unique positive solution [23], which is the minimizer of $J[f]$. Of special interest in the nonlinear optics context is the case when $d = 2$ and $\sigma = 1$, corresponding to the NLS (1). In this case, the minimizer $R(r)$, the so-called *Townes soliton*, is the ground-state solution of

$$R'' + \frac{1}{r}R' - R + R^3 = 0, \qquad R'(0) = 0, \quad R(\infty) = 0. \tag{20}$$

Therefore, the critical power when $d = 2$ is $N_c = 2\pi \int_0^{\infty} R^2\, rdr \;\approx\; 11.7$.

4. Biharmonic NLS. We now extend the NLS theory presented in section 3 to the biharmonic NLS

$$i\psi_t(\mathbf{x}, t) + \epsilon\Delta^2\psi + |\psi|^{2\sigma}\psi = 0, \qquad \epsilon = \pm 1. \tag{21}$$

We say that ψ becomes singular at $t = T_c$ if $||\psi(\cdot, t)||_{H^2} < \infty$ for $0 \leq t < T_c$ and if $\lim_{t\to T_c} ||\psi(\cdot, t)||_{H^2} = \infty$, where

$$||f||_{H^2} = \sqrt{||f||_2^2 + ||\nabla f||_2^2 + ||\Delta f||_2^2}.$$

Power and Hamiltonian conservation are given by (14) with

$$H = -\epsilon||\Delta\psi||_2^2 - \frac{1}{\sigma+1}||\psi||_{2\sigma+2}^{2\sigma+2}.$$

4.1. Waveguide solutions. We can look for waveguide solutions of (21) of the form $\psi = e^{i\lambda^4 t}R_{B,\lambda}(\mathbf{x})$. In that case, $R_{B,\lambda}$ satisfies

$$-\lambda^4 R_{B,\lambda} + \epsilon\Delta^2 R_{B,\lambda} + R_{B,\lambda}^{2\sigma+1} = 0. \tag{22}$$

LEMMA 4.1. *Two necessary conditions for existence of nontrivial solutions to (22) in H^2 are $\epsilon < 0$ and $\sigma < 4/(4-d)$.*

[4] i.e., the nontrivial solution with the smallest power.

Proof. If we multiply (22) by $R_{B,\lambda}$ and integrate by parts, we get that

$$-\lambda^4||R_{B,\lambda}||_2^2 + \epsilon||\Delta R_{B,\lambda}||_2^2 + ||R_{B,\lambda}||_{2\sigma+2}^{2\sigma+2} = 0. \tag{23}$$

Similarly, if we multiply (22) by $\mathbf{x}\cdot\nabla R_{B,\lambda}$ and integrate by parts, we get that

$$\lambda^4||R_{B,\lambda}||_2^2 + \epsilon\left(\frac{4}{d}-1\right)||\Delta R_{B,\lambda}||_2^2 - \frac{1}{\sigma+1}||R_{B,\lambda}||_{2\sigma+2}^{2\sigma+2} = 0. \tag{24}$$

If we multiply (24) by $(4/d-1)$ and subtract from (23), we get that

$$\lambda^4||R_{B,\lambda}||_2^2 = \frac{4-(d-4)\sigma}{4(\sigma+1)}||R_{B,\lambda}||_{2\sigma+2}^{2\sigma+2}.$$

If we multiply (24) by $(\sigma+1)$ and add to (23), we get that

$$\lambda^4\sigma||R_{B,\lambda}||_2^2 = -\epsilon\frac{4-(d-4)\sigma}{d}||\Delta R_{B,\lambda}||_2^2.$$

The necessary conditions follow from the requirement that the right-hand side of the last two equations must be positive. ∎

From now on, we consider only the case when $\epsilon=-1$ in (22), i.e.,

$$-\lambda^4 R_{B,\lambda}(\mathbf{x}) - \Delta^2 R_{B,\lambda} + R_{B,\lambda}^{2\sigma+1} = 0. \tag{25}$$

If we add (23) and (24), we get that

$$H(R_{B,\lambda}) = \frac{\sigma d-4}{4(\sigma+1)}||R_{B,\lambda}||_{2\sigma+2}^{2\sigma+2},$$

showing that

$$\begin{cases} H(R_{B,\lambda}) < 0, & \sigma d < 4, \\ H(R_{B,\lambda}) = 0, & \sigma d = 4, \\ H(R_{B,\lambda}) > 0, & \sigma d > 4. \end{cases}$$

Let $R_{B,\lambda}$ be a solution of (25). Then $R_{B,\lambda}(\mathbf{x}) = \lambda^{2/\sigma}R_B(\lambda\mathbf{x})$, where $R_B := R_{B,1}$ is the solution of

$$-R_B(\mathbf{x}) - \Delta^2 R_B + R_B^{2\sigma+1} = 0. \tag{26}$$

It is easy to see that

$$||R_{B,\lambda}||_2^2 = \lambda^{(4/\sigma-d)}||R_B||_2^2, \qquad H(R_{B,\lambda}) = \lambda^{4+(4-\sigma d)/\sigma}H(R_B). \tag{27}$$

Thus the critical case $\sigma d = 4$ has the two unique properties that $H(R_B)=0$ and

$$||R_{B,\lambda}(\mathbf{x})||_2^2 = ||R_B(\mathbf{x})||_2^2. \tag{28}$$

In addition, from (27), it follows that

$$\frac{d}{d\lambda}||R_{B,\lambda}||_2^2 > 0 \quad \Longleftrightarrow \quad \sigma d < 4,$$

suggesting that the waveguide solutions $\psi_\lambda(\mathbf{x},t) = e^{i\lambda^4 t}R_{B,\lambda}(\mathbf{x})$ are stable if and only if $\sigma d < 4$.

4.2. Existence. As in the case of the NLS (2), estimates of the linear biharmonic operator of (21) (see [6]) imply the so-called Strichartz estimates, which in turn imply local well-posedness in H^2. Therefore, a priori bounds of $\|\psi\|_{H^2}$ imply global existence in the biharmonic NLS (21).

THEOREM 4.2. *The following conditions are sufficient for global existence in* (21)*:*

1. $\epsilon > 0$.
2. $\epsilon < 0$, *and* $\sigma d < 4$.
3. $\epsilon < 0$, $\sigma d = 4$, *and* $||\psi_0||_2^2 < N_c^B$, *where*

$$N_c^B = \left(\frac{\sigma+1}{B_{\sigma,d}}\right)^{1/\sigma}. \tag{29}$$

Proof. When $\epsilon = 1$, Hamiltonian conservation leads to the a priori bound

$$||\Delta\psi||_2^2 \le |H(0)|.$$

In addition, using integration by parts and the Cauchy–Schwarz inequality,

$$||\nabla\psi||_2^2 \le \int |\psi||\Delta\psi|\, d\mathbf{x} \le ||\Delta\psi||_2||\psi||_2. \tag{30}$$

Since $||\psi||_2^2$ is conserved, we conclude that $||\psi||_{H^2}$ is globally bounded. Hence, when condition 1 holds, all solutions of the biharmonic NLS exist globally.

When $\epsilon = -1$, from Hamiltonian conservation we have that

$$||\Delta\psi||_2^2 = H(0) + \frac{1}{\sigma+1}||\psi||_{2\sigma+2}^{2\sigma+2}. \tag{31}$$

In order to bound $||\psi||_{2\sigma+2}^{2\sigma+2}$, we make use of the Gagliardo–Nirenberg inequality [17, 18, 28]

$$||f||_{2\sigma+2}^{2\sigma+2} \le B_{\sigma,d}||\Delta f||_2^{\sigma d/2}||f||_2^{2+2\sigma-\sigma d/2}, \tag{32}$$

where

$$\begin{cases} 0 \le \sigma, & d \le 4, \\ 0 \le \sigma < 4/(d-4), & d > 4, \end{cases} \tag{33}$$

and $B_{\sigma,d}$ is a constant which depends on d and σ. From relations (31) and (32) and power conservation, we get that

$$||\Delta\psi||_2^2 \le H(0) + \kappa||\Delta\psi||_2^{\sigma d/2}, \tag{34}$$

where κ is a positive constant whose value is given by

$$\kappa = \frac{B_{\sigma,d}}{\sigma+1}||\psi_0||_2^{2+2\sigma-\sigma d/2}. \tag{35}$$

Therefore, conditions 2 and 3 imply that $||\Delta\psi||_2^2$ is globally bounded. ∎

We recall that the NLS (13) is called focusing or defocusing when diffraction and nonlinearity are working against or with each other, respectively. Since the Laplacian is a negative operator, this corresponds to $\mu > 0$ and to $\mu < 0$, respectively. Since the biharmonic operator is positive, this suggests that $\epsilon > 0$ is the *defocusing biharmonic NLS* and $\epsilon < 0$ is the *focusing biharmonic NLS*. Thus Theorem 4.2 shows that *the critical exponent/dimension for singularity formation in the focusing biharmonic NLS* (21) *is given by* $\sigma d = 4$.

The calculation of the optimal constant $B_{\sigma,d}$ in inequality (32) is based on minimization of the functional

$$J_B[f] := \frac{||\Delta f||_2^{\sigma d/2}\ ||f||_2^{2+2\sigma-\sigma d/2}}{||f||_{2\sigma+2}^{2\sigma+2}} \tag{36}$$

over all functions $0 \not\equiv f \in H^2$. However, unlike the case of the NLS, where $J[|f|] = J[f]$ for all $f \in H^1$, one cannot assume that the minimizer of J_B (if it exists) is positive, because it is not true that $J_B[|f|] = J_B[f]$ for all $f \in H^2$. In fact, if there is a minimizer, it oscillates between positive and negative values (see (39)). Therefore, one cannot follow [32] and use Steiner symmetrization to show that the minimizer is radially symmetric and then use radial symmetry to prove the existence of a minimizer in H^2.

Let us assume that the minimum of $J_B[f]$ over all functions $0 \not\equiv f \in H^2$ is attained.[5] In this case, following [32], if we calculate the variational derivative of $J_B[f]$, we find that, in the critical case when $\sigma d = 4$ (see Appendix A),

$$\frac{1}{B_{\sigma,d}} = \frac{1}{\sigma+1}||R_B||_2^{2\sigma}, \qquad \sigma = 4/d,$$

where R_B is the ground-state solution of

$$-\Delta^2 R_B - R_B + R_B^{8/d+1} = 0, \tag{37}$$

i.e., the solution of (37) with the smallest power. Therefore, we have the following result.

LEMMA 4.3. *Let* $\epsilon = -1$ *and* $\sigma d = 4$. *Then the critical power for singularity formation in the biharmonic NLS* (21) *is given by*

$$N_c^B = ||R_B||_2^2, \tag{38}$$

where R_B *is the ground-state solution of* (37).

Let us also assume that the ground-state is radially symmetric, i.e., $R_B = R_B(r)$. WKB analysis of (37) shows that

$$R_B \sim ce^{-r/\sqrt{2}}\cos(r/\sqrt{2}), \qquad r \gg 1. \tag{39}$$

Therefore, unlike the ground-state of the NLS, *the ground-state solution of* (37) *is neither positive nor monotonic.*

4.3. Numerical calculation of waveguides. When $d = 1$ and $\sigma = 4$, the equation for $R_B(r)$ is given by

$$-R_B^{(4)}(r) - R_B + R_B^9 = 0 \tag{40}$$

subject to

$$R_B'(0) = R_B'''(0) = R_B(\infty) = R_B'(\infty) = 0.$$

[5]The question of whether the minimizer is unique is also open.

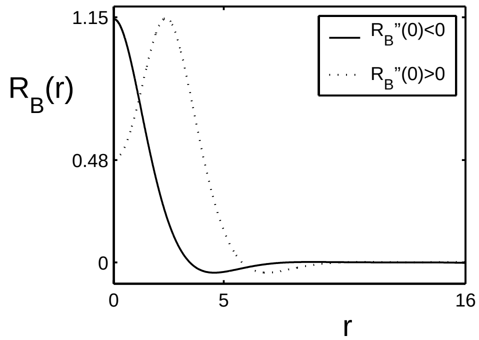

Fig. 1. *The two nontrivial solutions of* (40).

Table 1
Parameters of two numerical solutions of (40) *which are shown in Figure* 1. *Top: Solid line. Bottom: Dotted line.*

$R_B(0)$	$R''_B(0)$	$\|\|R_B\|\|_2^2$	$J_B(R_B)$
1.141176	-0.743742	2.986793	15.91661
0.4830812	0.4829379	6.290556	313.1744

Our numerical calculations (see Appendix B) suggest that this nonlinear boundary-value problem has *two* nontrivial solutions in H^2 (see Figure 1). Both solutions are indeed neither positive nor monotonically increasing. Inspection of the parameters of these two solutions in Table 1 shows that the ground-state is the one which is monotonically decreasing near the origin. Thus $\min_f J_B[f] \approx 15.9$[6] and

$$B_{4,1} = 0.5973586.$$

This calculation shows that the critical power for singularity formation in (44) is

$$N_c^B = ||R_B||_2^2 = 2.986793. \tag{41}$$

We note that, with less than 1% relative error, $\min_f J_B[f] \approx 50/\pi$, $B_{4,1} \approx \pi/50$, and $N_c^B \approx (250/\pi)^{1/4}$.

4.4. Biharmonic NLS on a bounded domain. Let us consider the biharmonic NLS on a smooth bounded domain $\Omega \in \mathbb{R}^d$ with Dirichlet boundary conditions, i.e.,

$$\begin{cases} i\psi_t(\mathbf{x},t) + \epsilon\Delta^2\psi + |\psi|^{2\sigma}\psi = 0, & t \geq 0, \quad \mathbf{x} \in \Omega, \\ \psi(\mathbf{x},t) = 0, & t \geq 0, \quad \mathbf{x} \in \partial\Omega, \end{cases} \tag{42}$$

where $\psi(\mathbf{x},0) = \psi_0(\mathbf{x}) \in H_0^2(\Omega)$. Then, as with the NLS on a bounded domain [10], the sufficient conditions for global existence for (21) are also sufficient for (42).

Theorem 4.4. *The following conditions are sufficient for global existence in* (42)*:*

1. $\epsilon > 0$.
2. $\epsilon < 0$, *and* $\sigma d < 4$.
3. $\epsilon < 0$, $\sigma d = 4$, *and* $||\psi_0||_2^2 < N_c^B$, *where* N_c^B *is given by* (38).

[6]For comparison, $J_B[e^{(-x^2)}] \approx 16.55$, $J_B[\text{sech}(x)] \approx 18.4$, and $J_B[\text{sech}^{1/2}(x)] \approx 22.3$.

In particular, *the (lower bound for the) critical power for singularity formation in the critical biharmonic NLS on bounded domains is the same as on* $\mathbb{R}^d$.

Proof. The proof of Theorem 4.4 is identical to that of Theorem 4.2, except that one makes use of the following Gagliardo–Nirenberg inequality for functions $f \in H_0^2(\Omega)$:

$$||f||_{L^{2\sigma+2}(\Omega)}^{2\sigma+2} \le B_{\sigma,d}(\Omega)||\Delta f||_{L^2(\Omega)}^{\sigma d/2}||f||_{L^2(\Omega)}^{2+2\sigma-\sigma d/2}, \tag{43}$$

where σ is given by (33). Therefore, the only nontrivial point is to show that, in the critical case when $\sigma d = 4$,

$$N_c^B(\Omega) = N_c^B(\mathbb{R}^d), \qquad \text{where} \quad N_c^B(\Omega) := \frac{\sigma+1}{B_{\sigma,d}(\Omega)}.$$

In general, the optimal constant in a Sobolev inequality depends also on the domain. Indeed, for general d and σ, it is only true that $B_{\sigma,d}(\Omega) \le B_{\sigma,d}(\mathbb{R}^d)$, where $B_{\sigma,d}(\mathbb{R}^d)$ are the optimal constants calculated in Appendix A. However, as in [10], one can use the scaling property (28) to prove that, in the critical case when $\sigma d = 4$, the optimal constants are independent of the domain, i.e.,

$$B_{4/d,d}(\Omega) = B_{4/d,d}(\mathbb{R}^d). \qquad \square$$

4.5. Blowup. There are only two known methods for proving blowup in the focusing NLS (1): the variance identity, which shows that solutions can blow up when $\sigma d \ge 2$ and $H(0) < 0$ and the lens (pseudoconformal) transformation, which, in the critical case when $\sigma d = 2$, can be used to construct exact blowup solutions. We do not know whether these results can be extended to the focusing biharmonic NLS. Therefore, at present, we rely on numerical simulations to see that solutions of the biharmonic NLS can blow up when $\sigma d \ge 4$.

In Figure 2A, we present numerical simulations of the critical biharmonic NLS ($\sigma = 4$ and $d = 1$)

$$i\psi_t(r,t) - \psi_{rrrr} + |\psi|^8\psi = 0, \tag{44}$$

with the initial conditions $\psi_0 = (1+\delta)R_B(r)$. The solution blows up for $\delta = 0.001$ but does not blow up for $\delta = -0.001$. These simulations provide support to the conjecture that the solution blows up for all $\delta > 0$ but does not blow up for all $\delta < 0$. If this conjecture is correct, we can conclude the following:

1. The waveguide solutions

$$\psi(r,t) = e^{i\lambda^4 t}\lambda^{1/2}R_B(\lambda r)$$

of the critical biharmonic NLS (44) are unstable.

2. The critical power (41) for singularity formation in the critical biharmonic NLS (44) is sharp in the sense that there exist solutions with input power slightly above the critical power which become singular.

When we solve the biharmonic NLS (44) with different initial profiles, we see that, as in the case of the critical NLS [11], the actual critical power is strictly above N_c^B yet only a few percent above N_c^B. For example, from Figures 2B–2D, we can conclude that the critical power for Gaussian initial conditions $\psi_0 = c \cdot \exp(-r^2)$ is between $1.003N_c^B$ and $1.004N_c^B$; for initial conditions $\psi_0 = c \cdot \text{sech}(r)$, it is between $1.01N_c^B$

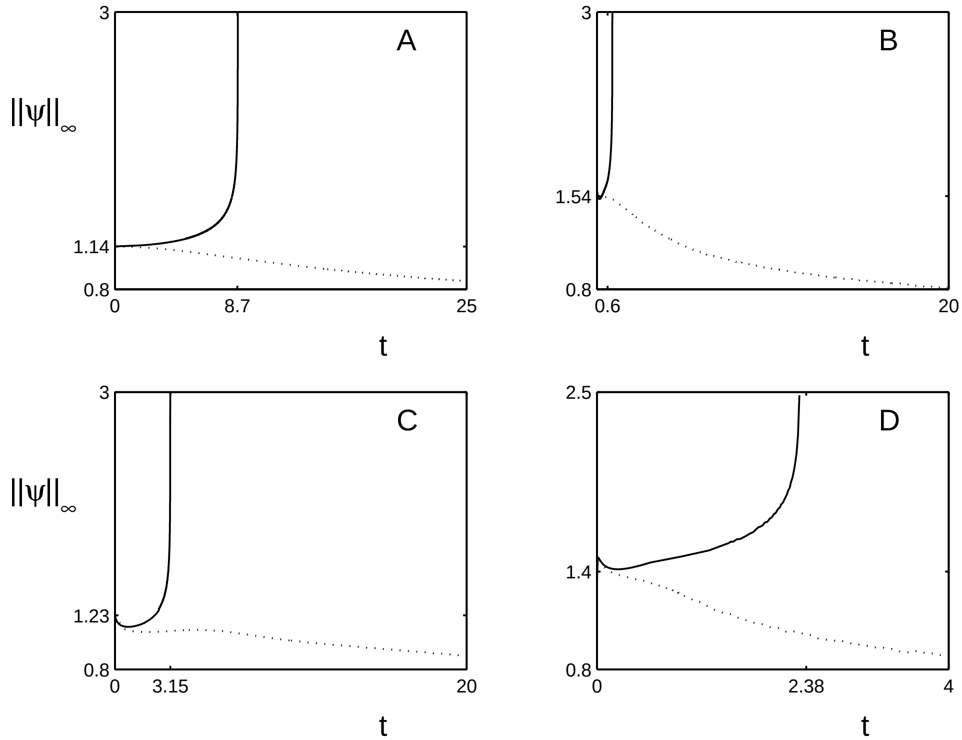

Fig. 2. *Calculation of lower bound (dots) and upper bound (solid line) for the critical power for blowup in* (44) *for various input profiles.* (A) $\psi_0 = cR_B(r)$*: Global existence when* $N(0) = 0.999Nc$*, blowup when* $N(0) = 1.001Nc$*.* (B) $\psi_0 = c\exp(-r^2)$*: Global existence when* $N(0) = 1.003Nc$*, blowup when* $N(0) = 1.004Nc$*.* (C) $\psi_0 = c \cdot \mathrm{sech}(r)$*: Global existence when* $N(0) = 1.01Nc$*, blowup when* $N(0) = 1.02Nc$*.* (D) $\psi_0 = c(1 + r^4)^{-1}$*: Global existence when* $N(0) = 1.005Nc$*, blowup when* $N(0) = 1.01Nc$*.*

and $1.02N_c^B$; and for initial conditions $\psi_0 = c \cdot (1 + r^4)^{-1}$, it is between $1.005N_c^B$ and $1.01N_c^B$.

In the case of the critical NLS, the asymptotic profile near the singularity is a modulated ground-state. This feature is closely related to the unique characteristics of critical self-focusing, such as power concentration [27] and sensitivity to small perturbations [14, 15]. Our numerical simulations show that, in the case of the biharmonic NLS (44), the solution also approaches a modulated ground-state near the singularity, i.e.,

$$|\psi| \sim L^{-1/2}(t) R_B(r/L(t)), \tag{45}$$

where R_B is the ground-state solution of (40) and the scaling function $L(t)$ vanishes at the blowup point. To show that, we plot $L^{1/2}(t)|\psi|$ as a function of $r/L(t)$, where $L(t)$ is determined from $L(t) = R_B^2(0)/|\psi(0,t)|^2$ (Figure 3). Although the initial conditions are not close to R_B, by the time the solution has focused to one-third of its initial width, the profile near the singularity is already close to the asymptotic profile (45).

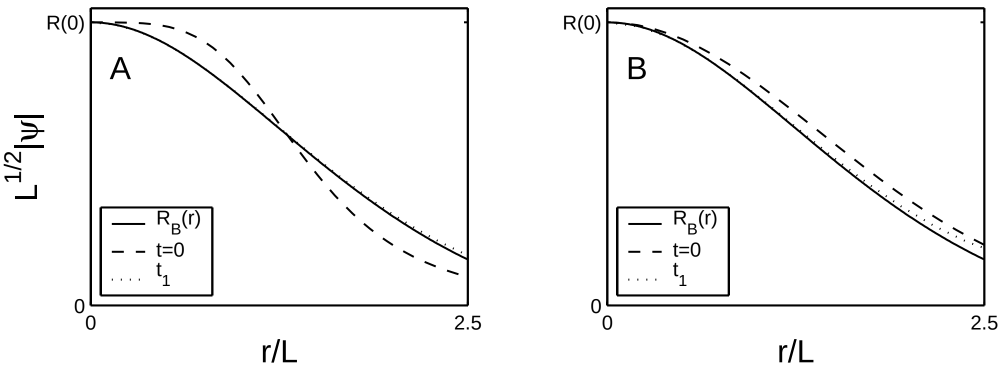

FIG. 3. *Convergence of blowup solutions of the critical biharmonic NLS* (44) *to the asymptotic profile* (45). (A) $\psi_0 = \sqrt{1.05 N_c^B \cdot 8/(3\sqrt{2}\pi)} \cdot (1+r^4)^{-1}$*. Here* $N(0) = 1.05 N_c^B$*,* $L(0) \approx 0.69$*, and* $L(t_1 = 0.175) \approx 0.375 L(0)$. (B) $\psi_0 = \sqrt{1.1 N_c^B \sqrt{2/\pi} \cdot e^{-r^2}}$*. Here* $N(0) = 1.1 N_c^B$*,* $L(0) \approx 0.50$*, and* $L(t_1 = 0.0465) \approx 0.464 L(0)$.

5. High-dispersion NLS. In this section, we sketch the extension of some of the results of section 4 to the NLS with qth-order dispersion

$$i\psi_t(t,\mathbf{x}) + \epsilon\Delta^q\psi + |\psi|^{2\sigma}\psi = 0, \qquad \psi(0,\mathbf{x}) = \psi_0(\mathbf{x}), \qquad \epsilon = \pm 1. \tag{46}$$

Power and Hamiltonian conservation are given by (14) with

$$H = \begin{cases} \epsilon||\nabla(\Delta^m\psi)||_2^2 - \frac{1}{\sigma+1}||\psi||_{2\sigma+2}^{2\sigma+2}, & q = 2m+1, \\ -\epsilon||\Delta^m\psi||_2^2 - \frac{1}{\sigma+1}||\psi||_{2\sigma+2}^{2\sigma+2}, & q = 2m. \end{cases}$$

Alternatively, we can write the Hamiltonian as

$$H = (-1)^{q-1}\epsilon||(-\Delta)^{q/2}\psi||_2^2 - \frac{1}{\sigma+1}||\psi||_{2\sigma+2}^{2\sigma+2},$$

where $||\Delta^{q/2}\psi||_2^2 = \int |\mathbf{k}|^q|\hat{\psi}|^2\, d\mathbf{k}$.

THEOREM 5.1. *The following conditions are sufficient for global existence in* (46)*:*

1. $\epsilon(-1)^q > 0$.
2. $\epsilon(-1)^q < 0$*, and* $\sigma d < 2q$.
3. $\epsilon(-1)^q < 0$*,* $\sigma d = 2q$*, and* $||\psi_0||_2^2 < [(\sigma+1)/D_{\sigma,d,q}]^{1/\sigma}$*, where* $D_{\sigma,d,q}$ *is the optimal constant in the inequality* (47).

Proof. When $\epsilon(-1)^q > 0$, from Hamiltonian conservation we have a priori bounds for $||\nabla(\Delta^m\psi)||_2^2$ when q is odd and for $||\Delta^m\psi||_2^2$ when q is even. Therefore, by standard arguments, the solutions exist globally.

When $\epsilon(-1)^q < 0$, we can combine the Gagliardo–Nirenberg inequality

$$||f||_{2\sigma+2}^{2\sigma+2} = D_{\sigma,d,q}||(-\Delta)^{q/2}f||_2^{\sigma d/q}||f||_2^{2\sigma+2-\sigma d/q} \tag{47}$$

with power and Hamiltonian conservation to get that

$$||(-\Delta)^{q/2}f||_2^2 \le H(0) + \frac{1}{\sigma+1}D_{\sigma,d,q}||(-\Delta)^{q/2}\psi||_2^{\sigma d/q}||\psi_0||_2^{2\sigma+2-\sigma d/q}.$$

Therefore, conditions 2 and 3 guarantee that $||(-\Delta)^{q/2}f||_2^2$ is globally bounded. ∎

Thus (46) is defocusing if $\epsilon(-1)^q > 0$ and focusing if $\epsilon(-1)^q < 0$. In the latter case, the critical exponent/dimension for collapse is $\sigma d = 2q$.

The optimal constants $D_{\sigma,d,q}$ can be calculated as in Appendix A. In addition, in critical dimension $\sigma d = 2q$, these optimal constants are independent of the domain.

6. Mixed dispersion. We are now ready to analyze the NLS with both low-order and high-order dispersion terms

$$i\psi_t(t,\mathbf{x}) + \Delta\psi + |\psi|^{2\sigma}\psi + \epsilon\Delta^2\psi = 0, \qquad \epsilon = \pm 1. \tag{48}$$

6.1. Rigorous analysis. We first note that power and Hamiltonian conservation in (48) are given by (14) with

$$H = -\epsilon||\Delta\psi||_2^2 + ||\nabla\psi||_2^2 - \frac{1}{\sigma+1}||\psi||_{2\sigma+2}^{2\sigma+2}. \tag{49}$$

We now show that the sufficient conditions for global existence in the biharmonic NLS (21) are also sufficient for the mixed-dispersion NLS (48).

THEOREM 6.1. *The following conditions are sufficient for global existence in* (48)*:*

1. $\epsilon > 0$.
2. $\epsilon < 0$*, and* $\sigma d < 4$.
3. $\epsilon < 0$*,* $\sigma d = 4$*, and* $||\psi_0||_2^2 < N_c^B$*, where* N_c^B *is given by* (38).

Proof. We first note that the estimates of the linear operator associated with (48) (see [6]) can be used to show that a priori bounds of $||\psi||_{H^2}$ imply global existence in (48).

When $\epsilon = 1$, from power and Hamiltonian conservation and inequality (30), we have that

$$||\Delta\psi||_2^2 + \frac{1}{\sigma+1}||\psi||_{2\sigma+2}^{2\sigma+2} = -H(0) + ||\nabla\psi||_2^2 \le -H(0) + ||\Delta\psi||_2||\psi_0||_2,$$

showing that $||\Delta\psi||_2$ is globally bounded. When $\epsilon = -1$, from Hamiltonian conservation, we have that

$$||\Delta\psi||_2^2 = H(0) + \frac{1}{\sigma+1}||\psi||_{2\sigma+2}^{2\sigma+2} - ||\nabla\psi||_2^2 \le H(0) + \frac{1}{\sigma+1}||\psi||_{2\sigma+2}^{2\sigma+2}.$$

Therefore, the proof is the same as that of Theorem 4.2. □

6.2. Asymptotic analysis. The a priori estimates in Theorem 6.1 lead to sufficient conditions for global existence for the mixed-dispersion NLS (3). These global estimates do not provide information, however, on the dynamics of solutions. Such information can be obtained when the biharmonic term is a small perturbation of the NLS. For example, as shown by Karpman and Shagalov, when $\epsilon < 0$ and $\sigma d < 2$, the small biharmonic term does not affect the stability of waveguide solutions.

We now consider the case where the biharmonic term is a small perturbation of the critical NLS ($\sigma = 1$, $d = 2$)

$$i\psi_t(x,y,t) + \Delta\psi + \epsilon\Delta^2\psi + |\psi|^2\psi = 0. \tag{50}$$

The critical case is especially interesting for the following reason. Unlike the case when $\sigma d < 2$, solutions of (50) can collapse in the absence of the biharmonic term, and the waveguides are unstable. However, in contrast with supercritical collapse, critical collapse can be arrested by infinitesimally small perturbations. This difference between critical and supercritical collapse explains the results of Karpman and

Shagalov that when $\sigma d = 2$, waveguides become stable for any small $\epsilon < 0$, whereas when $2 < \sigma d < 4$, waveguides become stable only when $(-\epsilon)$ is sufficiently large [21].

We now analyze the effect of a small biharmonic term using *modulation theory*, which is an asymptotic theory for analyzing the effects of small perturbations on critical self-focusing [14, 15]. As we shall see, our results agree with those of Karpman and Shagalov for (50). In addition, we show that the generic propagation dynamics for $\epsilon < 0$ is focusing-defocusing oscillations. We also analyze the case when $\epsilon > 0$, which was not considered by Karpman and Shagalov.

Modulation theory is based on the observation that, after some propagation has taken place, the collapsing part of the beam rearranges itself as a modulated Townesian, i.e., $|\psi| \sim L^{-1}(t)R(r/L(t))$, where $R(r)$ is the Townes soliton (20). Therefore, self-focusing dynamics is described by the modulation variable $L(t)$, which is proportional to beam-width and also to 1/(on-axis amplitude). In particular, $L \to 0$ and $L \to \infty$ correspond to blowup and to complete defocusing, respectively.

Let us rewrite (50) as a perturbed NLS

$$i\psi_t(x, y, t) + \Delta\psi + |\psi|^2\psi + \epsilon F[\psi] = 0, \tag{51}$$

where $F[\psi] := \Delta^2\psi$. Modulation theory for (51) is based on the following assumptions:

1. The solution ψ is close to a modulated Townes soliton, i.e.,

$$\psi(r, t) \sim \psi_R(r, t) := \frac{1}{L(t)} R(\rho) e^{iS}, \tag{52}$$

where $R(\rho)$ is the Townes profile (20) and

$$\rho(r, t) := \frac{r}{L(t)}, \quad S(r, t) := \tau(t) + \frac{r^2 L_t(t)}{4L(t)}, \quad \tau_t(t) := \frac{1}{L^2(t)}. \tag{53}$$

2. Let $\beta(t) = -L^3 L_{tt}$. Then $|\beta| \ll 1$.
3. The perturbation term is small, i.e., $|\epsilon F| \ll |\Delta\psi|$ and $|\epsilon F| \ll |\psi|^3$.

Under these assumptions, self-focusing dynamics of the perturbed NLS (51) is described, to leading order, by [14, Proposition 4.1]

$$L_{tt}(t) = -\frac{\beta}{L^3}, \qquad \beta_t(t) = \frac{\epsilon}{2M}\left(f_{1,t} - 4f_2 \right), \tag{54}$$

where

$$\begin{aligned} f_1(t) &= \frac{L}{\pi} \mathrm{Re} \int F[\psi_R][R(\rho) + \rho R'(\rho)] e^{-iS} \, dxdy, \\ f_2(t) &= \frac{1}{2\pi} \mathrm{Im} \int F[\psi_R]\psi_R^* \, dxdy. \end{aligned} \tag{55}$$

Application of modulation theory to (50) yields the following result.

PROPOSITION 6.2. *Self-focusing dynamics of the solution of* (50) *is given, to leading order, by the reduced system*

$$L_{tt}(t) = -\frac{\beta}{L^3}, \qquad \beta_t(t) = \frac{\epsilon C_{\mathrm{bihar}} N_c}{2M} \left(\frac{1}{L^2} \right)_t, \tag{56}$$

where $M = \frac{1}{4}\int_0^\infty \rho^2 R^2 \rho \, d\rho \approx 0.55$ *and* $C_{\mathrm{bihar}} \approx 12$.

Proof. See Appendix C. □

Thus, in the case of the critical NLS, the small biharmonic perturbation leads to the same generic reduced equations (and therefore has the same effect) as that of nonlinear saturation, nonparaxiality, and vectorial effects [12, 13, 14, 26].

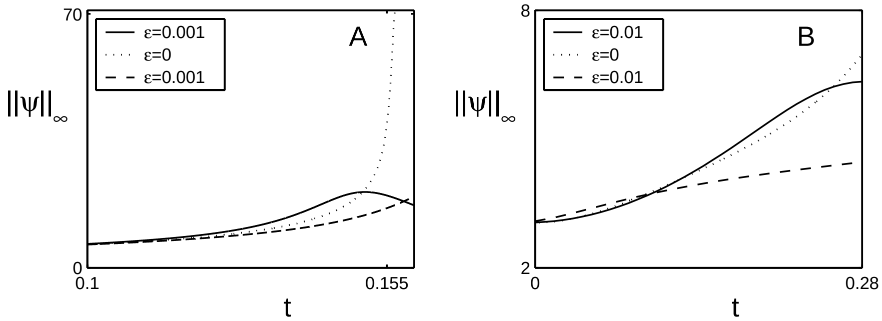

FIG. 4. *Dynamics of* $\max_r |\psi(r,t)|^2$ *for solutions of* (50) *with Gaussian initial conditions* (58). (A) $N(0) = 2N_c$. (B) $N(0) = 1.25N_c$.

6.2.1. Initial stage. We first consider the effect of the small biharmonic term in (50) during the initial stage of propagation. If we integrate (56) once, we get that

$$\beta(t) = \frac{\epsilon C_{\text{bihar}} N_c}{2M} \frac{1}{L^2} + \beta_0, \qquad \beta_0 = \beta(0) - \frac{\epsilon C_{\text{bihar}} N_c}{2ML^2(0)}. \tag{57}$$

When $\epsilon > 0$, we see from (57) that β increases as $L \to 0$. Therefore, the biharmonic perturbation accelerates self-focusing. Conversely, when $\epsilon < 0$, β decreases as $L \to 0$, and the biharmonic perturbation slows down self-focusing. Thus *a small biharmonic term is initially focusing when* $\epsilon > 0$ *and defocusing when* $\epsilon < 0$.

The last conclusion appears to contradict the global existence results in section 6.1, which show that the biharmonic term is defocusing when $\epsilon > 0$ and focusing when $\epsilon < 0$. There is no contradiction, however, because, as far as global existence is concerned, the Laplacian is a lower-order term, and the competition is between the biharmonic term and the nonlinearity. In contrast, in (50), the biharmonic term is a lower-order term, and the competition is between diffraction and nonlinearity. In that case, the biharmonic term is focusing when it acts with[7] the nonlinearity and vice versa. Thus, *the role of the biharmonic term (focusing or defocusing) depends on both its sign and whether it is small or large*.

To confirm this prediction of modulation theory, we solve (50) with Gaussian initial conditions

$$\psi(r, t = 0) = 2\sqrt{N(0)}\, e^{-r^2}, \tag{58}$$

whose input power is $N(0)$. These initial conditions lead to blowup in the NLS (1) when $N(0) \geq 1.018N_c$ [11]. In Figure 4 we see that, indeed, during the initial stage, self-focusing is faster when $\epsilon > 0$ compared with the NLS (2), which, in turn, is faster than when $\epsilon < 0$.

In order to analyze subsequent propagation, we separate the two cases $\epsilon < 0$ and $\epsilon > 0$.

6.2.2. $\epsilon < 0$. We can follow [9, 15] and integrate (56) to get

$$(y_t)^2 = -\frac{4H_0}{My}(y_M - y)(y - y_m), \qquad y(t) := L^2(t), \tag{59}$$

[7] i.e., has the same sign in the Hamiltonian (49) as

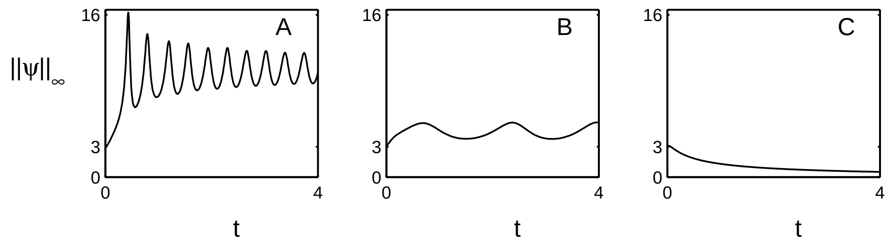

FIG. 5. *Peak amplitude of solutions of* (50)*,* (58) *with* $N(0) = 1.25N_c$*.* (A) $\epsilon = -0.001$*,* (B) $\epsilon = -0.01$*,* (C) $\epsilon = -0.1$*.*

where

$$
(60) \qquad \begin{aligned}
y_m &\approx -\frac{M\beta_0}{-2H_0}\left(1-\sqrt{1+4\delta}\right) \sim \frac{-\epsilon N_c C_{\text{bihar}}}{4M\beta_0}\left[1+O(\delta)\right],\\
y_M &\approx \frac{M\beta_0}{-2H_0}\left(1+\sqrt{1+4\delta}\right) \sim \frac{M\beta_0}{-H_0}\left[1+O(\delta)\right],
\end{aligned}
$$

$\delta = -\epsilon N_c C_{\text{bihar}} H_0/4M^2\beta_0^2$ and $H_0 \approx H(0)$.

We recall that a necessary condition for blowup in the NLS (1) is that the input power be above critical, i.e., $N(0) > N_c$. In modulation theory variables, this condition amounts to $\beta(0) \approx [N(0) - N_c]/M > 0$ [14]. However, when $\beta(0) > 0$ and $\epsilon < 0$, we see from (59) and (60) that $y(t) \geq y_m > 0$. Therefore, blowup is arrested by the high-order dispersion, and the minimal width is $L_m \sim L(0)\sqrt{|\epsilon| N_c C_{\text{bihar}}/4M\beta_0}$. Even at this stage, the magnitude of the higher-order dispersion term in (50) is $O(\beta)$ smaller than that of the NLS terms $\Delta\psi$ and $|\psi|^2\psi$, providing an a posteriori justification for treating this term as a small perturbation. Therefore, *when* $\epsilon < 0$*, modulation theory remains valid for a long time*.

A sufficient condition for blowup in the unperturbed NLS (2) is $H(0) < 0$. However, from (59) and (60) we see that, if $\beta_0 > 0$ and $H(0) < 0$, then $y_m \leq y(t) \leq y_M$; i.e., arrest of blowup is followed by focusing-defocusing oscillations. When nonadiabatic radiation is added to (56), the oscillations decay with propagation [9].

The focusing-defocusing oscillations that are predicted by modulation theory can be seen in numerical simulations of (50) with Gaussian initial conditions (58) with power $N(0) = 1.25N_c$ and $N(0) = 2N_c$ (Figures 5 and 6, respectively). In both cases, as $(-\epsilon)$ increases, the effect of the defocusing biharmonic term increases and the peak height of the oscillations decreases, in agreement with (60). In particular, when $|\epsilon|$ is sufficiently large, the biharmonic term dominates over diffraction, and the pulse simply defocuses as its propagation is dominated by a defocusing biharmonic NLS.

6.2.3. $\boldsymbol{\epsilon > 0}$**.** We have already seen that, when $\epsilon > 0$, the reduced system (56) predicts that high-order perturbation accelerates self-focusing. In fact, it is easy to see that if the solution of the reduced system (56) with $\epsilon = 0$ (which corresponds to the NLS (1)) blows up at time T_c (i.e., $L(T_c) = 0$), then the solution of the reduced system (56) with $\epsilon > 0$ blows up even faster (i.e., $L(T^*) = 0$ and $0 < T^* < T_c$). The prediction that self-focusing is initially accelerated is confirmed numerically in Figure 4. However, if we run the simulations of Figure 4 for a longer time, we can see that

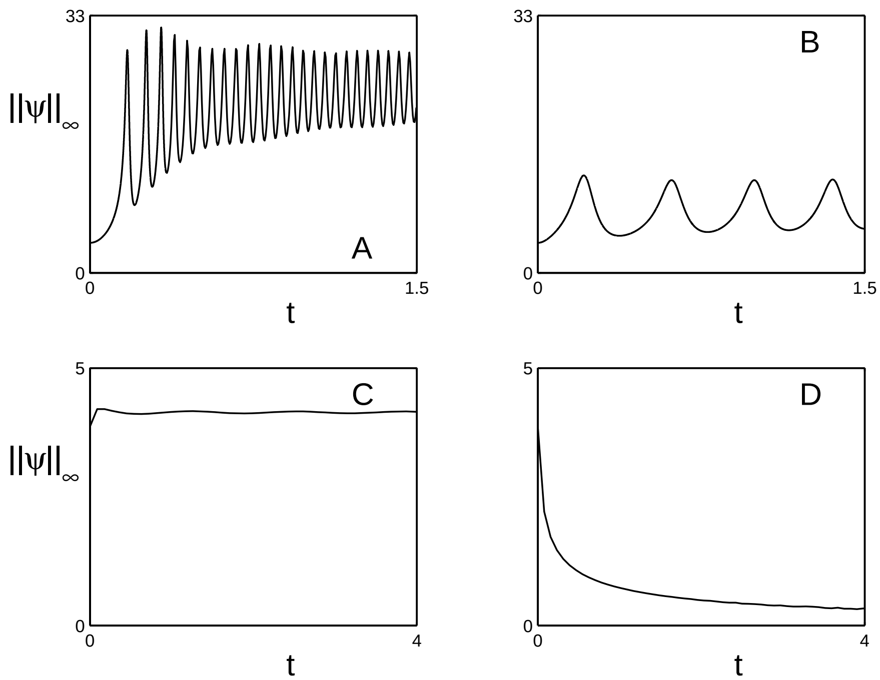

FIG. 6. *Peak amplitude of solutions of* (50), (58) *with* $N(0) = 2N_c$. (A) $\epsilon = -0.001$, (B) $\epsilon = -0.01$, (C) $\epsilon = -0.1$, (D) $\epsilon = -1$.

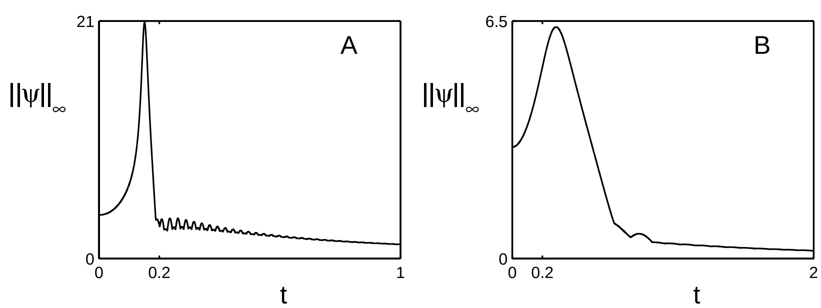

FIG. 7. *Peak amplitude of solutions of* (50), (58). (A) $\epsilon = 0.001$ *and* $N(0) = 2N_c$. (B) $\epsilon = 0.01$ *and* $N(0) = 1.25N_c$

focusing is arrested, after which the solution simply diffracts to infinity[8] (Figure 7).

In order to explain this disagreement, we recall that modulation theory is derived under the assumption that $|\beta| \ll 1$. However, from (57) we see that $\beta(t) \to \infty$ as $L \to 0$. Therefore, as the solution focuses, the validity of the assumption $\beta \ll 1$ breaks down, and (57) no longer describes the dynamics of the PDE (50). Roughly speaking, the validity of modulation theory breaks down when $\beta \sim 1$, which corresponds to $L \sim \sqrt{\epsilon}$.

The breakdown of modulation theory can also be explained at the PDE level as

[8]We verified that the small oscillations during the defocusing stage are not a numerical artifact repeating these simulations with different grid sizes and over larger computational domains.

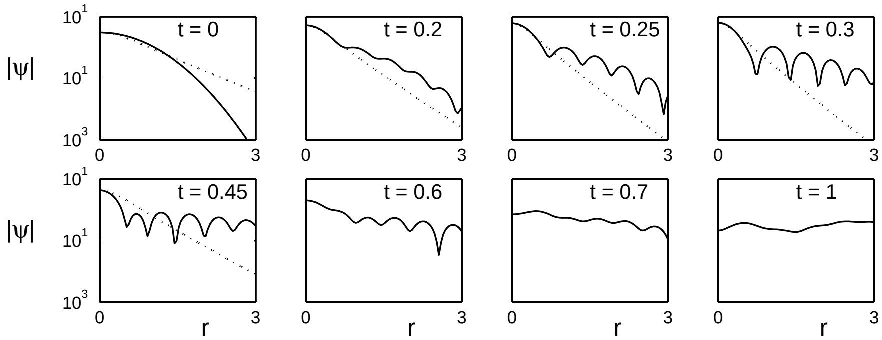

Fig. 8. *Spatial profile of the solution of Figure* 7B *(solid line) and a modulated Townes profile (dotted line).*

follows. Modulation theory is derived under the assumption that the perturbation term is small compared with the Laplacian. As $L \to 0$, the biharmonic term increases faster than the Laplacian. Therefore, roughly speaking, the validity of modulation theory breaks down when $\epsilon\Delta^2\psi$ becomes of comparable magnitude to $\Delta\psi$, which again corresponds to $L \sim \sqrt{\epsilon}$.

To see the breakdown of modulation theory numerically, we plot in Figure 8 the profile of the solution at various times with the corresponding modulated Townesian profile. As can be seen, as the solution initially focuses ($0 \le t \le 0.3$), the inner part of the solution approaches a Townesian profile, while the rest of the beam undergoes strong radiation. For $t \ge 0.3$, the solution simply diffracts to infinity and bears no resemblance to the Townes profile. At this stage, ψ is no longer close to the Townes profile, which is a key assumption of modulation theory.

We now analyze the dynamics after the validity of modulation theory breaks down. We have already seen that, at this stage, $\epsilon\Delta^2\psi$ is no longer a small term. As the solution continues to focus, $\epsilon\Delta^2\psi$ eventually becomes dominant over $\Delta\psi$, and the dynamics is captured, to leading order, by the defocusing biharmonic NLS

$$i\psi_t + \epsilon\Delta^2\psi + |\psi|^2\psi = 0, \qquad \epsilon > 0. \tag{61}$$

Since solutions of (61) simply diffract to infinity, the overall dynamics of solutions of (50) with $0 < \epsilon \ll 1$ is a single focusing-defocusing cycle. Clearly, if ϵ is sufficiently nonsmall, the dynamics is captured by (61) from the beginning. In that case, there is no initial focusing, and the solution diffracts to infinity. The above predictions are confirmed with simulations of (50) with Gaussian initial conditions (58) with power $N(0) = 1.25N_c$ and positive values of ϵ. Indeed, when ϵ is sufficiently small, there is a single focusing-defocusing cycle, whereas for $\epsilon \ge 0.1$ there is no initial focusing stage, and the solution simply defocuses (see Figure 9).

7. Fiber arrays revisited. Our rigorous global existence analysis of the mixed-dispersion NLS (48) does not extend to the NLS (12) because of the anisotropy of its biharmonic term. One can expect that the overall effect of $\epsilon\psi_{xxxx}$ (with $\epsilon > 0$) would be similar to that of the full biharmonic term; i.e., collapse is initially accelerated but later arrested, resulting in a single focusing-defocusing cycle. In addition, the solution should lose its radial symmetry. These predictions are confirmed numerically in Figures 10 and 11. One can clearly see that the mechanism for the arrest of collapse

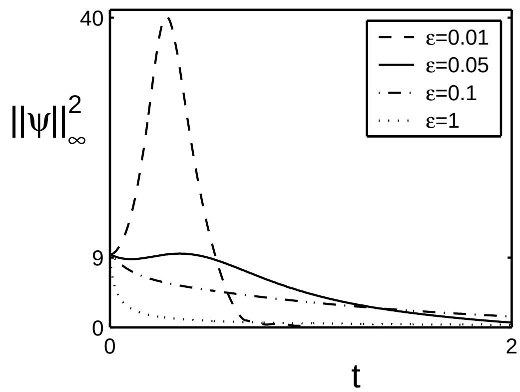

FIG. 9. *Solutions of* (50) *with different values of* ϵ *and* $N(0) = 1.25N_c$.

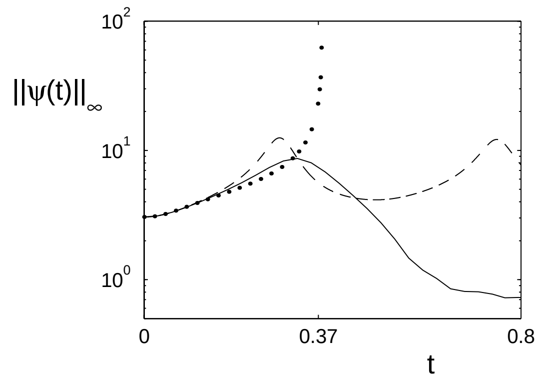

FIG. 10. *Solutions of* (2) *(dots), of* (10) *with* $n = 0, \pm 1, \dots, \pm 27$ *and* $h = \sqrt{12\epsilon}$ *(dashes), and of* (12) *(solid) with the same* ϵ *and initial condition as in Figure* 7B.

is strong radiation in the x direction due to the ψ_{xxxx} term.[9]

Because of the anisotropy of (12), proving global existence rigorously requires a more delicate analysis than for (48) and is, at present, an open problem. In [14], it was shown that application of modulation theory to the fiber arrays NLS (12) leads to the reduced equations

$$L_{tt}(t) = -\frac{\beta}{L^3}, \qquad \beta_t(t) = \frac{\epsilon |C_1|}{2M} \left(\frac{1}{L^2} \right)_t, \tag{62}$$

where $|C_1| \approx 9N_c/2$. Equations (62), which have the same form as (56), predict the initial acceleration of self-focusing, but they cease to be a valid approximation once $\epsilon\psi_{xxxx}$ becomes comparable to ψ_{xx}. Thus the addition of ψ_{xxxx} further destabilizes the already unstable waveguide solutions. In contrast, analysis and numerical simulations of the semidiscrete NLS (10) show that it has stable waveguide solutions [1, 2, 3, 4, 24, 33]. This inconsistency was resolved in [14] by keeping one more term

[9] In order to avoid reflections from the numerical boundaries, the computational domain should be larger in the x direction. For example, in Figure 11, the computational domain is $-40 \le x \le 40$ and $-12 \le y \le 12$ (see section 8).

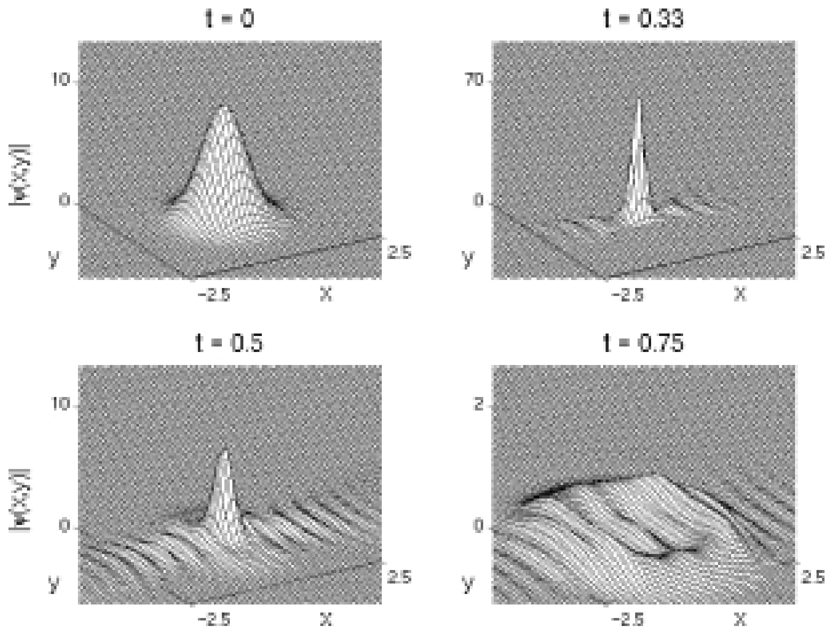

FIG. 11. *3D view of the solution of* (12) *in Figure* 10.

in the Taylor expansion in (11), in which case the equation for the propagation in fiber arrays is

$$i\psi_t(x,y,t) + \Delta\psi + |\psi|^2\psi + \epsilon\psi_{xxxx} + \frac{2}{5}\epsilon^2\psi_{xxxxxx} = 0. \tag{63}$$

Application of modulation theory to (63) shows that, indeed, the addition of $\epsilon^2\psi_{xxxxxx}$ stabilizes the waveguide solutions and leads to focusing-defocusing oscillations.

8. Radiation and numerical boundary conditions. In our simulations of the (2+1)D equations (2) and (12), we use a finite-difference scheme on a rectangular Cartesian grid with fourth-order accuracy in space. Time-stepping is implemented using a fourth-order Runge–Kutta method. We impose zero-Dirichlet boundary conditions at the outer boundaries. Because these boundary conditions are reflecting rather than absorbing, special care is taken to assure that the computational domain is sufficiently large so that reflections from the numerical boundaries have no effect.

Because the dispersion relation for the linear biharmonic Schrödinger equation is $\omega = k^4$, rather than $\omega = k^2$ for the linear Schrödinger equation, the propagation of the higher modes in the biharmonic NLS (21) and the mixed-dispersion NLS (48) is much faster than in the NLS (2) (see Figure 12). Therefore, in simulations of (21) and (48), we use substantially larger domains than in comparable NLS simulations. For example, in Figure 7A, the computational domain is $0 \le r \le 200$. For comparison, a similar simulation of the NLS requires a much smaller computational domain, e.g., $0 \le r \le 10$.

9. Final remarks. In this paper, we present the first systematic study of the biharmonic NLS (21), an equation which received little attention until now. We show that some properties of the NLS (13) extend to the biharmonic NLS while others do

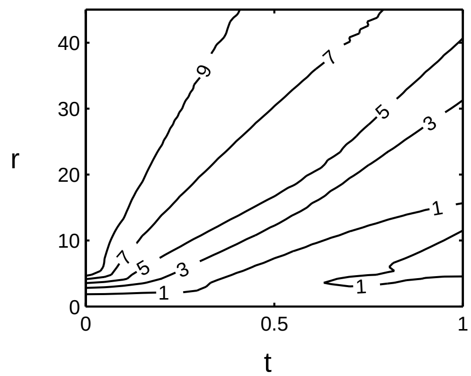

FIG. 12. *Contour plots of* $\log_{10}|\psi(t,r)|$ *for the solution of Figure* 7B.

not. Naturally, there are various important questions on the biharmonic NLS that are not addressed in this study such as a local existence theory, proof of existence of a (radially symmetric?) minimizer for the functional J_B (36), blowup rate in the critical and supercritical cases $\sigma d \geq 4$, a modulation theory for the effect of small perturbations in the critical case $\sigma d = 4$, etc.

Our analysis of the mixed-dispersion NLS (3) leads to the conclusion that when both diffraction and high-dispersion are present, their qualitative effect depends on their relative magnitudes. Thus the sufficient conditions for global existence for the biharmonic NLS are also sufficient for the mixed-dispersion NLS (3). In that case, the biharmonic term is focusing when $\epsilon < 0$ and defocusing when $\epsilon > 0$. When, however, the biharmonic term is a small perturbation of the critical NLS, it is stabilizing (defocusing) when $\epsilon < 0$ and destabilizing (focusing) when $\epsilon > 0$.

Our results for the mixed-dispersion NLS (3) do not extend directly to the NLS with nonparaxial effects (7), because it includes additional small terms. The quintic term in (7) has the same as the biharmonic one, i.e., stabilizing waveguides and arresting critical collapse [26]. Asymptotic and numerical results show that the remaining terms do not change these effects [9, 12].

Appendix A. Optimal constants $B_{\sigma,d}$. Because the optimal constants $B_{\sigma,d}$ have not been calculated before, we present the calculation for a general d and σ and not just for the case when $\sigma d = 4$. We first note that

$$\frac{1}{B_{\sigma,d}} = \inf_{0 \not\equiv f \in H^2} J_B[f]. \tag{64}$$

The Euler–Lagrange equation for the minimizer f of (36) can be derived from the condition that

$$\frac{d}{d\epsilon} J_B[f + \epsilon g]|_{\epsilon=0} = 0 \qquad \text{for all } g \in H^2,$$

which gives

$$\frac{\sigma d}{2\|\Delta f\|_2^2} \int \Delta f \Delta g \, d\mathbf{x} + \frac{2 + 2\sigma - \sigma d/2}{\|f\|_2^2} \int f g \, d\mathbf{x} - \frac{2\sigma + 2}{\|f\|_{2\sigma+2}^{2\sigma+2}} \int f^{2\sigma+1} g \, d\mathbf{x} = 0. \tag{65}$$

Let $f^{\alpha,\beta}(\mathbf{x}) = \beta f(\alpha \mathbf{x})$. Then it is easy to see that $J_B[f^{\alpha,\beta}] = J_B[f]$. Let us choose α_0 and β_0 so that $\|f^{\alpha_0,\beta_0}\|_2 = \|\Delta f^{\alpha_0,\beta_0}\|_2 = 1$. Therefore, $J_B[f^{\alpha_0,\beta_0}] = 1/\|f^{\alpha_0,\beta_0}\|_{2\sigma+2}^{2\sigma+2}$. To keep our notations clear, we denote from now on f^{α_0,β_0} by f. Equation (65) for the minimizer f can be rewritten as

$$\int g \left[\frac{\sigma d}{4} \Delta^2 f + \left(1 + \frac{\sigma}{4}(4-d) \right) f - J_B[f](\sigma+1) f^{2\sigma+1} \right] d\mathbf{x} = 0.$$

Since this equation is valid for all $g \in H^2$,

$$\frac{\sigma d}{4} \Delta^2 f + \left(1 + \frac{\sigma}{4}(4-d) \right) f - J_B[f](\sigma+1) f^{2\sigma+1} = 0. \tag{66}$$

In order to "remove" the unknown constant $J_B[f]$ from this equation, we note that

$$f(\mathbf{x}) = ((\sigma+1) J_B[f])^{-1/2\sigma} R_{B,\lambda_0} \left(\sqrt{\frac{4}{\sigma d}} \mathbf{x} \right), \tag{67}$$

where $R_{B,\lambda}$ is a solution of (25) and $\lambda_0 = 1 + \sigma(4-d)/4 > 0$. In addition, since $\|f\|_2 = 1$, it follows from (67) and (27) that

$$J_B[f] = \frac{1}{\sigma+1} \left(\frac{\sigma d}{4} \right)^{\sigma d/2} \|R_{B,\lambda_0}\|_2^{2\sigma} = \frac{1}{\sigma+1} \left(\frac{\sigma d}{4} \right)^{\sigma d/2} \lambda_0^{(4-\sigma d)} \|R_B\|_2^{2\sigma}, \tag{68}$$

where R_B is a solution of (26). In general, (26) can have more than one solution. However, in light of (68), the minimum of J_B is attained at R_B, the ground-state solution of (26), and

$$\min_{0 \not\equiv f \in H^2} J_B[f] = \frac{1}{\sigma+1} \left(\frac{\sigma d}{4} \right)^{\sigma d/2} \lambda_0^{(4-\sigma d)} \|R_B\|_2^{2\sigma}.$$

We thus have the following result.

LEMMA A.1. *The optimal constant in the Gagliardo–Nirenberg inequality* (32) *is given by*

$$B_{\sigma,d} = \left(\frac{\sigma d}{4} \right)^{-\sigma d/2} \lambda_0^{(\sigma d-4)} \frac{\sigma+1}{\|R_B\|_2^{2\sigma}},$$

where R_B is the ground-state solution of (26).

In the critical case when $\sigma d = 4$, this result simplifies as follows.

LEMMA A.2. *Let $\sigma = 4/d$. Then the optimal constant in the Gagliardo–Nirenberg inequality* (32) *is given by*

$$B_{\sigma,d} = \frac{\sigma+1}{\|R_B\|_2^{2\sigma}},$$

where R_B is the ground-state solution of (37).

Appendix B. Calculation of waveguides. Equation (40) is a nonlinear boundary value problem. We solve it using a shooting method; i.e., we search for the values of $R_B(0)$ and $R_B''(0)$ such that the solution will decay at infinity.[10] To simplify the search for $R_B(0)$ and for $R_B''(0)$, we multiply (40) by R_B', use the identity

[10] Since $R_B(r)$ is even, all odd derivatives of R_B vanish at $r = 0$.

$R_B^{(4)}R_B' = [R_B^{(3)}R_B' - R_B''^2/2]'$, integrate, and use the condition of decay at infinity to get that

$$-R_B^{(3)}R_B' + \frac{1}{2}R_B''^2 - \frac{1}{2}R_B^2 + \frac{1}{10}R_B^{10} = 0.$$

Substituting $r = 0$ gives

$$R_B''(0) = \pm\sqrt{R_B^2(0) - \frac{1}{5}R_B^{10}(0)}.$$

Therefore, we need only to search for $R_B(0)$ and only in the domain $0 < R_B(0) \leq 5^{1/8} \approx 1.22$.

Appendix C. Proof of Proposition 6.2. In order to apply modulation theory to (51), we first note from (51) and (55) that

$$f_2 = \frac{1}{2\pi}\text{Im}\int \psi_R^* \Delta^2 \psi_R \, dxdy \overset{\text{IBP}}{=} \frac{1}{2\pi}\text{Im}\int |\Delta^2\psi_R|^2 \, dxdy = 0, \tag{69}$$

where "IBP" stands for integration by parts. From (55) we also get that

$$f_1(t) = \frac{L}{\pi}\text{Re}\int (\Delta^2\psi_R)[R(\rho) + \rho R'(\rho)]e^{-iS} \, dxdy. \tag{70}$$

The calculation of this integral can be considerably simplified by using the following observation.

Lemma C.1. *Let $|\beta| \ll 1$. Then*

$$\Delta^2\psi_R(r/L) = L^{-1}e^{iS}(\Delta^2 R(r/L))\left[1 + O(|\beta|^{1/2})\right]. \tag{71}$$

Proof. We first obtain from (52) that

$$\psi_{R,x} = L^{-1}(R_x + iRS_x)e^{iS}. \tag{72}$$

In addition, from (53), we have that

$$S_x = \frac{1}{2}\rho L_t \cos\theta, \quad R_x = \frac{R'\cos\theta}{L}. \tag{73}$$

Because $\beta = L^3 L_{tt} \ll 1$, we have that $LL_t = O(|\beta|^{1/2}) \ll 1$. Therefore, from (72) and (73), we obtain that

$$\psi_{R,x} = L^{-1}e^{iS}R_x\left[1 + O(|\beta|^{1/2})\right].$$

Repeated application of this argument gives (71). ∎

Using (71) and (20) in (70) and integrating by parts, we get that

$$\begin{aligned} f_1(t) &\sim \frac{1}{\pi}\int (\Delta^2 R(r/L))[R(\rho) + \rho R'(\rho)] \, dxdy \\ &\overset{\text{IBP}}{=} \frac{1}{\pi}\int (\Delta R(r/L))\Delta(\rho R)' \, dxdy \\ &\overset{(20)}{=} \frac{2}{L^2}\int (R - R^3)\left(\partial_{\rho\rho} + \frac{1}{\rho}\partial_\rho\right)(\rho R)' \rho d\rho. \end{aligned} \tag{74}$$

Integration by parts shows that

$$\int (R - R^3)\left(\partial_{\rho\rho} + \frac{1}{\rho}\partial_\rho\right)(\rho R)'\,\rho d\rho = -\int R^{'2}\,\rho d\rho + 3\int (R^2 R^{'2} - \rho R R^{'3})\,\rho d\rho. \tag{75}$$

It is well known that

$$\int R^{'^2}\,\rho d\rho = N_c. \tag{76}$$

In [12], it is shown that

$$3\int R^2 R^{'2}\,\rho d\rho = I_6 - 2N_c, \qquad 3\int \rho R R^{'3}\,\rho d\rho = 3N_c - I_6, \tag{77}$$

where $I_6 := \int R^6\,\rho d\rho$. Substituting (75), (76), and (77) in (74), we obtain that

$$f_1(t) \sim \frac{C_{\rm bihar} N_c}{L^2}, \qquad C_{\rm bihar} = \frac{4I_6}{N_c} - 12.$$

Numerical evaluation of I_6 shows that it can be approximated with $I_6 \approx 6N_c$ with about 1% relative accuracy [12]. Therefore, we arrive at

$$f_1(t) \sim -\frac{12N_c}{L^2}. \tag{78}$$

Equation (56) follows from (54), (69), and (78). $\square$

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