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Self-gravitating discs and the Sobolev inequality

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ABSTRACT

We estimate the minimal mass of self-gravitating polytropic discs using the famous Sobolev inequality. This resembles the well-known mass formula for Lane–Emden stars. For ideal gas with the polytropic index $n = 3$, the minimal mass is not smaller than the Jeans mass. The accuracy of the estimate is verified in a number of numerical examples. The bound works well for heavy self-gravitating discs and is less useful for light discs.

Key words: accretion, accretion discs – gravitation – hydrodynamics.

1 INTRODUCTION

Models of static spherically symmetric configurations of selfgravitating polytropic fluids can be reduced to the analysis of the so-called Lane–Emden equations. They fall into the class of non-linear ordinary differential equations. Their solutions can be obtained by means of simple numerical methods or even analytically for some polytropic exponents.

In contrast to that, equations describing axially symmetric models of self-gravitating polytropic fluids undergoing a stationary rotation are in general intractable analytically (but see Odrzywołek 2003, for an approximation scheme). They can be written in the form of the Poisson equation for the gravitational potential and equations of hydrodynamics that can be formally integrated to yield an algebraic relation between the specific enthalpy, angular velocity and the gravitational potential. Another possibility, less useful computationally, but revealing the mathematical structure of the problem, is to express them as a single nonlinear elliptic equation for the specific enthalpy (or the mass density) with Dirichlet conditions imposed on the a priori unknown boundary of the disc.

We derive analytical results on rotating discs using simple physical information and certain functional analytic methods. There exists the so-called Sobolev inequality (see e.g. Rosen 1971). In this paper, we demonstrate that it can be used in order to estimate the mass of self-gravitating toroids. The approach works for those rotation laws for which the centrifugal potential Φ_c satisfies $\Delta\Phi_c \le 0$ (here Δ is the Laplacian). The equation of state of fluid is $p = K\rho^{1+1/n}$, where p is the pressure, ρ the density and $n \ge 3$. In analogy to the well-known Lane–Emden case, the estimate requires the knowledge of the maximal density and temperature of the configuration.

The bound is also valid for rotating polytropic stars. It yields rigorously, for ideal gas with the polytropic index $n = 3$, that the mass of stationary systems is not smaller than the Jeans mass. That is probably the first rigorous derivation of the Jeans inequality.

There are physically interesting situations, where it is important to distinguish between the mass of the whole accretion system consisting of a central object and a surrounding disc, and the mass of the central object itself (see e.g. Lodato & Bertin 2003; Kuo et al. 2010). Our estimate of the mass of the disc constrains some measurable quantities and can be helpful in making this distinction.

This paper is organized as follows. In the next section, all relevant equations are displayed. Section 3 gives the lower bound for the mass of a self-gravitating disc. In Section 4, we discuss the spherically symmetric case. There emerges a striking similarity between the expression for the mass of a Lane–Emden star and the bound that is proved in Section 3. Section 5 compares results of the numerical solutions describing axially symmetric discs and the analytical estimate of this paper. This comparison shows that our approach is particularly robust in the strongly non-linear regime, when the disc masses are much larger than the central mass. The final section contains a brief summary.

2 NOTATION AND EQUATIONS

Consider a disc of perfect fluid rotating around a central point mass. In this case, stationary Euler equations can be written as

$$(\boldsymbol{U} \cdot \nabla)\boldsymbol{U} = -\nabla\Phi - \frac{1}{\rho}\nabla p. \tag{1}$$

Here $\boldsymbol{U}$ denotes the fluid velocity, Φ is the gravitational potential, ρ denotes the density and p is the pressure of the fluid. For a self-gravitating disc, the gravitational potential satisfies

$$\Delta\Phi = 4\pi G\rho, \tag{2}$$

where G is a gravitational constant.

Let (r, ϕ, z) denote cylindrical coordinates. We will consider purely rotating, axially symmetric configurations, so that $\boldsymbol{U} = \omega\partial_\phi$, where ω is the angular velocity.

In order to find solutions of the above system of equations, it is customary to assume a fixed form of the rotation law $\omega = \omega(r)$ and the equation of state $p = p(\rho)$. There is a vast literature on the numerical solutions of such problems (see e.g. Stoeckly 1965;

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Ostriker & Mark 1968; Eriguchi 1978; Eriguchi & Müller 1985). In this paper, we are interested in establishing general analytic bounds on the mass of the disc.

In what follows, we will specialize to the polytropic equation of state $p = K\rho^{1+1/n}$, where K and n are constants. Let us introduce the specific enthalpy h, so that $\mathrm{d}h = \mathrm{d}p/\rho$. For the polytropic equation of state $h = K(1+n)\rho^{1/n}$.

Computing the divergence of equation (1) yields

$$\Delta h = -4\pi G\rho + \frac{1}{r}\partial_r(r^2\omega^2) = -Ch^n - \Delta\Phi_\mathrm{c}, \tag{3}$$

where we have introduced the centrifugal potential

$$\Phi_\mathrm{c} = -\int^r \mathrm{d}r' r' \omega^2(r'), \tag{4}$$

and a constant $C = 4\pi G/[K(1+n)]^n$.

Note that equation (3) can be further simplified by assuming the so-called v-const rotation law, that is, $\omega = v_0/r$, where v_0 is a constant. In this case, the centrifugal term on the right-hand side vanishes and we have

$$\Delta h = -Ch^n. \tag{5}$$

This simple form does not imply that the rotation does not influence the structure of the disc. It only means that Φ_c is a harmonic function inside the volume occupied by a disc. Let us point out that equation (5) is still difficult to solve, because the boundary condition $h = 0$ is to be posed on an unknown boundary of the disc.

In this paper, we will reserve the symbol Ω for the domain in $\mathbb{R}^3$ occupied by the disc. The disc boundary will be denoted by $\partial\Omega$.

Equation (5) is also valid for a static polytropic star. In that case, if we assume isotropy, introduce spherical coordinates and properly rescale variables, it reduces to the well-known Lane–Emden equation.

3 ESTIMATES OF THE DISC MASS

Assume that $\Delta\Phi_\mathrm{c} \le 0$. For the class of rotation laws of the form $\omega = \mathrm{constant}/r^p$, this implies that $p \le 1$, so that the v-const rotation is a limiting case.

Let us multiply both sides of equation (3) by h and integrate over Ω. It is easy to observe that

$$-\int_\Omega \mathrm{d}^3x h\Delta h = \int_\Omega \mathrm{d}^3x|\nabla h|^2 = C\int_\Omega \mathrm{d}^3x h^{n+1} + \int_\Omega \mathrm{d}^3x h\Delta\Phi_\mathrm{c} \le C\int_\Omega \mathrm{d}^3x h^{n+1}, \tag{6}$$

where the left-hand side has been integrated by parts, and we have used the fact that $h = 0$ on $\partial\Omega$. Further steps are adapted from Malec (1988). The last integral in equation (6) can be estimated making use of the Hölder inequality. For $n > 1$, we have

$$\int_\Omega \mathrm{d}^3x h^{n+1} = \int_\Omega \mathrm{d}^3x h^{n-1}h^2 \le \|h^{n-1}\|_{L^{3/2}(\Omega)}\|h^2\|_{L^3(\Omega)}.$$

Finally, with the help of the Sobolev inequality

$$\|h\|_{L^6(\Omega)} \le C(3,2)\,\|\nabla h\|_{L^2(\Omega)}\,,$$

we arrive at

$$\|\nabla h\|^2_{L^2(\Omega)} \le CC^2(3,2)\|h^{n-1}\|_{L^{3/2}(\Omega)}\,\|\nabla h\|^2_{L^2(\Omega)}\,.$$

Thus,

$$\|h^{n-1}\|_{L^{3/2}(\Omega)} = \left[\int_\Omega \mathrm{d}^3x h^{3(n-1)/2}\right]^{2/3} \ge \frac{1}{CC^2(3,2)}.$$

The Sobolev constant $C(3,2) = 4^{1/3}/(\sqrt{3}\pi^{2/3})$ is a universal number in $\mathbb{R}^3$ (cf. Rosen 1971; Talenti 1976). The specific enthalpy h can be extended to a function defined on $\mathbb{R}^3$ by setting $h = 0$ outside Ω. Such an extension belongs to $W_0^{1,2}(\mathbb{R}^3)$, that is, the closure of the set of compactly supported $C^\infty(\mathbb{R}^3)$ functions in the Sobolev space $W^{1,2}(\mathbb{R}^3)$.

The mass of a disc is given by

$$M = \int_\Omega \mathrm{d}^3x\rho = \frac{C}{4\pi G}\int_\Omega \mathrm{d}^3x h^n.$$

Let I be

$$I = \int_\Omega \mathrm{d}^3x h^{3(n-1)/2} = \int_\Omega \mathrm{d}^3x h^{(n-3)/2}h^n.$$

For $n > 3$, the value of I can be estimated as

$$I < h_{\max}^{(n-3)/2}\int_\Omega \mathrm{d}^3x h^n,$$

where $h_{\max}$ denotes the maximum value of the enthalpy within the disc. The reversed inequality holds for $n < 3$.

A combination of the above results gives the lower bound on the mass of the disc in the form

$$M > \left[4\pi G\sqrt{C}C^3(3,2)h_{\max}^{(n-3)/2}\right]^{-1} = \left[\frac{K(1+n)}{4\pi G}\right]^{3/2}\rho_{\max}^{-(n-3)/2n}C^{-3}(3,2) \tag{7}$$

valid for $n > 3$ and $\Delta\Phi_\mathrm{c} \le 0$. Here, similarly, $\rho_{\max}$ denotes the maximum density within the disc.

The obtained result can be also understood as a bound for the maximum temperature in the gas configuration. For the ideal gas, $T = p\mu m_\mathrm{p}/(\rho k_\mathrm{B})$, where μ is the mean molecular weight, m_p denotes the mass of the proton and k_B is the Boltzmann constant. Inequality (7) can be now written as

$$T_{\max} < 4\pi G\mu m_\mathrm{p}M^{2/3}\rho_{\max}^{1/3}C^2(3,2)/\left[(1+n)k_\mathrm{B}\right],$$

where $T_{\max}$ denotes the maximal temperature of gas.

Yet another consequence of equation (7) can be obtained for the value $n = 3$ of the polytropic index. Let $\bar\rho$ and $\bar T$ denote volume-averaged mass density and temperature, respectively. For the ideal gas, one gets, after simple calculations involving a Hölder inequality,

$$\bar T \le K\frac{\mu m_\mathrm{p}}{k_\mathrm{B}}\bar\rho^{1/n}.$$

Inserting that into equation (7) yields the estimate

$$M > \left(\frac{k_\mathrm{B}}{\pi G\mu m_\mathrm{p}}\right)^{3/2}\frac{\bar T^{3/2}}{\sqrt{\bar\rho}}C^{-3}(3,2),$$

or, writing C^{-3} (3, 2) explicitly,

$$M > \frac{3\sqrt{3\pi}}{4}\left(\frac{k_\mathrm{B}}{G\mu m_\mathrm{p}}\right)^{3/2}\frac{\bar T^{3/2}}{\sqrt{\bar\rho}} \equiv M_\mathrm{S}. \tag{8}$$

The Jeans mass is usually expressed as

$$M_\mathrm{J} = \beta\left(\frac{k_\mathrm{B}}{G\mu m_\mathrm{p}}\right)^{3/2}\frac{\bar T^{3/2}}{\sqrt{\bar\rho}},$$

where β is a constant (dependent on the convention of the Jeans mass) of the order of 1/2. Bound systems should possess a mass $M > M_\mathrm{J}$, according to a derivation that traditionally suffers from severe gaps. Note that the Jeans mass M_J is close to M_S, the right-hand side of equation (8); indeed, $M_\mathrm{S} = M_\mathrm{J} \times 3\sqrt{3\pi}/(4\beta)$. Therefore, for bound systems, we rigorously obtain $M > 3\sqrt{3\pi}/(4\beta)M_\mathrm{J}$.

4 LANE–EMDEN STARS

Inequality (7) is obviously satisfied for static polytropic stars, described entirely in terms of the so-called Lane–Emden functions.

Under the assumption of the spherical symmetry, equation (3) can be reduced to the Lane–Emden equation

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)+\theta^n=0,$$

where

$$\xi=\sqrt{\frac{4\pi G\rho_{\max}^{1-1/n}}{K(1+n)}}\,r,$$

$\rho=\rho_{\max}\theta^n$ and r denotes the distance from the centre of the star. Since $\rho_{\max}=\rho(r=0)$, we have $\theta(\xi=0)=1$. A textbook exposition of the theory of the Lane–Emden equation can be found in Chandrasekhar (1958).

Let the first zero of θ (if present) be denoted by ξ_0. If the Lane–Emden function corresponding to a given index n has no zeros, we assume $\xi_0=\infty$. A radius corresponding to ξ_0 will be denoted by R. The mass of the star can be computed as

$$M=\int_0^R 4\pi r^2\rho\,\mathrm{d}r=\left[\frac{K(1+n)}{4\pi G}\right]^{3/2}\rho_{\max}^{-(n-3)/2n}\hat{M}, \qquad (9)$$

where

$$\hat{M}=\int_0^{\xi_0}4\pi\xi^2\theta^n\mathrm{d}\xi=-4\pi\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)_{\xi_0}.$$

Since analytical expressions for θ are only known for $n=0$, 1 and 5, the values of $\hat{M}$ have to be computed numerically.

The similarity between equations (7) and (9) is remarkable. We see that for $n>3$, there must be $\hat{M}>C^{-3}(3,2)$.

The general bound on $\hat{M}$ can be obtained by observing that $\hat{M}$ attains its minimum value for $n=0$. In this case, the solution θ is known analytically and $\hat{M}=\hat{M}_{\min}=16\sqrt{6}\pi/5=24.625$. Numerical values of $\hat{M}$ for different indices n are shown on Fig. 1. The factor $\hat{M}$ is close to $C^{-3}(3,2)$ for n close to zero, which is outside the validity zone of our estimate. The solutions of the Lane–Emden equations are, however, relatively easy to be obtained numerically, and this case can only be treated as an illustration of the bound given by equation (7). In the next section, we will present some numerical results for the self-gravitating discs.

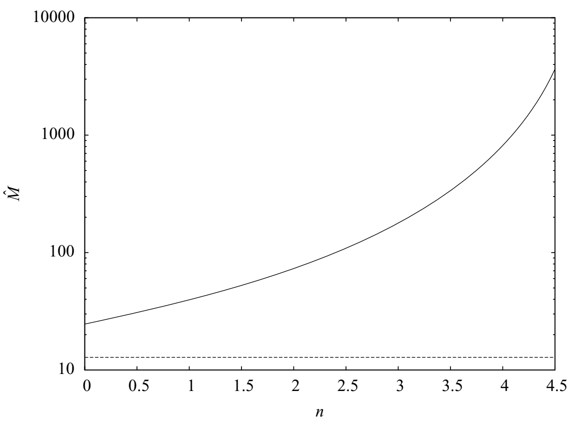

Figure 1. The integral $\hat{M}$ as a function of n. The horizontal line represents the value of $C^{-3}(3,2)=12.821$.

5 SELF-GRAVITATING DISCS

In this section, we shall test the quality of the estimate (7) by comparing it with appropriate numerical solutions describing self-gravitating rotating discs. The numerical method used here follows the Self-Consistent Field scheme introduced by Ostriker & Mark (1968) and used by many authors and in many variants (cf. Clement 1974; Komatsu et al. 1989; Nishida et al. 1992; Mach & Malec 2010). Here we will only sketch the main idea.

Equations (1) can be formally integrated yielding

$$h+\Phi+\Phi_c=\tilde{C}, \qquad (10)$$

where Φ_c is given by equation (4) and Φ satisfies equation (2). The integration constant $\tilde{C}$ is important and its value depends on the solution. Gravitational potential Φ can be expressed in terms of the Green function for the Laplace operator

$$\Phi(\boldsymbol{x})=-\frac{GM_c}{|\boldsymbol{x}|}-G\int_\Omega \mathrm{d}^3x'\frac{\rho(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}, \qquad (11)$$

where M_c denotes the central mass. In the implementation of the numerical method, the singularity of the Green function at $\boldsymbol{x}'=\boldsymbol{x}$ causes a problem that we avoid in the standard way – by expanding the above integral in terms of Legendre polynomials.

We assume that the domain Ω ranges from $r=r_{\rm in}$ to $r=r_{\rm out}$ in the equatorial plane and the maximal density within the disc reaches a fixed value $\rho_{\max}$. The rotation law is $\omega(r)=v_0/r$ and the equation of state has the form $p=K\rho^{1+1/n}$.

The choice of the initial parameters is to a large extent arbitrary. We fix values of $r_{\rm in}$, $r_{\rm out}$, n, $\rho_{\max}$ and M_c, whereas constants $\tilde{C}$, v_0 and K are computed, once the solution is known. For v_0 we have the formula

$$v_0^2=\frac{\Phi(r_{\rm out},z=0)-\Phi(r_{\rm in},z=0)}{\ln(r_{\rm out}/r_{\rm in})},$$

where we use the fact that the enthalpy h vanishes on $\partial\Omega$. Values of $\tilde{C}$ and K can be obtained from equation (10) taken at $(r_{\rm in},z=0)$ [or $(r_{\rm out},z=0)$] and the point where $\rho=\rho_{\max}$, respectively.

The structure of the disc is obtained by an iterative procedure. One starts by assuming an initial density distribution in the toroidal shape ranging from $r_{\rm in}$ to $r_{\max}$ with the a priori given maximum value $\rho_{\max}$. The gravitational potential is then found from the formula (11). In the next step, we compute constants v_0, $\tilde{C}$ and K corresponding to the assumed density distribution. Finally, equation (10) can be used to obtain a new approximate solution for h and the corresponding distribution of ρ. If equation (10) gives a negative value of h in some region of interest, then we assume that $h=0$ there. In this way, a new shape of the disc is obtained. This three-step procedure is iterated until a satisfactory convergence is reached.

The disadvantage of such approach is that a good spatial resolution is required in order to obtain accurate solutions, and computations of the gravitational potential Φ given by equation (11) become time-consuming.

An example of a disc solution obtained by the procedure described above is shown in Fig. 2. The solution was obtained for $r_{\rm in}=1\,\mathrm{R}_\odot$, $r_{\rm out}=100\,\mathrm{R}_\odot$, $n=3$, $\rho_{\max}=0.1\,\mathrm{g\,cm^{-3}}$ and $M_c=1\,\mathrm{M}_\odot$. The corresponding mass of the disc is equal to $M=1.8\times10^3\,\mathrm{M}_\odot$ and the bound given by equation (7) is $M>3.8\times10^2\,\mathrm{M}_\odot$. Here $\mathrm{M}_\odot$ and $\mathrm{R}_\odot$ denote the solar mass and solar radius, respectively.

Table 1 summarizes results obtained for a couple of numerical solutions and analytic estimates. The first three columns show values of the initial parameters: the inner radius $r_{\rm in}$, the outer radius $r_{\rm out}$ and the maximal mass density $\rho_{\max}$, respectively. The last two

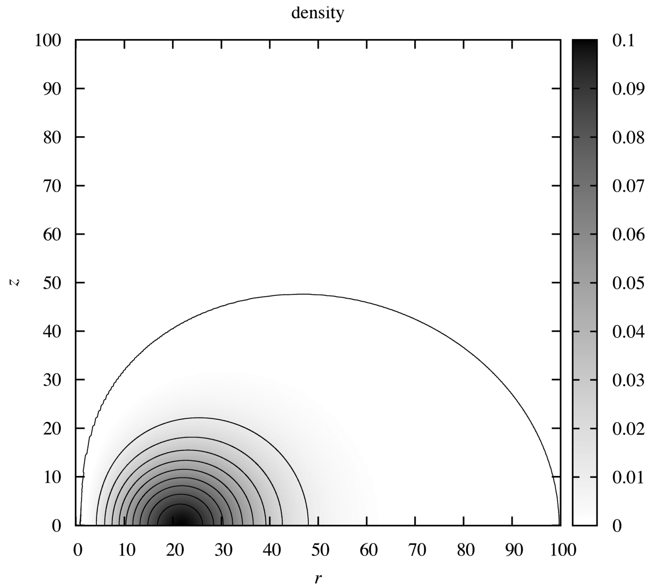

Figure 2. An example of the density distribution in the self-gravitating rotating disc. The plot shows a cross-section of the upper hemisphere in a meridian plane. Spatial dimensions are expressed in solar radii $R_\odot$. The density is grey-scale coded in g cm^{-3}.

Table 1. Disc masses and their lower bounds computed from equation (7). Configurations listed in the table were computed for $n = 3$ and $M_c = 1\,M_\odot$.

r_{in} ($R_\odot$)	r_{out} ($R_\odot$)	ρ_{max} (g cm^{-3})	M ($M_\odot$)	Mass bound ($M_\odot$)
5	10	0.1	1.4	5.1×10^{-2}
1	10	1	25	2.7
50	100	10^{-2}	1.8×10^{2}	7.3
500	1000	10^{-5}	1.8×10^{2}	7.3
2500	5000	10^{-6}	2.2×10^{3}	92
1	1000	10^{-4}	1.5×10^{3}	3.8×10^{2}
1	100	0.1	1.8×10^{3}	3.8×10^{2}
1	5000	10^{-5}	1.8×10^{4}	5.5×10^{3}

columns display the mass of a disc and its lower bound derived from equation (7).

The obtained inequality is never saturated. Relatively heavy discs, with the mass exceeding the central mass by three to four orders of magnitude, have masses close to the bound of equation (7). It is clear that the accuracy of the functional-analytic bound increases with the increase in the mass of the disc, and thus with the increase in the self-gravity. The exact numerical mass and the lower mass bound differ by a factor of 3 for heavy discs, and by less than two orders of magnitude for light discs.

6 CONCLUSIONS

We derive, using certain functional inequalities, an analytic lower bound for the mass of self-gravitating axially symmetric stationary configurations of polytropic fluids. It is valid for polytropic indices $n \geq 3$, both for rotating stars and for accretion discs with centrifugal potentials satisfying the condition $\Delta \Phi_c \leq 0$. This class of potentials is quite general and includes several common types of rotation, with the rigid $\omega =$ constant and the v-const rotations. The accuracy of the analytic bound increases with the mass of the disc. The exact numerical mass and the mass bound differ by a factor of 3 for large disc masses and by less than two orders of magnitude for lighter discs.

The bound on the mass is given in terms of the polytropic index, maximal density and maximal temperature of the gas. The obtained expression is strikingly similar to the mass formula of the Lane–Emden stars. One can use this result in order to obtain the Jeans inequality for equilibrium ideal gas discs and non-spherical stars, for the polytropic index $n = 3$. We believe that this is the first rigorous derivation of the Jeans mass and of the Jeans inequality for stationary systems.

It is well known that in some active galactic nuclei, rotation curves indicate the existence of massive discs, and their measured mass can be significantly larger than the actual core mass (Lodato & Bertin 2003). Our result can be applied in order to constrain mass and density parameters of these systems.

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