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SELF-INDUCED MOTION OF LINE DEFECTS

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Abstract. The evolution of the 2-d Ginzburg-Landau functional under the Schrodinger and the diffusion dynamics is considered. We construct solutions $u(x, t)$, $u \in \mathbf{R}^2$, $x \in \mathbf{R}^3$, such that the vector field u vanishes along a singular curve γ. Equations of motion for $\gamma(t)$ are derived by the method of matched asymptotic expansions.

I. Introduction. Singularities in the solutions of field equations play a fundamental role in mathematical physics. The location of the singularities and the behavior of the solution in their neighborhood are crucial to the understanding of the physical system. Point charges in electrodynamics and vortices in fluid mechanics are two classic examples. There are many other examples in modern physics: defect in liquid crystals [2], vortices in superfluidity [7], point defects in convection patterns [3], and monopoles in the Yang-Mills theory [5]. There are two types of singularities: one in which the field blows up (like the electromagnetic field near a point charge), and another in which the field vector vanishes. The second type is often called a (topological) defect.

Static solutions near singularities have been extensively studied in all the examples above. Much less is known about the dynamics of singularities. A notable exception is the theory of vortex dynamics, which is very useful in fluid mechanics.

In this paper we consider two prototype field equations with solutions having a defect along some curve embedded in $\mathbf{R}^3$. Our first model is the nonlinear Schrodinger-Ginzburg-Landau equation (NSGL)

$$-iu_t = \Delta u + u(1 - |u|^2) \qquad u: \mathbf{R}^3 \times (0, \infty) \to \mathbf{R}^2. \tag{1.1}$$

This equation was first written down by Ginzburg and Landau [7] as a phenomenological description of ideal Bose gas, where $|u|^2$ represents the gas density. A derivation of (1.1) from first principles is discussed by Feynmann [4]. The second model is the Landau-Stuart (LS) equation

$$u_t = \Delta u + u(1 - |u|^2). \tag{1.2}$$

This is a generic amplitude equation describing instabilities in dissipative systems near bifurcation points [6]. Dynamics of point defects in two dimensions for these

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equations has been recently studied by Neu [8]. He assumed that the defects are far from each other, and derived equations of motion for them.

We shall assume that the defect is located not at a point but along a slowly varying curve. This line defect is similar to a vortex line in ideal fluids. We shall obtain equations for the self-induced motion of that line.

In Sec. II we treat Eq. (1.1) and show that the singular curve evolves in the direction of its bi-normal with a velocity depending on its curvature and on the rotation number of u around it. While Eq. (1.2) is similar to (1.1) in many aspects, there are a few essential differences between the two. We therefore give a separate analysis of (1.2) in Sec. III.

We use the method of matched asymptotic expansions. The idea is to construct approximate solutions, both near the defect and far from it. Matching the two approximations yields an equation of motion for the singular curve. This method has already been used successfully in studying the dynamics of domain walls [9] and point defects [8]. For practical reasons we replace the assumption that the line defect varies slowly in space, by scaling space and time in Eqs. (1.1) and (1.2). The scaled equations are (2.1) and (3.1) below.

II. The nonlinear Schrodinger-Ginzburg-Landau equation. Let $u^\varepsilon(\mathbf{x}, t) \in \mathbf{R}^2$ be the solution of the following problem,

$$\begin{gathered} -iu_t^\varepsilon = \Delta u^\varepsilon + \varepsilon^{-2} u^\varepsilon(1 - |u^\varepsilon|^2), \qquad \mathbf{x} \in \mathbf{R}^3,\ t > 0,\ 0 < \varepsilon \ll 1, \\ u^\varepsilon(\mathbf{x}, 0) = f(\mathbf{x}). \end{gathered} \tag{2.1}$$

We assume that there exists a curve γ_0, either closed or extending to infinity, in $\mathbf{R}^3$ such that $\arg(f(\mathbf{x}))$ increases by $2\pi\Gamma$, for some integer Γ, in one positive revolution around it. Γ is called the topological degree, or the rotation number of γ, and we write $d_\gamma[f] = \mathrm{rot}_\gamma[\arg(f)] = \Gamma$. It follows that $f(\mathbf{x})$ must vanish on γ. Since the degree is an invariant of the flow, there is a curve $\gamma(t)$ such that $u^\varepsilon(\mathbf{x}, t)$ vanishes there, and $d_\gamma[u^\varepsilon] = \Gamma$.

We shall now construct the asymptotic expansion of u^ε. There are three relevant regions:

(i) *Outer region*: distance $(\mathbf{x}, \gamma(t)) \geq O(1)$. Here we expand

$$u_{\text{out}}^\varepsilon = v^0(\mathbf{x}, t) + \varepsilon v^1(\mathbf{x}, t) + \varepsilon^2 v^2(\mathbf{x}, t) + \dots. \tag{2.2a}$$

Substituting (2.2a) in (2.1) we get

$$|v^0| = 1, \tag{2.3}$$

$$v^0(v^0\overline{v}^1 + v^1\overline{v}^0) = 0. \tag{2.4}$$

Thus $v^0 + v^1 = (1 - i\varepsilon R)e^{i\psi}$ for some real functions R, ψ. Observe that $|v^0 + \varepsilon v^1| = 1 + O(\varepsilon^2)$. Therefore we can rewrite the expansion (2.2) in the form

$$u_{\text{out}}^\varepsilon = e^{iS} + \varepsilon^2 v^2. \tag{2.2b}$$

Then the $O(\varepsilon^{-2})$ and the $O(\varepsilon^{-1})$ terms in the expansion of (2.1) are automatically satisfied, and the $O(1)$ term yields two equations for s:

$$\Delta s = 0 \tag{2.5}$$

$$s_t = -(\nabla s)^2 - 2\rho, \tag{2.6}$$

where $\rho = \mathrm{Re}\{v^2\}$. The system (2.5), (2.6) is analogous to the equations for ideal fluids, where the phase s plays the role of the velocity potential, and 2ρ stands for the pressure. This is not surprising, since (1.1) first came up in the context of superfluids [7].

(ii) *Intermediate region*: distance $(\mathbf{x}, \gamma(t)) = O(\delta)$. We introduce an intermediate region in order to capture some influence of γ on the global solution without entering yet the line's core. A new parameter δ is chosen such that

$$1 \gg \delta \gg \varepsilon^p \qquad \forall p > 0. \tag{2.7}$$

Our final result will be independent of δ. Let $P(\sigma)$ be an arbitrary point on $\gamma(\sigma)$ where σ is a parametrization of γ. We then introduce scaled variables

$$\mathbf{X} = \frac{\mathbf{x} - P}{\delta}, \qquad T = \frac{t}{\delta^2}. \tag{2.8}$$

Thus (2.1) becomes

$$-iu_T = \Delta_X u + \varepsilon^{-2}\delta^2 u(1 - |u|^2), \tag{2.9}$$

and u depends on ε and δ. We now choose a new system of coordinates centered at P, such that the new axes coincide at P with the Frenet trihedron there, i.e., $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = (\mathbf{n}, \mathbf{b}, \mathbf{t})$. Hence $\mathbf{n}$ is the normal to γ, $\mathbf{b}$ is the bi-normal, and $\mathbf{t}$ is the tangent to γ. Therefore γ has the following local representation near P.

$$\gamma = \left(\frac{\kappa}{2}\delta Z^2, 0, Z\right) + O(\delta^2), \tag{2.10}$$

where κ is the curvature of γ at P. Denoting the solution in the intermediate region by u_{inter} and expanding it as in the outer region, one gets

$$u_{\mathrm{inter}} = e^{iS} + O\left(\frac{\varepsilon}{\delta}\right)^2, \tag{2.11}$$

with S satisfying

$$\Delta_X S = 0. \tag{2.12}$$

We seek a solution of the form $S = S\left(\frac{\mathbf{x}-\gamma}{\delta}\right)$. Using (2.10) we find

$$\tilde{\Delta} S - \delta\kappa S_X = O(\delta^2), \tag{2.13}$$

where $\tilde{\Delta} = (\partial^2/\partial X^2) + (\partial^2/\partial Y^2)$. This equation is supplemented by the two conditions:

$$\mathrm{rot}_p[S] = \Gamma, \qquad \nabla S \to 0 \quad \text{as } X^2 + Y^2 \to \infty. \tag{2.14}$$

The first condition expresses the topological singularity of γ, and the second states that the line defect is immersed in a "vacuum." We need an approximation to S

which is exact through $O(\delta)$. Thus we have to solve (2.13) with the right-hand side replaced by zero. The trouble is that S is a multiple valued function. Therefore we present in the Appendix several methods to solve equations of the form

$$\Delta S + \alpha \cdot \nabla S = 0, \qquad \mathrm{rot}_0[S] = \Gamma, \qquad \nabla S \to 0 \quad \text{as } X^2 + Y^2 \to \infty.$$

In the special case of (2.13), the solution is:

$$S = R\int_0^{\Phi} \left[\frac{\partial G}{\partial R}(R, \Phi') - \delta\kappa\cos(\Phi')G(R, \Phi')\right] d\Phi' + S_0 + O(\delta^2). \tag{2.15}$$

Here

$$G = -\Gamma e^{(\delta\kappa/2)X} K_0\left(\frac{\delta\kappa}{2}R\right), \tag{2.16}$$

S_0 is an arbitrary constant, K_0 is the modified Bessel function of order zero, $R^2 = X^2 + Y^2$, and $\Phi = \tan^{-1}(\frac{X}{Y})$.

(iii) *Inner region*: distance$(\mathbf{x}, \gamma(t)) = O(\varepsilon)$. Near the core of the line defect we use the scaling $\boldsymbol{\xi} = (\mathbf{x} - P)/\varepsilon$ with $\boldsymbol{\xi} = (\xi, \eta, \zeta)$. The local form of γ near P is similar to (2.10):

$$\gamma = \left(\frac{\kappa}{2}\varepsilon\zeta^2, 0, \zeta\right) + O(\varepsilon^2), \tag{2.17}$$

where we again choose the coordinate axes to agree with the Frenet trihedron. We seek an inner expansion of the form:

$$u_{\text{inner}} = U^0 + \varepsilon U^1 + \cdots, \qquad \text{with } U^i = U^i\left(\frac{\mathbf{x} - \gamma(t)}{\varepsilon}, t\right). \tag{2.18}$$

While (2.18) is an expansion in integer powers of ε, we let each U^i depend on smaller order functions of ε, such as $(\log\varepsilon)^{-1}$. The substitution of (2.18) into (2.1) yields the following equations for U^0 and U^1.

$$\overline{\Delta} U^0 + U^0(1 - |U^0|^2) = 0 \tag{2.19}$$

$$L[U^1] = i\overline{\nabla} U^0 \cdot \dot{\gamma} + \kappa U^0_{\xi}, \tag{2.20}$$

where $\overline{\nabla} = \partial_\xi e_1 + \partial_\eta e_2$,

$$L[W] = \overline{\Delta} W + (1 - |U^0|^2)W - U^{0\,2}\overline{W} - |U^0|^2 W, \tag{2.21}$$

and $\dot{\gamma}$ is the velocity of P in the (ξ, η) plane. Our goal is to find $\dot{\gamma}$. This will be achieved through the solvability condition for the nonhomogeneous equation (2.20). For this purpose, let v be any solution of the homogeneous equation, i.e., $L[v] = 0$. Multiplying (2.20) by $\overline{v}$ and integrating over a large disc of radius A, we get

$$\mathrm{Re}\left[i\dot{\gamma}\int_{\Omega_A} \overline{\nabla} U^0 \overline{v} + \kappa\int_{\Omega_A} U^0_\xi \overline{v}\right] = \mathrm{Re}\left[\int_{\partial\Omega_A} \overline{v}\frac{\partial U^1}{\partial r} - U^1\frac{\partial \overline{v}}{\partial r}\right], \tag{2.22}$$

where $r^2 = \xi^2 + \eta^2$. Now, the operator L is just the variation of the nonlinear operator in (2.19). Since (2.19) is an equation with constant coefficients, it follows that any derivatives of U^0 is in the kernel of L. Thus the two independent solutions needed for the two components of $\dot{\gamma}$ are U^0_ξ and U^0_η.

We now show that explicit expressions for $\dot{\gamma}$ can be read off from (2.22) without solving (2.19) and (2.20) for U^0 and U^1. The idea is to match U^0 and U^1 with the solution (2.11), (2.15) in the intermediate region. To do so we expand the intermediate solution. From (2.15) we get

$$S = \Gamma\Phi - \delta\frac{\kappa\Gamma}{2}\sin\Phi R[\ln(\delta\kappa R) + \alpha] + S_0 + O(\delta^2). \tag{2.23}$$

Here we used $K_0(\zeta) = -\ln\frac{\zeta}{2} - C + \cdots$ (C is Euler's constant), and $\alpha = \ln 4 + C - 1$. Thus,

$$u_{\text{inner}} \sim e^{i\Gamma\Phi + S_0}\left\{1 - i\varepsilon\frac{\kappa\Gamma}{2}\sin(\phi)r[\ln(\varepsilon\kappa r) + \alpha] + O(\varepsilon^2)\right\}, \qquad |\xi| \gg 1, \tag{2.24}$$

where (r, ϕ) are polar coordinates in the (ξ, η) plane. Therefore we assume U^0 has the following form

$$U^0 = e^{i\Gamma\phi + S_0}h(r), \tag{2.25}$$

with h satisfying

$$\frac{1}{r}\frac{d}{dr}(rh(r)) - \frac{\Gamma^2}{r^2}h + h - h^3 = 0, \qquad h(0) = 0, \; h(\infty) = 1. \tag{2.26}$$

This equation has been given particular attention in [7]. In addition to (2.25) we can use (2.24) to get U^1 for large values of r:

$$U^1 \sim -i\frac{\kappa\Gamma}{2}e^{i(\Gamma\phi + S_0)}\sin(\phi)r[\ln(\varepsilon\kappa r) + \alpha] \quad \text{as } r \gg 1. \tag{2.27}$$

We first choose $v = \partial U^0/\partial\xi$ in (2.22). Performing the integrations, we obtain

$$-\pi\Gamma\dot{\gamma}_2 + \pi\kappa\Gamma^2\ln A + \pi D_A\Gamma^2\kappa = \pi\kappa\Gamma^2\ln(\varepsilon K A) + \pi\kappa\Gamma^2\left(\alpha + \frac{1}{2}\right) + O\left(\frac{\ln A}{A}\right), \tag{2.28}$$

where the constant D_A is given by

$$D_A = \frac{1}{\Gamma^2}\left[\int_0^1\left(rh'^2 + \frac{\Gamma^2}{r}h^2\right)dr + \int_1^A\left(rh'^2 + \frac{\Gamma^2(h^2 - 1)}{r}\right)dr\right]. \tag{2.29}$$

Taking now $A \to \infty$ and setting $\lim_{A\to\infty} D_A = D$ we get

$$\dot{\gamma}_2 = \kappa\Gamma\ln\frac{1}{\varepsilon} - \Gamma\kappa\ln\kappa - \Gamma\kappa\left(\alpha + \frac{1}{2} - D\right). \tag{2.30}$$

Next, taking $v = \partial U^0/\partial\zeta$ in (2.22) we find

$$\dot{\gamma}_1 = 0. \tag{2.31}$$

REMARKS. 1. From (2.26) it follows that $h(r) \sim 1 + O(1/r^2)$ as $r \to \infty$. Therefore, $\lim_{A\to\infty} D_A$ exists.

2. Equation (2.30) gives the velocity induced at each point P on γ by a small neighborhood of γ around it. The interaction of P with other parts of γ comes in as a correction to the boundary condition at infinity in (2.14). It contributes an $O(1)$ correction to (2.30) [10].

The evolution of the line defect $\gamma(t)$ is described by Eqs. (2.30), (2.31). The leading term in (2.30) is similar to the formula derived by Batchelor [1] for the dynamics of a narrow vortex tube in an ideal fluid. That is, to leading order in ε, $\gamma(t)$ evolves like a vortex tube of radius ε and vortex strength $4\pi\Gamma$.

If γ_0 is a circle, then $\gamma(t)$ will move in the direction normal to the circle's plane, and the circle will not change. Another curve, whose shape is invariant under the flow (2.30), (2.31), is the helix.

III. The Landau-Stuart equation. Let $u^\varepsilon(\mathbf{x}, t) \in \mathbf{R}^2$ be the solution of

$$u_t^\varepsilon = \Delta u^\varepsilon + \varepsilon^{-2} u^\varepsilon (1 - |u^\varepsilon|^2), \qquad \mathbf{x} \in \mathbf{R}^3, \qquad t > 0, \ 0 < \varepsilon \ll 1, \qquad u^\varepsilon(\mathbf{x}, 0) = f(\mathbf{x}). \tag{3.1}$$

We assume that there exists a smooth curve γ_0 such that $d_\gamma[f] = \Gamma$. We proceed as in Sec. II to find an equation of motion for the singular curve $\gamma(t)$. We use the same method and the same notation as in the previous section.

(i) *Outer region*: distance $(\mathbf{x}, \gamma(t)) \geq O(1)$. This is a special case of a more general problem solved by Rubinstein, Sternberg, and Keller [11]. They have found

$$u^\varepsilon(\mathbf{x}, t) = v^0 + O(\varepsilon^2), \qquad v^0 = e^{is}, \tag{3.2}$$

and $s(\mathbf{x}, t)$ satisfies

$$s_t = \Delta s. \tag{3.3}$$

(ii) *Intermediate region*: distance $(\mathbf{x}, \gamma(t)) = O(\delta)$. We assume the following asymptotic form for u^ε:

$$u^\varepsilon = v^0 + \left(\frac{\varepsilon}{\delta}\right)^2 v^1 + \dots, \qquad v^i = v^i\left(\frac{\mathbf{x} - \gamma(t)}{\delta}\right). \tag{3.4}$$

Substituting (3.4) for (3.1) and using (2.10) we get

$$v^0 = e^{iS}, \tag{3.5}$$

$$\tilde{\Delta} S - \delta \kappa S_X + \delta \tilde{\nabla} S \dot{\gamma} = O(\delta^2), \tag{3.6}$$

where $\tilde{\nabla} = \partial_X e_1 + \partial_Y e_2$. We wish to solve (3.6), together with conditions (2.14), exactly through $O(\delta)$. Applying Neu's method (see the Appendix) we get:

$$S = R \int_0^\Phi \left[\frac{\partial H}{\partial R} - \delta(\kappa - \dot{\gamma}_1) H \cos\Phi' + \delta \dot{\gamma}_2 H \sin\Phi'\right] d\Phi' + S_0 + O(\delta^2), \tag{3.7}$$

$$H = -\Gamma \exp\left[\frac{\delta}{2}(\kappa - \dot{\gamma}_1) X - \frac{\delta}{2}\dot{\gamma}_2 Y\right] K_0\left[\frac{\delta}{2}((\kappa - \dot{\gamma}_1)^2 + \dot{\gamma}_2^2)^{1/2} R\right]. \tag{3.8}$$

(ii) *Inner region*: distance $(\mathbf{x}, \gamma(t)) = O(\varepsilon)$. We consider an expansion of the form (2.18). The leading terms U^0 and U^1 satisfy:

$$\bar{\Delta} u^0 + U^0(1 - |U^0|^2) = 0, \tag{3.9}$$

$$L[U^1] = -\bar{\nabla} U^0 \cdot \dot{\gamma} + \kappa U_\xi^0, \tag{3.10}$$

where $\overline{\nabla} = \partial_\xi e_1 + \partial_\eta e_2$, and L is defined in (2.21). The solvability condition for (3.10) now reads:

$$\operatorname{Re}\left[-\dot{\gamma}\int_{\Omega_A} \nabla U^0 \overline{v} + \kappa \int_{\Omega_A} U^0_\xi \overline{v}\right] = \operatorname{Re}\left[\int_{\partial\Omega_A} \overline{v}\frac{\partial U^1}{\partial r} - U^1 \frac{\partial \overline{v}}{\partial r}\right]. \tag{3.11}$$

Here Ω_A is a disc of radius A centered at the origin P, and v is in the kernel of L.

Now, U^0 is given again by (2.25) and (2.26). Matching the second terms in u_{inter} and u_{inner}, and using (3.8), we obtain

$$U^1 \sim -ie^{i(\Gamma\phi + S_0)}\frac{\Gamma r}{2}[\ln(\varepsilon b r) + \alpha][\sin(\phi)(\kappa - \dot{\gamma}_1) + (\cos\phi - 1)\dot{\gamma}_2] \quad \text{as } r \gg 1, \tag{3.12}$$

where

$$b = [(\kappa - \dot{\gamma}_1)^2 + \dot{\gamma}_2^2]^{1/2}. \tag{3.13}$$

We first choose $v = \partial U^0/\partial\eta$ in (3.11). Using (2.25), (2.26), (3.12), and (2.29) we get

$$-\pi\Gamma^2\dot{\gamma}_2 \ln A - \pi D_A \dot{\gamma}_2 = -\pi\Gamma^2\dot{\gamma}_2\ln(\varepsilon b A) - \pi\Gamma^2\left(\alpha + \frac{1}{2}\right)\dot{\gamma}_2 + O\left(\frac{\ln A}{A}\right). \tag{3.14}$$

Thus

$$\dot{\gamma}_2 = 0. \tag{3.15}$$

Next, we take in (3.11) $v = \partial U^0/\partial\xi$. Carrying out the integration we obtain

$$\pi(\kappa - \dot{\gamma}_1)(\Gamma^2 \ln A + D_A) = \pi(\kappa - \dot{\gamma}_1)\left[\Gamma^2\ln(\varepsilon\kappa A) + \Gamma^2\left(\alpha + \frac{1}{2}\right)\right] + O\left(\frac{\ln A}{A}\right). \tag{3.16}$$

Therefore

$$\dot{\gamma}_1 = \kappa. \tag{3.17}$$

The dynamics of $\gamma(t)$ is given by Eqs. (3.15) and (3.17). This kind of evolution is called *flow by curvature.* If γ_0 is a planar curve, then $\gamma(t)$ will evolve only in that plane. Flow by curvature of planar closed curves has been extensively studied in recent years (see [9] and references therein). An important property of this evolution law is that it shrinks curves. It is a simple exercise to show that a closed curve γ in $\mathbf{R}^3$ evolving according to (3.15) and (3.17) satisfies

$$\frac{dL}{dt} = -\int_0^{L(t)} \kappa^2(s, t)\, ds, \tag{3.18}$$

where s is the arclength parametrization of γ, and $L(t)$ is the length of γ at time t.

Now, Eq. (3.1) is a gradient flow in function space with the associated energy $\frac{1}{2}\nabla u^2 + \varepsilon^{-2}(|u|^2 - 1)^2$. The only source of energy is the line defect γ, whose energy per unit length is $O(\ln\frac{1}{\varepsilon})$. Since the defect itself is an invariant of the flow, the only way to reduce energy is to shorten the length.

Evolution of curves according to (3.15), (3.17) or (2.30), (2.31) is a very interesting and difficult problem. Its emergence here as a canonical problem in field theory calls for a detailed study of it.

Appendix. Elliptic phase equations in the plane. Consider the following equation:

$$\Delta S + \boldsymbol{\alpha} \cdot \nabla S = 0, \qquad \mathbf{x} \in \mathbf{R}^2; \tag{A.1}$$

$$\mathrm{rot}_0(S) = \Gamma, \qquad \nabla S \to 0 \quad \text{as } x^2 + y^2 \to \infty. \tag{A.2}$$

Here $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ is a constant vector. We refer to equations like (A.1) as elliptic phase equations. The difficulty in solving phase equations is that the solutions are multiple-valued functions. One way to resolve that problem was suggested by Rubinstein, Sternberg, and Keller [11]. Their idea is to write

$$S = \Gamma\theta + \sigma, \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right). \tag{A.3}$$

Then σ is a single-valued function solving

$$\Delta\sigma + \boldsymbol{\alpha} \cdot \nabla\sigma = -\boldsymbol{\alpha} \cdot \nabla\theta. \tag{A.4}$$

The decomposition (A.2) has been used successfully in [11] to solve parabolic phase equations.

In the particular case considered here, i.e., (A.1), there is a more convenient method, which was invented by J. Neu [8]. We shall use Neu's method to write, explicitly, the solution of (A.1). For this purpose, let G be the fundamental solution of $\Delta + \alpha\nabla$:

$$\Delta G + \boldsymbol{\alpha} \cdot \nabla G = 2\pi\Gamma\delta(\mathbf{x}). \tag{A.5}$$

G can be easily computed, and we find:

$$G = -\Gamma e^{(-\alpha \cdot x)/2} K_0\left(\frac{|\alpha|}{2} r\right), \qquad r = (x^2 + y^2)^{1/2}, \tag{A.6}$$

where K_0 is the modified Bessel function of order zero. Now, from (A.1), (A.2), and (A.5) it follows that S and G are related through

$$\begin{aligned} S_x &= -G_y + \alpha_2 G, \\ S_y &= G_x + \alpha_1 G. \end{aligned} \tag{A.7}$$

Integrating (A.7) we obtain

$$S = r\int_0^\theta \left(\frac{\partial G}{\partial r} + \alpha_2 \sin(\phi) G + \alpha_1 \cos\phi G\right) d\phi + S_0, \tag{A.8}$$

where S_0 is an arbitrary constant.

We mention yet a third method for solving (A.1), (A.2). Consider Eq. (A.1) in the slit plane, which is left from the whole plane, after the positive x axis has been removed from it. We then replace condition (A.2) by a jump discontinuity along the removed ray. Thus we solve

$$\Delta S + \boldsymbol{\alpha} \cdot \nabla S = -2\pi\left[\frac{\partial\delta(y)}{\partial y} + \alpha_2\delta(y)\right] H(x), \tag{A.9}$$

where $H(x)$ is the Heaviside function.

The first and third method can be applied to other types of phase equations. We also point out that Eq. (A.9) is particularly suitable for Fourier transform analysis.

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