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Self maps of homogeneous spaces

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Introduction

This paper arose out of an attempt to understand the following problem of Lazarsfeld [L]:

Problem 1. *Suppose G is a semi-simple algebraic group over $\mathbf{C}$, $P \subset G$ a maximal parabolic subgroup, $Y = G/P$. Let $f: Y \to X$ be a finite, surjective morphism of degree > 1 to a smooth variety X; then is $X \cong \mathbf{P}^n$? ($n = \dim X = \dim Y$)*

Lazarsfeld (*loc. cit.*) answers this in the affirmative when $Y = \mathbf{P}^n$, using the proof by S. Mori [M] of Hartshorne's conjecture. The general case seems to be open even for Grassmann varieties.

In this paper, we show (see Proposition 2): *if $Y = G/P$ is as above and $f: Y \to Y$ is a finite self map of degree > 1, then $Y \cong \mathbf{P}^n$.*

More generally, we prove the following:

Theorem. *Let G be a simply connected, semi-simple algebraic group over $\mathbf{C}$. Let $P \subset G$ be a parabolic subgroup, and let $Y = G/P$ be the homogeneous space. Let $f: Y \to Y$ be a generically finite morphism. Then there exist parabolic subgroups $P_0, P_1, \ldots, P_m$ of G containing P, and a permutation σ of $\{1, 2, \ldots, m\}$ such that:*

(i) *there are isomorphisms $G/P_i \cong \mathbf{P}^{n_i}$ for $i \geqq 1$, for some integers $n_i > 0$, such that $n_{\sigma(i)} = n_i$ for all i.*
(ii) *there is a finite morphism $\pi_i: \mathbf{P}^{n_i} \to \mathbf{P}^{n_i}$ for each $i > 0$.*
(iii) *the natural morphism*

$$Y \to G/P_0 \times G/P_1 \times \ldots \times G/P_m$$

is an isomorphism, under which $f: Y \to Y$ corresponds to the product $f_0 \times f_1 \times \ldots \times f_m$, where $f_0: G/P_0 \to G/P_0$ is an isomorphism and f_i the composite

$$G/P_{\sigma(i)} \cong \mathbf{P}^{n_i} \xrightarrow{\pi_i} \mathbf{P}^{n_i} \cong G/P_i\,.$$

We also show that Problem 1 has an affirmative answer if Y is a smooth quadric hypersurface of dimension $\geqq 3$ (Proposition 8); this includes the case of Grassmannian $Y = \mathbf{G}(2, 4)$. We also show:

Proposition 6. *Let $k \leqq n, 2 \leqq l \leqq m$ be integers, such that there exists a finite surjective morphism between Grassmann varieties*

$$f\colon \mathbf{G}(k, k+n) \to \mathbf{G}(l, l+m)$$

Then $k = l, m = n$ and f is an isomorphism.

In the spirit of Lazarsfeld's problem, we pose the following:

Problem 2. *Let $f\colon A \to X$ be a finite surjective morphism from a simple abelian variety A over $\mathbf{C}$ to a smooth variety X. Suppose that f is not étale. Then is $X \cong \mathbf{P}^n$? ($n = \dim A = \dim X$).*

This is easily proved for $\dim A \leqq 2$.

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1. Proof of the theorem

One of the tools used in the proof is the following "Bertini theorem" due to M.V. Nori.

Proposition 1. *Let $Y = G/P$ be a homogeneous space for a simply connected algebraic group over $\mathbf{C}$, and let $\pi\colon \tilde{Y} \to Y$ be a finite, surjective morphism with branch divisor $B \subset Y$. Let Z be an irreducible variety, $f\colon Z \to Y$ a non-constant morphism. Then either*

(i) *for a non-empty Zariski open subset $U \subset G$,*

$$Z_g = (Z \times_{g(f)} \tilde{Y})_{\text{red}}$$

is irreducible for all $g \in U$ (here $g(f)\colon Z \to Y$ is obtained from f by pointwise translation by g on Y), or

(ii) *there exists a closed subgroup P' of G, containing P, such that if $h\colon Y \to G/P'$ is the quotient map, then the composite $h \circ f\colon Z \to G/P'$ is constant, and for some non-zero effective divisor $D \subset G/P'$, $h^{-1}(D)$ is a component of B.*

Proof. This result is implicit in [N]. We only need the special case when P is a maximal parabolic subgroup of a simply connected semi-simple group G, so that only the possibility (i) can occur. We give the proof in this case, leaving the general case to the reader.

Replacing Z by its normalization, we are reduced to the case when Z is normal. Let

$$B_g = g(f)^{-1}(B) = f^{-1}(g^{-1}B) \subset Z$$

where $g^{-1}B$ is the translate of B by g^{-1}. If $p\colon Z_g \to Z$ is the natural map, then $Z_g - p^{-1}(B_g)$ is Zariski dense in Z_g, when it is non-empty (which it is for all g lying in some non-empty Zariski open subset of G). Since

$$Z_g - p^{-1}(B_g) \to Z - B_g$$

is an étale covering space, $Z_g - p^{-1}(B_g)$ is normal; hence to prove that it is irreducible, it suffices to prove that it is connected. To prove connectedness, it suffices to prove that the map on fundamental groups

$$g(f)_*: \pi_1(Z - B_g) \to \pi_1(Y - B)$$

is surjective (we omit the base points in the notation).

Consider the surjective morphism

$$\mu: G \times Z \to Y, \quad \mu(g, z) = g(f)(z).$$

The variety $G \times Z$ is connected, and the fibres of μ are principal P-bundles over Z, and in particular are connected. Hence, if $W = (G \times Z) - \mu^{-1}(B)$ then

$$\mu_*: \pi_1(W) \to \pi_1(Y - B)$$

is surjective. If $\varphi: W \to G$ is induced by the projection, then the scheme theoretic fibre

$$\varphi^{-1}(g) = \{g\} \times (Z - B_g).$$

Hence it suffices to prove that for a non-empty Zariski open set $U \subset G$, the inclusion of the fibre of φ induces a surjection

$$\pi_1(\varphi^{-1}(g)) \to \pi_1(W), \quad \forall g \in U.$$

By lemma (1.5) of [N], there is a non-empty Zariski open subset $U \subset G$ such that $\varphi^{-1}(U) \to U$ is a fibre bundle for the complex ("classical") topology; further, for all $g \in U$, there is an exact sequence

$$\pi_1(\varphi^{-1}(g)) \to \pi_1(W) \to \pi_1(G) \to 0$$

under the following additional hypothesis: there is a codimension 2 subvariety $L \subset G$ such that $U \subset (G - L)$, and for each $g \in (G - L)$, the scheme theoretic fibre $\varphi^{-1}(g)$ is non-empty and has a smooth point. Since G is simply connected

$$\pi_1(\varphi^{-1}(g)) \to \pi_1(W)$$

is surjective if

$$T = \{g \in G \mid \varphi^{-1}(g) \text{ is empty}\}$$

has codimension $\geqq 2$ in G (since $\varphi^{-1}(g)$ is an open subset of the normal variety Z, it has smooth points if it is non-empty).

Now $\varphi^{-1}(g)$ is empty $\Leftrightarrow Z = B_g \Leftrightarrow g(f)(Z) \subset B$. Let $k: G \to G/P = Y$ be the quotient map, and let $k^{-1}(B) = B'$, $k^{-1}f(Z) = Z'$. Then

$$g(f)(Z) \subset B \Leftrightarrow gZ' \subset B',$$

so that

$$T = \{g \in G \mid gZ' \subset B'\}$$

where $B' \subset G$ is a divisor. Replacing Z', B' by suitable translates (which replaces T by an isomorphic subvariety of G), we may assume that the identity element e of G lies in $Z' \subset B'$. Then $T \subset B'$. If T_0 is an irreducible component of T which is a divisor in G, then

$$T_0 = T_0 e \subset T_0 Z' \subset B',$$

and the Zariski closure of $T_0 Z'$ in G is irreducible. Hence $T_0 = T_0 Z'$ is a divisor, and

$$Z' \subset \{g \in G \mid T_0 g = T_0\} = P' .$$

But P is a proper subset of Z' and hence also of P', and P' is a closed subgroup of G. Since P is a maximal parabolic subgroup of G, we have $P' = G$, a contradiction. Hence $\operatorname{codim}_G T \geqq 2$ and this completes the proof. □

To prove the Theorem, we first prove it in the following special case:

Proposition 2. *Let $P \subset G$ be a maximal parabolic subgroup, $Y = G/P$, and $f\colon Y \to Y$ a non-constant morphism. Then either f is an isomorphism, or $Y \cong \mathbf{P}^n$.*

Proof. Since $\operatorname{Pic} Y = \mathbf{Z}$, generated by the class of a very ample divisor (see [B]), it follows that f is finite. To show that it is an isomorphism, it suffices to prove that it has degree 1. By the theorem of S. Mori [M], if the tangent sheaf T_Y is ample, then $Y \cong \mathbf{P}^n$. Using this, we show that if $Y \not\cong \mathbf{P}^n$, then $\deg f = 1$.

If $y \in Y = G/P$, then $y = gP$ for some $g \in G$, and we may identify the tangent space at y,

$$T_{y,Y} = \operatorname{Lie} G/\operatorname{Ad}(g) \operatorname{Lie} P .$$

Thus we have a natural map $\operatorname{Lie} G \to H^0(Y, T_Y)$ whose image generates T_Y at every point, and we have a surjection of locally free sheaves

$$(\operatorname{Lie} G) \otimes_{\mathbf{C}} \mathcal{O}_Y \to T_Y .$$

This gives a closed embedding of Y-schemes

$$\mathbf{P}_Y(T_Y) \to \mathbf{P}_Y((\operatorname{Lie} G) \otimes_{\mathbf{C}} \mathcal{O}_Y) = Y \times \mathbf{P}(\operatorname{Lie} G) ,$$

giving rise to the diagram

$$\begin{array}{ccc} \mathbf{P}_Y(T_Y) & \xrightarrow{\alpha} & \mathbf{P}(\operatorname{Lie} G) \\ \beta \downarrow & & \\ Y & & \end{array}$$

(where α, β are induced by the projections on $Y \times \mathbf{P}(\operatorname{Lie} G)$). The morphism α restricts to a linear embedding on each fibre of β.

Lemma 3. *Let Y be a projective variety, $\mathscr{E}$ a locally free sheaf on Y, such that there is a surjection*

$$V \otimes_{\mathbf{C}} \mathcal{O}_Y \to \mathscr{E}$$

for a finite dimensional vector space V, giving rise to a diagram

$$\begin{array}{ccc} \mathbf{P}_Y(\mathscr{E}) & \xrightarrow{p} & \mathbf{P}(V) \\ q \downarrow & & \\ Y & & \end{array}$$

Let $Z \subset Y$ be an irreducible subvariety and $r \geqq 1$ an integer. Then the following are equivalent:

(i) *$\mathscr{E} \otimes \mathcal{O}_Z$ has a trivial direct summand of rank r*
(ii) *there is a Zariski open set $U \subset Z$, such that for every irreducible curve $C \subset Z$ which meets U, $\mathscr{E} \otimes \mathcal{O}_C$ has a trivial direct summand of rank r*
(iii) *the linear subspace*

$$\bigcap_{y \in Z} p(q^{-1}(y)) = \bigcap_{y \in Z} p(\mathbf{P}(\mathscr{E}_y)) \subset \mathbf{P}(V)$$

has dimension $\geqq r - 1$.

Proof. Clearly (iii) is equivalent to the existence of a surjection $\varphi: V \to L$, with $\dim L = r$, such that for each $y \in Z$, φ factors through the quotient $V \to \mathscr{E}_y$. This is equivalent to the existence of a factorization

$$\begin{array}{ccc} V \otimes \mathcal{O}_Z & \xrightarrow{\varphi \otimes 1} & L \otimes \mathcal{O}_Z \\ \searrow & & \nearrow \\ & \mathscr{E} \otimes_{\mathcal{O}_Y} \mathcal{O}_Z & \end{array}$$

i.e. $\mathscr{E}|_Z$ has a trivial quotient of rank r (since Z is irreducible projective, a surjection $V \otimes \mathcal{O}_Z \to L \otimes \mathcal{O}_Z$ must have the form $\varphi \otimes 1$ for some φ). Since $\mathscr{E}$ is generated by global sections, this is equivalent to $\mathscr{E}$ having a trivial direct summand of rank r. Thus (i) $\Leftrightarrow$ (iii), and (i) $\Rightarrow$ (ii). To prove that (ii) $\Rightarrow$ (iii), suppose that (iii) does not hold i.e. the intersection of linear spaces defined in (iii) has dimension $< r - 1$ (if $r = 1$, we take this to mean it is empty). Then there is a finite set of points $y_1, \ldots, y_m \in Z$ such that

$$\dim\left(\bigcap_{i=1}^{m} \mathbf{P}(T_{y,Y})\right) < r - 1 .$$

We can then find an irreducible curve $C \subset Z$ such that C meets U and contains all of the y_i. Then by (i) $\Leftrightarrow$ (iii) applied to C, we see that $\mathscr{E} \otimes \mathcal{O}_C$ does not have a trivial direct summand of rank r, so that (ii) does not hold for Z. This completes the proof of the lemma. □

Since

$$(\alpha, \beta): \mathbf{P}(T_Y) \to \mathbf{P}(\operatorname{Lie} G) \times Y$$

is an embedding, we see that for any $x \in \mathbf{P}(\operatorname{Lie} G)$, the map β induces an isomorphism of $\alpha^{-1}(x)$ onto its image in Y. In particular, if $D \subset \alpha^{-1}(x)$ is an irreducible curve, then $C = \beta(D)$ is an isomorphic curve, such that

$$x \in \bigcap_{y \in C} \mathbf{P}(T_{y,Y}) .$$

Thus $T_Y \otimes \mathcal{O}_C$ has a trivial direct summand. The tangent sheaf T_Y is not ample (since we have assumed that Y is not isomorphic to $\mathbf{P}^n$) and hence α is not finite. So there are curves C on Y as above.

On the other hand, the top Chern class $c_n(T_Y) \in H^{2n}(Y, \mathbf{Z}) \cong \mathbf{Z}$ equals the topological Euler characteristic of Y, which is non-zero (for example this follows from the fact that Y has a cell decomposition with even dimensional cells). Thus T_Y does not have any trivial direct summands.

Let

$$A = \{Z \subset Y | Z \text{ is irreducible, and } T_Y \otimes \mathcal{O}_Z \text{ has a trivial direct summand}\} .$$

Then by the discussion above, A contains some curves, while $Y \notin A$; further, by lemma 3,

$$Z \in A \Leftrightarrow \exists x \in \mathbf{P}(\text{Lie } G) \text{ such that } Z \subset \beta(\alpha^{-1}(x)) .$$

Let $m = \max \{\dim Z | Z \in A\}$, and let

$$S = \{Z \in A | \dim Z = m\} .$$

Then $1 \leqq m < n = \dim Y$, and each $Z \in S$ is an irreducible component of $\beta(\alpha^{-1}(x))$ for some $x \in \mathbf{P}(\text{Lie} G)$.

Let $W = \alpha(\mathbf{P}(T_Y))$; then the morphism $\mathbf{P}(T_Y) \to W$ has a flattening stratification (see [Mu]), so that the set $\{\deg Z | Z \in S\}$ is finite. Hence from the theory of the Chow variety, we see that the non-empty set of cohomology classes

$$\text{Cl}[S] = \{[Z] \in H^{2n-2m}(Y, \mathbf{Z}) | Z \in S\}$$

is finite.

Lemma 4. *The map on cohomology groups*

$$f^*: H^{2n-2m}(Y, \mathbf{Z}) \to H^{2n-2m}(Y, \mathbf{Z})$$

maps Cl$[S]$ *into itself.*

Proof. If $Z \in S$, then any translate $gZ \in S$, and

$$[Z] = [gZ] \in H^{2n-2m}(Y, \mathbf{Z}) .$$

From Proposition 1, it follows that for each $c \in \text{Cl}[S]$, there exists $Z \in S$ with $[Z] = c$, such that $f^{-1}(Z)_{\text{red}} = Z'$ is irreducible, and represents the inverse image $f^*(Z)$ as a cycle (i.e. the scheme theoretic inverse image $f^{-1}(Z)$ is reduced at the generic point of Z'). We claim $Z' \in S$. Since $\dim Z' = m$, it suffices to prove that $Z' \in A$. Since Z is not contained in the branch locus of f, the map of locally free sheaves

$$T_Y \otimes \mathcal{O}_{Z'} \xrightarrow{df \otimes 1} f^* T_Y \otimes \mathcal{O}_Z ,$$

is an isomorphism at the generic point of Z'. As $Z \in S$, $f^* T_Y \otimes \mathcal{O}_Z$, has a trivial direct summand; hence there is a map $T_Y \otimes \mathcal{O}_{Z'} \to \mathcal{O}_{Z'}$ which is generically surjective, hence surjective (since $T_Y \otimes \mathcal{O}_{Z'}$ is generated by global sections). Hence $Z' \in S$.

But clearly

$$f^*(c) = f^*[Z] = [f^*Z] = [Z'] \in \text{Cl}[S] .$$

□

Since Cl$[S]$ is finite, it follows that some iterate $f^k = f \circ f \circ \ldots \circ f$ has the property that $(f^k)^*(c) = c$ for some non-zero $c \in \text{Cl}[S]$. Hence, in order to show that $\deg f = 1$, we may assume without loss of generality that $f^* c = c$.

Let $h \in H^2(Y, \mathbf{Z})$ be the Chern class c_1 of the ample generator of Pic $Y = \mathbf{Z}$. Then $H^2(Y, \mathbf{Z}) = \mathbf{Z} \cdot h$, and $H^{2n}(Y, \mathbf{Q}) = \mathbf{Q} \cdot h^n$. Also $f^* h^n = (\deg f) \cdot h^n$, so that $f^* h = (\deg f)^{1/n} \cdot h$. Now $c \cup h^m = [Z] \cup h^m = d \cdot h^n$, where $d = \deg Z / \deg Y$ is a positive rational. Thus

$$f^*(c \cup h^m) = f^*(d \cdot h^n) = (\deg f) d \cdot h^n .$$

On the other hand,

$$f^*(c \cup h^m) = f^* c \cup (f^* h)^m = c \cup (\deg f)^{m/n} \cdot h^m = (\deg f)^{m/n} d \cdot h^n .$$

Hence $\deg f = (\deg f)^{m/n}$, where $m = \dim Z < n$. Thus $\deg f = 1$, and this shows that f is an isomorphism. This completes the proof of Proposition 2. □

We now prove the Theorem in the general case, when $Y = G/P$, P is any parabolic subgroup, and $f \colon Y \to Y$ is a finite self-map. Let $P' \supset P$ be a parabolic subgroup, and let $\mathscr{L} \in \operatorname{Pic} Y$ be the pullback to Y of a very ample invertible sheaf $\mathscr{L}'$ on G/P', under the natural map

$$Y = G/P \to G/P' .$$

From the theory of dominant weights (see [B]), $H^0(Y, f^* \mathscr{L})$ gives a base-point free linear system on Y, such that for a unique parabolic subgroup $\tau(P')$ of G which contains P, the morphism

$$Y \to \mathbf{P}(H^0(Y, f^* \mathscr{L}))$$

is identified with the natural map

$$Y = G/P \to G/\tau(P')$$

composed with a projective embedding of the latter by a complete linear system. The map

$$f^* \colon H^0(Y, \mathscr{L}) \to H^0(Y, f^* \mathscr{L})$$

gives rise to a diagram with surjective arrows

$$\begin{array}{ccc} Y & \longrightarrow & G/\tau(P') \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \longrightarrow & G/P' \end{array}$$

(this diagram defines the map f'; the horizontal arrows are the natural ones).

Let $\mathfrak{p}$ be the set of parabolic subgroups of G containing P. Then $\mathfrak{p}$ is a finite set (see [B]), which is an ordered lattice with respect to the partial order given by inclusion. Fix a very ample $\mathscr{L}' \in \operatorname{Pic}(G/P')$ for each $P' \in \mathfrak{p}$. Then the above construction yields a map of sets $\tau \colon \mathfrak{p} \to \mathfrak{p}$.

Lemma 5. *$\tau \colon \mathfrak{p} \to \mathfrak{p}$ is an isomorphism of ordered lattices.*

Proof. We must show that τ is bijective (i.e. that it is injective, as $\mathfrak{p}$ is finite), and preserves the partial order i.e. $P' \subset P'' \Rightarrow \tau(P') \subset \tau(P'')$.

We first remark that for any $P' \in \mathfrak{p}$,

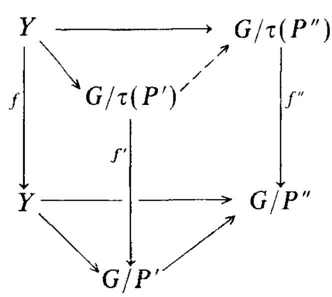
$$Y \to G/\tau(P') \xrightarrow{f'} G/P'$$

is the Stein factorization of the composite

$$\tilde{Y} \to Y \to G/P'$$

so that $\tau\colon \mathfrak{p} \to \mathfrak{p}$ is independent of the choices $\mathscr{L}' \in \operatorname{Pic}(G/P')$. Since the fibres of $\tilde{Y} \to G/\tau(P')$ are connected and $G/\tau(P')$ is smooth, this remark will follow if we prove that f' has finite fibres. If $x \in G/P'$, and $Z \subset Y$ is its inverse image under the natural map $Y = G/P \to G/P'$, then $\mathscr{L} \otimes \mathcal{O}_Z \cong \mathcal{O}_Z$, so that, $f^*\mathscr{L} \otimes \mathcal{O}_{f^{-1}(Z)} = \mathcal{O}_{f^{-1}(Z)}$. Hence each connected component of $f^{-1}(Z)$ is mapped to a point by the linear system associated to $f^*\mathscr{L}$, i.e. by the natural morphism $\tilde{Y} \to G/\tau(P')$. But $(f')^{-1}(x)$ consists of the finite set of images in $G/\tau(P')$ of connected components of $f^{-1}(Z)$, and thus is finite.

In particular, $\dim G/P' = \dim G/\tau(P')$, so that $\dim P' = \dim \tau(P')$. If $P' \subset P''$, then the natural map $Y \to G/P''$ factors through the natural map $Y \to G/P'$. By the functoriality of the Stein factorization, there is a unique map $G/\tau(P') \to G/\tau(P'')$ making the following diagram commute:



Since $\tilde{Y} \to G/\tau(P')$ and $\tilde{Y} \to G/\tau(P'')$ are the natural maps, this means $\tau(P'') \subset \tau(P'')$. Thus τ preserves the partial ordering on $\mathfrak{p}$.

Next, if $P', P'' \in \mathfrak{p}$, the natural map $G/P \to G/P' \times G/P''$ factors into the natural maps $G/P \twoheadrightarrow G/P' \cap P'' \hookrightarrow G/P' \times G/P''$. Consider the diagram

$$\begin{array}{ccc} \tilde{Y} & \longrightarrow & G/\tau(P') \times G/\tau(P'') \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f' \times f''} \\ Y & \longrightarrow & G/P' \times G/P'' \end{array}$$

which yields the diagram with surjective arrows

$$\begin{array}{ccc} \tilde{Y} & \longrightarrow & G/\tau(P') \cap \tau(P'') \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ Y & \longrightarrow & G/P' \cap P'' \end{array}$$

(this diagram defines $\bar{f}$). Note that $f' \times f''$, and hence $\bar{f}$, is finite. Since $\tau(P') \cap \tau(P'') \in \mathfrak{p}$ is connected, the fibres of $\tilde{Y} \to G/\tau(P') \cap \tau(P'')$ are connected. Thus we see that

$$\tau(P' \cap P'') = \tau(P') \cap \tau(P'')$$

by the uniqueness of the Stein factorization. Hence if

$$\tau(P') = \tau(P'') = \tau(P') \cap \tau(P'')$$

then $\dim P' = \dim P'' = \dim P' \cap P''$ since τ preserves dimensions. Since P', P'', $P' \cap P''$ are all connected they are equal. □

Let $\mathscr{M} \subset \mathrm{p}$ be the subset of maximal parabolic subgroups $P' \subset G$ which contain P. From lemma 5, τ restricts to a bijection on $\mathscr{M}$. Let $\mathscr{M}_1 \subset \mathscr{M}$ be the subset consisting of parabolics P' such that for each $j > 0$, if $P'' = \tau^{j-1}(P')$, then

$$f'': G/\tau(P'') \to G/P''$$

is an isomorphism. Clearly $\tau(\mathscr{M}_1) = \mathscr{M}_1$. If $P' \in \mathscr{M} - \mathscr{M}_1$, then for some $j > 0$, if $P'' = \tau^{j-1}(P')$,

$$f'': G/\tau(P'') \to G/P''$$

has degree > 1. But τ is a bijection of a finite set, so that for some $n \geqq j$, $\tau^n(P') = P'$; thus the composite

$$G/P' = G/\tau^n(P') \to G/\tau^{n-1}(P') \to \ldots \to G/\tau(P') \to G/P'$$

is a finite self map of degree > 1. Hence, by proposition 2, we have $G/P' \cong \mathbf{P}^{n'}$ for some n'.

Let

$$P_0 = \bigcap_{P' \in \mathscr{M}_1} P', \quad X = G/P_0$$

and let $\mathscr{M} - \mathscr{M}_1 = \{P_1, \ldots, P_m\}$; then $\tau(P_i) = P_{\sigma(i)}$ for some permutation σ of $\{1, \ldots, m\}$, and $G/P_i \cong \mathbf{P}^{n_i}$ for some integer $n_i > 0$, for $i = 1, \ldots, m$. Since $\dim G/P' = \dim G/\tau(P')$, we have $n_i = n_{\sigma(i)}$ for all i. Since $\tau(\mathscr{M}_1) = \mathscr{M}_1$, we have $\tau(P_0) = P_0$ by lemma 5. Let

$$f_0: X = G/P_0 = G/\tau(P_0) \to G/P_0 = X\ ,$$

and

$$f_i: G/P_{\sigma(i)} = G/\tau(P_i) \to G/P_i, \quad i > 0\ ,$$

be the maps induced by f as constructed above. Then f_0 is an isomorphism by the choice of $\mathscr{M}_1$. For each $i > 0$, f_i can be written as a composite

$$G/P_{\sigma(i)} \cong \mathbf{P}^{n_i} \xrightarrow{\pi_i} \mathbf{P}^{n_i} \cong G/P_i$$

where π_i is a finite self map of the projective space. Then we have a commutative square

$$\begin{array}{ccc} Y & \longrightarrow & X \times \prod_{i=1}^{m} G/P_i \\ \Big\downarrow f & & \Big\downarrow f_0 \times \prod_{i=1}^{m} f_i \\ Y & \longrightarrow & X \times \prod_{i=1}^{m} G/P_i \end{array}$$

where the horizontal maps are closed embeddings, since (see [B])

$$P = \bigcap_{P' \in \mathcal{M}} P' .$$

Thus to finish the proof of the theorem, we only need to show that

$$\Psi: Y = G/P \to X \times \prod_{i=1}^{m} G/P_i$$

is surjective (and hence an isomorphism). Replacing f by an iterate does not change the subset $\mathcal{M}_1 \subset \mathcal{M}$; hence to show that Ψ is an isomorphism, we may replace f by an iterate so that, without loss of generality, we may assume that σ is the identity permutation. Thus f_i: $G/P_i \to G/P_i$ is a finite self map, which is an isomorphism for $i = 0$, and a map $\mathbf{P}^{n_i} \to \mathbf{P}^{n_i}$ of degree > 1 for $1 \leqq i \leqq m$.

Fix an integer $j \in \{1, \ldots, m\}$. If F is a fibre of the natural map

$$Y \xrightarrow{p} X \times \prod_{i \neq j} G/P_i$$

then $\dim F > 0$ and F maps isomorphically to its image $\bar{F}$ under the natural map

$$Y \xrightarrow{q} G/P_j = \mathbf{P}^{n_j} .$$

Further, p, q are G-equivariant (for the left G-action), so that the translate gF (which is another fibre of p) maps isomorphically under q to $g\bar{F}$. From proposition 1 applied to f_j: $\mathbf{P}^{n_j} \to \mathbf{P}^{n_j}$, $f_j^{-1}(g\bar{F})$ is irreducible and has multiplicity 1 as a cycle, for all g in a non-empty Zariski open set in G. If F' is an irreducible component of $f^{-1}(F)$, then from the commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\ p\ } & X \times \prod_{i \neq j} G/P_i \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f_0 \times \prod_{i \neq j} f_i} \\ Y & \xrightarrow{\ p\ } & X \times \prod_{i \neq j} G/P_i \end{array}$$

we see that F' is also a fibre of p (since $\dim F' = \dim F$, and the non-empty fibres of p are precisely the translates gF). Now $q(F') = \bar{F}' \subset \mathbf{P}^{n_j}$ is contained in the inverse image $f_j^{-1}(\bar{F})$. Thus, replacing F by a translate gF, so that $f_j^{-1}(\bar{F})$ is irreducible, we have $f_j^{-1}(\bar{F}) = \bar{F}'$ as a cycle. But $\bar{F}'$ is a translate of $\bar{F}$. Hence if $s = \dim F$,

$$[\bar{F}] = [\bar{F}'] = f_j^*[\bar{F}] \in H^{2n_j - 2s}(\mathbf{P}^{n_j}, \mathbf{Z}) .$$

Since $\deg f_j > 1$, this forces $s = n_j$ i.e. $\bar{F} = \mathbf{P}^{n_j}$. Hence $Y \subset X \times \prod_{i=1}^{m} \mathbf{P}^{n_i}$ is the inverse image of its projection to $X \times \prod_{i \neq j} \mathbf{P}^{n_i}$, for all $j \in \{1, \ldots, m\}$. Since $Y \to X$

induced by Ψ is just the natural surjection $G/P \to G/P_0$, we see that Ψ is a bijection. This completes the proof of the theorem. □

2. Maps between Grassmann varieties

Let $\mathbf{G}(k, N)$ denote the Grassmann variety of k-dimensional quotients of an N-dimensional vector space over $\mathbf{C}$.

Proposition 6. *Let $k \leqq n, 2 \leqq l \leqq m$ be integers, such that there exists a finite surjection morphism between Grassmann varieties*

$$f: \mathbf{G}(k, k+n) \to \mathbf{G}(l, l+m).$$

Then $k = l$, $m = n$ and f is an isomorphism.

Proof. Let $Z \subset \mathbf{G}(l, l+m)$ be an irreducible subvariety such that $T_{\mathbf{G}(l,l+m)} \otimes \mathcal{O}_Z$ has a trivial direct summand of rank r. Then an analogous statement holds for any translate of Z by $\mathrm{GL}_{l+m}(\mathbf{C})$ (regarding $\mathbf{G}(l, l+m)$ as a homogeneous space for $\mathrm{GL}_{l+m}(\mathbf{C})$). Replacing Z by a translate, we may assume that Z is not contained in the branch locus of f. Then if Z' is any irreducible component of $f^{-1}(Z)_{\mathrm{red}}$, we see that $f^* T_{\mathbf{G}(l,l+m)} \otimes \mathcal{O}_{Z'}$ has a trivial direct summand of rank r. Further, the natural map (induced by df)

$$T_{\mathbf{G}(k,k+n)} \otimes \mathcal{O}_{Z'} \to f^* T_{\mathbf{G}(l,l+m)} \otimes \mathcal{O}_{Z'}$$

is an injection of locally free sheaves which is an isomorphism at the generic point of Z'. Hence, $T_{\mathbf{G}(k,k+n)} \otimes \mathcal{O}_{Z'}$ has a map to $\mathcal{O}_{Z'}^{\oplus r}$ which is generically surjective; since $T_{\mathbf{G}(k,k+n)}$ is generated by global sections, $T_{\mathbf{G}(k,k+n)} \otimes \mathcal{O}_{Z'}$ has a trivial direct summand of rank r. As a consequence, if $\mathbf{G}(l, l+m)$ has a subvariety Z of dimension d such that $T_{\mathbf{G}(l,l+m)} \otimes \mathcal{O}_{Z'}$ has a trivial direct summand of rank r, then $\mathbf{G}(k, k+n)$ also has such a subvariety of dimension d. We will use this to prove that $k = l, m = n$.

Let s, t be integers with $1 \leqq s \leqq l, 1 \leqq t \leqq m$. We have an embedding

$$X = \mathbf{G}(l-s, l+m-s-t) \to \mathbf{G}(l, l+m) = Y$$

which we may describe as follows: on X we have the universal quotient

$$\mathcal{O}_X^{\oplus l+m-s-t} \to \mathcal{Q}$$

where $\mathcal{Q}$ is locally free of rank $l-s$. This yields a quotient which is the composite

$$\mathcal{O}_X^{\oplus l+m} = \mathcal{O}_X^{\oplus l+m-s-t} \oplus \mathcal{O}_X^{\oplus s} \oplus \mathcal{O}_X^{\oplus t} \to \mathcal{Q} \oplus \mathcal{O}_X^{\oplus s} \oplus \mathcal{O}_X^{\oplus t} \to \mathcal{Q} \oplus \mathcal{O}_X^{\oplus s}.$$

By the universal property of $\mathbf{G}(l, l+m)$ this corresponds to the above morphism $X \to Y$. Let $Z_{s,t}$ denote the image. The universal exact sequence on $\mathbf{G}(l, l+m)$ restricted to $Z_{s,t}$ is the direct sum of the universal exact sequence on $\mathbf{G}(l-s, l+m-s-t)$ with the split sequence

$$0 \to \mathcal{O}_{Z_{s,t}}^{\oplus t} \to \mathcal{O}_{Z_{s,t}}^{\oplus s+t} \to \mathcal{O}_{Z_{s,t}}^{\oplus s} \to 0.$$

Thus, $T_{\mathbf{G}(l,l+m)} \otimes \mathcal{O}_{Z_{s,t}}$ has a trivial direct summand of rank st. This is also true of any translate of $Z_{s,t}$ under $\mathrm{GL}_{l+m}(\mathbf{C})$.

Lemma 7. *Let Z be an irreducible subvariety of $\mathbf{G}(l, l+m)$. Suppose that $T_{\mathbf{G}(l,l+m)} \otimes \mathcal{O}_Z$ has a trivial direct summand of rank r. Then there exists integers s, t with $1 \leqq s \leqq l$, $1 \leqq t \leqq m$ and $st \geqq r$, such that some $\mathrm{GL}_{l+m}(\mathbf{C})$ translate of $Z_{s,t}$ contains Z.*

Proof. Let $V = \mathbf{C}^{l+m}$, so that $\mathbf{G}(l, l+m)$ parametrizes l-dimensional quotients of V. If $x \in \mathbf{G}(l, l+m)$, then there is a corresponding l-dimensional quotient

$$V \to V/W_x$$

where W_x is of dimension m. Then the tangent space to $\mathbf{G}(l, l+m)$ at x is

$$T_x = \mathrm{Hom}(W_x, V/W_x) .$$

There is a surjection φ_x: $\mathrm{End}(V) \to T_x$ corresponding to a surjection of locally free sheaves

$$\mathcal{O}_{\mathbf{G}(l, l+m)} \otimes_{\mathbf{C}} \mathrm{End}(V) \to T_{\mathbf{G}(l, l+m)}$$

which gives rise to a morphism

$$\mathbf{P}(T_{\mathbf{G}(l, l+m)}) \to \mathbf{P}(\mathrm{End}(V)) ,$$

whose restriction to $\mathbf{P}(T_x)$ is induced by φ_x. Identifying $\mathrm{End}(V)$ with its dual space, we may identify the projective space $\mathbf{P}(\mathrm{End}(V))$ with the space of lines in $\mathrm{End}(V)$; then the subspace $\mathbf{P}(T_x)$ is the space of lines in $\mathrm{Hom}(V/W_x, W_x)$. Here we identify $\mathrm{Hom}(V/W_x, W_x)$ with

$$\{A \in \mathrm{End}(V) | \mathrm{im}\, A \subset W_x \subset \ker A\} .$$

Now if $Z \subset \mathbf{G}(l, l+m)$ is a subvariety, then by lemma 3,

$$T_{\mathbf{G}(l, l+m)} \otimes \mathcal{O}_Z \text{ has a trivial direct summand of rank } r$$

$$\Leftrightarrow \dim\left(\bigcap_{x \in Z} \mathrm{Hom}(V/W_x, W_x) \right) \geqq r ,$$

where the intersection is taken in $\mathrm{End}(V)$. But if

$$V_1 = \bigcap_{x \in Z} W_x,\ V_2 = \sum_{x \in Z} W_x ,$$

then $0 \subset V_1 \subset V_2 \subset V$, and

$$\bigcap_{x \in Z} \mathrm{Hom}(V/W_x, W_x) = \{A \in \mathrm{End}(V) | \mathrm{im}\, A \subset V_1 \subset V_2 \subset \ker A\}$$

$$= \mathrm{Hom}(V/V_2, V_1) .$$

If $t = \dim V_1$, $s = \dim V/V_2$, then

$$Z' = \{x \in \mathbf{G}(l, l+m) | V_1 \subset W_x \subset V_2\}$$

is a translate of $Z_{s,t}$ which clearly contains Z; also

$$\bigcap_{x \in Z'} \mathrm{Hom}(V/W_x, W_x) = \bigcap_{x \in Z} \mathrm{Hom}(V/W_x, W_x) = \mathrm{Hom}(V/V_2, V_1)$$

which has dimension st. Since the middle term has dimension $\geqq r$, we get $st \geqq r$. □

As a corollary, we observe that if $Z \subset \mathbf{G}(l, l+m)$ is such that $T_{\mathbf{G}(l, l+m)} \otimes \mathcal{O}_Z$ has a trivial direct summand, then

$$\dim Z \leqq \dim Z_{1,1} = \dim \mathbf{G}(l-1, l+m-2) = (l-1)(m-1) .$$

Hence if

$$f: \mathbf{G}(k, k+n) \to \mathbf{G}(l, l+m), \quad k \leqq n, 2 \leqq l \leqq m ,$$

is a finite morphism, $f^*(Z_{1,1})$ has an irreducible component $Z \subset \mathbf{G}(k, k+n)$ of dimension $(l-1)(m-1)$, such that $T_{\mathbf{G}(k, k+n)} \otimes \mathcal{O}_Z$ has a trivial direct summand. Applying lemma 7 to $\mathbf{G}(k, k+n)$, we see that

$$(k-1)(n-1) \geqq (l-1)(m-1) , \tag{1}$$

Since f is finite, we have $kn = lm$. Hence this implies

$$l + m \geqq k + n . \tag{2}$$

If equality holds, then since $kn = lm$, we must have $k = l, m = n$ and hence from the Theorem, f is an isomorphism. Hence it suffices to prove that strict inequality in (2) leads to a contradiction. Now

$$l + m > k + n \tag{3}$$

$$\Rightarrow (l+m)^2 > (k+n)^2$$

$$\Rightarrow (m-l)^2 > (n-k)^2; \quad \text{using } lm = kn$$

$$\Rightarrow m - l > n - k . \tag{4}$$

From (3) and (4),

$$m > n \geqq k > l \geqq 2 . \tag{5}$$

Let t_0 be the positive integer such that $t_0 l \geqq k$, while $(t_0 - 1)l < k$; then $t_0 \geqq 2$, as $k > l$. Let s be any integer satisfying

$$\frac{(k - t_0 - 1)m + 1}{k} \leqq s \leqq \frac{(k - t_0 + 1)m - 1}{k} . \tag{6}$$

Since $m > k$, there are at least two integers s satisfying (6). Then $(t_0 - 1)l \leqq (k-1)$ implies that

$$k - t_0 - 1 \geqq (l-1)t_0 - l \geqq 2(l-1) - l = l - 2 \geqq 0 ;$$

also, $(k - t_0 - 1) \leqq (k-1)$, so that

$$ks \leqq (k-1)m - 1 < km .$$

Hence for any s satisfying (6),

$$1 \leqq s \leqq (m-1) . \tag{7}$$

Consider the subvariety $Z_{1, m-s} \subset \mathbf{G}(l, l+m)$, for s satisfying (6). The subvariety is isomorphic to $\mathbf{G}(l-1, l-1+s)$, so it has dimension $(l-1)s$; also

$T_{\mathbf{G}(l,l+m)} \otimes \mathcal{O}_{Z_{1,m-s}}$ has a trivial direct summand of rank $m-s$. Hence there exists an irreducible component $Z \subset f^{-1}(Z_{1,m-s}) \subset \mathbf{G}(k, k+n)$ such that $\dim Z = (l-1)s$, and $T_{\mathbf{G}(k,k+n)} \otimes \mathcal{O}_Z$ has a trivial direct summand of rank $m-s$. By lemma 7 applied to $\mathbf{G}(k, k+n)$, there exist integers r, t with $1 \leqq t \leqq k-1$, $1 \leqq r \leqq n-1$ such that $Z \subset Z_{t,n-r} \subset \mathbf{G}(k, k+n)$. Now $Z_{t,n-r}$ has dimension $(k-t)r$ and $T_{\mathbf{G}(k,k+n)} \otimes \mathcal{O}_{Z_{t,n-r}}$ has a trivial direct summand of rank $t(n-r)$. Thus, t, r satisfy the system of inequalities:

$$r(k-t) \geqq s(l-1) \tag{8}$$

$$t(n-r) \geqq m-s\,. \tag{9}$$

Now $k-t \geqq 1$, so that (8) implies that

$$r \geqq \frac{s(l-1)}{k-t}\,, \tag{10}$$

while (9) implies that

$$r \leqq \frac{tn-m+s}{t}\,. \tag{11}$$

Combining (10) and (11), we obtain

$$(k-t)(tn-m+s) \geqq (l-1)ts\,.$$

Substituting $n = lm/k$, we get

$$(k-t)\left(\frac{tlm}{k} - m + s\right) \geqq (l-1)ts$$

$$\Rightarrow (k-t)\left(\frac{tlm}{k} - m\right) \geqq (tl-k)s$$

$$\Rightarrow ((k-t)m - ks)(tl-k) \geqq 0\,. \tag{12}$$

If $t > t_0$, then $tl - k > 0$, so that (12) yields $(k-t)m \geqq ks$. From (6), we get

$$(k-t)m \geqq (k-t_0-1)m + 1$$

$$\Rightarrow (t_0 - t + 1)m \geqq 1$$

$$\Rightarrow (t_0 + 1 - t) > 0$$

contradicting $t > t_0$.

If $t < t_0$, then $tl - k < 0$, and so (12) yields $(k-t)m \leqq ks$. From (6), we get

$$(k-t)m \leqq (k-t_0+1)m - 1$$

$$\Rightarrow (t - t_0 + 1)m \geqq 1$$

$$\Rightarrow (t + 1 - t_0) > 0$$

contradicting $t < t_0$.

Hence we must have $t = t_0$. Now if $t_0 l - k > 0$, then (12) again gives

$$(k-t_0)m \geqq ks\,. \tag{13}$$

But the interval

$$\left(\frac{(k-t_0)m}{k}, \frac{(k-t_0+1)m-1}{k}\right]$$

contains an integer s (which then satisfies (6)), since

$$(k-t_0+1)m-1-(k-t_0)m \geqq k\ ,$$

and this contradicts (13).

Hence, we are forced to choose $t = t_0$, where $t_0 l = k$. In this case, the inequalities (8) and (9) become (using $lm = kn = t_0 ln$, so that $m = t_0 n$)

$$r(t_0 l - t_0) \geqq s(l-1)$$

$$t_0(n-r) \geqq t_0 n - s$$

which yield the pair of inequalities

$$(rt_0 - s)(l-1) \geqq 0$$

$$0 \geqq rt_0 - s\ .$$

Since $l \geqq 2$, this forces $rt_0 = s$, so that t_0 divides s; also $t_0 \geqq 2$. But there are atleast two consecutive integers s satisfying (6); so we may choose s satisfying (6) but with $t_0 \nmid s$. Hence, in all cases, for some value of s satisfying (6), it is impossible to find any r, t with $1 \leqq t \leqq k-1$, $1 \leqq r \leqq n-1$ such that (8) and (9) hold. This proves Proposition 8. □

3. Remarks on Lazarsfeld's problem

In this section we show:

Proposition 8. *Let Y be a smooth quadric hypersurface,* $\dim Y = n \geqq 3$, *and $f\colon Y \to X$ be a finite surjective morphism of degree > 1 to a smooth variety X; then X is isomorphic to $\mathbf{P}^n$.*

Proof. We begin by reviewing the results of Mori [M]. He proves (Theorem 6 of [M]) that if X is a smooth, projective variety of dimension n such that the inverse of the canonical sheaf K_X^{-1} is ample, then for each $P \in X$, there is a non-constant morphism $u\colon \mathbf{P}^1 \to X$ with $\deg u^*(K_X^{-1}) \leqq n+1$, such that $P \in u(\mathbf{P}^1)$.

Now fix $P \in X$, and let $* \in \mathbf{P}^1$ be a fixed point. Assume that K_X^{-1} is ample, and let

$$d = \min\{\deg u^*(K_X^{-1}) | u\colon (\mathbf{P}^1, *) \to (X, P);\ u \text{ is non-constant}\}\ .$$

Then $d \leqq n+1$, and any $u\colon \mathbf{P}^1 \to X$ achieving this minimal degree is birational to its image. Let V be a connected component of $\underline{\mathrm{Hom}}^d((\mathbf{P}^1, *), (X, P))$, the scheme of morphisms $u\colon \mathbf{P}^1 \to X$ of degree d. Then $G = \mathrm{Aut}(\mathbf{P}^1, *)$ acts on V. Let $\tilde{V}$ be an irreducible component of the normalization of V. Then the G action on $\tilde{V}$ is proper and free, with a geometric quotient $\gamma\colon \tilde{V} \to W$, where W is a normal projective variety and γ is a principal G-bundle.

Assume further that for all $u \in V$,

$$H^1(\mathbf{P}^1, u^* T_X \otimes \mathcal{O}_{\mathbf{P}^1}(-1)) = 0\ .$$

Then from the Riemann-Roch theorem,

$$\dim H^0(\mathbf{P}^1, u^* T_X \otimes \mathcal{O}_{\mathbf{P}^1}(-1)) = d\,,$$

and Mori's arguments show that V is smooth of dimension d; hence W is smooth (and projective) of dimension $d-2$. Further, if $u^* T_X$ is ample for all $u \in V$, then

$$d = n+1, \quad W \cong \mathbf{P}^{n-1}, \quad \text{and } X \cong \mathbf{P}^n\,.$$

We now specialize to the situation when there is a finite surjective morphism $f\colon Y \to X$, where Y is a smooth quadric of dimension $n \geqq 3$. Let $B \subset X$ be the branch locus, and $R \subset Y$ be the ramification locus so that

$$f^* K_X \otimes K_Y^{-1} \cong \mathcal{O}_Y(-R) \text{ and } B = f(R)$$

(since Y is simply connected and every automorphism of Y has fixed points by the Lefschetz fixed point formula, R and B are effective and non-zero). As Pic $Y = \mathbf{Z}$, $\mathcal{O}_Y(R)$ is ample, and K_Y^{-1} is ample since Y is a quadric, K_X^{-1} is ample, and Mori's results apply.

Let $U = f(Y - R)$, so that $X - U \subset B$. Let $P \in U$, and let $Q \in Y - R$ with $f(Q) = P$. Then if $u\colon (\mathbf{P}^1, *) \to (X, P)$ is a curve such that $d = \deg u^* K_X^{-1}$, and C is the normalization of any irreducible component of $f^{-1}(u(\mathbf{P}^1))$ which passes through Q, then we have a diagram

$$\begin{array}{ccc} C & \xrightarrow{v} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbf{P}^1 & \xrightarrow{u} & X \end{array}$$

There is a map of locally free sheaves

$$v^* T_Y \to h^* u^* T_X$$

which is an isomorphism at the generic point of C, as $C \not\subset R$. Now

$$u^* T_X \cong \mathcal{O}_{\mathbf{P}^1}(m_1) \oplus \ldots \oplus \mathcal{O}_{\mathbf{P}^1}(m_n)\,,$$

with $m_1 \leqq \ldots \leqq m_n$; since T_Y is generated by global sections, we see that $h^* u^* T_X$ is generated at the generic point of C by its global sections, so that $m_i \geqq 0$ for all i. Further,

$$v^* T_Y \cong \mathcal{O}_C^{\oplus r} \oplus \mathcal{E}$$

where $\mathcal{E}$ is an ample locally free sheaf, and $r \geqq 0$. Hence, $m_i > 0$ for all $i > r$. Also, the inclusion of sheaves $T_{\mathbf{P}^1} \to u^* T_X$ shows that $m_n \geqq 2$. In any case, $H^1(\mathbf{P}^1, u^* T_X \otimes \mathcal{O}_{\mathbf{P}^1}(-1)) = 0$, so that V is smooth of dimension d and W is smooth of dimension $d-2$.

Lemma 9. *If Y is a smooth quadric of dimension $n \geqq 3$, and $v\colon C \to Y$ a non-constant morphism from an irreducible projective curve C, then either $v^* T_Y$ is ample, or $v(C)$ is a line contained in Y and in this case,*

$$v^*(T_Y) \cong \mathcal{O}_C \oplus \mathcal{E}\,,$$

where $\mathcal{E}$ is ample.

Proof. We note that Y is the space of isotropic lines in a quadratic space. As in section 1, we have a natural morphism

$$\mathbf{P}_Y(T_Y) \to \mathbf{P}(\operatorname{Lie} G),$$

where G is the corresponding orthogonal group. We may then identify $\operatorname{Lie} G$ with the space of skew-symmetric matrices, and for any $p \in Y$ the tangent space $T_{p,Y} = \operatorname{Hom}(p, p^{\perp}/p)$. From this it follows easily that, for any $p, q \in Y$ the linear subspaces $\mathbf{P}(T_{p,Y})$ and $\mathbf{P}(T_{q,Y})$ of $\mathbf{P}(\operatorname{Lie} G)$ intersect if and only if the lines p and q are orthogonal. Thus, from lemma 3, if $v(C) \subset Y$ is an irreducible curve such that $v^*(T_Y)$ has a trivial direct summand, then $v(C)$ lies in the projective space of an isotropic subspace of $\mathbf{P}^{n+1}$, i.e. for some $t > 0$,

$$v(C) \subset \mathbf{P}^t \subset Y \subset \mathbf{P}^{n+1}.$$

For a linear subspace $\mathbf{P}^t \subset Y$, we have the diagram with exact rows and columns

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 \to T_{\mathbf{P}^t} & \to & T_Y|_{\mathbf{P}^t} & \to & N & \to & 0 \\
\| & & \downarrow & & \downarrow & & \\
0 \to T_{\mathbf{P}^t} & \to & T_{\mathbf{P}^{n+1}}|_{\mathbf{P}^t} & \to & \mathcal{O}_{\mathbf{P}^t}(1)^{\oplus n+1-t} & \to & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & \mathcal{O}_{\mathbf{P}^t}(2) & = & \mathcal{O}_{\mathbf{P}^t}(2) & \to & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

The middle row is split exact since $H^1(\mathbf{P}^t, T_{\mathbf{P}^t}(-1)) = 0$, hence so is the top row. Tensoring the last column with $\mathcal{O}_{\mathbf{P}^t}(-1)$ we see that

$$N \cong \Omega^1_{\mathbf{P}^t}(2) \oplus \mathcal{O}_{\mathbf{P}^t}(1)^{\oplus(n-2t)}.$$

Hence,

$$T_Y|_{\mathbf{P}^t} \cong \Omega^1_{\mathbf{P}^t}(2) \oplus \mathcal{E}$$

where $\mathcal{E}$ is ample. Thus $v^*(T_Y)$ has a trivial direct summand if and only if $v^*(\Omega^1_{\mathbf{P}^t}(2))$ has one. So it suffices to show that this is possible only if $v(C)$ is a line in $\mathbf{P}^t$. Taking duals, if

$$v^*(T_{\mathbf{P}^t}(-1)) \cong v^*(\mathcal{O}_{\mathbf{P}^t}(1)) \oplus \mathcal{F},$$

then $\mathcal{F}$ is generated by global sections and has trivial determinant and is thus trivial. But this clearly implies that $v(C)$ is a line, in which case we may take $t = 1$, and this yields the second conclusion. □

If $C \subset Y$ is a line then $r = 1$ and $\deg v^* T_Y = n$. Hence, in any case, $m_1 \geqq 0$, $m_i \geqq 1$ for $i > 1$, and $m_n \geqq 2$, so that $d \geqq n$, and $d - 2 = \dim W > 0$. Thus there are infinitely many distinct rational curves through P with minimal degree d.

We now consider two cases.

Case 1. *For some* $P \in U$, $\{u \in V \mid u^* T_X$ is not ample$\}$ *consists of atmost finitely many G orbits.*

In this case, $d = n + 1$, and W is smooth and projective of dimension $n - 1$. If $u \in V$ such that $u^* T_X$ is ample, then

$$u^* T_X \cong \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus n-1} \oplus \mathcal{O}_{\mathbf{P}^1}(2) .$$

Hence u is an immersion. On the other hand, if $u^* T_X$ is not ample, then we have a diagram

$$\begin{array}{ccc} C & \xrightarrow{v} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbf{P}^1 & \xrightarrow{u} & X \end{array}$$

where v is the embedding of a line in Y through $Q \in (f^{-1}(P) - R)$. Then $C \to X$ is unramified at $v^{-1}(Q)$, and so u is unramified at $h(v^{-1}(Q))$. This is valid for each irreducible component C of $f^{-1}(u(\mathbf{P}^1))$ through Q, and so u is unramified at $* \in \mathbf{P}^1$.

Hence, if we fix a non-zero tangent vector $t \in T_{*,\mathbf{P}^1}$ then the assignment $u \mapsto du(t)$ gives a morphism

$$\eta\colon V \to (T_{P,X} - \{0\}) \cong \mathbf{A}^n - \{0\}$$

which yields a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\eta} & \mathbf{A}^n - \{0\} \\ {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle \pi} \\ W & \xrightarrow{\delta} & \mathbf{P}^{n-1} \end{array}$$

As in Mori's paper [M], we see that if $u \in V$ such that $u^* T_X$ is ample, then η is smooth along the G orbit of u in V, and so δ is étale at $\gamma(u) \in W$. By assumption, this means that δ is étale outside a finite set. Since $n \geqq 3$ this means that δ is étale, and hence an isomorphism. On the other hand, if $u \in V$ such that $u^* T_X$ is not ample, then

$$H^1(\mathbf{P}^1, u^* T_X \otimes \mathcal{O}_{\mathbf{P}^1}(-2)) \neq 0 .$$

The Zariski tangent space to the fibre of η at u is

$$H^0(\mathbf{P}^1, u^* T_X \otimes \mathcal{O}_{\mathbf{P}^1}(-2))$$

which has dimension > 1; hence η is not smooth at u. But $\delta \circ \gamma$ is a principal G-bundle so that η is a principal G_1-bundle, where G_1 is the subgroup of G fixing the tangent vector t. Hence η is smooth, and so $u^* T_X$ is ample for all $u \in V$. As in [M], this implies that $X \cong \mathbf{P}^n$.

Case 2. *For each* $P \in U$, $\{u \in V \mid u^* T_X$ is not ample$\}$ *consists of infinitely many G orbits.*

Since there are only a finite number of lines in Y joining distinct points of $f^{-1}(P)$, we see that there exists $u \in V$ such that $u^* T_X$ is not ample, and in the diagram

$$\begin{array}{ccc} C & \xrightarrow{v} & Y \\ h\downarrow & & \downarrow f \\ \mathbf{P}^1 & \xrightarrow{u} & X \end{array}$$

C is a line such that $C \cap f^{-1}(P) = \{Q\}$. Since f is unramified at $Q, f \circ v$ is birational, and h is an isomorphism. Thus

$$n + 1 \geqq \deg h^* u^* K_X^{-1} = \deg v^* K_Y^{-1} + \deg v^* \mathcal{O}_Y(R)$$
$$= n + \deg v^* \mathcal{O}_Y(R) \geqq n + 1 .$$

Hence we must have $d = n + 1$, and $\mathcal{O}_Y(R) = \mathcal{O}_Y(1)$. In particular, R is a hyperplane section of Y, and is reduced and irreducible; so f is simply ramified (has ramification index two) at the generic point of R. Thus $\dim V = n + 1$, and $\dim W = n - 1$.

As in case 1, we see that, by fixing a non-zero tangent vector $t \in T_{*,\mathbf{P}^1}$, we obtain a diagram

$$\begin{array}{ccc} V & \xrightarrow{\eta} & \mathbf{A}^n - \{0\} \\ \gamma\downarrow & & \downarrow \pi \\ W & \xrightarrow{\delta} & \mathbf{P}^{n-1} \end{array}$$

where for $u \in V$ such that $u^* T_X$ is ample, δ is étale at $\gamma(u)$.

The cone of lines in Y through Q is parametrized by a smooth quadric hypersurface $Z \subset \mathbf{P}(T^*_{Q,Y}) \cong \mathbf{P}(T^*_{P,Y})$. Since for any line $C \subset Y, \deg f^* K_X^{-1} \otimes \mathcal{O}_C = n + 1$, we see that $f|_C$ is birational for any line C meeting $f^{-1}(U)$. Thus we obtain a morphism $\zeta: Z \to W$ such that the composite

$$\delta \circ \zeta: Z \to \mathbf{P}^{n-1} = \mathbf{P}(T^*_{P,X})$$

is the natural embedding. Clearly the non-étale locus of δ is contained in $\zeta(Z)$. Hence δ is a finite morphism between smooth varieties, and its non-étale locus is a divisor, which must equal $\zeta(Z)$ if δ is not an isomorphism. As in case 1, if δ is an isomorphism, then $u^* T_X$ is ample for all $u \in V$, contradicting the hypothesis of case 2. Hence for every line $C \subset Y$ through $Q, f^* T_X \otimes \mathcal{O}_C$ is not ample on C.

We claim that $U = X - B$ i.e. $f^{-1}(B) = R$. If not, we can find $P \in U$ with Q, Q' in $f^{-1}(P)$, where $Q \notin R$, and $Q' \in R$, such that P is a smooth point of B, and f is simply ramified at Q'. We can find a line $C \subset Y$ through Q such that $C_1 = f(C)$ is smooth at P and transverse to B at P. Then we can find another line $C' \subset Y$ through Q' which maps birationally to C_1, since every irreducible component of $f^{-1}(C_1)$ must be a line. However simple ramification at Q' implies that $df(T_{Q',Y}) \subset T_{P,B}$. Since C_1 is transverse to B at P, this is a contradiction.

Now R is a hyperplane section of the smooth quadric Y of dimension $\geqq 3$; hence $\pi_1(Y-R)=0$ (this is clear if R is singular as $Y-R\cong \mathbf{A}^n$; if R is smooth, this follows from the facts (i) $\pi_1(Y-R)$ is abelian, and (ii) $H_1(Y-R,\mathbf{Z})=0$). Thus $Y-R$ is the universal covering space of $X-B$. In particular there is a finite group H of automorphisms of Y, which acts freely on $Y-R$, such that $X=Y/H$ (the automorphisms in H of $Y-R$ extend to Y as Y is the normalization of X in $\mathbf{C}(Y)$). Since f is simply ramified at the generic point of R, the inertia group of the corresponding discrete valuation on $\mathbf{C}(Y)$ has order two. The involution σ generating this inertia group extends to the ambient projective space $\mathbf{P}^{n+1}$, fixes the hyperplane spanned by R and has no other fixed points on Y. Thus σ has one other isolated fixed point in $\mathbf{P}^{n+1}-Y$ and the quotient map $Y\to Y/\langle\sigma\rangle$ is induced by the projection from this fixed point. Thus we have a factorization

$$\begin{array}{ccc} Y & \longrightarrow & \mathbf{P}^n \\ & {\scriptstyle f}\searrow \quad \swarrow & \\ & X & \end{array}$$

From the result of Lazarsfeld mentioned in the introduction (see [L]), we must have $X\cong \mathbf{P}^n$, so that T_X is ample on every curve in X, contradicting the hypothesis of case 2. □

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Note added in proof

1. Problem 2 has been answered affirmatively by O. Debarre.
2. The following observation by P. Polo and M.S. Raghunathan can be used to strengthen the Theorem. *If G is any simple connected, semi-simple algebraic group over* $\mathbf{C}$ *such that $G/P\cong \Pi G/P_i$, where P, P_i's are parabolic subgroups; then $G=\Pi G_i$ and there are parabolic groups $Q_i\subset G_i$ such that $P_i=p_i^{-1}(Q_i)$, where p_i: $G\to G_i$ is the projection.*
3. The Theorem has the following corollary: *Let G be a semi-simple, simply connected algebraic group over an algebraically closed field k with* $\operatorname{char} k=p>0$, *and let X be a projective homogeneous variety for G. Suppose X lifts to a smooth and proper scheme $\chi\to$* Spec *$W(k)$ over the Witt vectors of k, such that the absolute Frobenius morphism of X lifts to a morphism of χ (covering the Frobenius on $W(k)$). Then $X\cong \Pi\mathbf{P}^{n_i}$.*