Self-organization without conservation: true or just apparent scale-invariance?

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Self-organization without conservation: true or just apparent scale-invariance?

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Abstract. The existence of true scale-invariance in slowly driven models of self-organized criticality *without* a conservation law, such as forest-fires or earthquake automata, is scrutinized in this paper. By using three different levels of description—(i) a simple mean field, (ii) a more detailed mean-field description in terms of a (self-organized) branching processes, and (iii) a full stochastic representation in terms of a Langevin equation—it is shown on general grounds that non-conserving dynamics does *not* lead to *bona fide* criticality. Contrary to the case for conserving systems, a parameter, which we term the 're-charging' rate (e.g. the tree-growth rate in forest-fire models), needs to be *fine-tuned* in non-conserving systems to obtain criticality. In the infinite-size limit, such a fine-tuning of the loading rate is easy to achieve, as it emerges by imposing a second separation of timescales but, for any finite size, a precise tuning is required to achieve criticality and a coherent finite-size scaling picture. Using the approaches above, we shed light on the common mechanisms by which 'apparent criticality' is observed in non-conserving systems, and explain in detail (both qualitatively and quantitatively) the difference with respect to true criticality obtained in conserving systems. We propose to call this *self-organized quasi-criticality* (SOqC). Some of the reported results are already known and some of them are new. We hope that the unified framework presented here will help to elucidate the confusing and contradictory literature in this field. In a forthcoming paper, we shall discuss the implications of the general results obtained here for models of neural avalanches in neuroscience for which self-organized scale-invariance in the absence of conservation has been claimed.

Keywords: percolation problems (theory), phase transitions into absorbing states (theory), self-organized criticality (theory), sandpile models (theory)

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1. Introduction: critical self-organization with and without conservation

Power-law distributions are quite common in Nature: earthquakes and microfractures, solar flares, weather records, snow avalanches, crackling, and $1/f$ noise are just a few examples of systems displaying scale-invariance [1]. Many of these, among many others, have been claimed to be critical, i.e. to lie at, or very close to, a critical point, which yields scale-invariance and the concomitant power-law distributions. It is worth stressing that power laws (or approximate power laws) emerging from the complex interactions of many-component systems placed in the vicinity of a critical point and, therefore, with diverging correlation lengths and the associated power-law decay of temporal and spatial

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correlations, should be clearly distinguished from other power-law distributions arising in many different contexts (word distributions, city populations, citations, etc). These latter can be generated by a wealth of *non-critical mechanisms*, for instance, multiplicative noise or fragmentation processes (see [2]–[4] for recent reviews), without the need to invoke criticality.

For a system to be critical, it does not suffice to have power-law distributed observables but, more crucially, it has to obey *finite-size scaling*: measurements at various system sizes (scales) can be related to each other by re-scaling variables and quantities in some specific 'scale-invariant' way (i.e. scaling collapses can be performed).

Given that, in standard phase transitions (both in equilibrium and away from it), a precise parameter tuning is required to reach criticality and generate power-law distributed quantities, an alternative explanation for the emergence of generic critical scale-invariance (i.e. criticality occurring without requiring fine-tuning) was historically much needed[1]. How does criticality emerge spontaneously?

In a seminal paper, Bak, Tang and Wiesenfeld (BTW) [5] introduced, back in 1987, the concept of *self-organized criticality* (SOC) [6]–[11] aimed at solving the previous conundrum. The research line opened by their breakthrough work continues to attract, more than twenty years after, a great deal of interest. From this perspective, the overwhelming activity generated in the last decade around *scale-free networks* [12] can be certainly considered as a prominent extension of preceding research on SOC.

A handful of mechanisms were proposed under the common name of SOC to justify the abundant presence of scale-invariance in the natural world. The most successful among them is the one exemplified by the original BTW *sandpile model*. Many variations of the BTW sandpile were proposed: the Manna sandpile [13], the Oslo ricepile model [14], and the Zhang model [15] are just a few of them. For the sake of completeness and for future reference, a brief description of the best-known prototypical SOC models is provided in appendix A. Other possible mechanisms, such as *extremal dynamics* [16], have been studied, but we shall not be concerned with them here (see [9]).

The idea inspiring sandpiles, ricepiles, and related SOC models is that many systems in Nature, when pushed/driven slowly, respond very irregularly with rapid rearrangements (avalanches, bursts) of a broad variety of sizes. The distribution of such avalanches is scale-invariant in many cases. Of course, sandpile SOC models are too simplistic to reproduce the detailed behavior of real sandpiles; they can be regarded as 'metaphors' capturing only some particular features of real systems in a stylized manner. Nevertheless, after many (partially) failed attempts [17], power laws for avalanche in real granular piles, obeying finite-size scaling, were experimentally measured [18]. Moreover, other physical situations, such as vortex avalanches in type-II superconductors, can be mimicked as sandpiles, and their critical properties rationalized in terms of these [19].

The dynamics of a generic sandpile model can be synthesized as follows: some type of 'energy' or 'stress' (sand grains) is progressively injected in discrete units (i.e. grains are

[1] A few mechanisms for generic scale-invariance exist, for instance: (i) the breaking of a continuous symmetry is well known to generate a 'zero-mass' (Goldstone) mode with generic power-law decaying correlations in all of the broken-symmetry phase, or (ii) the presence of built-in quenched disorder can lead to generic power laws. The systems described in section 1 do not fit within these scenarios. Other possibilities are conservation laws in conjunction with a non-equilibrium which are more closely related to the content of this paper. A nice discussion of this can be found in [20].

dropped singly) at the sites of a spatially extended system (usually a two-dimensional square lattice) at a slow timescale. Whenever a certain threshold of local energy is overcome, the corresponding site becomes unstable and its accumulated energy is redistributed at a much faster timescale among its neighbor sites. These, in their turn, can become unstable, and trigger a cascade of rearrangements, i.e. an *avalanche* or outburst of activity. Local redistribution rules are conserving in sandpiles: energy does not disappear, but only diffuses around. Open boundaries are customarily considered to allow for energy release from the system. Once all activity ceases (i.e. the avalanche stops) new energy is injected (i.e. a grain is dropped) into the system, and so on, until a statistically stationary state is reached. In such a steady state, avalanches are scale-invariant, i.e. their sizes/times are power-law distributed up to a maximum scale imposed by the system size: the system 'self-organizes' to a critical point [6]–[10], [20]–[22]. Moreover, the associated power spectrum exhibits $1/f^{\alpha}$ noise [23].

For the sake of generality, let us fix a nomenclature common to all models discussed in this paper: '*energy*' refers to the accumulated and transported magnitude and '*activity*' describes 'energy above threshold'. In some cases, sites below threshold will be subclassified into two groups: *critical*, which can become active upon receiving one input of energy (for instance, one grain), and *stable*, which cannot [24].

An essential ingredient of SOC is *slow driving* (driving and dynamics operating at two infinitely separated timescales [6]–[8], [20, 25], i.e. avalanches are instantaneous relative to the timescale of driving). Such an infinite separation is usually achieved by driving the system only when all activity has stopped, but not during avalanches. For any *finite* separation of timescales, a finite characteristic (time/size) scale appears; hence, slow driving is a crucial requirement for generic scale-invariance to emerge [20], [26]–[28].

It was soon emphasized that *energy conservation* is also a key player for criticality to emerge in sandpile models [20, 29, 30]. Note that by 'conserving system' one can refer either to models with bulk conserving dynamics and boundary dissipation, or to cases with a bulk dissipation rate vanishing in the limit of large system size [31]. Various relatively simple arguments were proposed for rationalizing the existence of true criticality in the steady state of *conserving* self-organized systems (some of them are briefly discussed in section 2). However, these arguments cannot be easily extended to similar non-conserving systems. In particular, different works have shown that the level of dissipation acts as a relevant parameter in the renormalization group sense: any degree of bulk dissipation breaks criticality in sandpile models [20, 30, 32].

Still, given the large variety of natural phenomena exhibiting (exact or approximate) scale-invariance in which some form of dissipation is inevitably present (i.e. systems without any obvious conserved quantity), alternative mechanisms for self-organization to criticality in the absence of conservation were needed to achieve a comprehensive picture of generic scale-invariance [20].

Two acclaimed non-conserving self-organized models, or better, two *families of models* proposed to fill the gap between theoretical understanding and empirical facts are earthquake and forest-fire models (see appendix A for definitions). These are highly non-trivial and interesting models with rich and complex phenomenology. Owing to the lack of solid theoretical arguments, analogous to the ones sketched above for conserving systems, and despite numerical evidence showing power laws for some decades, the existence of true generic scale-invariance in them has long been controversial.

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It is beyond the scope of this paper to review exhaustively the large body of interesting literature devoted to non-conserving SOC models, some aspects of which remain unsettled. But, let us just underline that the state-of-the-art is, as documented in section 2, that none of the considered non-conserving models is truly critical; they just show 'apparent scale-invariance' or 'dirty criticality' for some decades.

However, this final conclusion has not been sufficiently stressed and it has certainly not permeated into the literature. This is likely due to the absence of a general theory, which may suggest that the results discussed above are specific to each particular model. Indeed, works continue to be published assuming or claiming true criticality for SOC non-conserving systems. For instance, in [33] an interesting and solvable 'non-conserving model of SOC' was proposed and studied analytically. In a more recent series of papers, Juanico and collaborators claim to have constructed different non-conserving self-organizing models with applications in various fields (neuroscience, population dynamics, etc) [34]. Also, in a recent work, Levina *et al* propose a non-conserving SOC model for *neural avalanches* to capture the apparent scale-free behavior of avalanches of activity observed experimentally in networks of cortical neurons [35]. An exhaustive analysis of this last model, as well as a study of the possible relation between SOC and neural avalanches, is left for a separate publication.

The aim of the present paper is to put together some previously existing results, scattered in the literature (although this is not intended to be an exhaustive review article) and, more importantly, to rationalize the conclusion that none of the above mentioned non-conserving models, or variations of them, exhibits true criticality, within a *unified framework*. To this end, we rely on different kinds of analytical arguments complemented by computer simulations. In passing, we shall report on a number of new results and present a critical discussion on the existence of true scale-invariance in Nature.

The rest of the paper is structured as follows. In section 2, we briefly review conserving and non-conserving models of SOC, as well as some arguments for justifying the existence of criticality in the first group. In the remaining sections, we elucidate the existence or non-existence of criticality in non-conserving systems using different approaches of increasing complexity. In particular, in section 3, we discuss a simple mean-field approach based on an energy balance equation; it is useful for illustrating some key concepts such as the 'loading mechanism'. In section 4, we study a self-consistent mean-field approximation, namely the so-called *self-organized branching process*; it serves as an adequate benchmark for scrutinizing the effects of dissipation and 'loading' in critical self-organization. In section 5, we present (and briefly review) the Langevin theory of conserving SOC systems. It constitutes a solid basis on which to implement dissipation and loading in a systematic way and for providing clear evidence on the lack of criticality in non-conserving models. Finally, the conclusions and a critical discussion of the implications of our main results are presented in section 6.

2. Conserving versus non-conserving models of SOC

2.1. Conservation and criticality

As stated above, different kinds of arguments of different nature justify asserting the existence of true criticality in conserving SOC models. Some of them are as follows:

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- The energy introduced into a pile at generic sites can reach the boundaries (and, thus, be dissipated) only by means of the diffusive transport of grains occurring during avalanches. Owing to this, and provided that a steady state exists, arbitrarily large avalanches (of all possible sizes) should exist for an arbitrarily large system size, yielding a power-law size distribution. In contrast, in the presence of non-vanishing bulk dissipation, energy disappears at some finite rate, and avalanches stop after some characteristic lifetime/size determined by the dissipation rate [20].
  This type of argument, even if commonly used in the literature, is (at best) incomplete, and can be misleading. It does not consider the possibility of having a characteristic scale larger than the system size, which would allow for avalanches to reach the boundaries. Actually, this is what happens in many real sandpiles (with inertial effects): energy is dissipated quasi-periodically in large system-wide avalanches rather than in scale-invariant avalanches (see the parts on experimental set-ups of sandpiles in [7, 8, 36] or [20]).
- From a more abstract viewpoint (not referring specifically to sandpiles or SOC), energy conservation follows from the existence of a continuous symmetry (in this case, *temporal translational invariance*) as a consequence of Noether's theorem [37]. This also holds the other way around: time translational invariance implies energy conservation. When energy conservation is violated, the corresponding symmetry is broken and a characteristic (finite) timescale appears generically.
- From a field theoretical perspective, in order to have scale-invariance, generic infrared divergences are required. But these are generically lost in the presence of a non-vanishing linear 'mass' term (adopting the field theory jargon), for instance, a dissipative term. In particular, if a term $-\epsilon\psi$ is introduced into the simplest mesoscopic equation for a diffusive field $\psi(\vec{x},t)$

$$\partial_t\psi(\vec{x},t) = D_\psi\nabla^2\psi(\vec{x},t) + \eta(\vec{x},t), \tag{1}$$

  where $\eta(\vec{x},t)$ is a zero-mean Gaussian white noise, a simple calculation reveals that the equal time two-point correlation function can be written as

$$\langle\psi(x,t)\psi(0,t)\rangle \sim \frac{\exp(-x/\xi)}{x^{d-2}}, \tag{2}$$

  for x much larger than the correlation length $\xi \sim 1/\sqrt{\epsilon}$. Accordingly, it is only for $\epsilon = 0$ that ξ diverges, the exponential cut-off disappears, and the correlation function decays algebraically; for any non-vanishing value of ϵ there is a *size-independent* exponential cut-off. Something similar occurs for other correlation functions.
  Note that the noise in equation (1) is not fully conserving, but only conserving on average (i.e. conservation needs only to hold on average to preserve scale-invariance [38]). The same conclusions can be also deduced for an equation analogous to equation (1) but with a strictly conserving noise (see [20, 26, 27]).
- Last but not least, a mesoscopic Langevin equation that captures the critical properties of stochastic sandpiles and related models with a conservation law has been proposed [21], [39]–[41]. It describes systems with many absorbing states (which correspond to the many stable microscopic configurations of a sandpile) and a conservation law. While a detailed description of this is left for a forthcoming section, we just stress here that it constitutes a sound field theoretical representation

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of conserving SOC, reproducing all critical exponents. The underlying mechanism of SOC, highlighted by this theory, cannot be straightforwardly generalized to non-conserving systems without including an additional fine-tuning (as we shall show in section 5).

In summary, there exist solid theoretical grounds underpinning the existence of true criticality in conserving self-organized systems. The same kinds of arguments cannot be easily extended to non-conserving systems. In particular, as stated already, non-conservative sandpiles have been explicitly shown to be non-critical.

2.2. Critical self-organization without conservation?

The two main prototypical non-conserving 'self-organized' models (or families of models), studied profusely in the literature, are (see appendix A):

- *Earthquake models* such as the Olami–Feder–Christensen (OFC) [42] and related ones [43]–[47], in which the degree of dissipation can be explicitly controlled, and for which conservation is preserved only for some specific choice of a parameter.
- *Forest-fire models* such as the Drossel–Schwabl automaton [48] (and earlier versions of it [49]), in which there is no conserved quantity whatsoever.

Together with bulk dissipation, these two models have a common key ingredient, absent in conserving systems: there is an increase of the 'background energy' at some (or at all) sites, occurring between avalanches; (i) the accumulated stress at each site grows continuously between quakes in earthquake models such as the OFC, and (ii) new trees grow between two consecutive fires in forest-fire models (see below).

The effect of these '*loading mechanisms*', as we shall call them generically, is to counterbalance the loss of 'energy' (grains, stress, trees) produced by dissipation and, in this manner, try to restore conservation on average and, thus, criticality [24, 33, 34]. However, let us caution that such a compensation needs to be exact and, therefore, unless a new mechanism giving rise to a perfect cancelation is devised, fine-tuning of the loading rate is the only obvious way in which conservation can be restored.

Let us now discuss in more detail these non-conserving archetypical models.

2.2.1. Earthquakes. The Olami–Feder–Christensen cellular automaton [42] is a simplified version of previously proposed fault dynamics models: the spring–block form of the Burridge–Knopoff model [43] and related *stick–slip models* [45, 46] designed to capture the essence of earthquakes (the Gutenberg–Richter law [50] for the distribution of magnitudes) as well as of similar systems with friction and jerky motion (see appendix A).

At each time step, the 'forces' $F(i, j)$ (or energies, to stick to our generic terminology), defined at each site of a two-dimensional lattice, are increased at a constant rate. Whenever F at any site reaches the threshold value, $F_{\rm thr} = 1$, it is reset to zero and the forces at its four nearest neighbors are increased by an amount $\delta F = \alpha F(i, j)$. This might trigger cascades of rearrangements, i.e. avalanches. Observe that the bulk dynamics is conserving only for $\alpha = 1/4$ in two dimensions.

Early computer simulations and theoretical results [42, 51, 52] seemed to support the existence of criticality for values of α as low as $\alpha \approx 0.05$. It was also reported early on

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that, imposing periodic boundary conditions, the OFC model enters a cycle of periodic configurations with no sign of criticality whatsoever [53, 27, 54]. This suggests that the bulk dynamics is profoundly influenced by boundary conditions. Actually, it was proposed that boundary-induced heterogeneity is essential to obtain partial synchronization between clusters of different sizes and that such a partial *synchronization or 'phase-locking' mechanism* is the basis of the OFC complex behavior [53, 27, 54].

The role of different features (for instance, changes in the boundary conditions, introduction of quenched disorder in the local rules, lattice topology, etc) in synchronization and their effects on the properties of the OFC model have been exhaustively analyzed in the literature [51], [55]–[62]. It has been also shown that results are affected by numerical precision [68]. The overall picture is that the 'synchronization mechanism', even if fascinating, is too fragile to be a solid explanation for generic emergence of criticality. A nice and fairly exhaustive review of the literature on the OFC model and variations of it can be found in [70].

On the analytical side, Bröker and Grassberger [63] and Chabanol and Hakim [64], in two independent papers, were able to calculate the energy distribution, the effective branching ratio, and the average avalanche size for a *random-neighbor* version of the OFC model, which turns out to be analytically solvable. Their main conclusion is that it is only in the conserving limit that the model becomes critical, while exponential cut-offs appear for any $\alpha \neq 1/4$. Similarly, de Carvalho and Prado claimed, relying on an effective branching ratio analysis [65], that the OFC model is only critical in the conserving limit (see also [66]).

Remarkably, it is also shown in [63] and [64] that the average avalanche size is distributed as a power law with a cut-off function, $\exp(4/(1-4\alpha))$, which diverges in a very fast way when the conserving limit is approached. This provides an explanation for the relative large power-law regimes observed even in the non-conserving case. It would certainly be nice to have extensions of this result to other non-mean-field-like systems.

With a similar line of reasoning, Kinouchi and Prado introduced the concept of '*robust criticality*' or '*almost criticality*' [67]: for a fixed dissipation rate, systems with a loading mechanism are closer to criticality than systems without it. The reason for this is simple: moderate loading partially compensates energy dissipation.

Finally, the most recent and exhaustive analyses by Miller and Boulter [69], Grassberger [53], and Drossel *et al* [70] conclude unambiguously, using a variety of arguments and large scale computer simulations, that the spatially extended version of the non-conserving OFC model is not critical.

As a consequence, the state-of-the-art is that, despite the apparent power-law distributions spanning for a few decades, the OFC model is not truly scale-invariant, except for its conservative limit. The question of whether real earthquakes are described or not by SOC models of this type or whether other kinds of mechanisms need to be invoked remains unsolved [71]–[73].

2.2.2. Forest-fires. The Drossel–Schwabl model [48] (see also [74]) is an improved version of an older forest-fire model [49] proposed to explain the apparent scale-invariance of real forest-fires [75] (see appendix A).

Three kinds of states are defined: $z = 0$ or empty, $z = 1$ or occupied by a tree, and $z = 2$ or burning tree. At each time step, new trees grow, at rate p, at randomly

chosen sites provided they were empty, and trees catch fire at a much smaller rate f. Fire propagates deterministically to neighboring occupied sites and, after burning, trees become empty sites. The relevant parameter is $1/\theta = p/f$,[2] and the model has been claimed to be critical provided that the double limit $f \to 0$, $p \to 0$ with $f/p \to 0$ is taken [48].

Observe that *a double separation of timescales* is imposed in the model definition: trees are born at a much faster rate than fires occur and fires propagate at a much faster pace than trees grow [48]. This is to be compared with the single timescale separation in sandpiles [48, 20]. We shall discuss later the consequences of such a double separation of timescales.

Analytical results and mappings into a branching process [76] first suggested some similarities with standard percolation models [48, 77, 78]. Given the limited analytical tractability of these models, the controversy about the existence of true criticality was mainly played out on the basis of computer simulations [48], [79, 78, 80]. For sufficiently large systems, anomalies were reported to appear; among them: (i) the repulsive character of the fixed point ($f/p = 0$), (ii) the coexistence of largely subcritical and supercritical clusters of trees, (iii) the existence of two length scales with different exponents in the system, (iv) the violation of standard scaling for the distribution of avalanche sizes $P(s)$, and (v) a pathological finite-size behavior [48], [78]–[81].

Finally, when 'massive' simulations of extremely large systems very close to the critical regime were accessible [82, 83], these anomalies turned into a lack of true critical behavior, besides the apparent scaling observed for a few decades: the self-organized stationary state of the Drossel–Schwabl model is not critical.

In the rest of the paper, we shall rationalize and generalize the above conclusion (i.e. the absence of *bona fide* criticality of earthquake and forest-fire SOC models) to generic non-conserving systems. To this end, we shall employ three different unified frameworks, as described in sections 3–5.

3. A simple mean-field approach

As already mentioned, the random-neighbor version of the OFC model has been solved analytically, with the conclusion that, except for the conserving limit, it is not critical, but generically subcritical [63, 64]. On the other hand, in a subsequent work, Pruessner and Jensen [33] considered a modified version of such a model in which, by including a different (stronger) loading mechanism, they showed that criticality can be restored in the infinite-size limit. In this section, we review the results of [33] and study the finite-size scaling of this and related mean-field systems.

The model in [33] is somewhere in between forest-fire and earthquake models. It is defined as follows: consider a set of N sites, each of them with an associated energy z_i (with z a continuous variable). As in the OFC model, three kinds of states exist: *stable*, with an energy $0 \leq z_i < 1 - \alpha$; *susceptible*, with $1 - \alpha \leq z_i < 1$; and *active* sites, with $z_i \geq 1$. The main difference with respect to the OFC model is that, between avalanches,

[2] We caution that $1/\theta$ is called θ in some papers.

driving and loading operate as independent mechanisms [33]:

- *Background loading*: in the spirit of forest-fire models [48, 80], $(1/\theta)$ sites are randomly chosen and their respective energies are increased to $1-\alpha$ (i.e. become susceptible) provided they were stable; otherwise, nothing happens.
- *Triggering of an avalanche*: A randomly chosen site, i, is activated ($z_i = 1$) provided it was susceptible.

The relaxation dynamics within avalanches is identical to that of the random-neighbor OFC model with m random neighbors: sites above threshold are emptied, $z = 0$, and the energy of its (randomly chosen) neighbors is increased by a fixed amount α. Conservation holds for $\alpha = 1/m$.

At a mean-field level, the condition for stationarity is given by the following energy balance equation [33]:

$$(1 - m\alpha) z_{\rm a} \langle s \rangle = (1 - z_{\rm s}) + (1/\theta)(1 - \alpha - z_{\rm st}) \frac{\zeta_{\rm st}}{\zeta_{\rm s}}, \tag{3}$$

where the different terms are as follows:

(i) lhs: for each relaxation event at site i, the amount of dissipated energy is $(1-m\alpha)z_i$; hence, the average dissipation during an avalanche is $(1 - m\alpha) z_{\rm a} \langle s \rangle$, where $\langle s \rangle$ is the average avalanche size and $z_{\rm a}$ the average energy of active sites.

(ii) rhs, first term: triggering increases the energy of the selected susceptible site, i, by an amount $1 - z_i$. The corresponding average increase is $1 - z_{\rm s}$, where $z_{\rm s}$ is the average energy of susceptible sites.

(iii) rhs, second term: every time the background is loaded, the energy of a stable site, i, is increased by an amount $(1 - \alpha - z_i)$. This is attempted $(1/\theta)$ times and the average number of triggering events before an avalanche is actually generated is $\zeta_{\rm st}/\zeta_{\rm s}$ (where $\zeta_{\rm st}$ and $\zeta_{\rm s}$ are the densities of stable and susceptible sites, respectively). The average increase of background energy per avalanche is, finally, $(1/\theta)(1 - \alpha - z_{\rm st})(\zeta_{\rm st}/\zeta_{\rm s})$, where $z_{\rm st}$ stands for the average energy of stable sites.

In this way, equation (3) establishes that, for a steady state to exist, the average dissipated energy should be compensated by the averaged energy increase of driving and loading. Now, imposing in equation (3) that $(1/\theta)$ diverges, one of the following two conditions must be obeyed for equation (3) to hold [33]:

(i) $(1 - m\alpha) = 0$, i.e. there is strict conservation, or
(ii) $\langle s \rangle$ diverges, which is a necessary condition for criticality.

In the second case, by studying the probability distribution function of z and using a mapping into a branching process, it has been shown not only that $\langle s \rangle$ diverges, but also that the system is *critical in the infinite-size limit* [33]. Moreover, as expected for a mean-field model, the avalanche size-distribution exponent is found to be $\tau = 3/2$ [6]–[8], [24].

However, as already pointed out in [33], for any finite system size, N, neither $1/\theta$ nor $\langle s \rangle$ is infinite. In such a case, a finite value of the parameter $1/\theta$ *must be fine-tuned* to some precise value, $1/\theta_{\rm c}(N)$, for equation (3) to hold. Such a value should diverge more slowly than N, in order to achieve the right limit $(1/\theta_{\rm c}(N))/N \to 0$ for $N \to \infty$, but there is no analytical prescription in [33] for how to fix it for each system size.

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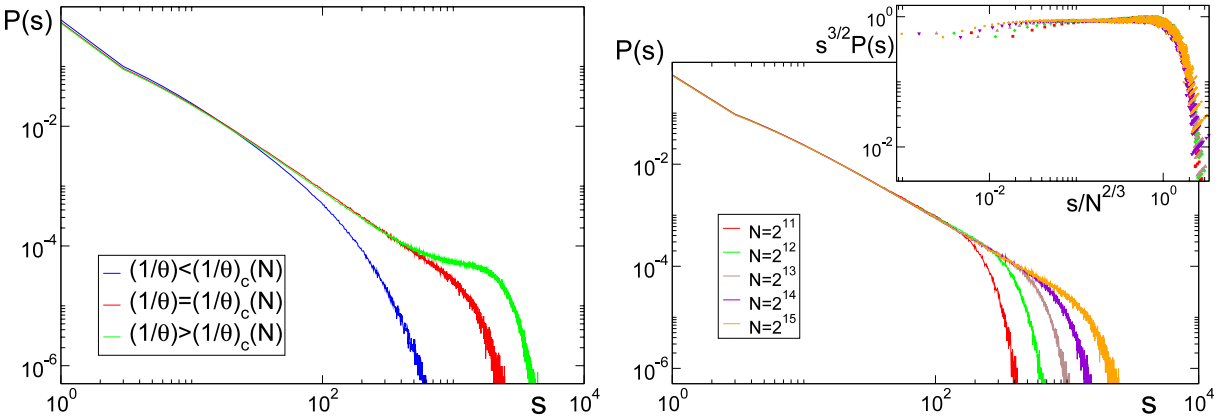

Figure 1. Left: avalanche size distribution for the random-neighbor model introduced by Pruessner and Jensen [33], for $N = 2^{15}$ and dissipation parameter $\alpha = 0.15$. Subcritical, almost critical (but still cut off by the system size) and supercritical distributions, corresponding to different values of the loading parameter ($1/\theta = 2$, 8, and 22, respectively) are shown. Right: avalanche size distribution at the *fine-tuned* critical values of $1/\theta_c(N)$ for different sizes (from 2^{11} to 2^{15}); the corresponding cut-offs grow with system size in a scale-invariant way. In the inset, the same data are collapsed onto a unique scaling curve by using equation (5).

This is a well-known problem, shared by forest-fire models, where the number of trees grown between two fires ($1/\theta$) is a parameter which needs to be carefully tuned for any finite size: too small values lead to subcritical fires, while too large values generate supercritical fires spanning the whole system (and generating a bump for large values in the size distribution).

This is graphically illustrated in figure 1 (left), where the avalanche size distribution for the Pruessner–Jensen model is plotted for a system with $N = 2^{15}$ sites, $\alpha = 0.15$, and three different values of $1/\theta(N)$; even if the values of $1/\theta(N)$ are large, the size distributions are not pure power laws: they are either subcritical (with an exponential cut-off) or supercritical (with a bump for large avalanches). Nevertheless, partial scaling is observed in any case. Given that the control parameter $1/\theta$ is an integer number, criticality cannot be tuned with arbitrary precision (especially for small system sizes) but, still, for each value of N it is possible to find an almost critical value of $1/\theta$, $1/\theta_c(N)$. In figure 1 (right) we show the avalanche size distribution at the fine-tuned critical point for different values of N. Increasing the system size, we have observed that such a value scales as $1/\theta_c(N) \sim N^{0.33(5)}$ suggesting

$$1/\theta_c(N) \sim N^{1/3}. \tag{4}$$

A collapse of the critical size distribution curves for different values of N (see the inset of figure 1) leads to $P(s, N) \sim s^{-1.5(1)} \exp(-s/N^{0.65(5)})$, compatible with

$$P(s, N) \sim s^{-3/2} \exp(-s/N^{2/3}), \tag{5}$$

where the mean-field exponent $\tau = 3/2$ is recovered.

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In section 5, we shall introduce a scaling Langevin theory for non-conserving SOC models, which explains in a straightforward way the two (formerly unknown) scaling laws, equations (4) and (5).

Let us comment on the peculiarity of the thermodynamic limit in this model (as well as in the Drossel–Schwabl forest-fire): the condition $1/\theta_c(N) \to \infty$ is automatically fulfilled by imposing the double separation of timescale discussed above, i.e. the second separation of timescales is tantamount to fine-tuning $\theta_c(\infty)$ to its critical value $\theta_c(\infty) = 0$. Note also that the infinite-size limit is somewhat pathological as the entire supercritical phase (as well as the critical point itself) collapses to a unique single point $\theta_c(\infty) = 0$. In contrast, for finite systems, a precise (not infinite) double separation of scales is required to set the system to the critical point, separating distinct subcritical and supercritical phases. This boils down to the need of fine-tuning $1/\theta_c(N)$ for each value of N to have a coherent finite-size scaling.

In contrast, in the conserving limit, large avalanches spanning the whole system are observed for any size and criticality is reached without resorting to careful tuning.

Summing up, even if the random-neighbor model studied in [33] exhibits infinite avalanches and is critical in the limit of infinite system size, it lacks a well-defined *finite-size scaling* and, therefore, it is not truly scale-invariant: for any finite system, deviations from criticality are observed if the control parameter $1/\theta(N)$ is not fine-tuned to a precise N-dependent critical value. In conclusion, this model does not qualify as a *bona fide* self-organized critical system.

In this respect, the situation for the RN-OFC model studied in [63, 64] is even worse: given that it lacks a parameter analogous to $1/\theta$ to be tuned, the degree of loading cannot be regulated and the model is generically subcritical for any non-vanishing dissipation rate even in the limit of large system size[3].

4. The self-organized mean-field approach: the self-organized branching process

In this section, we complement the mean-field approach of the previous one by exploiting the *self-organized branching process* introduced by Zapperi *et al* in [84]. First, in section 4.1, we introduce this approach for a conserving sandpile model (the Manna cellular automaton). Then, following also Zapperi *et al*, in section 4.2 we move on to analyze dissipative models, showing that they are generically subcritical. Finally, as a last step, in section 4.3 we implement a loading mechanism in the self-organized branching process which captures the essence of earthquake and forest-fire models, and explore under which circumstances the resulting model is critical.

4.1. The conserving case

In sufficiently high spatial dimensions (i.e. in the mean-field regime), avalanches in sandpiles rarely visit the same site twice; activity patterns are mostly tree-like. An avalanche can be seen as a *branching process* [76] in which an individual (*ancestor*) creates

[3] More specifically, assuming that in the steady state F is homogeneously distributed in the interval $[0, 1]$ (which is a good approximation [63]), the average loading per site between two consecutive avalanches is of the order of $1/N$, and hence the global loading of the whole system is of the order of 1. This cannot compensate the dissipation of large avalanches, and the system is damned to remain subcritical for any non-vanishing dissipation rate.

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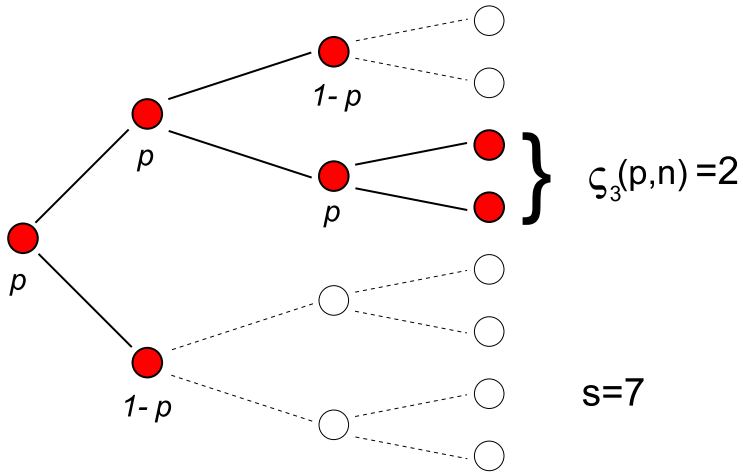

Figure 2. Left: sketch of the propagation of an avalanche in a self-organized branching process of size $N = 2^{3+1} - 1 = 15$. The avalanche has size $s = 7$ (red spots), and has reached the border (i.e. there are two occupied sites in the last generation; $\varsigma(p, n) = 2$). This figure was adapted from [84].

a fixed number k of *descendants* with probability $p(k)$. The average number of descendants per ancestor, $\sigma = \sum_k kp(k)$, is called the *branching ratio*. For $\sigma > 1$ avalanches propagate indefinitely (supercritical phase), for $\sigma < 1$ they stop after a typical number of generations (subcritical phase), while the process is critical in the marginal case, $\sigma_c = 1$ [76].

In this *static branching process*, fixing, without loss of generality, the number of descendants to $k = 2$, a given active site branches into two active ones with probability p or has no offspring with probability $1 - p$ (see figure 2), yielding $\sigma = 2p$ and a critical value $p_c = 1/2$.

Let us consider, as a simple example, the Manna sandpile with critical threshold $z_{thr} = 2$ (see appendix A), and map its mean-field version into a *self-organized branching process*, in which p itself is a dynamical variable [84]. In the Manna dynamics, each grain arriving at site i can either generate activity (energy above threshold) if previously $z_i = 1$, or not if $z_i = 0$. An avalanche in high spatial dimensions can be, therefore, seen as a branching process with $p = P(z = 1)$, i.e. the background energy density is nothing but the branching ratio [85]. A *generation* is defined as the set of sites probed for activation at each time step; after m generations there are $N = 2^{m+1} - 1$ sites involved (see figure 2). For computer simulations, we fix a maximum number of generations m and impose (to mimic boundary dissipation) that at the mth generation all grains are lost. Note that, apart from such a dissipative boundary, the bulk dynamics is conserving.

Each avalanche modifies the background in which the next avalanche is to be started; i.e. it changes the value of $P(z = 1)$. Thus, the branching probability p becomes a fluctuating variable, as illustrated in figure 3. In the left part of the figure, the value of p is plotted as a function of the avalanche number for different system sizes, while in the right figure the statistically stationary distribution of values of p is represented for various sizes; the width of the distributions decreases with increasing size and can be made as small as wanted.

To recover analytically these computational observations, let $Z(n)$ be the total number of grains in the system after n avalanches; then, $p(n) = Z(n)/N$. If $\varsigma(p, n)$ is the number

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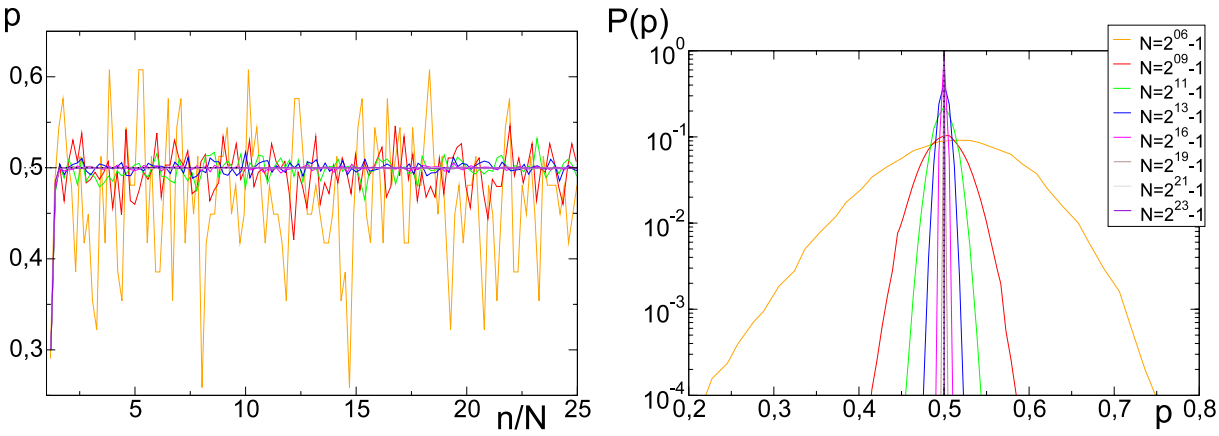

Figure 3. Left: evolution of the branching probability $p(n)$ in a computer simulation of the self-organized branching process as a function of the number of avalanches per site, for different sizes N (ranging from $2^6 - 1$ to $2^{23} - 1$). In all cases, p fluctuates around $p^* = 1/2$; the larger the system, the smaller the amplitude of fluctuations. Right: stationary probability distribution functions of values of p for various sizes (same color code in both figures); the width decreases with increasing system size and vanishes in the thermodynamic limit.

of grains dissipated at the mth (last) generation of the nth avalanche, in order to have stationarity, the following balance equation:

$$Z(n+1) = Z(n) + 1 - \varsigma(p, n) \tag{6}$$

or

$$p(n+1) = p(n) + \frac{1 - \varsigma(p, n)}{N} \tag{7}$$

must hold. The average number of grains dissipated at the boundary is $\langle \varsigma(p, n) \rangle = (2p)^m$ (2^m sites at the boundary, each one occupied with probability p [76]). For each avalanche, $\varsigma(p, n) = (2p)^m + \eta(p, n)$, where $\eta(p, n)$ is a Gaussian white noise. Plugging this into equation (7) and taking the continuum limit for n, one can formally write

$$\frac{\mathrm{d}p}{\mathrm{d}n} = \frac{1 - (2p)^m}{N} + \frac{\eta(p, n)}{N}, \tag{8}$$

whose deterministic part has a stable fixed point at $p^* = 1/2 = p_c$. Accordingly, in the thermodynamic limit (in which the effect of fluctuations can be neglected [84]), the dynamics attracts p to its critical value (see figure 3 left) and the width of the fluctuations of p around $p = p^* = p_c$ decreases with increasing system size (see figure 3 right). This simple (conserving) branching process self-organizes to its critical point.

4.2. The dissipative case

Borrowing still from Lauritsen *et al* [86], let us introduce a non-vanishing bulk dissipation rate into the Manna model and, as a consequence, into its self-organized branching process representation. Each offspring (not necessarily in the last generation) is removed from the

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system with probability ϵ; the effective branching probability becomes $q = p(1-\epsilon)$, and the criticality condition is $q = 1/2$, or

$$p_c = \frac{1}{2(1-\epsilon)}, \tag{9}$$

and $\langle \varsigma(q,n) \rangle = (2q)^m = (2p(1-\epsilon))^m$. Equation (7) transforms into

$$p(n+1) = p(n) + \frac{1 - \varsigma(q,n) - \kappa(q,n)}{N}, \tag{10}$$

where $\kappa(q,n)$ is the total amount of grains dissipated in the bulk. A simple calculation, reproduced in appendix B, leads to the following equation for the evolution of p:

$$\frac{\mathrm{d}p}{\mathrm{d}n} = A(p;\epsilon;N) + \frac{\eta(p,n)}{N}, \tag{11}$$

where

$$A(p;\epsilon;N) = \frac{1-(2p(1-\epsilon))^m}{N} - \frac{p\epsilon}{N\left[(1-p)+p\epsilon\right]} \times \left[1 + \frac{1-(2p(1-\epsilon))^{m+1}}{1-2p(1-\epsilon)} - 2\left(2p(1-\epsilon)\right)^m\right] \tag{12}$$

and, as above, the noise amplitude is dependent on N. After some simple algebra and omitting the noise term, equation (11) can be rewritten as

$$\frac{\mathrm{d}p}{\mathrm{d}n} = \frac{1-x^m}{(1-x)\ N}(1-2p), \tag{13}$$

with $x = 2p(1-\epsilon)$. It is straightforward to check that the only stable fixed point of equation (13) is $p^* = 1/2$.[4] Therefore, as $p_c = 1/2(1-\epsilon)$, the self-organized dynamics leads to a *subcritical* point $p^* = 1/2 < p_c$; the fixed-point branching ratio is less than unity, and the process propagates only for a finite number of generations for any non-vanishing value of ϵ [86]. It is only in the conserving limit, $\epsilon = 0$, that the self-organized value and the critical point coincide; otherwise there is *self-organization to a subcritical point*.

4.3. Dissipation and loading

In a recent series of papers [34], Juanico *et al* introduced a background dynamics into the self-organized branching model with dissipation. These authors consider a dissipative version of the Manna sandpile rules in the following way: with probability α, an active site transfers two grains to two different randomly chosen neighbors; with probability β, only one grain is transferred to one neighbor while the other one is dissipated; finally, with probability $\epsilon = 1-\alpha-\beta$, the two toppling grains are dissipated. For simplicity and to allow easy comparison with the calculations above, we fix $\beta = 0$; the critical branching probability becomes $p_c = 1/2(1-\epsilon) = 1/2\alpha$.

[4] Note that, for $p = 1/2$, typically there is just one surviving path arriving at the mth generation; for this the probability of not dissipating until the end is $(1-\epsilon)^m$, while the corresponding boundary dissipation can be seen to be $1-(1-\epsilon)^m$; hence, they sum to 1 and cancel the contribution of the starting seed in the balance equation.

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A *background dynamics* is implemented in [34] by introducing a rate λ for a stable site to be turned into a critical one ($z = 0 \to z = 1$), and a rate ϑ for the opposite transformation. From now on, and without loss of generality, we restrict ourselves to the 'loading' process (which increases the energy) and fix $\vartheta = 0$. Neglecting the noise term, it is straightforward to arrive at the following evolution equation for p:

$$\frac{\mathrm{d}p}{\mathrm{d}n} = (1-p)\lambda + \frac{1-x^m}{(1-x)N}(1-2p), \tag{14}$$

with $x = 2p\alpha$. It has a stable fixed point at

$$\lambda = \frac{x^m - 1}{(1-x)N}\frac{1-2p}{1-p}. \tag{15}$$

Note that, at p_c, x is equal to 1 and, thus, if λ is *fine-tuned* to

$$\lambda_\mathrm{c} = \frac{m}{N}\frac{2p_\mathrm{c}-1}{1-p_\mathrm{c}}, \tag{16}$$

then the fixed point of equation (14) becomes p_c, i.e. the pair $(\lambda_\mathrm{c}, p_\mathrm{c})$, with λ_c given by equation (16) and $p_\mathrm{c} = 1/2\alpha$, fulfills equation (15) and the self-organized branching process becomes critical. On the other hand, fixing $\lambda > \lambda_\mathrm{c}$ (resp. $\lambda < \lambda_\mathrm{c}$) the fixed point becomes supercritical (resp. subcritical) as illustrated in figure 4. In conclusion, by *carefully tuning* the loading parameter to exactly compensate the effect of dissipation, the system self-organizes to its critical point. This process, requiring an explicit parameter tuning, *cannot be called bona fide self-organization*. Finally, note that both the critical loading parameter λ_c and the driving rate $1/N$ vanish in the large system size limit ($m \to \infty, N \to \infty$), while the ratio 'loading over driving' ($\lambda_\mathrm{c} = m(2p_\mathrm{c}-1)/(1-p_\mathrm{c})$) diverges. These conditions are analogous to those usually imposed on forest-fire models ($p, f \to 0, p/f \to \infty$). The calculation above illustrates that such conditions are necessary but not sufficient for achieving criticality in the absence of a conservation law in any finite system: a size-dependent fine-tuning is also required.

Even if Juanico *et al* claim to have designed critical self-organized models, all branching processes studied by them [34] are similar in spirit to the example above: close inspection of their rules reveals an underlying parameter fine-tuning in all the different variations they study. For example, for the full model described above (i.e. with $\beta \neq \vartheta \neq 0$), Juanico *et al* explicitly make the 'convenient' choice of parameters

$$\frac{\vartheta}{\lambda} = 2\alpha + \beta - 1 = \frac{1}{p_\mathrm{c}} - 1, \tag{17}$$

analogous to equation (16). Actually, in [34], equation (17) is obtained by fine-tuning the N-independent terms to the critical point, while the last term in the flow equation, proportional to $1/N$, is neglected. Not surprisingly, the resulting model converges to its critical point in the thermodynamical limit; but, requiring careful tuning, it cannot be properly called critical self-organization.

Summing up, in this section we have illustrated that the conserving self-organized branching process shows asymptotically critical dynamics, while its dissipative counterpart self-organizes to a subcritical point. Introducing a loading mechanism, dissipation can be compensated and criticality restored *if and only if* the loading rate is fine-tuned to a precise size-dependent value. Otherwise, the system self-organizes generically either to a subcritical point or to supercritical one.

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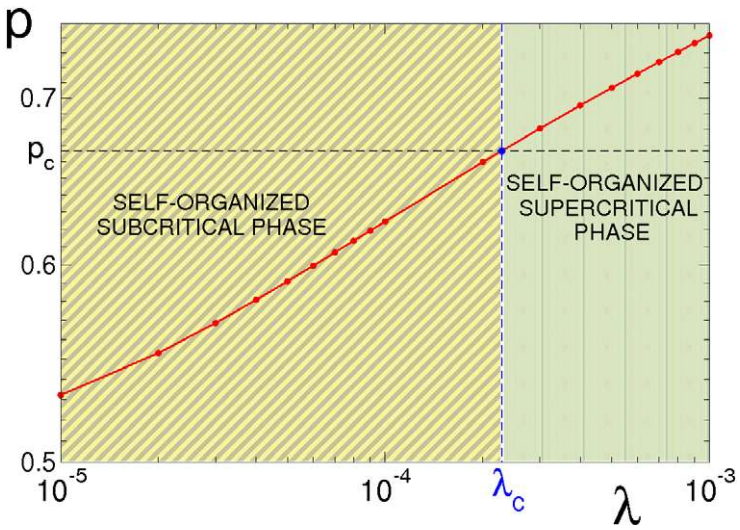

Figure 4. Fixed point of the evolution of equation (14) in the self-organized branching process as a function of λ, for a fixed dissipation parameter $\epsilon = 0.25$. Note that only for $\lambda = \lambda_c(\epsilon) \approx 2.288 \times 10^{-4}$ does the fixed point correspond to the critical value of p, $p_c(\epsilon) = 1/2(1-\epsilon) = 2/3$. Otherwise, the system self-organizes to a subcritical or to a supercritical point.

5. A full description: Langevin theory of SOC

Having already studied two different mean-field-like approaches, in this section we discuss the complete Langevin theory of self-organizing systems. This theory explains the origin of the underlying critical point beyond mean-field level, its universality in any dimension, as well as the key mechanism producing SOC.

First (section 5.1), we review the existing Langevin picture of conserving SOC models based on an absorbing phase transition. Then (section 5.2), we introduce bulk dissipation and extend the theory to non-conserving systems. Finally (section 5.3), a loading mechanism is introduced to elucidate the behavior of non-conserving self-organized systems. We emphasize the substantial differences with respect to the conserving case.

5.1. Langevin theory with conservation

The main idea for constructing a stochastic theory of conserving SOC is 'regularizing' sandpiles (and related systems) by switching off both boundary dissipation and slow driving [21, 24, 39, 40, 87]. In this way, the total amount of sand or 'energy', E, in the pile becomes a conserved quantity, and can be retained as a control parameter. Indeed, in the so-defined '*fixed-energy ensemble*' and for large values of E, the system is in an *active phase* with never-ending relaxation events. In contrast, for small values of E it gets trapped with certainty into some *absorbing state* [88] where all dynamics ceases (i.e. all sites are below threshold). Separating these two regimes there is a critical energy, E_c, at which an absorbing phase transition takes place. In this way self-organized criticality is related to a standard phase transition [21, 89, 90].

It has been shown [21, 40, 91] that such a critical value, E_c, coincides with the stationary energy density to which the original self-organizing sandpile converges. In

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other words, the energy around which the standard sandpile (i.e. including slow driving and boundary dissipation) fluctuates is the critical point of the 'fixed-energy sandpile'. Furthermore, the width of fluctuations becomes smaller and smaller with increasing system size (see the left part of figure 5), guaranteeing that in the thermodynamic limit the original sandpile self-organizes to criticality.

This connection between 'driven/dissipative' systems and their 'fixed-energy' counterparts permits us to relate avalanche exponents to standard critical exponents (see [92] for scaling relations) and to rationalize the critical properties of SOC systems from the broader point of view of the standard non-equilibrium (absorbing state) phase transition [21, 40, 93].

Using this approach, it has been established that stochastic sandpiles *do not* belong to the robust *directed percolation* (DP) *class*, prominent among absorbing phase transitions, but to the so-called 'conserving-DP' (C-DP hereafter) or Manna class. This class is characterized by the coupling of activity to a static conserved field representing the conservation of sand grains [21, 40, 94, 95]. The field theory or set of mesoscopic Langevin equations proposed on phenomenological grounds for describing this class is

$$\begin{aligned} \partial_t \rho(\vec{x},t) &= a\rho - b\rho^2 + \omega\rho E + D\nabla^2\rho + \sigma\sqrt{\rho}\eta(\vec{x},t), \\ \partial_t E(\vec{x},t) &= D_E\nabla^2\rho(\vec{x},t), \end{aligned} \tag{18}$$

where $\rho(\vec{x},t)$ is the activity field (characterizing the density of grains above threshold), $E(\vec{x},t)$ is the locally conserved energy field, a, b, ω, D, σ, and D_E are parameters and $\eta(\vec{x},t)$ is a Gaussian white noise. Some dependences on $(\vec{x},t)$ have been omitted to lighten the notational burden.

Note that in the C-DP class, two fields are required for a Langevin representation: the activity field representing grains/energy/force *above threshold* and the background or energy field describing the local amount of grains/energy/force. Nevertheless, it is possible to stick to a single-field description by integrating out the energy equation. This generates two extra terms for the activity equation

$$\omega\rho E(\vec{x},0) - \omega\rho\int_0^t \mathrm{d}t'\,\nabla^2\rho(t'). \tag{19}$$

The second, history-dependent (non-Markovian) term describes the tendency of sites that have been less active than their neighbors in the past to be more susceptible to activation (e.g. in a sandpile, if the neighbors of a given site have toppled, the site is very likely to overcome the threshold). The single equation for the activity reads

$$\partial_t\rho(\vec{x},t) = (a+\omega E(\vec{x},0))\rho - b\rho^2 - \omega\rho\int_0^t \mathrm{d}t'\,\nabla^2\rho(\vec{x},t') + D\nabla^2\rho + \sigma\sqrt{\rho}\eta(\vec{x},t); \tag{20}$$

non-Markovianity is the price to pay for removing the energy field[5].

The C-DP class described either by equations (18) or by equation (20) has a critical dimension $d_c = 4$ and embraces not only stochastic sandpiles, but also (among other examples) some conserving reaction–diffusion systems, for which the equations above can be explicitly derived from the microscopic dynamics [94, 95].

[5] The term, $\omega\rho E(\vec{x},0)$ contributes to the linear ('mass') term. Even if this term looks like a quenched variable, the local 'mass' for the activity is not a quenched variable at all; the term $\omega\rho[\int_0^t \mathrm{d}t'\,\nabla^2\rho(t')]$ converts the effective 'mass' at each site, $a+\omega E(\vec{x},t)$, into an annealed variable, evolving with the dynamics.

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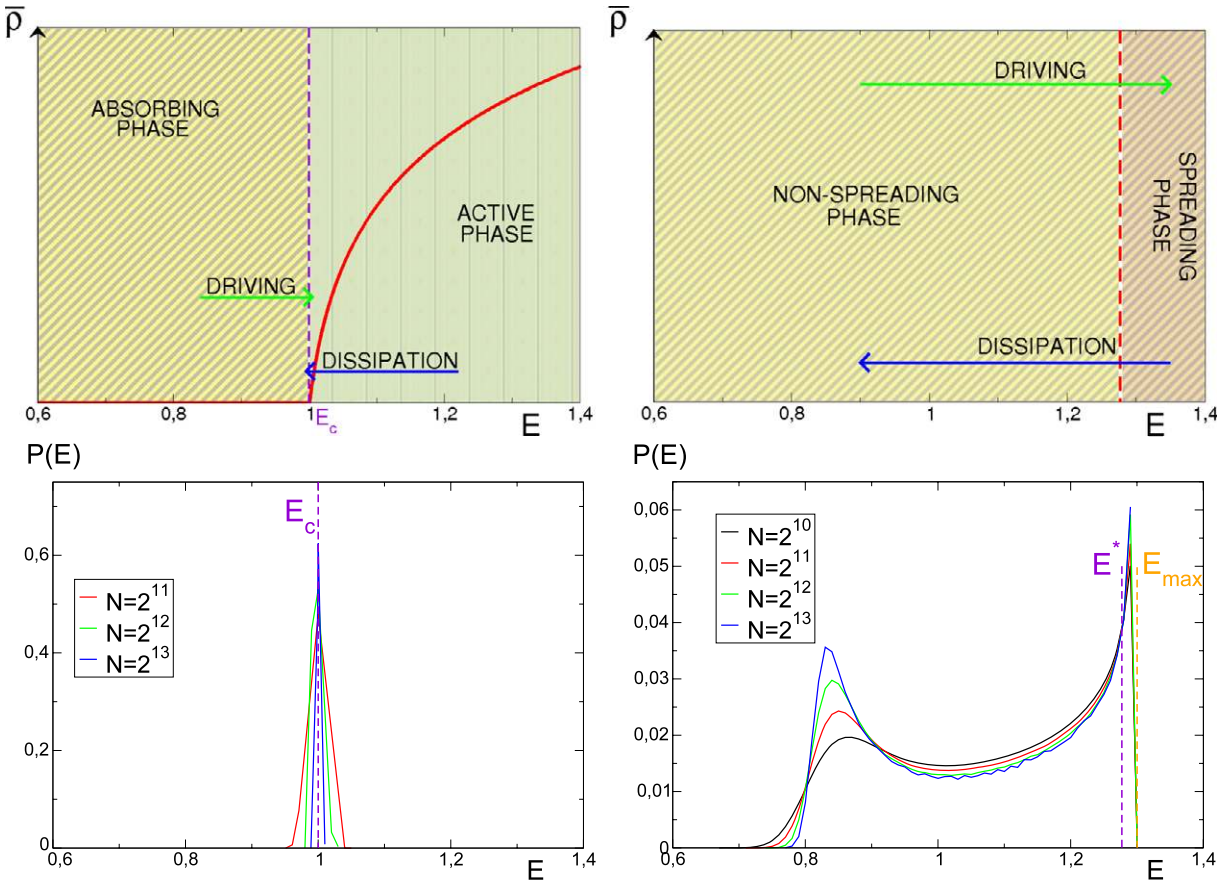

Figure 5. Upper part: sketched diagrams illustrating the mechanism of self-organization for conserving dynamics (left panels) and for non-conserving dynamics with loading (right panels). Left, upper: in the 'fixed-energy' ensemble, there are an active phase and an absorbing phase, separated by a critical point, E_c. Slow driving and dissipation make sandpiles (and related conserving systems) fluctuate around their associated fixed-energy counterpart critical point. Left, lower: the distribution of E during avalanches is plotted for simulations of the conserving Langevin theory, equations (18), using various system sizes ($2^{11}, 2^{12}$, and 2^{13}). Other parameter values are $a = 0.423$, $b = \omega = 1$, $D = D_2 = 0.25$, $\sigma = \sqrt{2}$; the distributions become progressively peaked around E_c upon increasing the system size (the value of E has been normalized with E_c for each size). Right: plots analogous to their counterparts to the left, but for non-conserving systems. Right, upper: in this case, there is no active phase, but just a phase in which perturbations/avalanches spread. Results shown lower right correspond to simulations of equations (22), using equations (25) and (26), with $E_{max} = 1.3$ (the rest of the parameters are as above). Observe the much broader distributions appearing in this case: the system hovers around the spreading critical point with large excursions into both the supercritical and the subcritical phases.

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Equations (18) can be studied either (i) in spatially extended systems, (ii) using random neighbors, or (iii) in a globally, all-to-all, coupled version in which the Laplacian is replaced by $\bar{\rho} - \rho(\vec{x}, t) \simeq (1/N) \sum_{y \neq x} [\rho(\vec{y}, t) - \rho(\vec{x}, t)]$. These last two are useful for constructing mean-field approximations of the full (spatially extended) theory.

5.1.1. The relation with other universality classes. First, note that the well-known theory for *directed percolation* (i.e. the Reggeon field theory [88]) is recovered upon fixing $\omega = 0$ in equations (18). The additional conserved field turns out to be a relevant perturbation altering the critical behavior of systems in the DP class [21, 40, 95].

On the other hand, for the sake of completeness, let us just briefly mention that the C-DP class is fully equivalent to the pinning/depinning transition of interfaces in random media, i.e. the *quenched Edwards–Wilkinson* case [22, 96, 97]. The absorbing (resp. active) phase maps into the pinned (resp. depinned) one. Exploiting the mapping between these two descriptions (of a unique underlying physics), critical exponents for equations (18) can be deduced from existing renormalization group results for interfaces [98].

It is worth stressing that, in terms of interfacial models, *conservation of energy is equivalent to interface translation invariance*; for instance, a dissipative term like $-\epsilon\rho$ introduced in equations (18) would map into a term $-\epsilon h$ (where $h = h(\vec{x}, t)$ is the interface height) in the quenched Edwards–Wilkinson equation, which breaks such an invariance.

In this respect, recent experimental evidence of self-organized critical behavior (including finite-size scaling), obtained for avalanches in type-II superconductors [19], gives critical exponents compatible with those of the C-DP class. Barkhausen noise [99] and acoustic emission in fracture [100] are other related examples.

5.1.2. Conserved SOC as an absorbing state transition. Within this framework, the way self-organized criticality works is as follows (see the left part of figure 5) [21, 40]: if the sandpile is in its absorbing phase ($E < E_c$, where E stands for the spatially averaged value of $E(\vec{x})$ in the steady state) then, owing to the driving mechanism, the energy is slowly increased until, eventually, the *active phase* ($E > E_c$) is reached. At this point avalanches are triggered and they re-structure the sandpile energy configuration. Avalanches may dissipate energy at the open boundaries, until eventually the system falls back into an *absorbing state*, the avalanche stops, and slow driving acts again, re-starting the cycle. In this way, the sandpile is expected to fluctuate around its critical point, E_c, with excursions to either the active or the absorbing phase as sketched in the upper left part of figure 5.

To have a numerical confirmation of this, equations (18) can be interpreted as in SOC, i.e. one can implement slow driving and dissipation at infinitely separated timescales, and integrate the equation using the efficient algorithm introduced in [101]. In particular, one considers an absorbing configuration ($\rho(\vec{x}, t = 0) = 0$) and open boundaries, then adds a small amount of activity/energy, δ, to a given site, $\vec{x_0}$:

$$\rho(\vec{x_0}, t = 0) \to \delta, \qquad E(\vec{x_0}, t = 0) \to E(\vec{x_0}, t = 0) + \delta; \tag{21}$$

this generates an avalanche, which evolves according to equations (18). Iterating this process, one obtains the distribution of values E sampled during avalanches, shown in the lower left panel of figure 5. Observe that, as dissipation and driving become arbitrarily small on increasing the system size (actually, they are infinitesimally small in the thermodynamic limit), the degree of penetration into the active and absorbing phases

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is arbitrarily small, the distribution of E becomes more and more peaked, and the system is arbitrarily close to its critical point. Moreover, the avalanche exponents measured by means of numerical integration of equations (18) at such a steady state coincide with (or can be related to) those obtained by performing standard fixed-energy simulations of equations (18) at its critical point [101, 97]. Note the obvious analogies between this picture and the self-organized branching process described above: E is the equivalent of p, i.e. the self-organized control parameter; the critical point E_c corresponds to the critical branching probability p_c.

The advantage of equations (18) as a theory for SOC with respect to the self-organized branching process is that, while this latter is a mean-field theory explaining qualitatively self-organization but failing to justify critical exponents in spatially extended systems, equations (18) give a full theory including fluctuations and spatial dimensionality. It provides accurate estimates for avalanche exponents in any dimension and opens the door to field theoretical analyses. Furthermore, equations (18), considered for a random-neighbor or all-to-all coupling, constitute a sound mean-field description of conserving SOC, equivalent to those in the preceding sections.

To end, note, once again, the essential role played by conservation in this theory. The underlying phase diagram sketched in the left part of figure 5 relies on the averaged energy being a control parameter. If there was a non-vanishing bulk dissipation, the energy would change continuously during avalanche evolution. In section 5.2, we shall explore how this affects the picture of SOC based on an absorbing phase transition.

5.2. Langevin theory with bulk dissipation

To tackle the problem of non-conservation within the absorbing state Langevin framework, we need to modify equations (18) to allow for bulk dissipation. Introducing in equations (18) the leading dissipative term, $-\epsilon\rho(\vec{x},t)$, and neglecting higher order corrections, the resulting set of equations becomes

$$\begin{aligned}\partial_t\rho(\vec{x},t) &= a\rho - b\rho^2 + \omega\rho E(\vec{x},t) + D\nabla^2\rho + \sigma\sqrt{\rho}\eta(\vec{x},t),\\ \partial_t E(\vec{x},t) &= D_E\nabla^2\rho(\vec{x},t) - \boldsymbol{\epsilon\rho}(\boldsymbol{\vec{x}},\boldsymbol{t}),\end{aligned} \tag{22}$$

which is, obviously, non-conserving owing to the activity-dependent energy leakage.

Integrating over time the equation for the energy field, the following extra terms for the activity equation are generated:

$$\omega\rho\left[E(\vec{x},0) + \int_0^t \mathrm{d}t'\,\nabla^2\rho(\vec{x},t') - \epsilon\int_0^t \mathrm{d}t'\,\rho(\vec{x},t')\right]. \tag{23}$$

The second term, dominant in equation (20), becomes a higher order correction here, i.e. it is irrelevant in the renormalization group sense as compared with the third, non-Markovian, term. The last one is well known to be the leading non-linearity in the *dynamical percolation* universality class [102]. From this perspective, it is no wonder that the critical behavior of some non-conserving SOC models (e.g. forest-fires) has been related to (dynamical) percolation in the literature [103]. Such a class, whose full (one-field) Langevin equation is

$$\partial_t\rho(\vec{x},t) = (a + \boldsymbol{\omega E}(\boldsymbol{\vec{x}},\boldsymbol{0}))\rho - b\rho^2 - \boldsymbol{\epsilon\omega\rho}\int_0^{\boldsymbol{t}}\mathbf{d}\boldsymbol{t}'\,\boldsymbol{\rho}(\boldsymbol{t}') + D\nabla^2\rho + \sigma\sqrt{\rho}\eta(\vec{x},t), \tag{24}$$

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describes the spreading properties of epidemics with immunization, etching of disordered solids [104], and some aspects of spreading in systems with many absorbing states *without a conservation law* (see [105] for more details). The term $-\epsilon\omega\rho\int_0^t \mathrm{d}t'\,\rho(t')$ implies that regions already visited by activity become less prompt at becoming active in the future. Owing to this term, equations (22) cannot sustain an active phase. However, even if it lacks a stable active phase, it exhibits a spreading phase transition separating a phase in which seeds of activity propagate indefinitely (in the form of rings of expanding activity; i.e. defining an 'annular growth' phase) from an absorbing phase in which they do not [102] (see the diagram in the right part of figure 5).

This spreading transition, whose critical dimension is $d_c = 6$, is controlled by the initial state in which the seed of activity is placed; indeed, the initial energy, $\omega E(\vec{x}, t = 0)\rho(\vec{x}, t)$, is a mass term in equation (23). If the initial value of E is large enough (i.e. for a favorable environment), then avalanches tend to propagate, while if it is small, they do not. Separating these two regimes there is a critical point for spreading propagation at some value $E^*(\vec{x}, 0)$.

In summary, the introduction of a non-vanishing dissipation rate affects in a relevant way the critical behavior of self-organizing systems; depending on the initial condition, the dissipative equations (22) can be in the propagating/supercritical or in the non-propagating/subcritical phase of a dynamical percolation phase transition.

5.3. Full theory: dissipation and loading

Now, we are in a good position to understand in depth the role of the loading mechanism in dissipative models of SOC, within the Langevin framework. To do so, let us complement equations (22) with a specific prescription for how to change the background energy field between avalanches, i.e. let us consider a loading rule, for instance

$$E(\vec{x}, t = 0) \to E(\vec{x}, t = 0) + \gamma(E_{\max} - E) \tag{25}$$

(where $E_{\max}$ is a parameter and E the average energy in the system), and a driving rule:

$$\rho(\vec{x_0}, t = 0) \to \delta. \tag{26}$$

Equations (25) and (26) define one possible loading mechanism for equations (22); other choices are, of course, possible (for instance, the loading mechanism could also act 'during' avalanches). The results presented in what follows are generic, essentially independent of such a choice.

As shown above, equations (22) can sustain avalanches propagating indefinitely (up to system size), provided that the initial energy is large enough. Therefore, considering large values of γ or of $E_{\max}$ in equation (25), the system becomes supercritical for avalanche propagation. In contrast, small initial densities lead to subcritical propagation.

To illustrate this and the forthcoming discussion, we have performed computer simulations of equations (22) using the parameter values specified in the caption of figure 5, and complemented these with the loading and driving rules of equations (25) and (26). For the sake of simplicity, we have considered N sites with an *all-to-all* (mean-field) coupling. To check the robustness of our conclusions we have also studied a *random-neighbor version*, obtaining very similar results (not shown). As before, the equation has been integrated using the algorithm introduced in [101].

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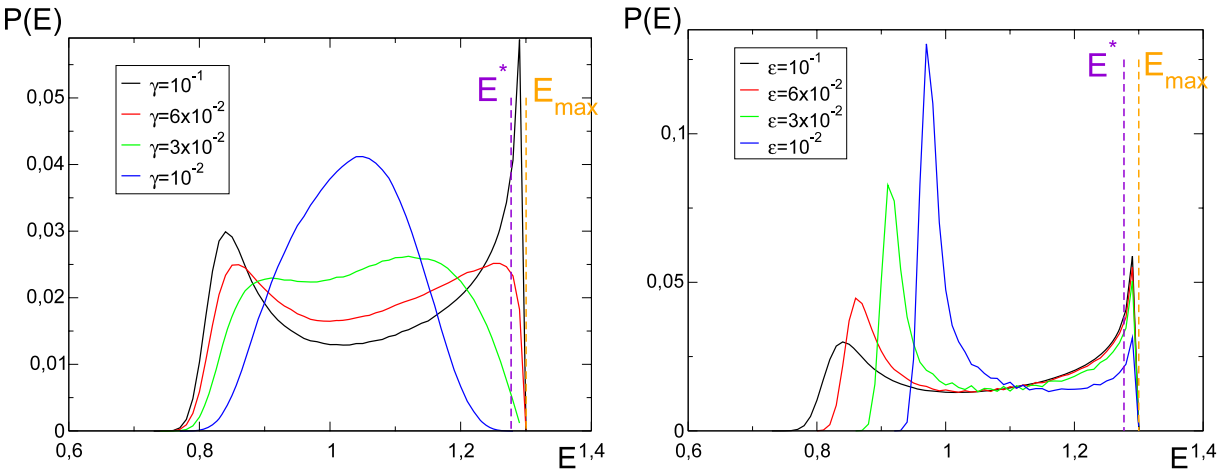

Figure 6. Histograms of E, sampled during avalanches, in a numerical integration of equations (22) for (left) $\epsilon = 0.1$ and different values of the loading parameter γ (as soon as $\gamma > \gamma_c$ the distribution develops a double-peak shape) and for (right) $\gamma = 0.1$ and different values of ϵ. Other parameter values are as in figure 5.

For a dissipation parameter $\epsilon = 0.1$ and fixing $E_{\max} = 1.3$, we find a critical point for spreading at some value of γ, $\gamma_c \approx 2 \times 10^{-2}$, which generates, on average, an initial energy $E^* \approx 1.277(5)$. At such a critical value, power laws for avalanche and spreading exponents are obtained. In complete analogy with figure 1, for smaller values of γ (and hence, smaller values of the initial average energy density) the avalanche size distribution has an exponential cut-off (subcritical), while for larger values the distribution develops a bump for large sizes (supercritical; results not shown).

We have constructed histograms of the average energy by sampling E during avalanches in computer simulations (see figure 6). Typically, for short times the system is in the right side of the distribution and, as the avalanche proceeds and dissipation acts, E moves progressively leftward. This shifting generates a *broad distribution of energy values for any system size*. After the avalanche stops, the system is 'loaded' again (equation (25)), a new avalanche is triggered (equation (26)), and so on. In this way, the system is kept hovering around the critical point, with a broad distribution of values, as illustrated in figures 5 and 6.

In figure 6 (left panel), histograms for $\epsilon = 0.1$ and various values of γ are plotted. As long as the loading is sufficiently strong ($\gamma > \gamma_c \approx 2 \times 10^{-2}$) the distribution develops a double-peak structure overlapping with both the propagating and the subcritical phases. However, for smaller values of γ ($\gamma = 10^{-2}$ in the figure), the loading is too small, and the histogram overlaps only with the subcritical phase. In figure 6 (right panel), histograms for $\gamma = 0.1$ and various values of ϵ are plotted; in all cases, there is a double-peak structure. Observe that, for large dissipation rates, the system gets deeper into the absorbing phase.

It is important to remark that (as illustrated in figure 5) the dynamics is rather different from its conserving counterpart: while, in the conserving case, fluctuations around E_c decrease in amplitude with system size (figure 5, left panels), in the non-conserving case the histograms remain broad, even in the thermodynamical limit (see figure 5, right panels); i.e. large variations around the critical spreading point persist for any system size.

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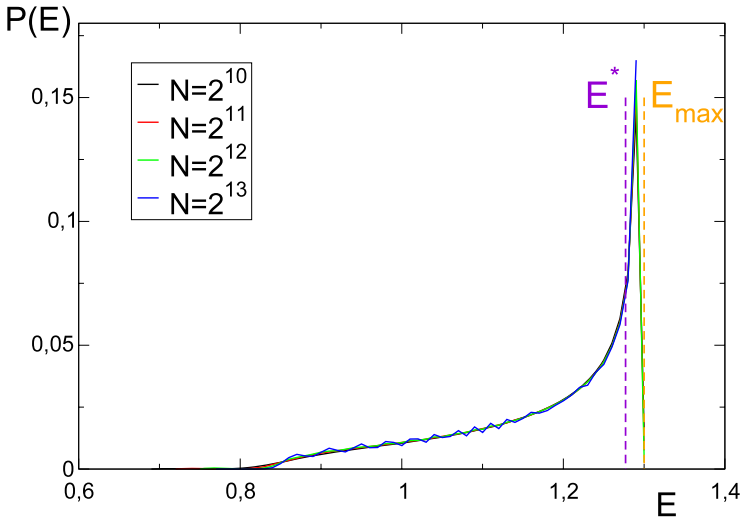

Figure 7. Histograms showing the distribution of initial conditions, E, for avalanche propagation for various system sizes in numerical simulations of equations (22) endowed with the loading mechanism equation (25) (parameter values are as in figure 5). Observe that the distribution remains broad in the thermodynamic limit.

We caution the reader that, within an avalanche, the process is *not stationary* (energy decreases) and, therefore, the histograms shown in figures 5 and 6 cannot be properly interpreted as probability distribution functions.

For this reason, we have also constructed *stationary* (steady state) histograms for (i) the distribution of the average initial energy for avalanches and (ii) the distribution of E values after avalanches. Any of these can be used, as well, to illustrate the differences from the conserving case. For example, figure 7 shows that the background in which avalanches are started is generically non-critical, but is broadly distributed around the critical point. Using this information, one can make the educated guess that the associated avalanche size distribution (or any other quantity measured for avalanches/spreading), with such a distribution of initial conditions, will be a convolution of different subcritical and supercritical curves weighted with the above distribution of initial energies. Let us emphasize the lack of any mechanism tuning the system to criticality: the energy is initially set to some arbitrary value (controlled by the parameter γ). If and only if the initial density is *fine-tuned* to the critical value for spreading of activity, $E^*(\vec{x}, 0)$, the system is at the critical point for avalanche spreading. Otherwise, for larger values of γ the system is initially supercritical, while for smaller values it is subcritical.

In conclusion, fine-tuning of the loading mechanism is required to have critical spreading in non-conserving systems, and it is controlled by a dynamical percolation critical point. The important point to stress is that, even if the initial condition is not critical for generic values of γ and $E_{\max}$, the mechanism of 'hovering around the critical point', illustrated in figure 5 (right panels), keeps the dynamics effectively not far from criticality, but not at criticality, for a broad range of parameter values. Furthermore, in contrast to the case for conservation (figure 5, left panels), the energy histograms do not tend to a delta-peak function for large system sizes. Large excursions into both the supercritical and the subcritical phases persist in the thermodynamic limit.

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5.3.1. Revisiting the mean-field Pruessner–Jensen model. Using this insight, we can now tackle an open question: how does $1/\theta_c(N)$ scale at criticality in the model discussed in section 3?

As $1/\theta_c(N)$ is a background energy, it corresponds to a mass term in the Langevin equation for the activity, equation (23). Therefore, to preserve scale-invariance, it needs to be scaled with N as a distance to the critical point and, therefore,

$$1/\theta_c(N) \sim \xi^{1/\nu}, \tag{27}$$

where ξ is the correlation length and ν the correlation length exponent, whose mean-field value in the dynamical percolation class is $\nu = 1/2$ [92].

At the upper critical dimension, $d_c = 6$ (where hyperscaling mean-field relations are expected to hold), ξ is limited by the system linear size, $N^{1/d} = N^{1/6}$, where N is the volume (i.e. total number of sites):

$$\xi \sim N^{1/6}. \tag{28}$$

Putting together these two last equations, we obtain the mean-field scaling result,

$$1/\theta_c(N) \sim N^{1/\nu d} = N^{1/3}. \tag{29}$$

This is in excellent agreement with the empirical result reported in equation (4). Analogously, the cut-off of the avalanche size distribution, s_c, needs to scale as

$$s_c \sim \xi^{D_f} \sim N^{D_f/d_c} = N^{2/3}, \tag{30}$$

where we have used the fractal dimension, $D_f = 4$, for mean-field dynamical percolation [92]. Again, this prediction is in perfect agreement with the numerical results for equation (5).

In conclusion, the loading parameter needs to be fine-tuned for each finite size to have true scale-invariance; its scaling is inherited from a dynamical percolation critical point.

5.3.2. Spatially extended systems. The numerical results reported in this section correspond, as already stressed, to an all-to-all (as well as random neighbors) coupling in equations (22). On the other hand, qualitatively similar results can be obtained for spatially extended systems: i.e. a broad distribution of the spatially averaged energy, hovering around the critical point. The main difference is that, obviously, the background energy becomes heterogeneously distributed in space and, therefore, the situation becomes much more involved.

Equations (22) spontaneously generate regions with higher and with lower values of E, which have different propensities to activity propagation: there are *locally* supercritical and subcritical regions. Patches where avalanches have passed are typically less likely to propagate new activity owing to the non-Markovian term in equations (22). The size distribution of such patches is, accordingly, inherited from the avalanche size distribution, creating a complex landscape for further avalanche propagation. This scenario is, of course, more complex than the mean-field one discussed above, but the essence of the phenomenology described remains unaltered: the (local) control parameter hovers around a dynamical percolation critical point, with fluctuations that do not vanish in the large system size limit.

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Observe that, in order to tune the system to criticality (as was done in the mean-field case) one would need in this case to define a more complicated loading mechanism which should get rid of the dynamically generated heterogeneities, leading to a homogeneous initial energy state, tuned exactly to its critical value. Using the language of [21], one needs to 'hire a babysitter' (or a 'gardener' using the forest-fire terminology [90]) to keep the spatially extended system sitting (everywhere) at criticality. Once such an efficient babysitter is at work, the initial condition is always at the (dynamical percolation) critical point in any dimension.

From this perspective, our theory provides additional support for the claim in [83] that the critical density of trees in a forest-fire model should coincide with the percolation critical density, i.e. in order to observe critical propagation in the forest-fire model one should tune the initial background (number of trees) to the corresponding dynamical percolation critical density.

As already pointed out by Grassberger some time ago [83], in the absence of an efficient gardener taking care of local fine-tuning, partial power laws are still observed in the *forest-fire model*. This is due to the existence of patches with different densities of trees, which appear with a broad spectrum of sizes. Each such patch lies at a different distance from the critical point. The convolution of avalanches propagating with such a variety of initial conditions originates a complex *pseudo-scaling picture* which, obviously, does not correspond to strict criticality. A similar picture applies also to the (open boundaries) OFC earthquake model [70], and to self-organized models of neural activity (as will be illustrated in a forthcoming paper).

Summing up: in this section, we have reviewed the standard absorbing state phase transition picture of SOC, underlining the special role played by conservation. Then, we introduced a bulk dissipation term and illustrated that the active phase disappears and the universality class of the spreading phase transition involved is changed from C-DP to *dynamical percolation*. Dissipation needs to be compensated by a loading mechanism (which controls the initial conditions in which avalanches are started) to keep the energy balance. If and only if the loading mechanism is perfectly fine-tuned to generate a precise initial energy density, true criticality is observed. Otherwise, the system just hovers around a dynamical percolation critical point, with large excursions into the propagating and the absorbing phases. In contrast to the situation for the conserving case, such fluctuations do not disappear in the thermodynamic limit. *Strictly speaking*, this mechanism of self-organization cannot be called critical; we propose to refer to it as *self-organized quasi-criticality* (SOqC). Last but not least, our approach provides a way to rationalize the finite-size scaling properties of non-conserving self-organized systems, such as earthquake or forest-fire models, and has allowed us to derive a number of previously unknown scaling relations.

6. Concluding remarks

We have shown, by using different levels of description, that non-conserving models of self-organized criticality, such as earthquake and forest-fire models, are not truly critical.

First, we have studied a simple mean-field theory, based on an energy balance equation, for a non-conserving model introduced by Pruessner and Jensen. It permits us to illustrate that, even if one can construct non-conserving models that seem critical

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in the thermodynamic limit, there is no systematic way to have a coherent finite-size scaling description of them: a precise *fine-tuning is required* for any finite size to observe criticality and to approach the thermodynamic limit in a scale-invariant way.

Second, we have revisited the mapping of high dimensional (mean-field) avalanching systems into a self-organized branching process, introduced by Zapperi *et al* some years ago. The underlying idea is that in high dimensions avalanches do not visit a given site twice and they can be described as a branching process, whose branching probability depends on the energy background. This allows us to write an evolution equation for the branching probability for models with slow driving and dissipation. While in the conserving case the branching probability converges to its critical value, in the presence of bulk dissipation the convergence is towards a *subcritical* point. Introducing a loading mechanism (which mimics the growth of new trees in forest-fire automata or the continuous building up of stress in earthquake models), we have shown that the fixed point towards which the system self-organizes can be either critical, subcritical, or supercritical. At variance with previous claims, a fine-tuning of the loading mechanism is required to reach criticality within this approach.

Third, we have introduced a full stochastic description of SOC systems in terms of Langevin equations. We have reviewed how conserving systems self-organize to a critical point with well-known critical exponents (in the conserved-directed-percolation universality class). In contrast, for dissipative systems with a loading term, the dynamics is found to *hover around* a critical point, with large excursions into the absorbing and the propagating phases, which do not disappear in the large system size limit. Therefore, such systems are not generically critical. Still, some traits of the underlying dynamical percolation critical point can be observed, depending on the loading parameter and system size, i.e. depending on how large the excursions into the absorbing/propagating phase are.

All these three approaches provide overwhelming evidence that conserving dynamics is a necessary condition for observing self-organization to criticality. However, in non-conserving (dissipative) systems equipped with a loading mechanism, a fine-tuning of a loading parameter is required to have the system sitting at a critical point. Otherwise, the system just hovers around a critical point, with broadly distributed fluctuations which do not disappear in the thermodynamic limit: for a broad range of parameters, non-conserving systems can be fluctuating in the vicinity of a critical point, but not at the critical point. We propose to call this self-organized quasi-criticality.

This conclusion extends to some other non-conserving models of SOC such as those for synchronization of integrate-and-fire oscillators described in [62, 106, 107], the model in [108], and the model of neural avalanches in [35] (as will be explained in a separate publication).

Is it sensible to refer to self-organized dissipative systems as 'critical'?

The answer to this question is mostly a matter of taste, and depends on what one wants to define as criticality. Being strict and calling critical only systems sitting at a critical point (allowing at most for fluctuations that vanish in the thermodynamic limit), *non-conserving systems are not truly critical models.*

Being more permissive, one could accept, in principle, the term 'critical' to refer to systems hovering around a critical point (with persistent excursions into the subcritical and supercritical regimes), which exhibit 'dirty scaling'.

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In order to avoid misunderstandings and misconceptions, we strongly favor the use of an alternative terminology like 'almost criticality' [65], 'pseudo-criticality', or, as we said, *self-organized quasi-criticality* (SOqC) to refer to non-conserving self-organized systems, and suggest restricting use of the term 'critical' to truly scale-invariant systems.

Actually, in many cases, strict criticality might not be required for explaining empirical (truncated) power-law distributions observed in the real world and, therefore, 'self-organized quasi-criticality' remains a useful concept, despite the somewhat inappropriate and certainly confusing use (and abuse) of the word 'critical' in the literature. In the light of this, one could reconsider the empirical observations discussed in section 1, as well as similar ones for which power-law distributions have been reported. A critical inspection of them reveals in many cases that empirical data are better described using truncated power laws rather than pure power laws [4].

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Appendix A. Basic models in a nutshell

For the sake of completeness, in this appendix we present some of the basic toy models of self-organized criticality. All of them are defined in a two-dimensional lattice (generalizations to d dimensions, and to random-neighbor or all-to-all couplings are straightforward).

A.1. Sandpile-like models

Consider a (height or 'energy') variable $z(i, j)$, which takes integer (non-negative) values at each site (i, j) of a two-dimensional lattice. The ingredients of sandpiles, ricepiles, and related models can be sketched as:

- *Slow driving*: a small input of energy is externally introduced into the system (grains are dropped), usually at a single site: $z(i, j) \to z(i, j) + 1$.
- *Activation*: a site receiving energy, stores it until a given *threshold*, $z_{\rm thr}$, is exceeded and the site is declared *active*; otherwise nothing happens and the system is driven again.
- *Relaxation (or 'toppling')*: Each active site redistributes (all or a fraction of) its accumulated energy among its neighbors. A relaxation event can trigger a chain reaction or *avalanche* by activating its neighbors and so forth.
- *Boundary dissipation*: when redistribution events reach the (open) boundaries of the system, energy is dropped off.

- *Iteration*: when activity has ceased, the avalanche stops, and a new external input is added. The driving/dissipation cycle is iterated until a statistically stationary state is reached.

For each specific model, the relaxation rules and some other details can change, giving rise to a zoo of models. We list some of the more commonly studied ones:

The deterministic Bak–Tang–Wiesenfeld sandpile model. The threshold is fixed to $z_c = 4$ ($2d$ in d dimensions). Active sites relax according to

$$\begin{aligned} &z(i,j) \to z(i,j) - 4, \\ &z(i',j') \to z(i',j') + 1, \qquad \text{for all } (i',j') \text{ n.n. of } (i,j), \end{aligned} \tag{A.1}$$

that is, an active site is emptied and its grains are *deterministically* redistributed amongst its nearest neighbors. The energy at sites out of the system is fixed to $z = 0$, enforcing boundary dissipation when boundary sites topple. Avalanches measured in such a steady state were originally claimed to be critical [5]. Later work showed that actually, owing to the *deterministic* nature of the model and, as a consequence, to the existence of many toppling invariants and the breakdown of ergodicity [109], the system is not truly critical but exhibits anomalous multi-scaling [110]. Other authors (see, for instance, [111]) suggested that avalanches do not obey any type of scaling whatsoever. To avoid the pathologies associated with deterministic rules, we focus throughout this paper on *stochastic* models.

The stochastic Manna model. The dynamics of the Manna model is similar to that of the deterministic BTW, but the threshold is fixed to $z_c = 2$ in any dimension and stochasticity is introduced in the redistribution rule [13]:

$$\begin{aligned} &z(i,j) \to z(i,j) - 2, \\ &z(i',j') \to z(i',j') + 1, \qquad \text{for two randomly chosen } (i',j') \text{ n.n. of } (i,j). \end{aligned} \tag{A.2}$$

This is the 'inclusive' version of the model. In its 'exclusive' version, the two grains are forced to go to different neighbors. Both of these versions self-organize the model to a critical point in the C-DP (or 'Manna') class in any dimension [13, 94, 95].

The stochastic Oslo model. This (ricepile) model has annealed random thresholds at each site: every time a site topples, a new threshold $z_{\text{thr}}(i,j) = 2, 3$ is randomly chosen with equal probabilities [14]. Redistribution of energy units (grains) is done as in the Manna model, in a stochastic fashion. These rules lead to self-organization to a critical point in the C-DP universality class with rather clean scaling [14, 94, 95].

A.2. The Olami–Feder–Christensen earthquake model

A continuous 'force' or 'energy' $F(i,j)$, randomly distributed between 0 and a threshold value, F_{thr}, is initially assigned to each site (i,j) of a two-dimensional lattice. During the driving step, the site with maximum force, $F_{i,j}^{\max}$, is identified, and the force of all sites is increased by $F_{\text{thr}} - F_{i,j}^{\max}$, creating (at least) one seed of activity (this is equivalent to advancing all sites at a fixed constant velocity until one of them, the maximum, reaches the threshold). In contrast to the sandpile models case, this driving affects all sites. An

active site relaxes according to

$$\begin{aligned} &F(i,j) \to 0, \\ &F(i',j') \to F(i',j') + \alpha F(i,j), \qquad \forall\ (i',j')\ \text{n.n. of}\ (i,j), \end{aligned} \tag{A.3}$$

where $\alpha \leq 1/4$ is a *bulk dissipation parameter*. The model is conserving only for $\alpha = 1/4$. The force at sites out of the system is fixed to $F = 0$, entailing boundary dissipation. The stationary state of such a system was claimed to be critical for a broad range of values of α [42], but recent analyses disprove such a claim (see for instance [53, 70] and section 1).

A.3. The Drossel–Schwabl forest-fire model

Sites of a two-dimensional lattice can be either empty ($z = 0$), occupied by a tree ($z = 1$), or occupied by a tree on fire ($z = 2$) [48]. The dynamics proceeds as follows:

- At every *empty* site, a tree grows with probability p, and the site becomes 'occupied': $z = 0 \to z = 1$.
- Initial spark: a tree not surrounded by any fire becomes a 'burning' tree (e.g. is struck by lightning) with probability f: $z = 1 \to z = 2$.
- A burning tree sets all of its nearest neighbors on fire, and it becomes empty: $z = 2 \to z = 0$.
- The 'fire avalanche' proceeds, burning all trees in contact with fires.
- When fire ceases the dynamical processes is re-started.

The relevant parameter is f/p which is sometimes called $1/\theta$ [80]. In the implementation of the model that we use, $1/\theta$ is the number of trees grown between two consecutive ignitions. Despite the initial claims and various forms of reported 'anomalous scaling', the most recent studies revealed the absence of generic scale-invariance [82, 83].

Appendix B

To compute the mean number of grains lost in the bulk in the dissipative self-organized branching process [84], let us first consider the number of offspring, φ, not occupied owing either to the absence of branching or to bulk dissipation:

$$\varphi = \sum_{j=1}^{m-1} (2\varsigma_j - \varsigma_{j+1}), \tag{B.1}$$

then, the average fraction corresponding to bulk dissipation is

$$\langle \kappa(q,n) \rangle = \langle \varphi \rangle \frac{p\epsilon}{(1-p) + p\epsilon}. \tag{B.2}$$

Noting that the avalanche size is $s = \sum_j \varsigma_j$ and

$$\sum_{k=0}^{m-1} \varsigma_k = \sum_{k=0}^{m} \varsigma_k - \varsigma_m = s - \varsigma_m \qquad \sum_{k=0}^{m-1} \varsigma_{k+1} = \sum_{l=1}^{m} \varsigma_l = s - 1, \tag{B.3}$$

then $\varphi = 1 + s - 2\varsigma_m$. Using the definition of s,

$$\langle s\rangle = \sum_{k=0}^{m}\langle\varsigma_k\rangle = \sum_{k=0}^{m}\left(2p(1-\epsilon)\right)^k = \frac{1-\left(2p(1-\epsilon)\right)^{m+1}}{1-2p(1-\epsilon)}. \tag{B.4}$$

Plugging equation (B.4) into the equation for φ,

$$\langle\varphi\rangle = \left[1 + \frac{1-\left(2p(1-\epsilon)\right)^{m+1}}{1-2p(1-\epsilon)} - 2\left(2p(1-\epsilon)\right)^m\right], \tag{B.5}$$

and, from this and equation (B.2),

$$\langle\kappa(q,n)\rangle = \left[1 + \frac{1-\left(2p(1-\epsilon)\right)^{m+1}}{1-2p(1-\epsilon)} - 2\left(2p(1-\epsilon)\right)^m\right]\frac{p\epsilon}{(1-p)+p\epsilon}, \tag{B.6}$$

leading, once the continuum limit for n has been taken and fluctuations included, to equation (11).

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