

# Self-organized partially synchronous dynamics in populations of nonlinearly coupled oscillators

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We analyze a minimal model of a population of identical oscillators with a nonlinear coupling — a generalization of the popular Kuramoto model. In addition to well-known for the Kuramoto model regimes of full synchrony, full asynchrony, and integrable neutral quasiperiodic states, ensembles of nonlinearly coupled oscillators demonstrate two novel nontrivial types of partially synchronized dynamics: self-organized bunch states and self-organized quasiperiodic dynamics. The analysis based on the Watanabe-Strogatz ansatz allows us to describe the self-organized bunch states in any finite ensemble as a set of equilibria, and the self-organized quasiperiodicity as a two-frequency quasiperiodic regime. An analytic solution in the thermodynamic limit of infinitely many oscillators is also discussed.

*Key words:* Coupled oscillators, oscillator ensembles, partial synchronization, quasiperiodicity

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## 1 Introduction

A model of all-to-all coupled limit cycle oscillators describes many natural phenomena in physics, chemistry, biology and social sciences. For ensembles of weakly interacting units, the description of the dynamics is often provided by the paradigmatic Kuramoto model of globally coupled phase oscillators [1,2]. This model explains self-synchronization and appearance of a collective mode in an ensemble of generally non-identical elements – the problem relevant for Josephson junction and laser arrays [3,4], electrochemical reactions [5], neuronal dynamics [6], social behavior [7–9], etc. In a widely-used analogy between Kuramoto synchronization transition and phase transitions in the equilibrium statistical mechanics, the width of the distribution of natural frequencies of interacting units plays a role of temperature, and the transition occurs at a certain critical value of the coupling constant that is roughly proportional to the temperature. The ordered phase is characterized by a

non-zero mean field (collective mode) that maintains the collective synchrony. Many relevant references to the Kuramoto model can be found in [10,11].

The physical reason for the Kuramoto transition is an attraction force between the oscillators. In the present context the terms “attraction” and “repulsion” mean that the interaction between two oscillators tends to synchronize them in phase or in antiphase, respectively. Attracting interaction in an ensemble tends to adjust the phases of elements so that they form a synchronous cluster. If the coupling constant is beyond some critical value, this tendency prevails over the desynchronizing effect due to the spread of oscillator frequencies. If the interaction between the oscillators is repulsive, an asynchronous state remains stable.

In this paper, following our brief communication [12], we describe and systematically discuss an extension of the Kuramoto model that demonstrates a novel transition from the synchronous state to a regime of partial synchronization, when the system settles in a self-organized fashion at the border between stable synchrony and asynchrony. Remarkably, the transition can be observed already in a population of *identical* units, what in the language of the theory of phase transitions corresponds to zero temperature. Thus, the transition to partial synchronization is an analogon to a quantum phase transition. Partial synchronization has been previously studied for coupled integrate-and-fire oscillators [13,14]; however, this system does not exhibit a transition from full to partial synchrony. We would also like to mention work [15], where a similar phenomenon was observed numerically.

The main physical effect that is responsible for the transition to partial synchrony is *nonlinearity of coupling*. We will discuss this crucial issue in more details in Section 2, while here we just briefly present the main idea. Roughly speaking, a nonlinear coupling means that the response of an oscillator to a strong forcing cannot be simply “upscaled” from its response to a weak forcing. For the simplest example, let us assume that weak forcing leads to “attraction”, while strong forcing leads to “repulsion” (another case of nonlinear coupling which remains attractive for all nonlinearities have been considered recently in [16,17]). Now recall that in a large, globally coupled ensemble each oscillator can be considered as forced by the mean field; the strength of the forcing is determined by the product of the coupling constant and the amplitude of the mean field (see [1,2] and discussion below). Hence, with an increase of the coupling constant, the forcing increases as well, unless at some point the interaction changes from the attractive to the repulsive one, and the initially synchronous state dissolves. However, this would reduce the mean field and, therefore, the effective forcing. It means that the interaction would again become attractive and therefore a completely asynchronous state would be also impossible. As a result, an intermediate state sets in, where the interaction is tuned exactly to the border between attraction and repul-

sion. This neutral state is a self-organized partially coherent state with the order parameter (amplitude of the mean field) between zero and one. Below we demonstrate that there are two types of such states, which we call “self-organized quasiperiodic state” and “self-organized bunch state”, respectively. We shall show that in the former case the dynamics is generally quasiperiodic.

The paper is organized as follows. We introduce a minimal model with nonlinear coupling in Section 2. The self-organized bunch state is described numerically and theoretically in Section 3. In Section 4 we present results of numerical simulations, illustrating the properties of the self-organized quasiperiodicity (SOQ) and of the transition to this partially synchronized state. Theory of the effect is developed in Section 5; this theory is heavily based on the seminal paper by Watanabe and Strogatz [18] where a full analysis of linearly coupled identical phase oscillators has been performed. We first describe the approach of [18] and then show how it can be extended to the case of nonlinear coupling. In a particular case, which is, however, relevant for large ensembles with homogeneous initial conditions, the resulting equations are rather simple and their bifurcation analysis can be performed analytically. We discuss our results in Section 6.

## 2 A minimal model for an ensemble of nonlinearly coupled identical phase oscillators

### 2.1 Linear vs. nonlinear coupling: a generalized Kuramoto model

We start by a brief description of the popular Kuramoto model of sine-coupled identical phase oscillators:

$$\dot{\varphi}_k = \omega_k + \frac{\varepsilon}{N} \sum_{j=1}^N \sin(\varphi_j - \varphi_k + \beta), \quad (1)$$

where  $\varphi_k$  is the phase of the  $k$ -th oscillator,  $N$  is the number of oscillators in the ensemble,  $\varepsilon$  is the strength of the interaction between each pair of oscillators,  $\beta$  is the phase shift, inherent to coupling, and  $\omega_k$  are natural frequencies of ensemble elements. The model with  $\beta \neq 0$  is also called Sakaguchi – Kuramoto model [19]. The model (1) can be reformulated in terms of the complex mean field, defined via

$$Z_1 = K e^{i\Theta} = \frac{1}{N} \sum_{k=1}^N e^{i\varphi_k}, \quad (2)$$

where  $K$  and  $\Theta$  are the amplitude and the phase of the mean field.  $K$  is also called the order parameter of the synchronization transition; obviously, it varies from zero in the absence of synchrony to one, if all elements have

identical phases. By means of simple manipulations, Eq. (1) can be re-written in the form

$$\dot{\varphi}_k = \omega_k + \varepsilon K \sin(\Theta - \varphi_k + \beta). \quad (3)$$

To explain the notion of the nonlinear coupling, let us interpret each of the Eqs. (3) as an equation of a phase oscillator, driven by a harmonic force with amplitude  $\delta = \varepsilon K$  and phase  $\Theta$ . Although the interaction is described by a nonlinear sine-function, we denote the coupling in this model as linear, since all the parameters in Eq. (3) are independent of  $\delta$ . Generally, the response of an oscillator to the forcing can depend on the amplitude of the latter. It means that the oscillator frequency as well as the parameters of the coupling function in Eq. (3) can depend on  $\delta = \varepsilon K$ . Considering these dependencies as a minimal possible nonlinear effect, we generalize Eq. (3) to

$$\dot{\varphi}_k = \omega_k(\varepsilon K) + R(\varepsilon K)\varepsilon K \sin(\Theta - \varphi_k + \beta(\varepsilon K)), \quad (4)$$

where the functions  $\omega_k$ ,  $R$ , and  $\beta$  tend to some constants as  $\varepsilon K \rightarrow 0$ . Model (4) is minimal, because here the dependence on  $\varphi_k$  remains purely harmonic; for a general nonlinear coupling see discussion in [12]. Hereafter we analyze only the case of identical oscillators, thus  $\omega_1 = \omega_2 = \dots = \omega$  and we can drop the index.

To summarize, we call the coupling nonlinear, if the parameters of the coupling function depend on the amplitude of the force that acts on the oscillator. We emphasize that the dependencies themselves can be linear.

## 2.2 *Effect of nonlinearity on the stability of the self-consistent synchronous solution*

Let us again interpret the basic model (4) as equations for oscillators, driven by the force with amplitude  $\delta = \varepsilon K$  and frequency  $\omega_{ext}$ , so that the driving phase is  $\Theta = \omega_{ext}t$ . Considering one oscillator and dropping the index, we write:

$$\dot{\varphi} = \omega(\delta) + R(\delta)\delta \sin(\omega_{ext}t - \varphi + \beta(\delta)). \quad (5)$$

Now we discuss synchronization properties of this nonlinearly forced oscillator. Writing an equation for the phase difference  $\varphi - \Theta$ , we easily find the phase locking region (Arnold tongue) on a plane of parameters  $(\delta, \omega_{ext})$ :

$$\omega(\delta) - R(\delta)\delta \leq \omega_{ext} \leq \omega(\delta) + R(\delta)\delta. \quad (6)$$

In addition to this standard problem of synchronization by an external force, in our context we have to take into account the *self-consistency condition*, namely, that the force itself is produced by the ensemble of synchronized

oscillators. This means that  $\varphi = \Theta$  and  $\dot{\varphi} = \omega_{ext}$ , what being substituted in (5) gives

$$\omega_{ext} = \omega(\delta) + R(\delta)\delta \sin \beta(\delta) . \quad (7)$$

This condition defines a line  $\delta = \delta(\omega_{ext})$  within the synchronization region (6) of a harmonically driven individual oscillator; this line corresponds to possible fully synchronized states of the ensemble. Stability of these states is determined by the condition  $\frac{d\dot{\varphi}}{d\varphi} = -R(\delta)\delta \cos \beta(\delta) < 0$ .

For simplicity of presentations let us consider the effects of different possible nonlinearities separately. We start with the case of linear coupling, when  $\omega = \text{const}$ ,  $\beta = \text{const}$ ,  $|\beta| \leq \pi/2$ , and  $R = \text{const} > 0$ ; it is illustrated in Fig. 1(a). Here the line of stable self-consistent solutions is a straight line. Next we discuss the nonlinearity in the frequency,  $\omega = \omega(\delta)$ , while  $\beta = \text{const}$ ,  $|\beta| \leq \pi/2$ , and  $R = \text{const} > 0$ . Now the Arnold tongue is skewed (Fig. 1(b)), and the line of stable self-consistent solutions is skewed as well. However, it is easy to see that this line remains within the synchronization region. Thus, in both considered cases the stable synchronous solution exists for all amplitudes  $\delta$ , only the frequency of this solution experiences a linear or nonlinear shift, respectively.

Two nontrivial cases leading to a breakdown of the collective synchrony are depicted in Figs. 1(c,d). In the first plot we illustrate a situation when  $\omega = \text{const}$ ,  $\beta = \text{const}$ ,  $|\beta| \leq \pi/2$ , and  $R(\delta)$  changes its sign with an increase of  $\delta$ , e.g.,  $R(\delta) = a_1 - a_2\delta^2$ , where  $a_{1,2}$  are positive constants. Here at the critical strength of the forcing  $\delta_c = \sqrt{a_1/a_2}$  the Arnold tongue shrinks to zero<sup>1</sup> and for  $\delta > \delta_c$  the self-consistent synchronous solution becomes unstable, because the coupling becomes repulsive. The same may happen in the case when  $\omega = \text{const}$ ,  $R = \text{const} > 0$ , but the phase shift  $\beta$  depends on the forcing strength  $\delta$ ; for definiteness in the example illustrated in Figs. 1(d) we assume that  $|\beta| \leq \pi/2$  for  $\delta \rightarrow 0$  and monotonically decreases with  $\delta$ . Now the Arnold tongue has a finite width for all amplitudes of the forcing, but the self-consistent solution (7) is stable only for  $\delta < \delta_c$ , where the critical value  $\delta_c$  is determined from the condition  $\beta(\delta) = \pm\pi/2$ .

Now we come back to the coupled oscillator model (4) and recall that the amplitude of the forcing is  $\delta = \varepsilon K$ , where  $K$  is the mean field amplitude, defined by Eq. (2). Now the discussed above conditions for the loss of stability of a self-consistent synchronous solution with  $\varphi_1 = \varphi_2 = \dots = \varphi_N$  and  $K = 1$  lead to the critical value of the parameter  $\varepsilon$  determined according to

$$R(\varepsilon_b) = 0 \quad \text{or} \quad \beta(\varepsilon_q) = \pm\pi/2 . \quad (8)$$

<sup>1</sup> See [20] for an analysis of the closing of Arnold tongues for forced weakly nonlinear oscillators.

Below we demonstrate that these two conditions determine critical values for transitions to bunch and quasiperiodic states, respectively. Therefore, we use indices  $b$  and  $q$  to denote the corresponding values of the critical coupling. The nontrivial dynamics beyond the criticality is illustrated by several examples in the next Section.

### 2.3 Further generalization

The model (4) can be further generalized, if we take into account that a dependence on the forcing amplitude can appear in the equations for state variables. Then in the corresponding phase equation the parameters of the coupling can generally be functions of two arguments: the amplitude of the mean field  $K$  and of the bifurcation parameter (coupling constant  $\varepsilon$ ), and not just functions of the product  $\varepsilon K$ . In the following we analyze such a general

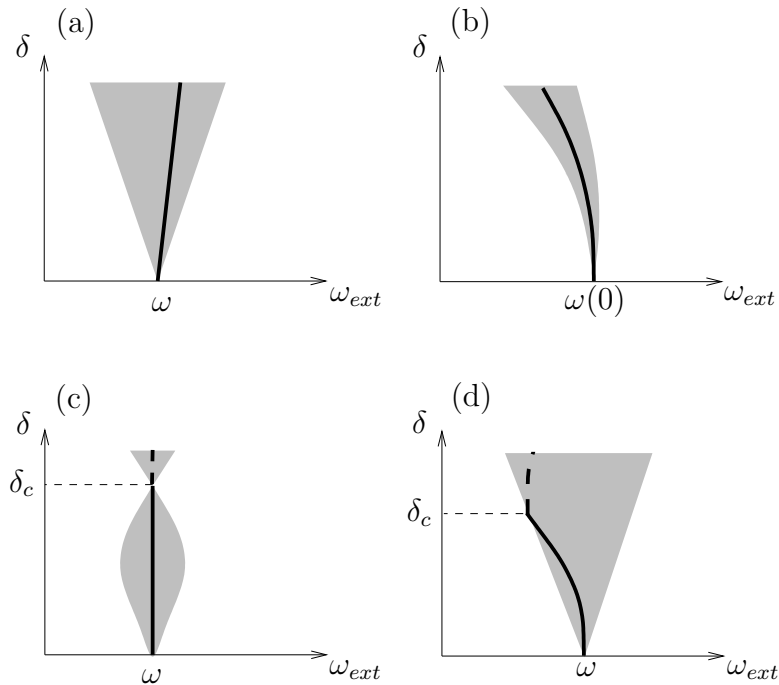


Fig. 1. Determination of self-consistent synchronous solutions for globally coupled oscillators. Arnold tongues (6) for individual oscillators, driven by a harmonic force with amplitude  $\delta$  and frequency  $\omega_{ext}$  are shown as shadowed regions. Bold (dashed) lines show the parameters, corresponding to stable (unstable) self-consistent solutions (7) for the ensemble dynamics. (a) Linear coupling. (b,c,d) Cases of nonlinear coupling, when different parameters of the model (5) depend on the amplitude  $\delta$  of the force. (b) Nonlinearity in the natural frequency  $\omega(\delta)$ . (c) Nonlinearity in the effective forcing  $R(\delta)$ . Here at some critical value  $\delta_c$  the tongue shrinks to zero. (d) Nonlinearity in the phase shift  $\beta(\delta)$ . Here at some critical value  $\delta_c$  the line of stable self-consistent synchronous solutions Eq. (7) reaches the boundary of the tongue; for  $\delta > \delta_c$  these solutions are unstable.

model [12] for the case of identical oscillators:

$$\dot{\varphi}_k = \omega(\varepsilon, K) + R(\varepsilon, K)\varepsilon K \sin(\Theta - \varphi_k + \beta(\varepsilon, K)). \quad (9)$$

For this model, as it follows from the arguments above, the critical values of the bifurcation parameter  $\varepsilon$  are determined by the conditions similar to (8):

$$R(\varepsilon_b, 1) = 0 \quad \text{or} \quad \beta(\varepsilon_q, 1) = \pm\pi/2. \quad (10)$$

Finally, we note that Kuramoto model (1) is a particular case of the Daido model [21–23]

$$\dot{\varphi}_k = \omega_k + N^{-1} \sum_j^N h(\varphi_j - \varphi_k), \quad (11)$$

where  $h(\cdot)$  is an arbitrary  $2\pi$ -periodic function. In its turn, model (9) is a particular case of the nonlinear generalization [12] of the Daido model; this generalization accounts for possible dependence of the interaction function on the generalized order parameters  $Z_n = N^{-1} \sum_j^N e^{in\varphi_j}$ .

#### 2.4 Nonlinear coupling: an example

We further motivate our generalization of the Kuramoto model to Eq. (9) by the following example. We consider an ensemble of Landau-Stuart oscillators, each described by a complex variable  $a_k$ , coupled nonlinearly via the mean field  $A = N^{-1} \sum_{k=1}^N a_k$ :

$$\dot{a}_k = (1 + i\omega)a_k - (1 + i\alpha)|a_k|^2 a_k + (\mu_1 + i\mu_2)A - (\eta_1 + i\eta_2)|A|^2 A, \quad (12)$$

where  $\mu_{1,2}$  and  $\eta_{1,2}$  are real parameters describing linear and nonlinear coupling, respectively. If these parameters are small, then in the first approximation we can neglect variations of the amplitudes of the oscillators. Looking for the solution in the form  $a_k = e^{i\varphi_k}$  and  $A = Ke^{i\Theta}$ , we obtain

$$\dot{\varphi} = \omega - \alpha + K[(\mu_1 - \eta_1 K^2) \sin(\Theta - \varphi) + (\mu_2 - \eta_2 K^2) \cos(\Theta - \varphi)]. \quad (13)$$

Introducing new notations  $\omega - \alpha \rightarrow \omega$  and

$$R^2 \varepsilon^2 = (\mu_1 - \eta_1 K^2)^2 + (\mu_2 - \eta_2 K^2)^2, \quad \tan \beta = \frac{\mu_2 - \eta_2 K^2}{\mu_1 - \eta_1 K^2}, \quad (14)$$

we finally obtain a particular case of Eq. (9), where the bifurcation parameter  $\varepsilon$  corresponds to one of the parameters  $\mu_{1,2}$ ,  $\eta_{1,2}$ , or to their combination.

Consider now a particular choice of parameters:  $|\eta_{1,2}| \ll |\mu_{1,2}|$  and  $\mu_1 \eta_1 + \mu_2 \eta_2 = 0$ . Then  $R^2 \varepsilon^2 \approx \mu_1^2 + \mu_2^2 = \text{const}$ ,  $\tan \beta \approx \frac{\mu_2}{\mu_1} + \left( \frac{\eta_1 \mu_2}{\mu_1^2} - \frac{\eta_2}{\mu_1} \right) K^2 =$

$\frac{\mu_2}{\mu_1^2} - \frac{\mu_1^2 + \mu_2^2}{\mu_1^2} \frac{\eta_2}{\mu_1} K^2$  and  $\beta \approx \arctan \frac{\mu_2}{\mu_1} - \frac{\eta_2}{\mu_1} K^2 = \beta_0 + \varepsilon^2 K^2$ . The model with  $R(\varepsilon, K) = \text{const}$  and  $\beta(\varepsilon, K) = \beta_0 + \varepsilon^2 K^2$  is used below for numerical illustration of self-organized quasiperiodic dynamics.

### 3 Self-organized bunch states

In the case when the nonlinearity in the coupling does not lead to the phase shift, but affects only the effective coupling strength  $R$  (like in Fig. 1c), the complete analysis of the dynamics is quite simple. For a particular example we consider purely real coupling in Eq. (13), i.e.  $\mu_2 = \eta_2 = 0$ . Let us fix  $\mu_1$  and take  $\eta_1$  as a bifurcation parameter, i.e.  $\varepsilon = \eta_1$ . From Eqs. (14) it follows that  $\beta = 0$ , the amplitude function takes the form  $R\varepsilon = \mu_1 - \varepsilon K^2$ , and the phase equation (9) reads

$$\dot{\varphi}_k = (\mu_1 - \varepsilon K^2) K \sin(\Theta - \varphi_k). \quad (15)$$

Here we set the frequency  $\omega$  to zero, what can be always done by a transformation to the reference frame, rotating with velocity  $\omega$ .

From (15) we easily obtain the equation for the evolution of the complex order parameter  $Z_1$  (see Eq. (2)):

$$\frac{dZ_1}{dt} = \frac{1}{2}(\mu_1 - \varepsilon|Z_1|^2)(Z_1 - Z_1^* Z_2), \quad (16)$$

where  $Z_2$  is the second generalized order parameter

$$Z_2 = K_2 e^{i\Theta_2} = \frac{1}{N} \sum_k e^{i2\varphi_k}. \quad (17)$$

Thus, the evolution of  $K$  obeys

$$\frac{dK}{dt} = \frac{1}{2}(\mu_1 - \varepsilon K^2) K (1 - K_2 \cos(\Theta_2 - 2\Theta)). \quad (18)$$

If we exclude a situation when the phases are initially organized in several clusters, then  $K_2 \leq 1$  reaches unity only together with  $K$  for the fully synchronous state  $\varphi_1 = \dots = \varphi_N$ . Therefore  $(1 - K_2 \cos(\Theta_2 - 2\Theta)) > 0$  for  $K < 1$  and  $(1 - K_2 \cos(\Theta_2 - 2\Theta)) = 0$  if  $K = K_2 = 1$ . This consideration allows us to give a qualitative picture of the dynamics.

Consider first the case  $\mu_1 > 0$ . Here the critical value of parameter  $\varepsilon$  at which the steady state  $K = 1$  bifurcates is  $\varepsilon_b = \mu_1$ . For  $\varepsilon < \varepsilon_b$  the fully synchronous



state with  $K = 1$  is stable, whereas for  $\varepsilon > \varepsilon_b = \mu_1$ , a new stable state with

$$K = \sqrt{\frac{\varepsilon_b}{\varepsilon}} \quad \text{for } \varepsilon \geq \varepsilon_b, \quad (19)$$

appears according to Eq. (18), see Fig. 2a. Thus, the system organizes itself in such a way that the coupling always vanishes and the system stays at the border between attraction and repulsion. We call this regime *self-organized bunch state*. Obviously, the above consideration can be done for the full Eq. (12): if with variation of a coupling parameter the nonlinear term compensates the linear one, the system settles on the border of stability of the synchronous regime, and the synchrony gets destroyed, cf. [24].

In the self-organized bunch state oscillators do not rotate (in the frame moving with frequency  $\omega$ ) and are generally non-uniformly distributed around the unit circle: this “static” distribution has a fixed mean field amplitude  $K$  but all other generalized order parameters  $Z_m$ ,  $m \geq 2$  are arbitrary, they depend on the initial conditions. This is illustrated in Fig. 3, where we show snapshots for model (15) for 3 different values of coupling and two different sets of initial conditions: a nearly uniform and a nearly identical initial distribution of the phases  $\varphi_k$ . For a nearly uniform distribution we took the phases as  $\varphi_k = 2\pi(k-1)/N$  and added a perturbation  $10^{-4}$  to one phase  $\varphi_1$ ; by the nearly identical initial conditions we mean a uniform distribution of phases along the arc  $0.02\pi$ . Parameters of the model are  $N = 1000$  and  $\mu_1 = 1$ .

Consider now the case  $\mu_1 < 0$ . For  $\varepsilon > \mu_1$  only the asynchronous state  $K = 0$  is stable, whereas for  $\varepsilon < \mu_1$  the system is bistable, i.e. both the fully synchronous ( $K = 1$ ) and the asynchronous ( $K = 0$ ) states are possible. Indeed, if the initial configuration has the mean field amplitude  $K > K_{cr}$  such that  $\mu_1 - \varepsilon K^2 > 0$ , then, according to Eq. (18), the system synchronizes. This yields the value of the critical mean field amplitude  $K_{cr} = \sqrt{\mu_1/\varepsilon}$ , see Fig. 2b. Here, again, besides the bistability in  $K$ , there exist a multistability with respect to initial conditions for asynchronous solutions.

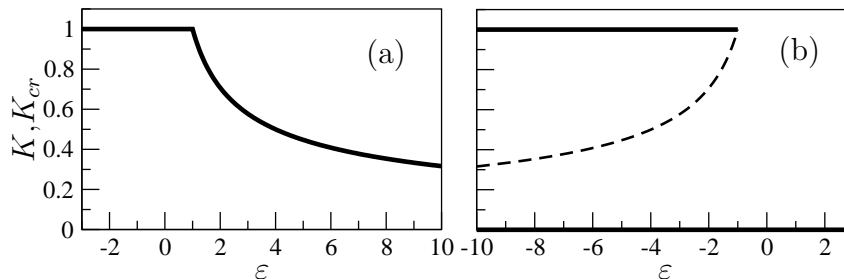


Fig. 2. Self-organized bunch states in the model (15). (a) Mean field amplitude  $K$  for positive  $\mu_1$  (here  $\mu_1 = 1$ ). (b) Mean field amplitude  $K$  for negative  $\mu_1$  (here  $\mu_1 = -1$ ). For  $\varepsilon < \mu_1$  we observe multistability: the initial states with  $K > K_{cr}(\varepsilon)$  synchronize;  $K_{cr}(\varepsilon)$  is shown by dashed line.

## 4 Self-organized quasiperiodic states: Numerical illustration

In this Section we analyze the case illustrated in Fig. 1(d), when the breakdown of synchrony occurs due to the nonlinear phase shift  $\beta(\varepsilon, K)$ . A complete description of the arising dynamical state will be given in Section 5, where the theory is developed. Here we provide a numerical illustration of the effect. For simulations we consider a particular case of the general model (4) with  $R = 1$ ,  $\omega = 0$  and  $\beta(\varepsilon, K) = \beta_0 + \varepsilon^2 K^2$  (see Section 2.4), i.e. the following model:

$$\dot{\varphi}_k = \varepsilon K \sin(\Theta - \varphi_k + \beta_0 + \varepsilon^2 K^2), \quad (20)$$

and simulate it for  $\beta_0 = 0.475\pi$ . According to the stability analysis above, the state of full synchrony  $\varphi_1 = \dots = \varphi_N$  becomes unstable at  $\varepsilon_c = \sqrt{0.025\pi} \approx 0.2802$ , where  $\beta(\varepsilon_c, 1) = \pi/2$ . Contrary to the bunch solutions, analyzed in Section 3, now beyond the transition we generally observe time-dependent, partially synchronous solutions where both the order parameter  $K$  as well as the growth rate of individual phases  $\dot{\varphi}_k$  vary in time. We first illustrate this in Fig. 4 for the case of  $N = 10$  oscillators. As follows from numerics, after a transient the amplitude of the mean field  $K$  becomes a periodic function of time with some period  $T_K$ . However, the complex mean field  $Z_1(t)$  is not periodic, because  $\Theta(t + T_K) \pmod{2\pi} \neq \Theta(t)$ . This is demonstrated in Fig. 5, where we plot the values  $K_i = K(t_i)$  at the moments of time when  $\Theta(t_i) \pmod{2\pi} = 0$ . This Poincaré map plot proves that the dynamics of the system is quasiperiodic.

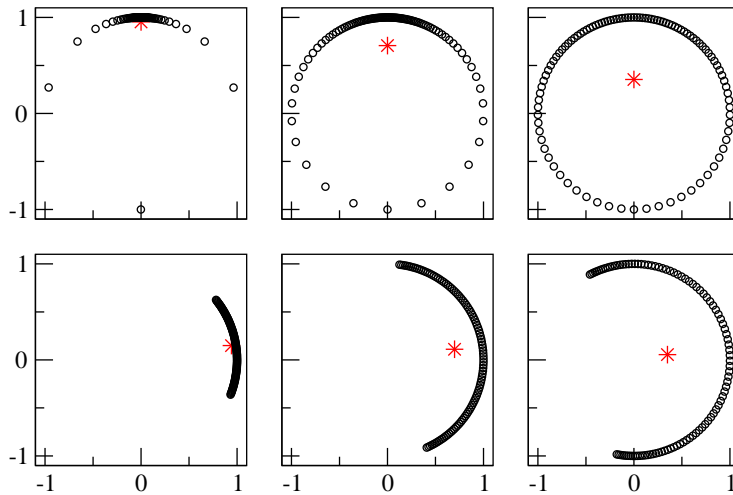


Fig. 3. Snapshots for the model (15) in the self-organized bunch state. Top and bottom row correspond to nearly uniform and nearly identical initial conditions, respectively (see text for details). Left, middle, and right column correspond to different values of coupling:  $\varepsilon = 1.1$ ,  $\varepsilon = 2$ , and  $\varepsilon = 8$ . Individual oscillators are shown by circles; mean field is shown by red star. For better visibility, only 100 oscillators out of  $N = 1000$  are shown.

We emphasize that the features of the oscillations  $K(t)$  strongly depend on the initial conditions. Numerically we found the oscillations of the mean field amplitude to be minimal for a nearly uniform initial distribution of the phases

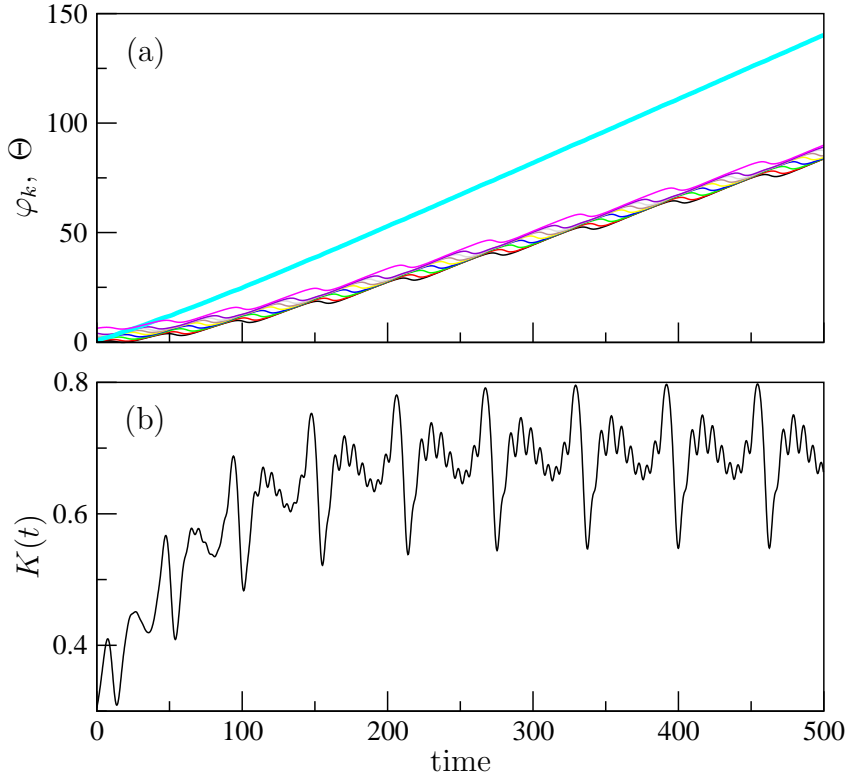


Fig. 4. Dynamics of 10 oscillators according to Eq. (20) for  $\varepsilon = 0.4$  and initial conditions  $\varphi_k = \frac{k^2}{100}2\pi$ . (a) Individual phases  $\varphi_k$  (solid lines) and the phase of the mean field  $\Theta$  (upper, bold, curve). (b) Time evolution of the amplitude of the mean field  $K(t)$ .

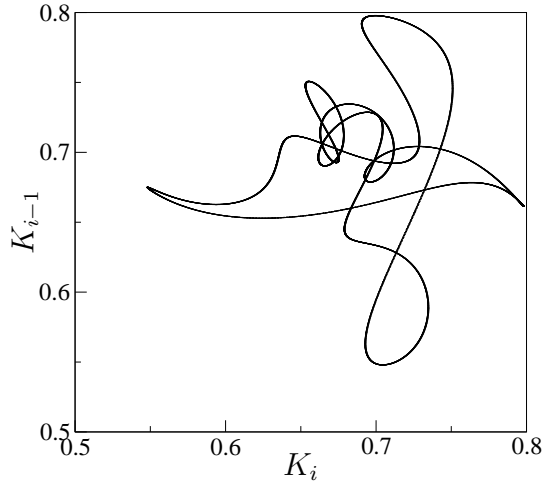


Fig. 5. The Poincaré map based on the dynamics of the mean field for ten oscillators. Points lie on a line, indicating that the dynamics is quasiperiodic with two incommensurate frequencies.

$\varphi_k(0)$ . To be precise, we took initially the phases as  $\varphi_k = 2\pi(k-1)/N$  and added a perturbation  $10^{-4}$  to  $\varphi_1$ . This case has been studied in more details for different number of oscillators in the ensemble:  $N = 10$ ,  $N = 100$ , and  $N = 1000$ . The results are presented in Fig. 6, where we show the averaged over time amplitude of the mean field amplitude  $\overline{K} = T^{-1} \int_0^T dt K(t)$ , its root mean square  $\text{rms}(K)$ , and the mean frequencies of one oscillator  $\omega_{osc} = T^{-1}(\varphi_k(T) - \varphi_k(0))$  and of the mean field  $\Omega = T^{-1}(\Theta(T) - \Theta(0))$  as functions of the coupling strength  $\varepsilon$ . As expected, for small coupling  $\varepsilon < \varepsilon_c$  we observe synchronous regime with  $K = 1$  and  $\omega_{osc} = \Omega$ , whereas a transition to a time-dependent state occurs at  $\varepsilon_q$ . Beyond this point, the time-averaged mean field amplitude  $\overline{K}$  decays monotonically. Its value between zero and one corresponds to a partially coherent state, where the phases of oscillators are scattered around the unit circle. However, the instantaneous distributions of phases are not uniform (cf. Figs. 3,4). The time variations of the mean field amplitude  $K$  are characterized by the root mean square  $\text{rms}(K)$ . The variations are mostly pronounced close to the transition point, and decay rapidly with the ensemble size  $N$ . This allows us to hypothesize that, for the considered initial conditions, the amplitude of the mean field in the thermodynamical limit is constant. This constant simply corresponds to a value for which  $\beta(\varepsilon, K) = \pi/2$ , what in our case means  $K = \varepsilon_q/\varepsilon$  (we do not show this curve in Fig. 6 since it overlaps with the numerical curves).

Another important feature of the partially synchronous regime is a discrepancy between the frequency of the mean field  $\Omega$  and that of individual units  $\omega_{osc}$  (Fig. 6c). Note that both frequencies seemingly smoothly depend on  $\varepsilon$  and therefore are, generally, incommensurate. It means that oscillators are generally in the quasiperiodic state.

Next we choose a different initial distribution of the phases, when they are almost identical, namely they are uniformly distributed along the arc  $0.02\pi$ . In Fig. 7 we compare the results for this initial conditions with those for almost uniform distribution, for  $N = 1000$ . One can see that while for a nearly uniform initial distribution of phases  $K \approx \text{const}$ , for a nearly identical initial phases there is a strong variation of  $K$  even in the thermodynamic limit. Another way of comparison is to look for time dependence (after a transient) of the mean field amplitude and of the instantaneous frequency, here for  $\varepsilon = 0.5$  (Fig. 8). Similar to the case of 10 oscillators, presented in Fig. 5, the mean field itself is quasiperiodic. This can be confirmed by plotting the Poincaré map in Fig. 9. The map was constructed for  $\Theta(t_i) \pmod{2\pi} = 0$ .

To illustrate the self-organized onset of quasiperiodic dynamics, we plot in Fig. 10 the transients, for both nearly uniform and nearly identical initial conditions. We see, that in the former case the amplitude monotonically increases unless it approaches a constant value  $\varepsilon_q/\varepsilon \approx 0.5605$ . Similarly, the nonlinear phase shift  $\beta$  approaches  $\pi/2$ .

Finally, we note that computation of the Lyapunov exponents shows that the system has one negative and  $N - 1$  vanishingly small exponents. These results will be explained in the Section 5.3 below.

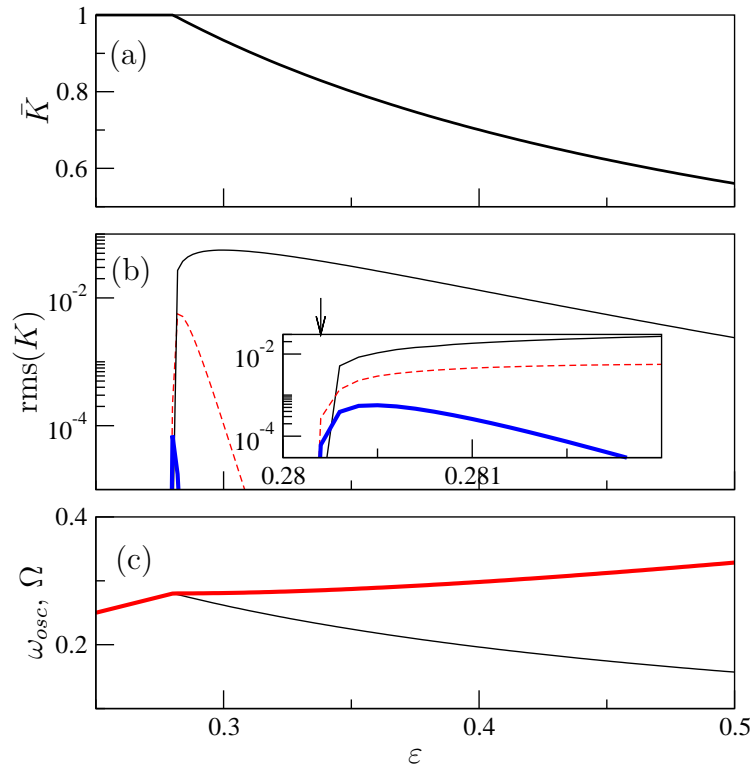


Fig. 6. Average value  $\bar{K}$  (a) and root mean square (b) of the mean field amplitude and frequencies of oscillators  $\omega_{osc}$  and of the mean field  $\Omega$  (c) as functions of the coupling constant  $\varepsilon$ , for nearly uniform initial conditions. The results in (a) and (c) are shown for  $N = 100$ ; the corresponding curves for  $N = 10$  and  $N = 1000$  differ by a third decimal and are therefore not shown here. Size effect manifests itself only in the fluctuations of the mean field (b); here the results for  $N = 10$ ,  $N = 100$ , and  $N = 1000$  are shown by solid, dashed and bold lines, respectively. The theoretical value of the critical coupling is indicated by an arrow. In (c) frequencies of the mean field and of one oscillator are shown by bold and solid lines, respectively.

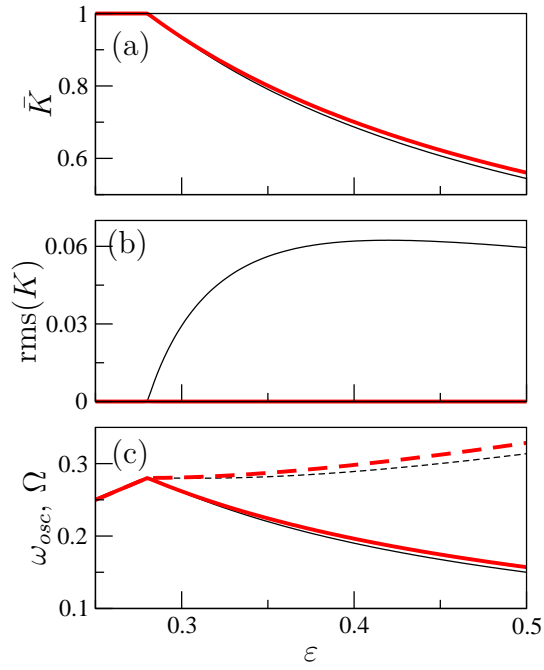


Fig. 7. Average value (a)  $\bar{K}$  and root mean square (b) of the mean field amplitude and frequencies of oscillators  $\omega_{osc}$  and of the mean field  $\Omega$  (c), as function of the coupling strength  $\varepsilon$  for nearly identical (black solid line) and nearly uniform (red bold line) initial conditions, for  $N = 1000$ . In (c) dashed lines show the frequencies of the mean field and solid lines those of individual oscillators.

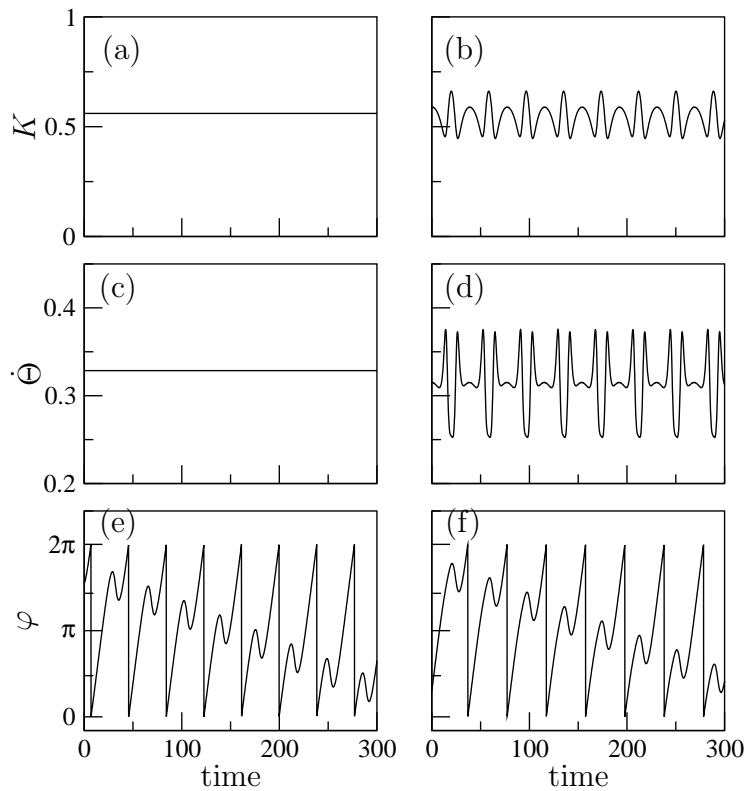


Fig. 8. Time dependence of the mean field amplitude (a,b), of the instantaneous frequency of the mean field (c,d), and of the phase of individual oscillators (e,f) for nearly uniform (left) and nearly identical (right) initial conditions;  $N = 1000$ . It is seen that uniform initial conditions lead to a harmonic mean field, whereas almost identical initial conditions yield a quasiperiodic mean field (see also Fig. 9). Individual oscillators are quasiperiodic in both cases.

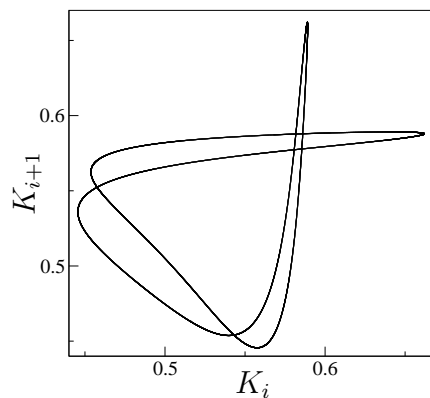


Fig. 9. Poincaré section for the mean field for the case of almost identical initial conditions,  $N = 1000$ , proves that the mean field is quasiperiodic.

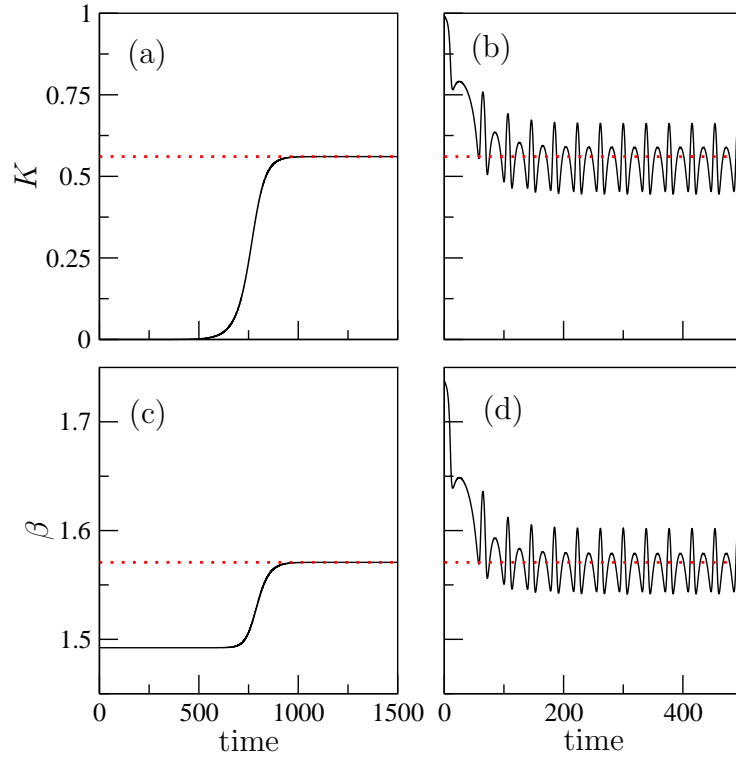


Fig. 10. Self-organized onset of quasiperiodic dynamics for nearly uniform (left) and nearly identical (right) initial conditions. In the first case, mean field amplitude  $K$  and phase shift  $\beta$  monotonically approach their critical values  $K = \varepsilon_q/\varepsilon$  and  $\beta = \pi/2$ ; in the second case they oscillate around these values. These critical values are shown by dashed lines.



## 5 Self-organized states: theory

In this Section we present a theoretical analysis of the self-organized collective dynamics. First we mention that a particular, however important, solution under the assumption of a harmonic mean field ( $K = \text{const}$ ,  $\Theta = \Omega t$ ) was treated in our previous publication [12]. Following the standard Kuramoto approach for linearly coupled oscillators (Eqs. (1)) and writing self-consistent equations for the mean field [2,10], we have found in [12] the quantities  $K(\varepsilon)$ ,  $\Omega(\varepsilon)$ , and  $\omega_{osc}(\varepsilon)$  for the nonlinear case (Eqs. (4)), for supercritical coupling  $\varepsilon > \varepsilon_c$ . Here we exploit the method developed by Watanabe and Strogatz (WS) [18], to analyze the ensembles of nonlinearly coupled oscillators in a general setting of a time-dependent mean field.

### 5.1 Watanabe-Strogatz theory

In the following analysis we concentrate on the effect of functions  $R(\varepsilon, K)$  and  $\beta(\varepsilon, K)$ , and take  $\omega(\varepsilon, K) = \text{const} = 0$ . For  $\omega = 0$  both the standard Kuramoto model (1) and the generalized model (9) can be rewritten as

$$\dot{\varphi}_k = g \cos \varphi_k + h \sin \varphi_k . \quad (21)$$

In the latter case the functions  $g$ ,  $h$  are

$$g = R\varepsilon K \sin(\Theta + \beta) , \quad h = -R\varepsilon K \cos(\Theta + \beta) . \quad (22)$$

The main result of Watanabe and Strogatz is that for any time-dependent functions  $g(t)$  and  $h(t)$ , the  $N$ -dimensional system (21) can be reduced to a three-dimensional system

$$\dot{\gamma} = -(1 - \gamma^2)(g \sin \Phi - h \cos \Phi) , \quad (23)$$

$$\gamma \dot{\Psi} = -\sqrt{1 - \gamma^2}(g \cos \Phi + h \sin \Phi) , \quad (24)$$

$$\gamma \dot{\Phi} = -g \cos \Phi - h \sin \Phi , \quad (25)$$

where  $\Psi$ ,  $\Phi$  are ‘‘global phases’’ and  $0 \leq \gamma \leq 1$  is ‘‘amplitude’’; the relation of these quantities to  $\Omega$ ,  $\omega_{osc}$ , and  $K$  is discussed below. The reduction can be achieved with the help of the following ansatz:

$$\varphi_k = \Phi(t) + 2 \arctan \left[ \sqrt{\frac{1 + \gamma(t)}{1 - \gamma(t)}} \tan \left[ \frac{1}{2}(\psi_k - \Psi(t)) \right] \right] , \quad k = 1 \dots N , \quad (26)$$

where  $\psi_k$  are constants. (Note that the limit  $\gamma \rightarrow 1$  exists, and therefore in the following we can consider also the value  $\gamma = 1$ .) WS have demonstrated [18]

that the set of constants  $\psi_k$  together with the solutions of Eqs. (23-25) yields a solution of Eq. (21) via transformation (26). To specify the solution, one has to determine the initial values of  $\gamma, \Phi, \Psi$  and the constants  $\psi_k$  from the given initial conditions  $\varphi_k$ ; this is discussed in detail in [18]. For the following it is important to note, that two additional constraints are imposed on the constants  $\psi_k$ :

$$\sum_{k=1}^N \cos \psi_k = \sum_{k=1}^N \sin \psi_k = 0 . \quad (27)$$

These conditions allow one to determine unambiguously the new variables  $\gamma(0), \Psi(0), \Phi(0), \psi_k$  from the initial conditions  $\varphi_k(0)$  (up to a remaining trivial arbitrary shift that can be attributed either to  $\Psi$  or to  $\psi_k$ ).

Now we discuss the relation between the mean field amplitude  $K$  and the amplitude variable  $\gamma$ ; this relation follows from Eq. (26). First we see that for  $\gamma = 0$  we have  $\varphi_k = \psi_k + \Phi - \Psi$ , what together with Eqs. (2,27) yields  $K = 0$ . For the other limiting case  $\gamma = 1$  we have  $\varphi_k = \Phi + \pi$ , and, hence,  $K = 1$ . Intermediate values  $0 < K < 1$  are determined by both  $\gamma$  and  $\Psi$ , i.e.  $K = K(\gamma, \Psi)$ . Thus the variable  $\gamma$  is roughly proportional to the amplitude of the mean field  $K$ .

To understand, how the frequencies of the oscillators and of the mean field are related to the new variables, let us give in Fig. 11 a graphical representation of the transformation (26). First we consider a nearly uniform distribution of  $\psi_k$ . At each moment of time the distribution of phases  $\varphi_k$  has a symmetrical hump centered at  $\Phi(t)$ . The width of the hump is determined by the variable  $\gamma$ : for  $\gamma = 0$  there is no hump at all and the distribution is flat, while for  $\gamma = 1$  the hump collapses to a  $\delta$ -function. Obviously, the center of the hump corresponds to the phase of the mean field, i.e.  $\Phi = \Theta$ . Thus, the averaged velocity of the hump is the frequency of the mean field,  $\Omega = \langle \dot{\Phi} \rangle$ . For a non-uniform distribution of  $\psi_k$  the hump is generally skewed so that  $\Phi = \Theta + \text{const}$ ; the relation for  $\Omega$  holds. Next, as follows from Eq. (26),  $\varphi_k$  grows by  $2\pi$  when  $\Psi$  decreases by  $2\pi$ . Hence, for the averaged phase growth we obtain  $\langle \dot{\varphi}_k \rangle = \omega_{osc} = \langle \dot{\Phi} \rangle - \langle \dot{\Psi} \rangle = \Omega - \langle \dot{\Psi} \rangle$ . Thus, generally the oscillator frequency differs from that of the mean field. In summary,

$$\Omega = \langle \dot{\Phi} \rangle , \quad \omega_{osc} = \Omega - \langle \dot{\Psi} \rangle . \quad (28)$$

For the analysis of Eqs. (23-25) it is convenient to introduce quantities  $S$  and  $T$  which are directly related to the amplitude of the mean field (see Eq. (3.3),

Eq. (4.10) and preceding unnumbered equations in [18]):

$$\begin{aligned} S(\gamma, \Psi) &= K \sin(\Theta - \Phi) = N^{-1} \sum_{k=1}^N \sin(\varphi_k - \Phi) = \\ &= N^{-1} \sum_{k=1}^N \frac{\sqrt{1 - \gamma^2} \sin(\psi_k - \Psi)}{1 - \gamma \cos(\psi_k - \Psi)}, \end{aligned} \quad (29)$$

$$\begin{aligned} T(\gamma, \Psi) &= -K \cos(\Theta - \Phi) = -N^{-1} \sum_{k=1}^N \cos(\varphi_k - \Phi) = \\ &= N^{-1} \sum_{k=1}^N \frac{\gamma - \cos(\psi_k - \Psi)}{1 - \gamma \cos(\psi_k - \Psi)}. \end{aligned} \quad (30)$$

For these quantities we have  $K^2 = T^2 + S^2$ . Furthermore, they can be represented via derivatives of a function  $H(\Psi, \gamma)$ :

$$H(\Psi, \gamma) = \frac{1}{N} \sum_{k=1}^N \log \left( \frac{1 - \gamma \cos(\psi_k - \Psi)}{\sqrt{1 - \gamma^2}} \right), \quad (31)$$

as

$$T = (1 - \gamma^2) \frac{\partial H}{\partial \gamma}, \quad S = -\frac{\sqrt{1 - \gamma^2}}{\gamma} \frac{\partial H}{\partial \Psi}. \quad (32)$$

Considered as a function of polar coordinates  $\gamma$  and  $\Psi$ ,  $H$  has a minimum  $H = 0$  at the origin  $\gamma = 0$  and tends to infinity on the unit circle  $\gamma = 1$ . A useful relation for this function (Eq. (4.9) in [18]) is

$$\frac{dH}{dt} = R \varepsilon K^2 \cos \beta. \quad (33)$$

With the help of the quantities  $S(\gamma, \Psi)$ ,  $T(\gamma, \Psi)$  and with account of Eq. (22), Eqs. (23-25) can be re-written as

$$\dot{\gamma} = (1 - \gamma^2) \varepsilon R (T \cos \beta + S \sin \beta), \quad (34)$$

$$\gamma \dot{\Psi} = \sqrt{1 - \gamma^2} \varepsilon R (T \sin \beta - S \cos \beta), \quad (35)$$

$$\gamma \dot{\Phi} = \varepsilon R (T \sin \beta - S \cos \beta). \quad (36)$$

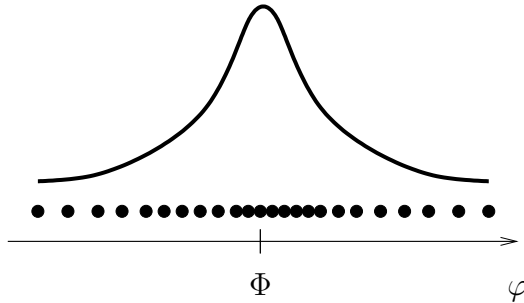


Fig. 11. Illustration of the transformation (26).

The first two equations represent a closed two-dimensional system for variables  $\gamma$  and  $\Psi$ ; this system is complemented by an equation for  $\Phi$ . The last two equations provide a relation  $\dot{\Psi} = \sqrt{1 - \gamma^2} \dot{\Phi}$ .

### 5.2 Results of WS for linear coupling

Up to now we have just reproduced the equations derived by Watanabe and Strogatz [18]. Here we briefly summarize their results for the linear coupling. For this case we have  $R = 1$ ,  $\varepsilon = \text{const}$ , and  $\beta = \text{const}$ . Then, as follows from Eq. (33), everything depends on the sign of  $\varepsilon \cos \beta$ . If  $\varepsilon \cos \beta < 0$  (this corresponds to the repulsion between the oscillators), the function  $H$  monotonically decreases, and, thus,  $\gamma \rightarrow 0$  and an incoherent state with zero mean field  $K = 0$  sets in. If  $\varepsilon \cos \beta > 0$  (attracting interaction), then  $H$  grows and a fully synchronous state with  $\gamma = 1$  establishes. This state corresponds to a stable fixed point in Eqs. (34,35). As follows from Eqs. (31,32), in this state  $S = 0$  and  $T = 1$ , what together with Eq. (36) yields  $\dot{\Phi} = \Omega = \varepsilon \sin \beta$ .

At the border between attraction and repulsion  $\cos \beta = 0$  (we exclude the trivial case  $\varepsilon = 0$ ) and the system (34,35) has an integral of motion. In fact, it can be rewritten as a Hamiltonian one; here one observes a periodic motion of  $\gamma$  and  $\Psi$ .

### 5.3 General results for nonlinear coupling

The main new feature that appears in the case of nonlinear coupling is that the product  $\varepsilon R(\varepsilon, K) \cos \beta(\varepsilon, K)$  can change its sign at some value of the bifurcation parameter. Suppose for definiteness that  $\varepsilon R(\varepsilon, K) \cos \beta(\varepsilon, K) > 0$  (attraction) for small  $K$  and  $\varepsilon R(\varepsilon, K) \cos \beta(\varepsilon, K) < 0$  (repulsion) for  $K \lesssim 1$ . Then both the asynchronous state with  $\gamma = K = 0$  and the synchronous one with  $\gamma = K = 1$  are unstable, and some intermediate, partially synchronous state with  $0 < \gamma < 1$  appears. This state corresponds either to a fixed point or to a limit cycle in the plane  $\gamma, \Psi$ . Though we cannot find these solutions explicitly for the general case, we can do it for an important particular case, which we consider below in the Section 5.4. However, several important conclusions can be drawn for the general case as well. Here again we distinguish between two self-organized states – the bunch and the quasiperiodic ones.

**Bunch states.** Suppose first that the partially synchronous case appears because the function  $R(\varepsilon, K(\gamma, \Psi))$  crosses zero for some  $K = K_0$ , while  $\cos \beta(\varepsilon, K_0) \neq 0$ . Then Eqs. (34,35) have a family of fixed points determined by the condition  $K(\gamma, \Psi) = K_0$ . Next, also  $\dot{\Phi} = 0$ , i.e. the mean field has the

same frequency, as individual oscillators. Hence, there exist a family of bunch states, corresponding to a family of equilibria in Eqs. (34-36), with different distributions of phases depending on initial conditions.

**Quasiperiodic state.** Now we consider the case  $R(\varepsilon, K) > 0$ . Then one can easily see that the only fixed points of Eqs. (34,35) are those with  $\gamma = 0$  or  $\gamma = 1$ . However, they are unstable, and therefore the system possesses at least one stable limit cycle with some period  $T$ ; this cycle has to wrap the origin. On this solution  $\gamma(t) = \gamma(t + T)$  and  $\Psi(t) + 2\pi = \Psi(t + T)$ . The change of the variable  $\Phi$  during the period is given by

$$\Phi(T) - \Phi(0) = \int_0^{2\pi} \frac{d\Psi}{\sqrt{1 - \gamma^2}}. \quad (37)$$

With account of (28) this gives the following relation between the frequencies

$$\frac{\omega_{osc}}{\Omega} = 1 - \frac{2\pi}{\int_0^{2\pi} \frac{d\Psi}{\sqrt{1 - \gamma^2}}} \quad (38)$$

that is in general irrational. Thus, the dynamics of the three-dimensional system Eqs. (34-36) lies on a torus, it is characterized by one negative and two zero Lyapunov exponents; all other exponents of the full system correspond to the constants  $\psi$  and are zeros. This explains the numerical findings of Section 4.

#### 5.4 Self-organized quasiperiodic state in the thermodynamic limit

The system (34-36) can be essentially simplified in the thermodynamic limit  $N \rightarrow \infty$ . In this case we have to replace the sums in (29,30) by the integrals,  $\frac{1}{N} \sum_{k=1}^N (\cdot) \rightarrow \int_{-\pi}^{\pi} \sigma(\psi)(\cdot) d\psi$ , where  $\sigma(\psi)$  is the normalized distribution density, determined by the initial conditions in the ensemble. Furthermore, for the particular case of uniform initial conditions,  $\sigma(\psi) = 1/2\pi$ , the integrals can be computed and we obtain

$$S = 0, \quad T = \frac{1 - \sqrt{1 - \gamma^2}}{\gamma}. \quad (39)$$

Next, recalling that  $S^2 + T^2 = K^2$  we obtain  $T = K$  and  $\gamma = 2K/(1 + K^2)$ . Substituting this into system (34-36) we reduce it to

$$\dot{K} = \frac{1}{2}K(1 - K^2)\varepsilon R(\varepsilon, K) \cos \beta(\varepsilon, K) , \quad (40)$$

$$\dot{\Psi} = \frac{1}{2}(1 - K^2)\varepsilon R(\varepsilon, K) \sin \beta(\varepsilon, K) , \quad (41)$$

$$\dot{\Phi} = \frac{1}{2}(1 + K^2)\varepsilon R(\varepsilon, K) \sin \beta(\varepsilon, K) . \quad (42)$$

Here it is sufficient to analyze only the first equation. It has two trivial steady states:  $K = 0$  corresponds to complete asynchrony and  $K = 1$  to full synchrony. Additionally, a nontrivial steady state with  $K = K_0$  can exist for some range of parameters, satisfying either  $R(\varepsilon, K_0) = 0$  or  $\cos \beta(\varepsilon, K_0) = 0$ . In the former case  $\dot{\Psi} = \dot{\Phi} = 0$ , what means that the mean field has the same frequency as each oscillator and the distribution of phases on the unit circle rotates as a whole, forming the bunch state as described above.

A stationary state of Eq. (40) with  $\cos \beta(K_0) = 0$ ,  $\sin \beta(K_0) = \pm 1$  is slightly less trivial. Here the frequency of the mean field is  $\dot{\Phi} = \pm \frac{1}{2}(1 + K_0^2)\varepsilon R(\varepsilon, K_0)$  and the frequency of an oscillator  $\omega_{osc} = \dot{\Phi} - \dot{\Psi} = \pm K_0^2 \varepsilon R(\varepsilon, K_0)$  deviates from that of the mean field. This solution is a circular stable limit cycle in the system (40,41), it corresponds to a stable two-frequency torus in the full system.

As an example, in Fig. 12 we show the bifurcation diagram for system (40-42) for a particular choice of functions  $R = \text{const} > 0$  and  $\beta = \beta_0 + \varepsilon^2 K^2$ . The bifurcation lines are defined by the relations  $\beta(\varepsilon, 1) = (2k - 1)\pi/2$  with  $k = \dots, -2, -1, 0, 1, 2, \dots$  and  $\beta(\varepsilon, 0) = \pm\pi/2$ . In addition to synchronous ( $K = 1$ ), asynchronous ( $K = 0$ ), and self-organized quasiperiodic regimes ( $0 < K < 1$ ) one observes different types of multistability.

### 5.5 Solution $K = \text{const}$ as an attractive state

We were able to provide a complete analysis of the self-organized quasiperiodic dynamics only in thermodynamical limit and only for the solution with a constant in time order parameter. Numerical simulations indicate that such a solution with  $K = \text{const}$  seemingly exists only in this limit  $N \rightarrow \infty$  and only for a uniform distribution of variables  $\psi$ . The latter can be implemented if the initial distribution of phases  $\varphi$  is chosen to be a uniform one, as has been done in the numerical examples in Section 4. Now we argue that this particular solution is however rather important. Below we present a numerical evidence that this state is in some sense “attractive” for general initial distributions of  $\varphi_k$ , if random perturbations of the dynamics are taken into account. In other

words, with the following example we illustrate a natural expectation that in the presence of noise, variables  $\psi_k$ , which are not constants any more, spread almost uniformly. Note that the random perturbations have to be different for different oscillators, because a common noise preserves the time invariance of  $\psi_k$ .

For this purpose we consider model (9) with  $R = 1$  and  $\beta = \frac{\pi}{2}\varepsilon^2 k^2$ , for  $\varepsilon = 1.5 > \varepsilon_q = 1$ . It is easy to see that for this model the amplitude of the time-independent mean field solution is  $K = 2/3$ . Initial distribution of phases was taken in the following way: for  $k < 3N/4$ ,  $\varphi_k = \frac{\pi}{2N}k + 0.1\frac{2\pi}{N}\sin(\sqrt{2}k)$  and for  $3N/4 \leq k \leq N$ ,  $1.2\pi + \frac{\pi}{2N}k + 0.1\frac{2\pi}{N}\sin(\sqrt{2}k)$ ; the number of oscillators was  $N = 200$ . First we have run a simulation with this set of initial phases; it yields a time-dependent solution with the mean field amplitude oscillating in a range  $0.5 \lesssim K \lesssim 0.75$ , shown in Fig. 13a. This state corresponds to a nonuni-

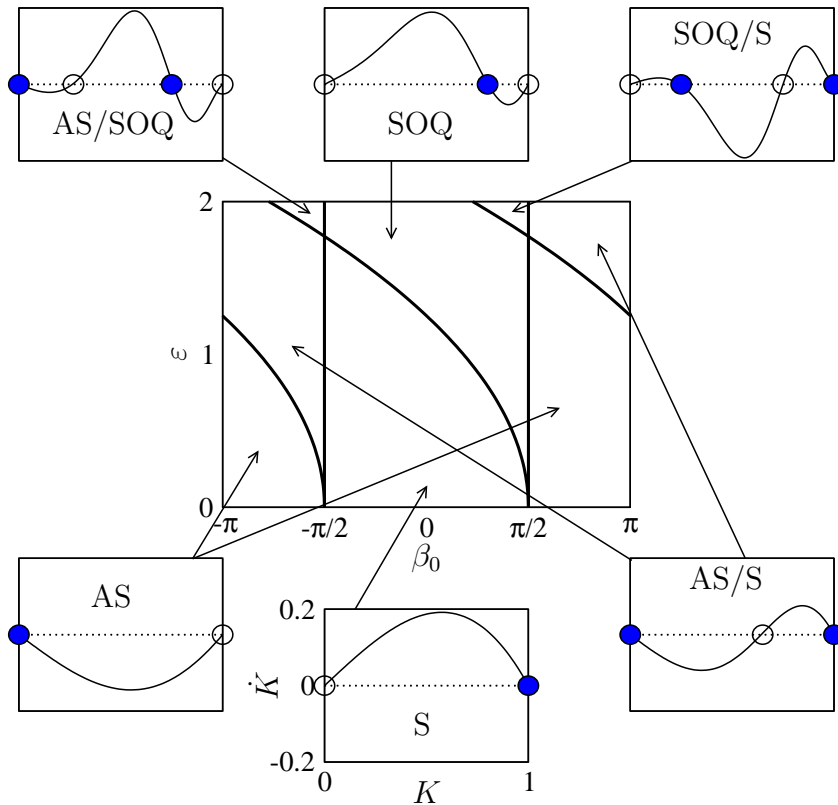


Fig. 12. The bifurcation diagram for system (40-42) on the plane  $(\beta_0, \varepsilon)$ , for the particular example  $R = \text{const}$  and  $\beta = \beta_0 + \varepsilon^2 K^2$ . Small panels depict  $\dot{K}$  vs  $K$ , according to Eq. (40). (Note, that arrows do not exactly point to the parameter values, used for computation of  $\dot{K}$  vs  $K$ , but to domain(s) where the qualitative behavior of this function is as shown in the panel.) Stable and unstable fixed points of this equation are shown by filled and open circles, respectively. Stable points with  $0 < K < 1$  correspond to SOQ state. S and AS mean synchronous and asynchronous regimes, correspondingly. AS/S denotes bistability synchrony – asynchrony, etc.

form distribution of the constants  $\psi_k$ , shown in Fig. 13b. We have checked the stability of this configuration by calculating the variables  $\psi_k$  numerically (see Sec. 4.2 in [18]) from the initial distribution of phases  $\varphi_k$  and from the distribution at the final integration time  $t = 2 \cdot 10^7$ , these sets of  $\psi_k$  practically coincide (the difference is less than 0.0145).

Next, we have run a simulation with the same set of initial conditions for  $\varphi_k$ , but with small random kicks added in the course of the evolution. At each kick  $\varphi_k \rightarrow \varphi_k + \delta\eta_k$ , where  $\eta_k$  are independent Gaussian random variables with unit variance and zero mean value; the interval between kicks was  $\Delta t = 2$ . The results for  $\delta = 5 \cdot 10^{-5}$  and  $\delta = 10^{-4}$  are shown in Fig. 13 c and e, respectively. We see, that on a very long time scale  $\sim 10^7$  the large-amplitude oscillations of the mean field amplitude decay and finally  $K$  fluctuates in a small range near  $K = 2/3$ . In the stochastically perturbed system, the quantities  $\psi_k$  are not constants of motion any more, but evolve slowly towards a nearly uniform distribution in the range  $(-\pi, \pi)$  (Figs. 13d,f). This numerical experiment shows that the state with a uniform distribution of  $\psi_k$  is weakly attracting if random perturbations are present.

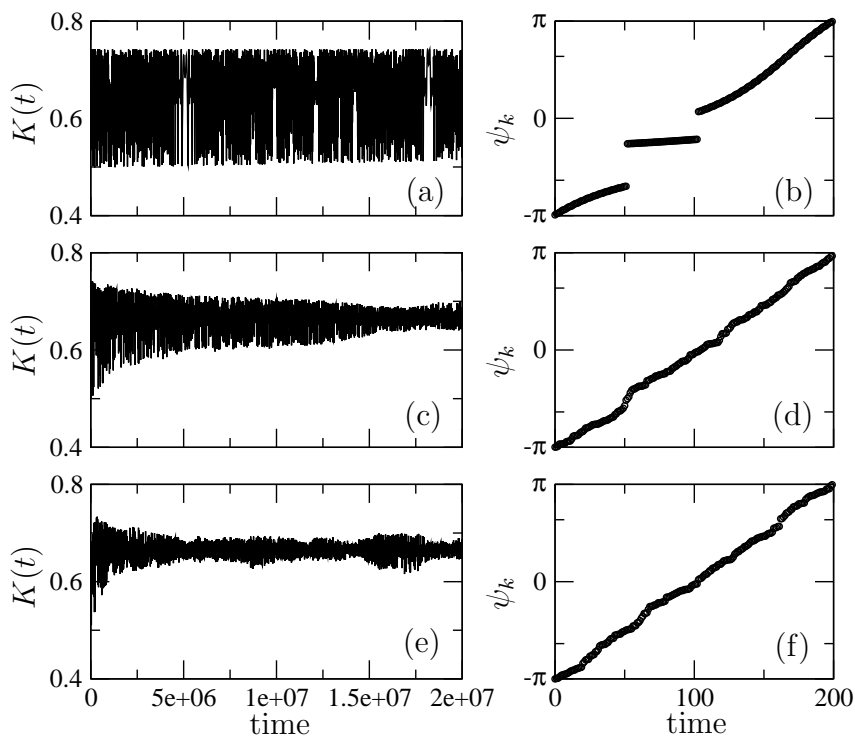


Fig. 13. (a,c,e) Evolution of  $K(t)$  for a particular nonuniform initial distribution of phases  $\varphi_k$ , obtained via direct numerical simulations, with additional random kicks. (b,d,f) The distributions of the variables  $\psi_k$  at the end of the integration interval. The amplitude of kicks is  $\delta = 0$  (a,b),  $\delta = 5 \cdot 10^{-5}$  (c,d), and  $\delta = 10^{-4}$  (e,f). For  $\delta = 0$  (b) the distribution is the same as the initial one; for  $\delta = 5 \cdot 10^{-5}$  and  $\delta = 10^{-4}$  the final distribution of  $\psi_k$  is nearly uniform.



## 6 Conclusions and outlook

In this paper we presented a complete analysis of the minimal model for an ensemble of nonlinearly coupled identical phase oscillators. The model itself is a direct generalization of the famous Kuramoto model to the case of a nonlinear coupling. The latter is not an exotic phenomenon, but naturally arises, e.g., in ensembles of elements, coupled via a common nonlinear unit [12] and can be expected in many cases, when the coupling is relatively strong. We emphasize here that the nonlinearity of coupling does not mean that one must go beyond the phase description of the dynamics. The nonlinear effects, illustrated in Fig. 1, can occur while the amplitude of the oscillator is enslaved and the phase description is well-justified. Nonlinearity of coupling can appear in the equations of motion explicitly, via higher powers of the mean field, like in Eq. (12), or implicitly. In the latter case the force acting on an oscillator is described by linear terms in the equations for state variables. However, because the force is strong enough, its effect can be nonlinear. Moreover, the effect of nonlinearity and an onset of quasiperiodic dynamics can be demonstrated for chaotic oscillators as well. The corresponding examples will be presented elsewhere.

Our theoretical analysis heavily relies on the seminal paper by Watanabe and Strogatz [18], who have found an analytical solution for a model of identical and identically forced oscillators. We have demonstrated, that when nonlinearity in the coupling is present, new quasiperiodic solutions of the Watanabe-Strogatz equations appear. These solutions describe the regime of self-organized quasiperiodicity in the population of oscillators, where individual oscillators are not locked by the mean field, but oscillate with a frequency incommensurate with that of the mean field. Similar states have been studied for integrate-and-fire oscillators in [13,14]. The self-organized bunch states correspond to a set of non-trivial equilibrium states in the Watanabe-Strogatz equations.

Our numerical and theoretical analysis was restricted to the case of identical oscillators. The corresponding desynchronization transition can be viewed at as a quantum phase transition. Preliminary numerical results show that quasiperiodic dynamics can be observed for the case of nonidentical oscillators as well, what corresponds to a transition at a finite temperature. This subject, however, requires a separate investigation. As another issue of ongoing research we mention, that in a more general context, the effects of nonlinear coupling appear to be relevant not only for populations of all-to-all coupled oscillators, but also for regular and complex networks.

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